Scale function approach to exit problems of refracted Lévy risk processes

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Master's thesis in mathematics

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2018

Abstract: Johannes Kohal, *Scale function approach to exit problems of refracted Lévy risk processes*, Master's thesis in mathematics, 54 pages, University of Jyväskylä, Department of Mathematics and Statistics, autumn 2018.

In this masters's thesis we start by introducing a family of functions called scale functions. Scale functions are defined with the help of Laplace transform and they are unique with respect to the corresponding spectrally negative Lévy process. Spectrally negative Lévy processes are Lévy processes which can only jump downwards. With the help of scale functions we inspect the behaviour of spectrally negative Lévy processes. Principally in the case of spectrally negative Lévy processes we are interested in problems concerning boundary crossings, called exit problems, of intervals $[a, b]$ and $[b, c]$, where $a, b, c \in \mathbb{R}$ and $a \leq b \leq c$. Hence we present results for exit problems of spectrally negative Lévy processes in which we use scale functions.

After this we start consider refracted Lévy processes and connect those to spectrally negative Lévy processes and scale functions. In our case a refracted Lévy process is defined such that it is a process which has been deducted by a linear drift when it is above a pre-specified level. Mathematically we define a refracted Lévy process to be the strong solution of the stochastic differential equation

$$\mathrm{d}U_t = \mathrm{d}X_t - \alpha \mathbb{1}_{\{U_t > b\}} \mathrm{d}t, \text{ for } t \geq 0,$$

where $X = (X_t)_{t \geq 0}$ is a spectrally negative Lévy process and $b \in \mathbb{R}$. For fixed $X_0 = x$ we have that a unique strong solution exists and the solution is a strong Markov process. If we define the spectrally negative Lévy process $Y = (Y_t)_{t \geq 0}$ such that $Y_t := X_t - \alpha t$, we get a stochastic differential equation which is equivalent to the earlier stochastic differential equation:

$$\mathrm{d}U_t = \mathrm{d}Y_t + \alpha \mathbb{1}_{\{U_t < b\}} \mathrm{d}t, \text{ for } t \geq 0.$$

We derive the equalities considering the exit problems of refracted Lévy process by using the scale functions of the corresponding spectrally negative Lévy processes X and Y.

In the main theorem we give the representations for the joint Laplace transforms of

$$\left(\kappa_a^-, \int_0^{\kappa_a^-} \mathbb{1}_{\{U_s < b\}} \mathrm{d}s \right) \text{ and } \left(\kappa_c^+, \int_0^{\kappa_c^+} \mathbb{1}_{\{U_s < b\}} \mathrm{d}s \right),$$

where κ_a^- and κ_c^+ are optional times of the exit problems of interval $[a, c]$ related to the refracted Lévy process $U = (U_t)_{t \geq 0}$ and $a \leq b \leq c$. From these representations we derive an expression for the expected time that the process U spends inside the set $[a, b)$ before crossing c or a. Finally we present a risk model, in which we can apply the main result. In the risk model we consider the set $[a, b)$ to be the red zone, which we consider to be the set inside of which the company, which financial situation the refracted Lévy process U represents, is in financial distress. This paper is based on the Jean-François Renaud's paper On the time spent in the red by a refracted Lévy risk process (J. Appl. Prob. **51**, 1171-1188 (2014)).

Tiivistelmä: Johannes Kohal, *Skaalafunktioiden käyttö taivutettujen Lévy-riskiprosessien poistumisongelmissa*, matematiikan pro gradu -tutkielma, 54 sivua, Jyväskylän yliopisto, Matematiikan ja tilastotieteen laitos, syksy 2018.

Opinnäytetyö aloitetaan esittelemällä skaalafunktiot (englanniksi *scale function*). Skaalafunktiot määritellään Laplace-muunnoksen avulla ja ne ovat yksikäsitteisiä annetun spektraalisti negatiivisen Lévy-prosessin suhteen. Spektraalisti negatiivinen Lévy-prosessi (englanniksi *spectrally negative Lévy process*) on Lévy-prosessi, jolla on vain negatiivisia hyppyjä. Skaalafunktioiden avulla tutkitaan spektraalisti negatiivisten Lévy-prosessien käyttäytymistä. Tässä työssä keskitytään pääasiassa tutkimaan spektraalisti negatiivisten Lévy-prosessien arvojoukkovälien $[a, b]$ ja $[b, c]$, missä $a, b, c \in \mathbb{R}$ ja $a \leq b \leq c$, päätepisteiden ylityksiin liittyviä ongelmia eli poistumisongelmia (englanniksi *exit problems*). Tästä syystä työssä esitetään tuloksia spektraalisti negatiivisten Lévy-prosessien poistumisongelmille, joissa käytetään skaalafunktioita.

Tämän jälkeen käsitellään taivutettuja Lévy-prosesseja ja nämä yhdistetään spektraalisti negatiivisiin Lévy-prosesseihin ja skaalafunktioihin. Taivutettu Lévy-prosessi (englanniksi *refracted Lévy process*) määritellään tässä työssä siten, että se on prosessi, jota taivutetaan alaspäin lineaarisen suoran avulla prosessin ollessa annetun arvon yläpuolella. Matemaattisesti taivuttu Lévy-prosessi määritellään olevan stokastisen differentiaaliyhtälön

$$\mathrm{d}U_t = \mathrm{d}X_t - \alpha \mathbb{1}_{\{U_t > b\}} \mathrm{d}t, \quad t \geq 0,$$

missä $X = (X_t)_{t \geq 0}$ on spektraalisti negatiivinen Lévy-prosessi ja $b \in \mathbb{R}$, vahva ratkaisu. Annetulla alkuarvolla $X_0 = x$ yksikäsitteinen ratkaisu on olemassa ja ratkaisu on vahva Markov-prosessi. Mikäli määritellään spektraalisti negatiivinen Lévy-prosessi $Y = (Y_t)_{t \geq 0}$ siten, että $Y_t := X_t - \alpha t$, saadaan yllä olevan stokastisen differentiaaliyhtälön kanssa ekvivalentti stokastinen differentiaaliyhtälö, joka on

$$\mathrm{d}U_t = \mathrm{d}Y_t + \alpha \mathbb{1}_{\{U_t < b\}} \mathrm{d}t, \quad t \geq 0.$$

Taivutettuihin Lévy-prosesseihin liittyvien poistumisongelmien yhtälöt johdetaan käyttämällä vastaavien spektraalisti negatiivisten Lévy-prosessien X ja Y skaalafunktioita.

Työn päätulos antaa satunnaismuuttujien

$$\left(\kappa_a^-, \int_0^{\kappa_a^-} \mathbb{1}_{\{U_s < b\}} \mathrm{d}s \right) \text{ ja } \left(\kappa_c^+, \int_0^{\kappa_c^+} \mathbb{1}_{\{U_s < b\}} \mathrm{d}s \right),$$

yhdistettyjen Laplace-muunnosten esitykset, missä κ_a^- ja κ_c^+ ovat taivutetun Lévy-prosessin $(U_t)_{t \geq 0}$ välin $[a, c]$ poistumisongelmiin liittyvät optionaaliset hetket (englanniksi *optional times*) ja $a \leq b \leq c$. Näistä esityksistä johdetaan yhtälö odotetulle ajalle, jonka prosessi U viettää välillä $[a, b)$ ennen arvon c ylittämistä tai arvon a alittamista. Lopuksi esitellään riskimalli, jossa voidaan soveltaa työn päätulosta. Riskimallissa väliä $[a, b)$ pidetään punaisena alueena. Kun prosessi U on tämän alueen sisällä, sanotaan, että yritys, jonka taloudellista tilannetta prosessi U mallintaa, on taloudellisessa ahdingossa. Tämä työ pohjautuu Jean-François Renaud'n työhön On the time spent in the red by a refracted Lévy risk process (J. Appl. Prob. **51**, 1171-1188 (2014)).

Contents

Introduction

In classical ruin theory it is usually assumed that ruin occurs if the given risk process falls too low. In many cases it is said that the risk process falls too low if it falls below 0. A way to refine the classical ruin definition is to define a boundary b such that ruin happens if the risk process spends too much time below b. If we consider the case that the given risk process describes the financial situation of some company it is natural to think that when the risk process is below b some kind of restructuring should happen. Since in classical ruin theory the restructuring is not considered, we want to build such a model where restructuring can occur. This gives us the motivation to study refracted Lévy processes.

A refracted Lévy process is introduced as a process which dynamics is changed while it is above a pre-specified level by subtracting off a fixed linear drift. Formally we define that the refracted Lévy process is the strong solution of the stochastic differential equation

$$\mathrm{d}U_t = \mathrm{d}X_t - \alpha \mathbb{1}_{\{U_t > b\}} \mathrm{d}t, \tag{0.1}$$

where b is now the pre-specified level above of which we subtract off the linear drift, and X is spectrally negative Lévy process. For fixed $X_0 = x$ we have that the strong solution exists and it is a strong Markov process. Another way to define the process U is

$$\mathrm{d}U_t = \mathrm{d}Y_t + \alpha \mathbb{1}_{\{U_t < b\}} \mathrm{d}t, \tag{0.2}$$

where $Y_t = X_t - \alpha t$. A consequence from (0.1) and (0.2) is that the process U is described by the process X when it is below b, and by the process Y when it is above b. Clearly the definitions (0.1) and (0.2) are equivalent when we define Y as above. The only difference is that on (0.1) the process U is refracted above b and on (0.2) it is refracted when it is below b.

The processes which are of the type as U are used to describe for example an insurance company's surplus. Refraction in this situation can thought to be the consequence from, for example, the following action: when the insurance company is in financial distress, that is when the process is above the level of which ruin occurs but below b, it increases its premiums. When the company is in financial distress, we say that the risk process U is in the red zone, which is the interval $[0, b)$ if we assume that ruin occurs when the risk process falls under 0, as in classical ruin theory.

The main results of this paper gives us representations for the joint Laplace transforms of

$$\left(\kappa_a^-, \int_0^{\kappa_a^-} \mathbb{1}_{\{U_s < b\}} \mathrm{d}s\right) \text{ and } \left(\kappa_c^+, \int_0^{\kappa_c^+} \mathbb{1}_{\{U_s < b\}} \mathrm{d}s\right), \tag{0.3}$$

where $a \le b \le c$, and $\kappa_a^- := \inf\{t \ge 0 : U_t < a\}$ and $\kappa_c^+ := \inf\{t \ge 0 : U_t > c\}$ are the optional times of the refracted Lévy process U. With help of the representations in (0.3) we derive the ruin probability of the risk process U.

To get knowledge about the behaviour of refracted Lévy processes, namely about the representations in (0.3), we need to inspect the corresponding spectrally negative Lévy processes. Spectrally negative Lévy processes are defined to be Lévy processes which can only jump downwards. This type of processes are very commonly used in risk theory. Because the refracted Lévy process changes its dynamics we are interested in exit problems of the corresponding spectrally negative Lévy processes, which are used in (0.1) and (0.2), with which we can investigate the boundary crossing, especially for the boundaries of the red zone. In the results for the exit problems we are constantly using a function class called scale functions. This class of functions is defined with help of Laplace transform. One of the properties that scale functions possess is that they are unique with respect to the corresponding spectrally negative Lévy process. This function class is well studied and has many applications in risk theory.

The paper is organised in the following way. On the first chapter we go through the basics of the probability theory and give the important base knowledge regarding to our main theory. In the beginning of the second chapter we introduce scale functions and give an example how to compute the scale function of given spectrally negative Lévy process. After that we start to consider the exit problems of spectrally negative Lévy processes and proceed with exit problems of refracted Lévy processes. In the final chapter, chapter three, we give our main result which gives us the time spent in the red zone. After that we present a risk model and define the probability of bankruptcy. In the computations of the probability of bankruptcy we apply the main result. This paper is mainly based on the paper of Jean-François Renaud [**10**]. Also the book of Kyprianou [**6**] and results from Kyprianou and Loeffen [**7**] are strongly used.

CHAPTER 1

Preliminaries

We start with introducing some of the basic concepts of probability theory. For more information about the basics of probability theory see for example [**4**], [**9**] and [**1**].

Definition 1.1 (σ-algebra). Let Ω be a non-empty set. A collection $\mathcal{F}$ of subsets of Ω is called a σ-algebra if the following assertions are true:

(i) $\Omega \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$.
(ii) If $A \in \mathcal{F}$ then $A^{\complement} \in \mathcal{F}$.
(iii) If $A_1, A_2, A_3 \ldots \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ such as in Definition 1.1 is called measurable space.

Definition 1.2 (Probability measure). Let $(\Omega, \mathcal{F})$ be a measurable space. A map $\mathbb{P} : \mathcal{F} \longrightarrow [-\infty, \infty]$ is called a probability measure if following assertions are true:

(i) $\mathbb{P}(A) \geq 0$ for all $A \in \mathcal{F}$.
(ii) $\mathbb{P}(\emptyset) = 0$.
(iii) $\mathbb{P}(\Omega) = 1$.
(iv) $\mathbb{P}\big(\bigcup_{n=1}^{\infty} A_n\big) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$ for $A_1, A_2, A_3, \ldots \in \mathcal{F}$ such that $A_i \cap A_j = \emptyset$ when $i \neq j$.

A map $\mathbb{P}$ from definition 1.2 is called measure, if the condition (*iii*) is omitted, and is usually denoted by μ in the literature. The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ such that it satisfies Definitions 1.1 and 1.2 is called probability space.

Definition 1.3 (σ-finite measure). Let $(\Omega, \mathcal{F})$ be a measurable space and μ be a measure on Ω. The measure μ is called σ-finite if there exists a countable set $(E_i)_{i \in I} \subseteq \mathcal{F}$ such that $\mu(E_i) < \infty$ for all $i \in I$ and $\bigcup_{i \in I} E_i = \Omega$.

Definition 1.4 (Filtration). Let $(\Omega, \mathcal{F})$ be a measurable space. The sequence of σ-algebras $(\mathcal{F}_t)_{t \geq 0}$ such that $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \geq 0$ and $\mathcal{F}_t \subseteq \mathcal{F}_s$ for all $s \geq t \geq 0$ is called a filtration of $\mathcal{F}$.

Definition 1.5 (Stochastic basis). A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $(\mathcal{F}_t)_{t \geq 0}$ of $\mathcal{F}$ is called stochastic basis and denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

Definition 1.6 (Random variable). Let $(\Omega, \mathcal{F})$ be a measurable space. A map $X : \Omega \longrightarrow \mathbb{R}$ is called a random variable if $\{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$ for all $B \in \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ denotes a Borel σ-algebra on $\mathbb{R}$.

We denote the expected value of a random variable X by $\mathbb{E}[X]$. The definition and the properties of the expected value, that are used in this paper, can be found for example in [**1**, Chapter 2]. For a random variable X and an event A, we define $\mathbb{E}[X; A] := \mathbb{E}[X \mathbb{1}_A]$.

Definition 1.7 (Stopping time). A map $\tau : \Omega \longrightarrow [0, \infty) \cup \{\infty\}$ is called a $(\mathcal{F}_t)_{t \geq 0}$-stopping time if $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Definition 1.8 (Optional time). A map $\tau : \Omega \longrightarrow [0, \infty) \cup \{\infty\}$ is called a $(\mathcal{F}_t)_{t \geq 0}$-optional time if $\{\omega \in \Omega : \tau(\omega) < t\} \in \mathcal{F}_t$ for all $t \geq 0$.

1.1. Processes and properties

In this section we define the processes that will be used. For more information about the used processes see for example [**2**]. Let us begin with the definition of a stochastic process.

Definition 1.9 (Stochastic process). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and I an arbitrary set. A family $X := \{X_t; t \in I\}$, where $X_t : \Omega \longrightarrow \mathbb{R}$ is a random variable for all $t \in I$, is called a stochastic process with index set I.

Remark 1.10. If $X_0(\omega) = x$ for all $\omega \in \Omega$ we denote the law of the process $X = (X_t)_{t\geq 0}$ by $\mathbb{P}_x$ and its corresponding expectation by $\mathbb{E}_x$.

From now on we use $[0, \infty)$ or the extended positive real line $[0, \infty]$ as the index set of stochastic processes. Let us now present the Lévy process. This process is used for example in risk modelling and in that context it is called a Lévy risk process. The following definition and other properties of Lévy processes can be found in [**2**, Chapter 3].

Definition 1.11 (Lévy process). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A right-continuous stochastic process $X := (X_t)_{t\geq 0}$ with $X_0 = 0$, which has left limits, is called a Lévy process if the following assertions are true:

(i) Independent increments: for every increasing sequence of times $t_0, t_1, \dots, t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.
(ii) Stationary increments: for any $h \geq 0$, the law of $X_{t+h} - X_t$ does not depend on $t \geq 0$.

The most well-known examples of Lévy processes are Brownian motion and Poisson process. They are defined as follows.

Definition 1.12 (Brownian motion). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A stochastic process $B := (B_t)_{t\geq 0}$, with $B_0 = 0$, is called Brownian motion if the following assertions are true:

(i) B is continuous.
(ii) B has independent increments.
(iii) $B_t - B_s \sim \mathcal{N}(0, t - s)$ for all $0 \leq s \leq t$.

Definition 1.13 (Poisson process). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(Y_i)_{i\geq 1}$ be a sequence of independent exponential random variables with parameter $\lambda > 0$ and $T_n = \sum_{i=1}^n Y_i$. The process $(N_t)_{t\geq 0}$ defined by

$$N_t := \sum_{n\geq 1} \mathbb{1}_{\{T_n \leq t\}} = \max\{n; T_n \leq t\}, \quad t \geq 0,$$

is called a Poisson process with intensity λ.

With use of the above two stochastic processes it is easy to get examples of other Lévy processes. In Lévy risk models the jump part of a given Lévy risk process can be described by a compound Poisson process. It is defined as follows.

Definition 1.14 (Compound Poisson process). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A stochastic process $X := (X_t)_{t\geq 0}$ defined as

$$X_t := \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0,$$

where jump sizes ξ_i are independent and identically distributed with distribution $f = \mathbb{P}_{\xi_1}$ and $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ, independent from $(\xi_i)_{i\geq 1}$, is called a compound Poisson process with intensity λ and jump size distribution f.

For the next theorem we define the natural filtration of a process X.

Definition 1.15 (Natural filtration). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X := (X_t)_{t\geq 0}$ be a stochastic process. The filtration $\mathbb{F}^X := \{\mathcal{F}_t^X; t \geq 0\}$, where $\mathcal{F}_t^X := \sigma\{X_s; 0 \leq s \leq t\}$, is called the natural filtration of the process X.

The next theorem follows from Definition 1.11. This is one of the most used properties of Lévy processes here.

Theorem 1.16 (Strong Markov property for a Lévy process). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X := (X_t)_{t\geq 0}$ a Lévy process and assume that τ is an $\mathbb{F}^X$-optional time. Define the process $Z := (Z_t)_{t\geq 0}$ by*

$$Z_t := \mathbb{1}_{\{\tau<\infty\}}(X_{t+\tau} - X_\tau).$$

Then on $\{\tau < \infty\}$ the process Z is independent of $\mathcal{F}_{\tau+}^X$ and Z has the same distribution as X.

Proof. See for example [**6**, Theorem 3.1]. □

What earlier theorem says is that the distribution of the given Lévy process does not change if we shift time and the conditional probability distribution of future states depends only upon the present state of the process. The general formulation of the strong Markov property is as follows. This is a modification of [**4**, Definition 6.2].

Definition 1.17 (Strong Markov property). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(M, \mathcal{M})$ a measurable space and $X := (X_t)_{t\geq 0}$ a stochastic process such that $X_t : \Omega \longrightarrow M$ for all $t \geq 0$ and it is adapted to the filtration $\mathbb{F}^X$. If for all $A \in \mathcal{M}$, $\mathbb{F}^X$-optional times τ and $t \geq 0$ it holds almost surely that

$$\mathbb{1}_{\{\tau<\infty\}}\mathbb{P}(X_{\tau+t} \in A|\mathcal{F}_{\tau+}^X) = \mathbb{1}_{\{\tau<\infty\}}\mathbb{P}(X_{\tau+t} \in A|X_\tau),$$

then X is called strong Markov.

1.2. Laplace transforms

In the next chapter we need the Laplace transform of a function. In general the range of a Laplace transform is a subset of the complex plane. Later on we see that in our case only the real valued codomain occurs. Let us begin with the definition of the Laplace transform.

DEFINITION 1.18 ([**9**], Chapter 1, Section 1.2). Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be a function. Then the Laplace transform of f is defined by

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-sx} f(x) \mathrm{d}x$$

for the range of values of $s \in \mathbb{C}$ for which the integral exists as a finite number.

If we have that F is the Laplace transform of f, then f is called the inverse Laplace transform of F. There are many known results for both Laplace transform and inverse Laplace transform which can be found in the literature. For a random variable X the Laplace transform is defined as follows.

DEFINITION 1.19 (Laplace (Laplace-Stieltjes) transform of a probability distribution or of a random variable). Let X be a non-negative random variable with distribution function

$$f_X(x) = \mathbb{P}(X \leq x).$$

The Laplace transform F_X of the distribution function f_X is defined for $s \geq 0$ by

$$F_X(s) = \int_0^\infty e^{-sx} \mathrm{d} f_X(x).$$

We shall see that Definition 1.19 also gives the "Laplace transform of a random variable X", when X has a density.

REMARK 1.20. For a non-negative random variable X one has that

$$F_X(s) = \mathbb{E}[e^{-sX}] = \int_0^\infty e^{-st} \mathrm{d}\mathbb{P}_X(t), \quad s \geq 0,$$

and $F_X(0) = 1$. For non-negative random variables, the Laplace transform $F_X(s)$ exists for all $s \geq 0$, and F_X uniquely describes the distribution of X. Suppose that X has a density $h_X(x) = f'_X(x)$. Then one has that

$$F_X(s) = \int_0^\infty e^{-sx} h_X(x) \mathrm{d}x, \quad s \geq 0. \tag{1.1}$$

This is the ordinary Laplace transform $\mathcal{L}\{h_X\}(s)$ from Definition 1.18 of the density function h_X.

1.3. Spectrally negative Lévy processes

In this section we consider the case that the Lévy process has no positive jumps. These type of processes are used for example in insurance risk models. In that case jumps usually present claims that a company gets. Let us begin the section with the following definition.

DEFINITION 1.21 (Spectrally negative Lévy process). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The process X is called a spectrally negative Lévy process if it satisfies the properties of Definition 1.11 and has no positive jumps.

Compound Poisson and gamma processes can be used to create spectrally negative Lévy processes. Later in this paper we are going to use a spectrally negative Lévy process which is of form $X_t = \mu t + \sigma B_t - S_t$, where $\mu, \sigma \in \mathbb{R}$, $(B_t)_{t \geq 0}$ is a Brownian motion and $(S_t)_{t\geq 0}$ is a compound Poisson process which jumps only up. Next we recall the moment generating function for a spectrally negative Lévy process.

PROPOSITION 1.22 (The moment generating function of a spectrally negative Lévy process). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(X_t)_{t\geq 0}$ a spectrally negative Lévy process. Then for X we have that

$$\mathbb{E}[e^{\lambda X_t}] < \infty \quad \text{and} \quad \mathbb{E}[e^{\lambda X_t}] = e^{t\psi(\lambda)}$$

for all $t, \lambda \geq 0$, where function $\psi : [0, \infty) \longrightarrow \mathbb{R}$ is of the form

$$\psi(\lambda) = \mu\lambda + \frac{\sigma^2\lambda^2}{2} + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x \mathbb{1}_{\{|x|<1\}})\Pi(\mathrm{d}x), \quad \lambda \geq 0, \tag{1.2}$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and Π is a σ-finite measure on $(0, \infty)$ such that $\int_0^\infty (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$.

PROOF. This follows from the Lévy-Khintchine formula for a Lévy process, which can be found in [**2**, Theorem 3.1], by change of variable $\lambda = \mathrm{i}\theta$. □

The function $\psi(\lambda)$ is called the Laplace exponent of the process X and it is unique and the triplet (μ, σ^2, Π) from Proposition 1.22 is called the Lévy triplet of the process X.

REMARK 1.23. The Lévy triplet (μ, σ^2, Π) uniquely defines the distribution of a Lévy process, if the same cut-off function $\mathbb{1}_{\{|x|<1\}}$ is used in the Lévy-Khintchine formula or in (1.2). The measure Π is called the Lévy measure. Notice that in general for Lévy processes, one would use the measure given by $-\Pi(-B)$, where $B \in \mathcal{B}(\mathbb{R}\backslash\{0\})$, as Lévy measure.

PROPOSITION 1.24. Assume that X is non-trivial spectrally negative Lévy process. For process X the Laplace exponent $\psi(\lambda)$ is strictly convex and $\lim_{\lambda\to\infty} \psi(\lambda) = \infty$ for $\lambda \geq 0$. Therefore there exists a function $\phi : [0, \infty) \to [0, \infty)$ defined by

$$\phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\} \tag{1.3}$$

so that $\psi[\phi(q)] = q$ for $q \geq 0$. This function is called inverse Laplace transform.

PROOF. One can verify the strict convexity of the Laplace exponent $\psi(\lambda)$ of a spectrally negative Lévy process by differentiating it twice and checking that $\psi''(\lambda) > 0$ for all $\lambda \geq 0$. This also implies that $\phi(q) = 0$ if and only if $q = 0$ and $\psi'(0+) > 0$. Note that convexity and $\lim_{\lambda\to\infty} \psi(\lambda) = \infty$ imply that there are at most two solutions to the equation $\psi(\lambda) = q$ for $\lambda \geq 0$. □

CHAPTER 2

Scale functions and exit problems

In this chapter we introduce scale functions. These functions will be our main tool when we start considering boundary crossings. After presenting the scale functions we continue with identities that create a connection between scale functions and exit problems. Our first observations considers the exit problems of spectrally negative Lévy processes. With the help of exit problems of spectrally negative Lévy processes we derive the equalities for exit problems of refracted Lévy processes. These identities are used to describe refracted Lévy risk processes. In the end of this chapter we give some useful results for the scale functions.

2.1. The concept of scale functions

Scale functions got their name from the analogous role they play in the identity

$$\mathbb{E}_x[e^{-q\tau_a^+}; \tau_a^+ < \tau_0^-] = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{2.1}$$

where τ_a^+ and τ_0^- are certain stopping times of the stochastic process $X := (X_t)_{t\geq 0}$ and $W^{(q)}$ is the q-scale function of X. This identity first appeared for the case $q = 0$ in [**11**]. Later in this chapter we prove a more general form of identity (2.1). Now let us begin with the definition of a scale function.

PROPOSITION 2.1 (Scale function). Let $X := (X_t)_{t\geq 0}$ be a spectrally negative Lévy process with the Lévy triplet (μ, σ^2, Π). Then there exists a unique function $W^{(q)} : \mathbb{R} \longrightarrow [0, \infty)$, $q \geq 0$, such that $W^{(q)}(x) = 0$ if $x < 0$, and otherwise $W^{(q)}$ is characterised as a strictly increasing and continuous function with Laplace transform

$$\int_0^\infty e^{-\lambda z} W^{(q)}(z)\mathrm{d}z = \frac{1}{\psi(\lambda) - q}, \quad \lambda > \phi(q), \tag{2.2}$$

where $\psi(\lambda)$ is the Laplace exponent of the process X and ϕ is defined as in (1.3).

PROOF. The proof can be found in [**6**, Theorem 8.1 (i)]. □

LEMMA 2.2. *For all $q \geq 0$, the initial value of $W^{(q)}(0) := \lim_{x\to 0^+} W^{(q)}(x)$ is*

$$W^{(q)}(0) = \begin{cases} \frac{1}{c}, & \text{when } \sigma = 0 \text{ and } \int_0^1 z\Pi(\mathrm{d}z) < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

where $c := \mu + \int_0^1 z\Pi(\mathrm{d}z)$.

PROOF. The proof can be found in [**6**, Lemma 8.6]. Note that in case $W^{(q)}(0) = \frac{1}{c}$ the process X has paths of bounded variation. Otherwise X has paths of unbounded variation. □

Occasionally we use the function $Z^{(q)}$ which is defined as

$$Z^{(q)}(x) := 1 + q\int_0^x W^{(q)}(z)\mathrm{d}z, \quad x \in \mathbb{R}. \tag{2.3}$$

The functions $\{W^{(q)} : q \geq 0\}$ and $\{Z^{(q)} : q \geq 0\}$ are called q-scale functions.
Now we give an example how to calculate the scale functions of a spectrally negative Lévy process.

Example 2.3 (Evaluating scale functions). This example follows from [**5**, Example 1.3]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X the spectrally negative Lévy process defined by

$$X_t := \sigma B_t + \mu t - \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0, \tag{2.4}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion, $(\xi_i)_{i=1}^{\infty}$ are independent, identically distributed random variables, which are exponentially distributed with parameter $\rho > 0$. The process $(N_t)_{t\geq 0}$ is a Poisson process independent from $(\xi_i)_{i=1}^{\infty}$ with intensity $a > 0$ and $\sigma > 0$ and $\mu \in \mathbb{R}$.

In order to calculate scale functions, we need first to find the Laplace exponent of the process X. We notice that the Lévy measure of X is finite, and hence we can replace the function $\mathbb{1}_{\{|x|<1\}}$ by $h(x) \equiv 0$ in the representation (1.2). This changes the value of μ into $\hat{\mu} = \mu + \int_0^1 x\Pi(\mathrm{d}x)$, but gives us a more simple expression of Laplace exponent of X, which is

$$\psi(z) = \hat{\mu} z + \frac{\sigma^2 z^2}{2} + \int_0^{\infty} (e^{-zx} - 1)\Pi(\mathrm{d}x) = \hat{\mu} z + \frac{\sigma^2 z^2}{2} - \frac{az}{\rho + z} \text{ for } z \geq 0. \tag{2.5}$$

We see that the equation $\psi(z) = q$ can be written as

$$\hat{\mu} z + \frac{\sigma^2 z^2}{2} - \frac{az}{\rho + z} - q = 0. \tag{2.6}$$

By considering the behavior of $\psi(z) - q$ we see that

$$\lim_{z\to\pm\infty} \psi(z) - q = \pm\infty, \quad \psi(-\rho) - q > 0 \quad \text{and} \quad \psi(0) - q < 0.$$

Therefore, there exist exactly three solutions for the equation (2.6). Clearly one solution is $\phi(q) \geq 0$, where ϕ is defined as in Proposition 1.24. Let the other solutions be $-\eta_1 < -\rho < -\eta_2 < 0 \leq \phi(q)$.

By using the solutions and partial fraction decomposition we get the representation

$$\frac{1}{\psi(z) - q} = \frac{1}{(z + \eta_1)(z + \eta_2)(z - \phi(q))} = \frac{C_1}{z + \eta_1} + \frac{C_2}{z + \eta_2} + \frac{C_3}{z - \phi(q)} \tag{2.7}$$

for some $C_1, C_2, C_3 \in \mathbb{R}$. Next we use the inverse Laplace transform which gives us that

$$W^{(q)}(x) = C_1 e^{-\eta_1 x} + C_2 e^{-\eta_2 x} + C_3 e^{\phi(q)x}. \tag{2.8}$$

Now we need to find C_1, C_2 and C_3. By using the second equality of (2.7) we get that

$$(z + \eta_2)(z - \phi(q))C_1 + (z + \eta_1)(z - \phi(q))C_2 + (z + \eta_1)(z + \eta_2)C_3 = 1.$$

Rewriting the equation into $az^2 + bz + c = 1$ leads us to the following simultaneous equations:

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ (\eta_2 - \phi(q))C_1 + (\eta_1 - \phi(q))C_2 + (\eta_1 + \eta_2)C_3 = 0 \\ \eta_1\eta_2 C_3 - \phi(q)\eta_2 C_1 - \phi(q)\eta_1 C_2 = 1. \end{cases}$$

Solving the equations gives that

$$\begin{cases} C_1 = [(\eta_1 - \eta_2)(\phi(q) + \eta_1)]^{-1} \\ C_2 = -[(\eta_1 - \eta_2)(\phi(q) + \eta_2)]^{-1} \\ C_3 = [(\phi(q) + \eta_1)(\phi(q) + \eta_2)]^{-1}. \end{cases}$$

This result combined with the first equality of (2.7) and the fact $\frac{\mathrm{d}}{\mathrm{d}z}\big(\psi(z) - q\big) = \psi'(z)$ gives us that $C_1 = \psi'(-\eta_1)^{-1}$, $C_2 = \psi'(-\eta_2)^{-1}$ and $C_3 = \psi'(\phi(q))^{-1}$.
Now we insert the expressions for C_1, C_2 and C_3 into the formula (2.8) and get that the scale functions of X are

$$W^{(q)}(x) = \frac{e^{-\eta_1 x}}{\psi'(-\eta_1)} + \frac{e^{-\eta_2 x}}{\psi'(-\eta_2)} + \frac{e^{\phi(q)x}}{\psi'(\phi(q))}, \quad x \geq 0$$

and

$$Z^{(q)}(x) = 1 - \frac{q(e^{-\eta_1 x} - 1)}{\eta_1 \psi'(-\eta_1)} - \frac{q(e^{-\eta_2 x} - 1)}{\eta_2 \psi'(-\eta_2)} + \frac{q(e^{\phi(q)x} - 1)}{\phi(q)\psi'(\phi(q))}, \quad x \geq 0.$$

One can calculate explicit expressions of $W^{(q)}(x)$ and $Z^{(q)}(x)$ by applying Cardano's formula for (2.6) and get the exact values of $-\eta_1$, $-\eta_2$ and $\phi(q)$. Recall that $\hat{\mu}$ is implied by the representation (2.5).

Other examples of scale functions can be found for example in [**3**, Chapter 2]. Let us now start considering the exit problems of the hitting times $\tau_c^+ := \inf\{t > 0 : X_t > c\}$ and $\tau_a^- := \inf\{t > 0 : X_t < a\}$. Note that in general the first hitting time of an open set is an optional time. For the proof see for example [**4**, Problem 2.6]. The next theorem gives us the first identity that connects scale functions to exit problems. It is a generalization of equality (2.1).

Theorem 2.4 (Solution to the two-sided exit problem $\tau_c^+ < \tau_a^-$). *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis and X a spectrally negative Lévy process and define the following optional times*

$$\tau_a^- := \inf\{t > 0 : X_t < a\}$$
$$\tau_c^+ := \inf\{t > 0 : X_t > c\}$$

for $a, c \in \mathbb{R}$. For all $a \leq x \leq c$ the solution to the two-sided exit problem for X is given by

$$\mathbb{E}_x[e^{-q\tau_c^+}; \tau_c^+ < \tau_a^-] = \frac{W^{(q)}(x-a)}{W^{(q)}(c-a)}.$$

Proof. We follow here the proofs of [**6**, Theorem 8.1 (i) and (iii)], and show the result for the case $q = 0$ and $\psi'(0+) > 0$, which implies that X drifts to $+\infty$. For case $x = c$, we have that

$$\mathbb{E}_x[e^{-q\tau_c^+}; \tau_c^+ < \tau_a^-] = 1 = \frac{W^{(q)}(c-a)}{W^{(q)}(c-a)}.$$

Hence we can assume, that $a \le x < c$. Let $\underline{X}_\infty := \inf_{t\ge 0} X_t$ and define the increasing function

$$x \longmapsto H(x-a) := \frac{\mathbb{P}_{x-a}(\underline{X}_\infty \ge 0)}{\psi'(0+)}. \tag{2.9}$$

Note that for process X we have that $\mathbb{P}(\underline{X}_\infty \le 0) = 1$. In this case, by using the tower property and the law of total probability, we get that

$$\begin{aligned} H(x-a)\psi'(0+) &= \mathbb{P}_{x-a}(\underline{X}_\infty \ge 0) = \mathbb{P}_x(\underline{X}_\infty \ge a) \\ &= \mathbb{P}_x(\inf_{t\ge 0} X_t \ge a) = \mathbb{E}_x\big[\mathbb{P}_x(\inf_{t\ge 0} X_t \ge a | \mathcal{F}_{(\tau_c^+)+})\big] \\ &= \mathbb{E}_x\big[\mathbb{1}_{\{\tau_c^+ < \tau_a^-\}} \mathbb{P}_x(\inf_{t\ge \tau_c^+} X_t \ge a, \inf_{t<\tau_c^+} X_t \ge a | \mathcal{F}_{(\tau_c^+)+})\big] \\ &\quad + \mathbb{E}_x\big[\mathbb{1}_{\{\tau_a^- < \tau_c^+\}} \mathbb{P}_x(\inf_{t\ge \tau_a^-} X_t \ge a, \inf_{t<\tau_a^-} X_t \ge a | \mathcal{F}_{(\tau_c^+)+})\big] \\ &= \mathbb{E}_x\big[\mathbb{1}_{\{\tau_c^+ < \tau_a^-\}} \mathbb{E}_x[\mathbb{1}_{\{\inf_{t\ge\tau_c^+} X_t \ge a, \inf_{t<\tau_c^+} X_t \ge a\}} | \mathcal{F}_{(\tau_c^+)+}]\big] \\ &\quad + \mathbb{E}_x\big[\mathbb{1}_{\{\tau_a^- < \tau_c^+\}} \mathbb{E}_x[\mathbb{1}_{\{\inf_{t\ge\tau_a^-} X_t \ge a, \inf_{t<\tau_a^-} X_t \ge a\}} | \mathcal{F}_{(\tau_c^+)+}]\big]. \end{aligned} \tag{2.10}$$

The fact that $\mathbb{1}_{\{\tau_c^+ < \tau_a^-\}}$ and $\mathbb{1}_{\{\tau_a^- < \tau_c^+\}}$ are $\mathcal{F}_{(\tau_c^+)+}$-measurable, see [**4**, Lemma 2.15], combined with the tower property, gives us, since $\mathcal{F}_{\tau_a^-} \subseteq \mathcal{F}_{(\tau_c^+)+}$ when $\tau_a^- < \tau_c^+$, that

$$\begin{aligned} &\mathbb{E}_x\big[\mathbb{1}_{\{\tau_c^+ < \tau_a^-\}} \mathbb{E}_x[\mathbb{1}_{\{\inf_{t\ge\tau_c^+} X_t \ge a, \inf_{t<\tau_c^+} X_t \ge a\}} | \mathcal{F}_{(\tau_c^+)+}]\big] \\ &\quad + \mathbb{E}_x\big[\mathbb{1}_{\{\tau_a^- < \tau_c^+\}} \mathbb{E}_x[\mathbb{1}_{\{\inf_{t\ge\tau_a^-} X_t \ge a, \inf_{t<\tau_a^-} X_t \ge a\}} | \mathcal{F}_{(\tau_c^+)+}]\big] \\ &= \mathbb{E}_x\big[\mathbb{E}_x[\mathbb{1}_{\{\tau_c^+ < \tau_a^-\}} \mathbb{1}_{\{\inf_{t\ge\tau_c^+} X_t \ge a, \inf_{t<\tau_c^+} X_t \ge a\}} | \mathcal{F}_{(\tau_c^+)+}]\big] \\ &\quad + \mathbb{E}_x\big[\mathbb{E}_x[\mathbb{1}_{\{\tau_a^- < \tau_c^+\}} \mathbb{1}_{\{\inf_{t\ge\tau_a^-} X_t \ge a, \inf_{t<\tau_a^-} X_t \ge a\}} | \mathcal{F}_{\tau_a^-}]\big]. \end{aligned}$$

From the definitions of τ_a^- and τ_c^+ it follows that the event $\{\inf_{t<\tau_a^-} X_t \ge a\} = \Omega$ and that $\{\tau_c^+ < \tau_a^-\} \subseteq \{\inf_{t<\tau_c^+} X_t \ge a\}$. Hence we get by using the strong Markov property of X and the fact, that $X_{\tau_c^+} = c$, that

$$\begin{aligned} &\mathbb{E}_x\big[\mathbb{E}_x[\mathbb{1}_{\{\tau_c^+ < \tau_a^-\}} \mathbb{1}_{\{\inf_{t\ge\tau_c^+} X_t \ge a, \inf_{t<\tau_c^+} X_t \ge a\}} | \mathcal{F}_{(\tau_c^+)+}]\big] \\ &\quad + \mathbb{E}_x\big[\mathbb{E}_x[\mathbb{1}_{\{\tau_a^- < \tau_c^+\}} \mathbb{1}_{\{\inf_{t\ge\tau_a^-} X_t \ge a, \inf_{t<\tau_a^-} X_t \ge a\}} | \mathcal{F}_{\tau_a^-}]\big] \\ &= \mathbb{E}_x\big[\mathbb{E}_x[\mathbb{1}_{\{\tau_c^+ < \tau_a^-\}} \mathbb{1}_{\{\inf_{t\ge\tau_c^+} X_t \ge a\}} | \mathcal{F}_{(\tau_c^+)+}]\big] + \mathbb{E}_x\big[\mathbb{E}_x[\mathbb{1}_{\{\tau_a^- < \tau_c^+\}} \mathbb{1}_{\{\inf_{t\ge\tau_a^-} X_t \ge a\}} | \mathcal{F}_{\tau_a^-}]\big] \\ &= \mathbb{E}_x\big[\mathbb{1}_{\{\tau_c^+ < \tau_a^-\}} \mathbb{E}_x[\mathbb{1}_{\{\inf_{t\ge\tau_c^+} X_t \ge a\}} | \mathcal{F}_{(\tau_c^+)+}]\big] + \mathbb{E}_x\big[\mathbb{1}_{\{\tau_a^- < \tau_c^+\}} \mathbb{E}_x[\mathbb{1}_{\{\inf_{t\ge\tau_a^-} X_t \ge a\}} | \mathcal{F}_{\tau_a^-}]\big] \\ &= \mathbb{E}_x\big[\mathbb{1}_{\{\tau_c^+ < \tau_a^-\}} \mathbb{E}_x[\mathbb{1}_{\{\inf_{s\ge 0}(X_{\tau_c^+ + s} - X_{\tau_c^+}) + c \ge a\}} | \mathcal{F}_{(\tau_c^+)+}]\big] + \mathbb{E}_x\big[\mathbb{1}_{\{\tau_a^- < \tau_c^+\}} \mathbb{E}_x[\mathbb{1}_{\{\inf_{t\ge\tau_a^-} X_t \ge a\}} | \mathcal{F}_{\tau_a^-}]\big] \\ &= \mathbb{E}_x\big[\mathbb{1}_{\{\tau_c^+ < \tau_a^-\}} \mathbb{E}_c[\mathbb{1}_{\{\inf_{s\ge 0} X_s \ge a\}}]\big] + \mathbb{E}_x\big[\mathbb{1}_{\{\tau_a^- < \tau_c^+\}} \mathbb{E}_x[\mathbb{1}_{\{\inf_{t\ge\tau_a^-} X_t \ge a\}} | \mathcal{F}_{\tau_a^-}]\big]. \end{aligned}$$

Since $\tau_a^- = \inf\{t > 0 : X_t < a\}$ we get that $\mathbb{E}_x[\mathbb{1}_{\{\inf_{t\geq\tau_a^-} X_t \geq a\}}|\mathcal{F}_{\tau_a^-}] = 0$ and therefore

$$\begin{aligned}
&\mathbb{E}_x\big[\mathbb{1}_{\{\tau_c^+<\tau_a^-\}}\mathbb{E}_c[\mathbb{1}_{\{\inf_{s\geq 0} X_s\geq a\}}]\big] + \mathbb{E}_x\big[\mathbb{1}_{\{\tau_a^-<\tau_c^+\}}\mathbb{E}_x[\mathbb{1}_{\{\inf_{t\geq\tau_a^-} X_t\geq a\}}|\mathcal{F}_{\tau_a^-}]\big]\\
&= \mathbb{E}_x\big[\mathbb{1}_{\{\tau_c^+<\tau_a^-\}}\mathbb{E}_c[\mathbb{1}_{\{\inf_{s\geq 0} X_s\geq a\}}]\big]\\
&= \mathbb{E}_x[\mathbb{1}_{\{\tau_c^+<\tau_a^-\}}]\mathbb{E}_c[\mathbb{1}_{\{\inf_{s\geq 0} X_s\geq a\}}]\\
&= \mathbb{E}_x[\mathbb{1}_{\{\tau_c^+<\tau_a^-\}}]\mathbb{P}_c(\inf_{s\geq 0} X_s \geq a)\\
&= \mathbb{E}_x[\mathbb{1}_{\{\tau_c^+<\tau_a^-\}}]H(c-a)\psi'(0+). \qquad (2.11)
\end{aligned}$$

From (2.10) and (2.11) it follows that

$$H(x-a) = \mathbb{E}_x[\mathbb{1}_{\{\tau_c^+<\tau_a^-\}}]H(c-a). \qquad (2.12)$$

If we can show that H is the scale function of X, the proof is done. For that we need the Wiener-Hopf factorization, which is introduced in [**6**, Chapter 6]. For the function H, we first notice that

$$\begin{aligned}
&\psi'(0+)\int_0^\infty e^{-\beta x}H(x)\mathrm{d}x = \int_0^\infty e^{-\beta x}\mathbb{P}_x(\inf_{t\geq 0} X_t \geq 0)\mathrm{d}x = \int_0^\infty e^{-\beta x}\mathbb{P}(\inf_{t\geq 0} X_t \geq -x)\mathrm{d}x\\
&= \int_0^\infty e^{-\beta x}\mathbb{P}(-\inf_{t\geq 0} X_t \leq x)\mathrm{d}x = \int_0^\infty e^{-\beta x}\mathbb{P}(\{-\inf_{t\geq 0} X_t \in (0,x]\} \cup \{-\inf_{t\geq 0} X_t = 0\})\mathrm{d}x\\
&= \int_0^\infty e^{-\beta x}\mathbb{P}(-\inf_{t\geq 0} X_t \in (0,x])\mathrm{d}x + \frac{\mathbb{P}(-\inf_{t\geq 0} X_t = 0)}{\beta}\\
&= \frac{\int_0^\infty e^{-\beta x}\mathrm{d}\mathbb{P}(-\inf_{t\geq 0} X_t \in (0,x])}{\beta} + \frac{\mathbb{P}(-\inf_{t\geq 0} X_t = 0)}{\beta}\\
&= \frac{\int_0^\infty e^{-\beta x}\mathrm{d}\mathbb{P}(-\inf_{t\geq 0} X_t \in [0,x])}{\beta} = \frac{\int_{[0,\infty)} e^{-\beta x}\mathrm{d}\mathbb{P}(-\inf_{t\geq 0} X_t \leq x)}{\beta}\\
&= \frac{\mathbb{E}[e^{\beta \underline{X}_\infty}]}{\beta},
\end{aligned}$$

where equality from third line to fourth line follows from integration by parts. From the Wiener-Hopf factorization, presented in [**6**, Theorem 6.16], it follows that

$$\mathbb{E}[e^{\beta \underline{X}_\infty}] = \psi'(0+)\frac{\beta}{\psi(\beta)},$$

where ψ is the Laplace exponent of X. Now by Proposition 2.1 function H is the scale function of X. Hence we have that $H(c-a) > 0$ for $c > a$. Now by dividing both sides of the equality (2.12) with $H(c-a)$ we get what we wanted and the proof is finished. For the cases $q = 0$ and $\psi'(0+) \leq 0$ see for example [**6**, Theorem 8.1]. □

For $q = 0$ Theorem 2.4 gives the probability, that a spectrally negative Lévy process X, with $X_0 = x$, hits the interval (c, ∞) before it hits $(-\infty, a)$. Later on we also use the following theorem which is generalization of the second part of Theorem 8.1 (*iii*) in [**6**].

THEOREM 2.5 (Solution to the two-sided exit problem $\tau_a^- < \tau_c^+$). *Assume the same setting as in Theorem 2.4. Then for all $a \leq x \leq c$*

$$\mathbb{E}_x[e^{-q\tau_a^-}; \tau_a^- < \tau_c^+] = Z^{(q)}(x-a) - \frac{Z^{(q)}(c-a)}{W^{(q)}(c-a)} W^{(q)}(x-a).$$

PROOF. Proof is similar to the proof of Theorem 2.4. For more details see for example the proof of [**6**, Theorem 8.1 (*iii*)]. □

2.2. Exit problems of refracted Lévy processes

We start this section with the definition of refracted Lévy process and continue with an existence and uniqueness result for refracted Lévy processes.

DEFINITION 2.6 (Refracted Lévy process). Let $X := (X_t)_{t\geq 0}$ be a spectrally negative Lévy process with triplet (μ, σ^2, Π). If the process $U := (U_t)_{t\geq 0}$ satisfies

$$U_t := X_t - \alpha \int_0^t \mathbb{1}_{\{U_s > b\}} \mathrm{d}s, \quad t \geq 0,$$

it is called a refracted Lévy process.

THEOREM 2.7 (Existence and uniqueness of refracted Lévy process). *The refracted Lévy process $U := (U_t)_{t\geq 0}$ is the unique strong solution to the stochastic differential equation*

$$\mathrm{d}U_t = \mathrm{d}X_t - \alpha \mathbb{1}_{\{U_t > b\}} \mathrm{d}t, \quad t \geq 0,$$

which is equivalent to

$$U_t = X_t - \alpha \int_0^t \mathbb{1}_{\{U_s > b\}} \mathrm{d}s, \quad t \geq 0, \tag{2.13}$$

where X, with $X_0 = 0$, is a spectrally negative Lévy process with triplet (μ, σ^2, Π), satisfying $0 \leq \alpha < \mu + \int_0^1 x\Pi(\mathrm{d}x)$, if X has paths of bounded variation. Moreover, U is a strong Markov process.

PROOF. For the existence of the unique strong solution see [**7**, Theorem 1, Remark 2 and Remark 3]. □

As it can be seen from Definition 2.6, a refracted Lévy process is a process whose dynamics changes by subtracting off a fixed linear drift whenever the process is above a pre-specified level b. When the refracted Lévy process is below the pre-specified level it is equal to the corresponding spectrally negative Lévy process.

From now on, we let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis, $X = (X_t)_{t\geq 0}$ be a spectrally negative Lévy process with triplet $(\mu, 0, \Pi)$, which has paths of bounded variation, and assume that ψ is its Laplace exponent. Define the scale functions $W^{(q)}$ and $Z^{(q)}$ to be the scale functions related to X.

For $0 \leq \alpha < \mu + \int_0^1 x\Pi(\mathrm{d}x)$, we define the process $Y = (Y_t)_{t\geq 0}$ such that

$$Y_t := X_t - \alpha t. \tag{2.14}$$

Since X is a spectrally negative Lévy process and has paths of bounded variation we clearly have that the process Y is also a spectrally negative Lévy process and has paths of bounded variation. Let Ψ be the Laplace exponent of Y and define the scale functions $\mathbb{W}^{(q)}$ and $\mathbb{Z}^{(q)}$ to be the scale functions related to Y.

For processes X and Y we define the following optional times:

$$\tau_b^+ := \inf\{t > 0 : X_t > b\} \text{ and } \nu_c^+ := \inf\{t > 0 : Y_t > c\} \tag{2.15}$$

$$\tau_a^- := \inf\{t > 0 : X_t < a\} \text{ and } \nu_b^- := \inf\{t > 0 : Y_t < b\} \tag{2.16}$$

for $a, b, c \in \mathbb{R}$.

We also define a refracted Lévy process $U := (U_t)_{t\geq 0}$ such that

$$U_t := Y_t + \alpha \int_0^t \mathbb{1}_{\{U_s < b\}} \mathrm{d}s. \tag{2.17}$$

From (2.14) and (2.17) it follows that

$$U_t = X_t - \alpha \int_0^t \mathbb{1}_{\{U_s > b\}} \mathrm{d}s. \tag{2.18}$$

For the process U we define optional times

$$\kappa_a^- := \inf\{t > 0 : U_t < a\} \text{ and } \kappa_c^+ := \inf\{t > 0 : U_t > c\}. \tag{2.19}$$

The following theorem gives us the first results for exit problems related to the process U.

THEOREM 2.8 ([**7**], Theorem 6). *Fix a Borel set $B \subseteq \mathbb{R}$.*

(i) *For $0 < \alpha < \mu + \int_0^1 x\Pi(\mathrm{d}x)$, $q \geq 0$ and $x, b \in [0, a]$,*

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^\infty e^{-qt}\mathbb{1}_{\{U_t\in B, t<\kappa_0^-\wedge\kappa_a^+\}}\mathrm{d}t\right] \\
&= \int_{B\bigcap[b,a]}\left\{\frac{W^{(q)}(x)+\alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}{W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\mathbb{W}^{(q)}(a-y)-\mathbb{W}^{(q)}(x-y)\right\}\mathrm{d}y \\
&\quad+\int_{B\bigcap[0,b)}\left\{\frac{W^{(q)}(x)+\alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}{W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\right. \\
&\quad\times\left(W^{(q)}(a-y)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z-y)\mathrm{d}z\right) \\
&\quad\left.-\left(W^{(q)}(x-y)+\alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z-y)\mathrm{d}z\right)\right\}\mathrm{d}y.
\end{aligned}$$

(ii) *For* $0 < \alpha < \mu + \int_0^1 x\Pi(\mathrm{d}x)$*,* $x, b \geq 0$ *and* $q > 0$

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^\infty e^{-qt}\mathbb{1}_{\{U_t\in B, t<\kappa_0^-\}}\mathrm{d}t\right] \\
&= \int_{B\bigcap[b,\infty)}\left\{\frac{W^{(q)}(x) + \alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}{\alpha\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z)\mathrm{d}z}e^{-\varphi(q)y} - \mathbb{W}^{(q)}(x-y)\right\}\mathrm{d}y \\
&+ \int_{B\bigcap[0,b)}\left\{\frac{\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z-y)\mathrm{d}z}{\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z)\mathrm{d}z}\left(W^{(q)}(x) + \alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z\right)\right. \\
&\left. - \left(W^{(q)}(x-y) + \alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z-y)\mathrm{d}z\right)\right\}\mathrm{d}y,
\end{aligned}$$

where $\varphi(q)$ *is the inverse Laplace exponent of the spectrally negative Lévy process related to* $\mathbb{W}^{(q)}$ *from Proposition 1.24.*

For the proof we need the following two theorems.

Theorem 2.9 ([**7**], Theorem 23). *For a spectrally negative Lévy process* X *let*

$$\tau_a^+ = \inf\{t > 0 : X_t > a\} \text{ and } \tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

(i) *For* $q \geq 0$ *and* $0 \leq x \leq a$ *we have*

$$\mathbb{E}_x[e^{-q\tau_a^+}\mathbb{1}_{\{\tau_0^- > \tau_a^+\}}] = \frac{W^{(q)}(x)}{W^{(q)}(a)}.$$

(ii) *For any* $a > 0$*,* $x, y \in [0, a]$*,* $q \geq 0$

$$\begin{aligned}
&\int_0^\infty e^{-qt}\mathbb{P}_x(X_t \in \mathrm{d}y, t < \tau_a^+ \wedge \tau_0^-)\mathrm{d}t \\
&= \left\{\frac{W^{(q)}(x)W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y)\right\}\mathrm{d}y.
\end{aligned}$$

(iii) *Let* $a > 0$*,* $x \in [0, a]$*,* $q \geq 0$ *and* f*,* g *be positive, bounded measurable functions. Further, suppose that* X *is of bounded variation or* $f(0)g(0) = 0$*. Then*

$$\begin{aligned}
&\mathbb{E}_x[e^{-q\tau_0^-}f(X_{\tau_0^-})g(X_{\tau_0^- -})\mathbb{1}_{\{\tau_0^- < \tau_a^+\}}] \\
&= \int_0^a \int_{(y,\infty)} f(y-\theta)g(y)\left\{\frac{W^{(q)}(x)W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y)\right\}\Pi(\mathrm{d}\theta)\mathrm{d}y.
\end{aligned}$$

Proof. Part (i) is a special case of Theorem 2.4. The proofs of other identities can be found in [**6**, Chapter 8]. □

Theorem 2.10 ([**7**], Theorem 16). *Suppose that* X *is a spectrally negative Lévy process that has paths of bounded variation and let* $0 < \alpha < c$*, where*

$c = \mu + \int_{(0,1)} x\Pi(\mathrm{d}x)$. *Then for* $v \geq u \geq m \geq 0$

$$\begin{aligned}
&\int_0^\infty \int_{(z,\infty)} W^{(q)}(z-\theta+m)\Pi(\mathrm{d}\theta)\left[\frac{\mathbb{W}^{(q)}(v-m-z)}{\mathbb{W}^{(q)}(v-m)}\mathbb{W}^{(q)}(u-m) - \mathbb{W}^{(q)}(u-m-z)\right]\mathrm{d}z \\
&= -\frac{\mathbb{W}^{(q)}(u-m)}{\mathbb{W}^{(q)}(v-m)}\left(W^{(q)}(v) + \alpha\int_m^v \mathbb{W}^{(q)}(v-z)W^{(q)\prime}(z)\mathrm{d}z\right) \\
&\quad + W^{(q)}(u) + \alpha\int_m^u \mathbb{W}^{(q)}(u-z)W^{(q)\prime}(z)\mathrm{d}z.
\end{aligned}$$

PROOF. The proof can be found in [**7**, Theorem 16]. □

Now we can go back to the proof of Theorem 2.8. For the proof recall that the process X has paths of bounded variation.

PROOF. The proof is derived in the same way as the proof of Theorem 6 in [**7**].
Part (i):
Define the function

$$V^{(q)}(x,B) := \mathbb{E}_x\left[\int_0^\infty e^{-qt}\mathbb{1}_{\{U_t\in B, t<\kappa_0^-\wedge\kappa_a^+\}}\mathrm{d}t\right]. \tag{2.20}$$

Let us first assume that $x \leq b \leq a$. We notice that

$$\begin{aligned}
V^{(q)}(x,B) &= \mathbb{E}_x\left[\int_0^\infty e^{-qt}\mathbb{1}_{\{U_t\in B, t<\kappa_0^-\wedge\kappa_a^+\}}\mathrm{d}t\right] \\
&= \mathbb{E}_x\left[\int_0^{\tau_b^+} e^{-qt}\mathbb{1}_{\{U_t\in B, t<\kappa_0^-\wedge\kappa_a^+\}}\mathrm{d}t\right] \qquad (2.21)\\
&\quad + \mathbb{E}_x\left[\int_{\tau_b^+}^\infty e^{-qt}\mathbb{1}_{\{U_t\in B, t<\kappa_0^-\wedge\kappa_a^+, \tau_b^+<\tau_0^-\}}\mathrm{d}t\right]. \qquad (2.22)
\end{aligned}$$

Since from (2.18) it follows that $U = X$ when $U \leq b$, we have that $\kappa_b^+ = \tau_b^+$ and $\kappa_0^- = \tau_0^-$. Also, because $x \leq b \leq a$, we have that $\kappa_b^+ \leq \kappa_a^+$. Hence we get for the component (2.21), by using the Fubini's theorem, that

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{\tau_b^+} e^{-qt}\mathbb{1}_{\{U_t\in B, t<\kappa_0^-\wedge\kappa_a^+\}}\mathrm{d}t\right] = \mathbb{E}_x\left[\int_0^{\tau_b^+} e^{-qt}\mathbb{1}_{\{U_t\in B, t<\tau_0^-\wedge\kappa_a^+\}}\mathrm{d}t\right] \\
&= \mathbb{E}_x\left[\int_0^{\tau_b^+} e^{-qt}\mathbb{1}_{\{X_t\in B, t<\tau_0^-\wedge\kappa_b^+\}}\mathrm{d}t\right] = \mathbb{E}_x\left[\int_0^{\tau_b^+} e^{-qt}\mathbb{1}_{\{X_t\in B, t<\tau_0^-\wedge\tau_b^+\}}\mathrm{d}t\right] \\
&= \int_0^\infty e^{-qt}\mathbb{P}_x(X_t\in B, t<\tau_0^-\wedge\tau_b^+)\mathrm{d}t. \qquad (2.23)
\end{aligned}$$

For the component (2.22) we get, by using the change of variable and the tower property, that

$$
\begin{aligned}
&\mathbb{E}_x\left[\int_{\tau_b^+}^{\infty} e^{-qt}\mathbb{1}_{\{U_t\in B, t<\kappa_0^-\wedge\kappa_a^+, \tau_b^+<\tau_0^-\}}\mathrm{d}t\right] \\
&= \mathbb{E}_x\left[\int_0^{\infty} e^{-q(s+\tau_b^+)}\mathbb{1}_{\{U_{s+\tau_b^+}\in B, s+\tau_b^+<\kappa_0^-\wedge\kappa_a^+, \tau_b^+<\tau_0^-\}}\mathrm{d}s\right] \\
&= \mathbb{E}_x\left[\mathbb{E}_x\left[\int_0^{\infty} e^{-q(s+\tau_b^+)}\mathbb{1}_{\{U_{s+\tau_b^+}\in B, s+\tau_b^+<\kappa_0^-\wedge\kappa_a^+, \tau_b^+<\tau_0^-\}}\mathrm{d}s\middle|\mathcal{F}_{(\tau_b^+)+}\right]\right] \\
&= \mathbb{E}_x\left[\mathbb{E}_x\left[\int_0^{\infty} e^{-qs}e^{-q\tau_b^+}\mathbb{1}_{\{U_{s+\tau_b^+}\in B, s+\tau_b^+<\kappa_0^-\wedge\kappa_a^+\}}\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}\mathrm{d}s\middle|\mathcal{F}_{(\tau_b^+)+}\right]\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}e^{-q\tau_b^+}\mathbb{E}_x\left[\int_0^{\infty} e^{-qs}\mathbb{1}_{\{U_{s+\tau_b^+}\in B, s+\tau_b^+<\kappa_0^-\wedge\kappa_a^+\}}\mathrm{d}s\middle|\mathcal{F}_{(\tau_b^+)+}\right]\right],
\end{aligned}
$$

where in the last equality we used that $e^{-q\tau_b^+}$ and $\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}$ are $\mathcal{F}_{(\tau_b^+)+}$-measurable. By using the strong Markov property of U and that $U_{\tau_b^+} = X_{\tau_b^+} = b$, since U can not jump upwards, we get that

$$
\begin{aligned}
&\mathbb{E}_x\left[\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}e^{-q\tau_b^+}\mathbb{E}_x\left[\int_0^{\infty} e^{-qs}\mathbb{1}_{\{U_{s+\tau_b^+}\in B, s+\tau_b^+<\kappa_0^-\wedge\kappa_a^+\}}\mathrm{d}s\middle|\mathcal{F}_{(\tau_b^+)+}\right]\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}e^{-q\tau_b^+}\mathbb{E}_x\left[\int_0^{\infty} e^{-qs}\mathbb{1}_{\{U_{s+\tau_b^+}-U_{\tau_b^+}+U_{\tau_b^+}\in B, s+\tau_b^+<\kappa_0^-\wedge\kappa_a^+\}}\mathrm{d}s\middle|\mathcal{F}_{(\tau_b^+)+}\right]\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}e^{-q\tau_b^+}\mathbb{E}_b\left[\int_0^{\infty} e^{-qs}\mathbb{1}_{\{U_s\in B, s<\kappa_0^-\wedge\kappa_a^+\}}\mathrm{d}s\right]\right].
\end{aligned}
$$

By using Fubini's theorem, we get that

$$
\begin{aligned}
&\mathbb{E}_x\left[\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}e^{-q\tau_b^+}\mathbb{E}_b\left[\int_0^{\infty} e^{-qs}\mathbb{1}_{\{U_s\in B, s<\kappa_0^-\wedge\kappa_a^+\}}\mathrm{d}s\right]\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}e^{-q\tau_b^+}\int_0^{\infty} e^{-qs}\mathbb{E}_b[\mathbb{1}_{\{U_s\in B, s<\kappa_0^-\wedge\kappa_a^+\}}]\mathrm{d}s\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}e^{-q\tau_b^+}\int_0^{\infty} e^{-qs}\mathbb{P}_b(U_s\in B, s<\kappa_0^-\wedge\kappa_a^+)\mathrm{d}s\right] \\
&= \mathbb{E}_x[\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}e^{-q\tau_b^+}]\int_0^{\infty} e^{-qs}\mathbb{P}_b(U_s\in B, s<\kappa_0^-\wedge\kappa_a^+)\mathrm{d}s \\
&= \mathbb{E}_x[\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}e^{-q\tau_b^+}]V^{(q)}(b,B). \qquad (2.24)
\end{aligned}
$$

Now, by using Theorem 2.9 for (2.23) and (2.24), $V^{(q)}(x,B)$ turns into

$$
\begin{aligned}
&V^{(q)}(x,B) \\
&= \int_0^{\infty} e^{-qt}\mathbb{P}_x(X_t\in B, t<\tau_0^-\wedge\tau_b^+)\mathrm{d}t + \mathbb{E}_x[\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}e^{-q\tau_b^+}]V^{(q)}(b,B) \\
&= \int_B\left(\frac{W^{(q)}(x)W^{(q)}(b-y)}{W^{(q)}(b)} - W^{(q)}(x-y)\right)\mathrm{d}y + \frac{W^{(q)}(x)}{W^{(q)}(a)}V^{(q)}(b,B). \qquad (2.25)
\end{aligned}
$$

To get an expression for $V^{(q)}(b,B)$ with help of W and $\mathbb{W}$, we need to continue and consider the case $b \le x \le a$. First we notice that before the hitting time ν_b^-, we have that $U = Y$. Hence for $V^{(q)}(x,B)$ it follows that

$$V^{(q)}(x,B)$$
$$= \mathbb{E}_x\left[\int_0^{\nu_b^- \wedge \nu_a^+} e^{-qt}\mathbb{1}_{\{Y_t \in B \bigcap [b,a]\}}\mathrm{d}t\right] \tag{2.26}$$
$$+ \mathbb{E}_x\left[\mathbb{1}_{\{\nu_b^- < \nu_a^+\}}\int_{\nu_b^-}^{\kappa_a^+ \wedge \kappa_0^-} e^{-qt}\mathbb{1}_{\{U_t \in B\}}\mathrm{d}t\right]. \tag{2.27}$$

For the component (2.26), by using Theorem 2.9 and the Fubini's theorem, we get that

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{\nu_b^- \wedge \nu_a^+} e^{-qt}\mathbb{1}_{\{Y_t \in B \bigcap [b,a]\}}\mathrm{d}t\right] \\
&= \mathbb{E}_x\left[\int_0^{\infty} e^{-qt}\mathbb{1}_{\{Y_t \in B \bigcap [b,a], t < \nu_b^- \wedge \nu_a^+\}}\mathrm{d}t\right] \\
&= \int_0^{\infty} e^{-qt}\mathbb{E}_x[\mathbb{1}_{\{Y_t \in B \bigcap [b,a], t < \nu_b^- \wedge \nu_a^+\}}]\mathrm{d}t \\
&= \int_0^{\infty} e^{-qt}\mathbb{P}_x(Y_t \in B \cap [b,a], t < \nu_b^- \wedge \nu_a^+)\mathrm{d}t \\
&= \int_{B\cap[b,a]}\left(\frac{\mathbb{W}^{(q)}(a-y)\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)} - \mathbb{W}^{(q)}(x-y)\right)\mathrm{d}y. \qquad (2.28)
\end{aligned}$$

For the component (2.27), by using a change of variable, the tower property and the fact that $e^{-q\nu_b^-}$ and $\mathbb{1}_{\{\nu_b^- < \nu_a^+\}}$ are $\mathcal{F}_{(\nu_b^-)+}$-measurable, it follows that

$$\begin{aligned}
&\mathbb{E}_x\left[\mathbb{1}_{\{\nu_b^- < \nu_a^+\}}\int_{\nu_b^-}^{\kappa_a^+ \wedge \kappa_0^-} e^{-qt}\mathbb{1}_{\{U_t \in B\}}\mathrm{d}t\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\nu_b^- < \nu_a^+\}}\int_{\nu_b^-}^{\infty} e^{-qt}\mathbb{1}_{\{U_t \in B, t < \kappa_a^+ \wedge \kappa_0^-\}}\mathrm{d}t\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\nu_b^- < \nu_a^+\}}\int_0^{\infty} e^{-q(s+\nu_b^-)}\mathbb{1}_{\{U_{s+\nu_b^-} \in B, s+\nu_b^- < \kappa_a^+ \wedge \kappa_0^-\}}\mathrm{d}s\right] \\
&= \mathbb{E}_x\left[\mathbb{E}_x\left[\mathbb{1}_{\{\nu_b^- < \nu_a^+\}}\int_0^{\infty} e^{-q(s+\nu_b^-)}\mathbb{1}_{\{U_{s+\nu_b^-} \in B, s+\nu_b^- < \kappa_a^+ \wedge \kappa_0^-\}}\mathrm{d}s \,\middle|\, \mathcal{F}_{(\nu_b^-)+}\right]\right] \\
&= \mathbb{E}_x\left[e^{-q\nu_b^-}\mathbb{1}_{\{\nu_b^- < \nu_a^+\}}\mathbb{E}_x\left[\int_0^{\infty} e^{-qs}\mathbb{1}_{\{U_{s+\nu_b^-} \in B, s+\nu_b^- < \kappa_a^+ \wedge \kappa_0^-\}}\mathrm{d}s \,\middle|\, \mathcal{F}_{(\nu_b^-)+}\right]\right].
\end{aligned}$$

Now, by using the strong Markov property, the fact that $U_{\nu_b^-} = Y_{\nu_b^-}$ by (2.17) and Fubini's theorem, we get that

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-q\nu_b^-}\mathbb{1}_{\{\nu_b^-<\nu_a^+\}}\mathbb{E}_x\left[\int_0^\infty e^{-qs}\mathbb{1}_{\{U_{s+\nu_b^-}\in B, s+\nu_b^-<\kappa_a^+\wedge\kappa_0^-\}}\mathrm{d}s\middle|\mathcal{F}_{(\nu_b^-)+}\right]\right]\\
&=\mathbb{E}_x\left[e^{-q\nu_b^-}\mathbb{1}_{\{\nu_b^-<\nu_a^+\}}\mathbb{E}_x\left[\int_0^\infty e^{-qs}\mathbb{1}_{\{U_{s+\nu_b^-}-U_{\nu_b^-}+U_{\nu_b^-}\in B, s+\nu_b^-<\kappa_a^+\wedge\kappa_0^-\}}\mathrm{d}s\middle|\mathcal{F}_{(\nu_b^-)+}\right]\right]\\
&=\mathbb{E}_x\left[e^{-q\nu_b^-}\mathbb{1}_{\{\nu_b^-<\nu_a^+\}}\mathbb{E}_{Y_{\nu_b^-}}\left[\int_0^\infty e^{-qs}\mathbb{1}_{\{U_s\in B, s<\kappa_a^+\wedge\kappa_0^-\}}\mathrm{d}s\right]\right]\\
&=\mathbb{E}_x\left[e^{-q\nu_b^-}\mathbb{1}_{\{\nu_b^-<\nu_a^+\}}\int_0^\infty e^{-qs}\mathbb{E}_{Y_{\nu_b^-}}[\mathbb{1}_{\{U_s\in B, s<\kappa_a^+\wedge\kappa_0^-\}}]\mathrm{d}s\right]\\
&=\mathbb{E}_x\left[e^{-q\nu_b^-}\mathbb{1}_{\{\nu_b^-<\nu_a^+\}}\int_0^\infty e^{-qs}\mathbb{P}_{Y_{\nu_b^-}}(U_s\in B, s<\kappa_a^+\wedge\kappa_0^-)\mathrm{d}s\right]\\
&=\mathbb{E}_x\left[e^{-q\nu_b^-}\mathbb{1}_{\{\nu_b^-<\nu_a^+\}}V^{(q)}(Y_{\nu_b^-},B)\right].
\end{aligned}$$

Since Y has paths of bounded variation it follows from the definition of ν_b^- that $Y_{\nu_b^-} < b$. Hence we can use the expression (2.25) for $V^{(q)}(Y_{\nu_b^-}, B)$, which gives us that

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-q\nu_b^-}\mathbb{1}_{\{\nu_b^-<\nu_a^+\}}V^{(q)}(Y_{\nu_b^-},B)\right]\\
&=\mathbb{E}_x\left[e^{-q\nu_b^-}\mathbb{1}_{\{\nu_b^-<\nu_a^+\}}\left(\int_B\left(\frac{W^{(q)}(b-y)W^{(q)}(Y_{\nu_b^-})}{W^{(q)}(b)}-W^{(q)}(Y_{\nu_b^-}-y)\right)\mathrm{d}y\right.\right.\\
&\quad\left.\left.+\mathbb{E}_{Y_{\nu_b^-}}[\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}e^{-q\tau_b^+}]V^{(q)}(b,B)\right)\right].
\end{aligned}$$

By using Theorem 2.9 (*iii*), for ν_b^-, ν_a^+ and process Y instead of τ_0^-, τ_a^+ and X so that $\mathbb{W}$ is used instead of W and $x-a$ and $a-b$ instead of x and a, for the equation above it follows that

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-q\nu_b^-}\mathbb{1}_{\{\nu_b^-<\nu_a^+\}}\left(\int_B\left(\frac{W^{(q)}(b-y)W^{(q)}(Y_{\nu_b^-})}{W^{(q)}(b)}-W^{(q)}(Y_{\nu_b^-}-y)\right)\mathrm{d}y\right.\right.\\
&\quad\left.\left.+\mathbb{E}_{Y_{\nu_b^-}}[\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}e^{-q\tau_b^+}]V^{(q)}(b,B)\right)\right]\\
&=\mathbb{E}_x\left[e^{-q\nu_b^-}\mathbb{1}_{\{\nu_b^-<\nu_a^+\}}\left(\int_B\left(\frac{W^{(q)}(b-y)W^{(q)}(Y_{\nu_b^-})}{W^{(q)}(b)}-W^{(q)}(Y_{\nu_b^-}-y)\right)\mathrm{d}y\right.\right.\\
&\quad\left.\left.+\frac{W^{(q)}(Y_{\nu_b^-})V^{(q)}(b,B)}{W^{(q)}(b)}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
(2.29)\quad &= \int_0^\infty \int_{(z,\infty)} \left\{ \int_B \left(\frac{W^{(q)}(b-y)}{W^{(q)}(b)} W^{(q)}(z-\theta+b) - W^{(q)}(z-\theta+b-y) \right) \mathrm{d}y \right. \\
&\quad \left. + \frac{V^{(q)}(b,B)}{W^{(q)}(b)} W^{(q)}(z-\theta+b) \right\} \left[\frac{\mathbb{W}^{(q)}(x-b)\mathbb{W}^{(q)}(a-b-z)}{\mathbb{W}^{(q)}(a-b)} \right. \\
&\quad \left. - \mathbb{W}^{(q)}(x-b-z) \right] \Pi(\mathrm{d}\theta)\mathrm{d}z.
\end{aligned}
$$

Note that above we can use the integration with respect to $\mathrm{d}z$ on $(0,\infty)$ instead of $(0, a-b)$ since $\mathbb{W}^{(q)}(a-b-z) = \mathbb{W}^{(q)}(x-b-z) = 0$ for $z > a-b$. To simplify the expression above let us denote

$$
A = \left[\frac{\mathbb{W}^{(q)}(x-b)\mathbb{W}^{(q)}(a-b-z)}{\mathbb{W}^{(q)}(a-b)} - \mathbb{W}^{(q)}(x-b-z) \right].
$$

We abbreviate (2.29) by D and write D as a sum and change the terms using Fubini:

$$
\begin{aligned}
D := &\int_0^\infty \int_{(z,\infty)} \left\{ \int_B \left(\frac{W^{(q)}(b-y)}{W^{(q)}(b)} W^{(q)}(z-\theta+b) - W^{(q)}(z-\theta+b-y) \right) \mathrm{d}y \right. \\
&\left. + \frac{V^{(q)}(b,B)}{W^{(q)}(b)} W^{(q)}(z-\theta+b) \right\} A \Pi(\mathrm{d}\theta)\mathrm{d}z \\
= &\int_0^\infty \int_z^\infty W^{(q)}(z-\theta+b)\Pi(\mathrm{d}\theta) \int_B \frac{W^{(q)}(b-y)}{W^{(q)}(b)} \mathrm{d}y A \mathrm{d}z \\
&- \int_0^\infty \int_z^\infty \int_B W^{(q)}(z-\theta+b-y)\mathrm{d}y \Pi(\mathrm{d}\theta) A \mathrm{d}z \\
&+ \int_0^\infty \int_z^\infty \frac{V^{(q)}(b,B)}{W^{(q)}(b)} W^{(q)}(z-\theta+b)\Pi(\mathrm{d}\theta) A \mathrm{d}z \\
= &\int_{B\cap[0,b)} \frac{W^{(q)}(b-y)}{W^{(q)}(b)} \int_0^\infty \int_z^\infty W^{(q)}(z-\theta+b)\Pi(\mathrm{d}\theta) A \mathrm{d}z \mathrm{d}y \\
&- \int_{B\cap[0,b)} \int_0^\infty \int_z^\infty W^{(q)}(z-\theta+b-y)\Pi(\mathrm{d}\theta) A \mathrm{d}z \mathrm{d}y \\
&+ \frac{V^{(q)}(b,B)}{W^{(q)}(b)} \int_0^\infty \int_z^\infty W^{(q)}(z-\theta+b)\Pi(\mathrm{d}\theta) A \mathrm{d}z.
\end{aligned}
$$

Now for (2.29), by taking into consideration that $W^{(q)}(b-y) = 0$ and $W^{(q)}(z-\theta+b-y) = 0$ for $y > b$, and using Theorem 2.10 twice by setting first that $u = x$, $m = b$ and $v = a$ and after that setting $m = b-y$, $u = x-y$ and $v = a-y$, we have that

(2.30)

$$
\begin{aligned}
D = &\int_{B\cap[0,b)} \frac{W^{(q)}(b-y)}{W^{(q)}(b)}\Bigg[-\frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)}\Big(W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z\Big) \\
&+W^{(q)}(x)+\alpha\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z\Bigg]\mathrm{d}y \\
&-\int_{B\cap[0,b)}\Bigg[-\frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)}\Big(W^{(q)}(a-y)+\alpha\int_{b-y}^{a-y} \mathbb{W}^{(q)}(a-y-z)W^{(q)\prime}(z)\mathrm{d}z\Big) \\
&+W^{(q)}(x-y)+\alpha\int_{b-y}^{x-y} \mathbb{W}^{(q)}(x-y-z)W^{(q)\prime}(z)\mathrm{d}z\Bigg]\mathrm{d}y \\
&+\frac{V^{(q)}(b,B)}{W^{(q)}(b)}\Bigg[-\frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)}\Big(W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z\Big) \\
&+W^{(q)}(x)+\alpha\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z\Bigg].
\end{aligned}
$$

Now $V^{(q)}(x,B)$ can be written by using (2.28), which coinsides with term (2.26), and (2.30), which is D and stands for (2.27). This leads us to the following equality

(2.31)

$$
\begin{aligned}
&V^{(q)}(x,B) \\
&=\int_{B\cap[b,a]}\Big(\frac{\mathbb{W}^{(q)}(a-y)\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)}-\mathbb{W}^{(q)}(x-y)\Big)\mathrm{d}y \\
&+\int_{B\cap[0,b)} \frac{W^{(q)}(b-y)}{W^{(q)}(b)}\Bigg[-\frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)}\Big(W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z\Big) \\
&+W^{(q)}(x)+\alpha\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z\Bigg]\mathrm{d}y \\
&-\int_{B\cap[0,b)}\Bigg[-\frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)}\Big(W^{(q)}(a-y)+\alpha\int_{b-y}^{a-y} \mathbb{W}^{(q)}(a-y-z)W^{(q)\prime}(z)\mathrm{d}z\Big) \\
&+W^{(q)}(x-y)+\alpha\int_{b-y}^{x-y} \mathbb{W}^{(q)}(x-y-z)W^{(q)\prime}(z)\mathrm{d}z\Bigg]\mathrm{d}y \\
&+\frac{V^{(q)}(b,B)}{W^{(q)}(b)}\Bigg[-\frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)}\Big(W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z\Big) \\
&+W^{(q)}(x)+\alpha\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z\Bigg].
\end{aligned}
$$

Setting $x=b$ in equation (2.31) causes several terms to vanish and leads us to a linear equation for $V^{(q)}(b,B)$ of the form

(2.32)
$$V^{(q)}(b,B) = A_1 + V^{(q)}(b,B)\Bigg[1 - \frac{\mathbb{W}^{(q)}(0)}{\mathbb{W}^{(q)}(a-b)W^{(q)}(b)} \times \left(W^{(q)}(a) + \alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z\right)\Bigg],$$

which we solve and get that

$$\begin{aligned}
&V^{(q)}(b,B)\\
&= \Bigg\{\int_{B\cap[b,a]}\left(\frac{\mathbb{W}^{(q)}(a-y)\mathbb{W}^{(q)}(0)}{\mathbb{W}^{(q)}(a-b)}\right)\mathrm{d}y + \int_{B\cap[0,b)}\frac{W^{(q)}(b-y)}{W^{(q)}(b)}\\
&\quad\times\Bigg[-\frac{\mathbb{W}^{(q)}(0)}{\mathbb{W}^{(q)}(a-b)}\left(W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z\right) + W^{(q)}(b)\Bigg]\mathrm{d}y\\
&\quad-\int_{B\cap[0,b)}\Bigg[-\frac{\mathbb{W}^{(q)}(0)}{\mathbb{W}^{(q)}(a-b)}\left(W^{(q)}(a-y)+\alpha\int_{b-y}^{a-y}\mathbb{W}^{(q)}(a-y-z)W^{(q)\prime}(z)\mathrm{d}z\right)\\
&\quad+W^{(q)}(b-y)\Bigg]\mathrm{d}y\Bigg\}\times\left\{\frac{W^{(q)}(b)\mathbb{W}^{(q)}(a-b)}{\mathbb{W}^{(q)}(0)[W^{(q)}(a)+\alpha\int_b^a\mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z]}\right\}
\end{aligned}$$

(2.33)
$$\begin{aligned}
&= \int_{B\cap[b,a]}\left(\frac{\mathbb{W}^{(q)}(a-y)W^{(q)}(b)}{W^{(q)}(a)+\alpha\int_b^a\mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\right)\mathrm{d}y - \int_{B\cap[0,b)}W^{(q)}(b-y)\mathrm{d}y\\
&\quad+\int_{B\cap[0,b)}\left[\frac{W^{(q)}(b)[W^{(q)}(a-y)+\alpha\int_{b-y}^{a-y}\mathbb{W}^{(q)}(a-y-z)W^{(q)\prime}(z)\mathrm{d}z]}{W^{(q)}(a)+\alpha\int_b^a\mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\right]\mathrm{d}y.
\end{aligned}$$

Now, by inserting (2.33) into (2.31) and simplifying the expression, we get that

(2.34)
$$\begin{aligned}
&V^{(q)}(x,B)\\
&= \int_{B\cap[b,a]}\left(\frac{\mathbb{W}^{(q)}(a-y)\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)} - \mathbb{W}^{(q)}(x-y)\right)\mathrm{d}y\\
&\quad+\int_{B\cap[0,b)}\frac{W^{(q)}(b-y)}{W^{(q)}(b)}\Bigg[-\frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)}\left(W^{(q)}(a)+\alpha\int_b^a\mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z\right)\\
&\quad+W^{(q)}(x)+\alpha\int_b^x\mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z\Bigg]\mathrm{d}y\\
&\quad-\int_{B\cap[0,b)}\Bigg[-\frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)}\left(W^{(q)}(a-y)+\alpha\int_{b-y}^{a-y}\mathbb{W}^{(q)}(a-y-z)W^{(q)\prime}(z)\mathrm{d}z\right)\\
&\quad+W^{(q)}(x-y)+\alpha\int_{b-y}^{x-y}\mathbb{W}^{(q)}(x-y-z)W^{(q)\prime}(z)\mathrm{d}z\Bigg]\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&- \int_{B\cap[b,a]} \frac{\mathbb{W}^{(q)}(a-y)\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)}\mathrm{d}y \\
&+ \int_{B\cap[b,a]} \frac{\mathbb{W}^{(q)}(a-y)[W^{(q)}(x)+\alpha\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z]\mathrm{d}y}{W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z} \\
&- \int_{B\cap[0,b)} \frac{W^{(q)}(b-y)[W^{(q)}(x)+\alpha\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z]\mathrm{d}y}{W^{(q)}(b)} \\
&+ \int_{B\cap[0,b)} \left[\frac{[W^{(q)}(a-y)+\alpha\int_{b-y}^{a-y} \mathbb{W}^{(q)}(a-y-z)W^{(q)\prime}(z)\mathrm{d}z]}{W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\right]\mathrm{d}y \\
&\times \Bigg[-\frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)}\left(W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z\right) \\
&+ W^{(q)}(x)+\alpha\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z\Bigg] \\
=& \int_{B\bigcap[b,a]} \left\{\frac{W^{(q)}(x)+\alpha\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}{W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\mathbb{W}^{(q)}(a-y) - \mathbb{W}^{(q)}(x-y)\right\}\mathrm{d}y \\
&+ \int_{B\bigcap[0,b)} \Bigg\{\frac{W^{(q)}(x)+\alpha\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}{W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z} \\
&\times \left(W^{(q)}(a-y)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z-y)\mathrm{d}z\right) \\
&- \left(W^{(q)}(x-y)+\alpha\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z-y)\mathrm{d}z\right)\Bigg\}\mathrm{d}y
\end{aligned}
$$

for all $b \le x \le a$. By inserting the expression (2.33) into (2.25) and noticing that

$$
\mathbb{1}_{\{x\ge b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z-y)\mathrm{d}z = 0
$$

and

$$
\mathbb{1}_{\{x\ge b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z = 0
$$

for all $x \le b$, we get that

$$
\begin{aligned}
&V^{(q)}(x,B) \\
=& \int_{B\bigcap[0,b)} \left\{\frac{W^{(q)}(x)W^{(q)}(b-y)}{W^{(q)}(b)} - W^{(q)}(x-y)\right\}\mathrm{d}y \\
&+ \frac{W^{(q)}(x)}{W^{(q)}(b)}\Bigg[\int_{B\cap[b,a]}\left(\frac{\mathbb{W}^{(q)}(a-y)W^{(q)}(b)}{W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\right)\mathrm{d}y - \int_{B\cap[0,b)} W^{(q)}(b-y)\mathrm{d}y \\
&+ \int_{B\cap[0,b)}\left[\frac{W^{(q)}(b)[W^{(q)}(a-y)+\alpha\int_{b-y}^{a-y} \mathbb{W}^{(q)}(a-y-z)W^{(q)\prime}(z)\mathrm{d}z]}{W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\right]\mathrm{d}y\Bigg]
\end{aligned}
$$

$$= \int_{B \bigcap [b,a]} \left\{ \frac{W^{(q)}(x)\mathbb{W}^{(q)}(a-y)}{W^{(q)}(a) + \alpha \int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z} \right\}$$
$$+ \int_{B \cap [0,b)} \left\{ \frac{W^{(q)}(x)W^{(q)}(b-y)}{W^{(q)}(b)} - \frac{W^{(q)}(x)W^{(q)}(b-y)}{W^{(q)}(b)} \right.$$
$$\left. + \frac{W^{(q)}(x)\big[W^{(q)}(a-y) + \alpha \int_{b-y}^{a-y} \mathbb{W}^{(q)}(a-y-z)W^{(q)\prime}(z)\mathrm{d}z\big]}{W^{(q)}(a) + \alpha \int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z} - W^{(q)}(x-y) \right\}$$

(2.35)

$$= \int_{B \bigcap [b,a]} \left\{ \frac{W^{(q)}(x) + \alpha \mathbb{1}_{\{x \ge b\}} \int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}{W^{(q)}(a) + \alpha \int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z} \mathbb{W}^{(q)}(a-y) - \mathbb{W}^{(q)}(x-y) \right\} \mathrm{d}y$$
$$+ \int_{B \bigcap [0,b)} \left\{ \frac{W^{(q)}(x) + \alpha \mathbb{1}_{\{x \ge b\}} \int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}{W^{(q)}(a) + \alpha \int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z} \right.$$
$$\times \left(W^{(q)}(a-y) + \alpha \int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z-y)\mathrm{d}z \right)$$
$$\left. - \left(W^{(q)}(x-y) + \alpha \mathbb{1}_{\{x \ge b\}} \int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z-y)\mathrm{d}z \right) \right\} \mathrm{d}y$$

for all $x \le b \le a$. From (2.20) and equalities (2.34) and (2.35) it follows that the part (i) holds for all $x, b \in [0, a]$ and the proof is done.

Part (ii): By using part (i) and letting $a \to \infty$, we get the wanted result by using the Monotone Convergence Theorem. Note that here we need the fact, presented in [**6**, Lemma 8.4], that scale functions can be written as

$$W^{(q)}(x) = e^{\phi(q)x} W_{\phi(q)}(x),$$

where ϕ is the inverse Laplace exponent of X, defined in Proposition 1.24 and function $W_{\phi(q)}(x) := \mathbb{P}_x^{\phi(q)}(\underline{X}_\infty \ge 0)$ stands for 0-scale function. The measure $\mathbb{P}_x^{\phi(q)}$ is the measure defined as

$$\left. \frac{\mathrm{d}\mathbb{P}_x^{\phi(q)}}{\mathrm{d}\mathbb{P}_x} \right|_{\mathcal{F}_t} = e^{\phi(q)X_t + \psi(\phi(q))t}.$$

□

The expected values, which we introduced in Theorem 2.8, give us the discounted measure of how long the process U stays inside a given set on average. Hence they are also called q-potential measures in the literature. Also the expected values, which are of the same form as in Theorem 2.8, are called q-potential measures. Let us now consider the case that the process U crosses one boundary before the other.

THEOREM 2.11 ([**7**], Theorem 4). *For* $q \ge 0$, $x, b \in [0, a]$ *and* $0 < \alpha < \mu + \int_0^1 x\Pi(\mathrm{d}x)$ *we have*

(i)

$$\mathbb{E}_x[e^{-q\kappa_a^+} \mathbb{1}_{\{\kappa_a^+ < \kappa_0^-\}}] = \frac{W^{(q)}(x) + \alpha \mathbb{1}_{\{x \ge b\}} \int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)\mathrm{d}y}{W^{(q)}(a) + \alpha \int_b^a \mathbb{W}^{(q)}(a-y)W^{(q)\prime}(y)\mathrm{d}y}.$$

(ii)

$$\mathbb{E}_x[e^{-q\kappa_0^-}\mathbb{1}_{\{\kappa_0^-<\kappa_a^+\}}]$$
$$= Z^{(q)}(x) + \alpha q \mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)}(y)\mathrm{d}y$$
$$- \frac{Z^{(q)}(a) + \alpha q\int_b^a \mathbb{W}^{(q)}(a-y)W^{(q)}(y)\mathrm{d}y}{W^{(q)}(a) + \alpha\int_b^a \mathbb{W}^{(q)}(a-y)W^{(q)\prime}(y)\mathrm{d}y}\left(W^{(q)}(x) + \alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)\mathrm{d}y\right).$$

Proof. Part (i): First we prove the equality for $q>0$. Let e_q, $q>0$, be an exponentially distributed random variable with rate q such that e_q is independent from U. Recall the optional times $\kappa_0^- := \inf\{t>0 : U_t<0\}$ and $\kappa_a^+ := \inf\{t>0 : U_t>a\}$ and that the corresponding spectrally negative Lévy processes X and Y have paths of bounded variation. From the definitions of κ_0^- and κ_a^+ it follows for the process U that

$$\{\underline{U}_t\geq 0\} = \{\inf_{0\leq s\leq t} U_s\geq 0\} = \{\kappa_0^->t\} \quad \text{and} \quad \{\overline{U}_t>a\} = \{\sup_{0\leq s\leq t} U_s>a\} = \{\kappa_a^+<t\}.$$

Hence it follows, by using the independence of U and e_q and the tower property, that

$$\begin{aligned}
\mathbb{P}_x(\underline{U}_{e_q}\geq 0, \overline{U}_{e_q}>a) &= \mathbb{E}_x\int_0^\infty \mathbb{1}_{\{\underline{U}_t\geq 0,\overline{U}_t>a\}}qe^{-qt}\mathrm{d}t \\
&= \mathbb{E}_x\int_0^\infty \mathbb{1}_{\{\kappa_0^->t,\kappa_a^+<t\}}qe^{-qt}\mathrm{d}t = \mathbb{E}_x\int_{\kappa_a^+}^\infty \mathbb{1}_{\{\kappa_0^->t\}}qe^{-qt}\mathrm{d}t \\
&= \mathbb{E}_x\left[\mathbb{E}_x\left[\int_{\kappa_a^+}^\infty \mathbb{1}_{\{\kappa_0^->t\}}(\mathbb{1}_{\{\kappa_0^-<\kappa_a^+\}}+\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}})qe^{-qt}\mathrm{d}t\middle|\mathcal{F}_{(\kappa_a^+)+}\right]\right].
\end{aligned} \tag{2.36}$$

Since $\{\kappa_a^+<t<\kappa_0^-\}\bigcap\{\kappa_0^-<\kappa_a^+\}=\emptyset$, the strong Markov property and measurability give us that

$$\begin{aligned}
&\mathbb{E}_x\left[\mathbb{E}_x\left[\int_{\kappa_a^+}^\infty \mathbb{1}_{\{\kappa_0^->t\}}(\mathbb{1}_{\{\kappa_0^-<\kappa_a^+\}}+\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}})qe^{-qt}\mathrm{d}t\middle|\mathcal{F}_{(\kappa_a^+)+}\right]\right] \\
&= \mathbb{E}_x\left[\mathbb{E}_x\left[\int_{\kappa_a^+}^\infty \mathbb{1}_{\{\kappa_0^->t\}}\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}qe^{-qt-q(\kappa_a^+-\kappa_a^+)}\mathrm{d}t\middle|\mathcal{F}_{(\kappa_a^+)+}\right]\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}e^{-q\kappa_a^+}\mathbb{E}_x\left[\int_{\kappa_a^+}^\infty \mathbb{1}_{\{\kappa_0^->t\}}qe^{-q(t-\kappa_a^+)}\mathrm{d}t\middle|\mathcal{F}_{(\kappa_a^+)+}\right]\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}e^{-q\kappa_a^+}\mathbb{E}_x\left[\int_{\kappa_a^+}^\infty \mathbb{1}_{\{\inf_{0\leq r\leq t}U_r\geq 0,t\geq\kappa_a^+\}}qe^{-q(t-\kappa_a^+)}\mathrm{d}t\middle|\mathcal{F}_{(\kappa_a^+)+}\right]\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}e^{-q\kappa_a^+}\mathbb{E}_x\left[\int_0^\infty \mathbb{1}_{\{\inf_{0\leq r\leq t}U_{r+\kappa_a^+}-U_{\kappa_a^+}+a\geq 0,t\geq 0\}}qe^{-qt}\mathrm{d}t\middle|\mathcal{F}_{(\kappa_a^+)+}\right]\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}e^{-q\kappa_a^+}\mathbb{E}_a\left[\int_0^\infty \mathbb{1}_{\{\inf_{0\leq r\leq s}U_r\geq 0\}}qe^{-qs}\mathrm{d}s\right]\right] \\
&= \mathbb{E}_x[\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}e^{-q\kappa_a^+}]\mathbb{E}_a\left[\int_0^\infty \mathbb{1}_{\{\inf_{0\leq r\leq s}U_r\geq 0\}}qe^{-qs}\mathrm{d}s\right] \\
&= \mathbb{E}_x[\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}e^{-q\kappa_a^+}]\mathbb{P}_a(\underline{U}_{e_q}\geq 0).
\end{aligned} \tag{2.37}$$

Combining (2.36) and (2.37) gives that

$$\mathbb{P}_x(\underline{U}_{e_q} \geq 0, \overline{U}_{e_q} > a) = \mathbb{E}_x[\mathbb{1}_{\{\kappa_a^+ < \kappa_0^-\}} e^{-q\kappa_a^+}]\mathbb{P}_a(\underline{U}_{e_q} \geq 0). \tag{2.38}$$

For the probabilities above we have that

$$\begin{aligned}
&\mathbb{P}_x(\underline{U}_{e_q} \geq 0, \overline{U}_{e_q} > a) = \mathbb{P}_x(\underline{U}_{e_q} \geq 0) - \mathbb{P}_x(\underline{U}_{e_q} \geq 0, \overline{U}_{e_q} \leq a) \\
&= q\int_0^\infty e^{-qt}\mathbb{P}_x(U_t \in \mathbb{R}, t < \kappa_0^-)\mathrm{d}t - q\int_0^\infty e^{-qt}\mathbb{P}_x(U_t \in [0,a], t < \kappa_0^- \wedge \kappa_a^+)\mathrm{d}t
\end{aligned}$$

and

$$\mathbb{P}_a(\underline{U}_{e_q} \geq 0) = q\int_0^\infty e^{-qt}\mathbb{P}_a(U_t \in \mathbb{R}, t < \kappa_0^-)\mathrm{d}t.$$

From Theorem 2.8 (ii) it follows that for all $q > 0$

$$\begin{aligned}
&\int_0^\infty e^{-qt}\mathbb{P}_a(U_t \in \mathbb{R}, t < \kappa_0^-)\mathrm{d}t \\
&= \big[W^{(q)}(a) + \alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z\big] \\
&\quad \times \Bigg\{ \int_{[b,\infty)} \Bigg[\frac{e^{-\varphi(q)y}}{\alpha\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z)\mathrm{d}z} - \frac{\mathbb{W}^{(q)}(a-y)}{W^{(q)}(a) + \alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\Bigg]\mathrm{d}y \\
&\quad + \int_{[0,b)} \Bigg[\frac{\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z-y)\mathrm{d}z}{\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z)\mathrm{d}z} - \frac{W^{(q)}(a-y) + \alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z-y)\mathrm{d}z}{W^{(q)}(a) + \alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\Bigg]\mathrm{d}y\Bigg\}.
\end{aligned} \tag{2.39}$$

Moreover Theorem 2.8 (i) and (ii) imply

$$\begin{aligned}
&\int_0^\infty e^{-qt}\mathbb{P}_x(U_t \in \mathbb{R}, t < \kappa_0^-)\mathrm{d}t - \int_0^\infty e^{-qt}\mathbb{P}_x(U_t \in [0,a], t < \kappa_0^- \wedge \kappa_a^+)\mathrm{d}t \\
&= \big[W^{(q)}(x) + \alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z\big] \\
&\quad \times \Bigg\{ \int_{[b,\infty)} \Bigg[\frac{e^{-\varphi(q)y}}{\alpha\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z)\mathrm{d}z} - \frac{\mathbb{W}^{(q)}(x-y)}{W^{(q)}(x) + \alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}\Bigg]\mathrm{d}y \\
&\quad + \int_{[0,b)} \Bigg[\frac{\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z-y)\mathrm{d}z}{\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z)\mathrm{d}z}\Bigg]\mathrm{d}y - \int_{[b,a]} \Bigg[\frac{\mathbb{W}^{(q)}(a-y)}{W^{(q)}(a) + \alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z} \\
&\quad - \frac{\mathbb{W}^{(q)}(x-y)}{W^{(q)}(x) + \alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}\Bigg]\mathrm{d}y \\
&\quad - \int_{[0,b)} \Bigg[\frac{W^{(q)}(a-y) + \alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z-y)\mathrm{d}z}{W^{(q)}(a) + \alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\Bigg]\mathrm{d}y\Bigg\}.
\end{aligned} \tag{2.40}$$

We notice that because of $\mathbb{W}^{(q)}(a-y)=0$ for $y>a$ we can extend the integration from $[b,a]$ to $[b,\infty)$. Hence expression (2.40) turns into

$$
\begin{aligned}
&\big[W^{(q)}(x)+\alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z\big]\\
&\quad\times\Big\{\int_{[b,\infty)}\Big[\frac{e^{-\varphi(q)y}}{\alpha\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z)\mathrm{d}z}-\frac{\mathbb{W}^{(q)}(x-y)}{W^{(q)}(x)+\alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}\Big]\mathrm{d}y\\
&\quad+\int_{[0,b)}\Big[\frac{\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z-y)\mathrm{d}z}{\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z)\mathrm{d}z}\Big]\mathrm{d}y-\int_{[b,\infty)}\Big[\frac{\mathbb{W}^{(q)}(a-y)}{W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\\
&\quad-\frac{\mathbb{W}^{(q)}(x-y)}{W^{(q)}(x)+\alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}\Big]\mathrm{d}y\\
&\quad-\int_{[0,b)}\Big[\frac{W^{(q)}(a-y)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z-y)\mathrm{d}z}{W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\Big]\mathrm{d}y\Big\}
\end{aligned}
$$

(2.41)

$$
\begin{aligned}
&=\big[W^{(q)}(x)+\alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z\big]\\
&\quad\times\Big\{\int_{[b,\infty)}\Big[\frac{e^{-\varphi(q)y}}{\alpha\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z)\mathrm{d}z}-\frac{\mathbb{W}^{(q)}(a-y)}{W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\Big]\mathrm{d}y\\
&\quad+\int_{[0,b)}\Big[\frac{\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z-y)\mathrm{d}z}{\int_b^\infty e^{-\varphi(q)z}W^{(q)\prime}(z)\mathrm{d}z}-\frac{W^{(q)}(a-y)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z-y)\mathrm{d}z}{W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\Big]\mathrm{d}y\Big\}.
\end{aligned}
$$

Substituting (2.39) and (2.41) into equation (2.38) gives after rearranging that

$$\mathbb{E}_x[\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}e^{-q\kappa_a^+}]=\frac{W^{(q)}(x)+\alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}{W^{(q)}(a)+\alpha\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z},$$

which is what we wanted. The result for $q=0$ follows from the limit behaviour of the case $q>0$ as $q\to 0$. Note here that for scale functions we have the representation

$$W^{(q)}(x)=e^{\phi(q)x}W_{\phi(q)}(x),$$

where ϕ is the inverse Laplace exponent of X. The representation is presented in [**6**, Lemma 8.4].

Part (ii): The proof is similar to the proof of part (i). For more details see for example the proof of [**8**, Theorem 7].

□

For the process U we define the following functions:

$$w^{(q)}(x;a):=W^{(q)}(x-a)+\alpha\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y-a)\mathrm{d}y \tag{2.42}$$

$$z^{(q)}(x;a):=Z^{(q)}(x-a)+\alpha q\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)}(y-a)\mathrm{d}y. \tag{2.43}$$

The function $Z^{(q)}$ in (2.43) is defined as in (2.3). Notice that if $x\leq b$, then $w^{(q)}(x;a)=W^{(q)}(x-a)$ and $z^{(q)}(x;a)=Z^{(q)}(x-a)$. In the next theorem we use the functions $w^{(q)}$ and $z^{(q)}$ to generalize Theorem 2.4. Hence we can think that these functions are the scale functions of the refracted Lévy process U. The proof of the next theorem follows [**10**, Theorem 2].

THEOREM 2.12 ([**10**], Theorem 2). *For $q \geq 0$ and $x, b \in [a, c]$, we have that*

$$\mathbb{E}_x[e^{-q\kappa_c^+}; \kappa_c^+ < \kappa_a^-] = \frac{w^{(q)}(x;a)}{w^{(q)}(c;a)}.$$

and

$$\mathbb{E}_x[e^{-q\kappa_c^+}; \kappa_a^- < \kappa_c^+] = z^{(q)}(x;a) - \frac{z^{(q)}(c;a)}{w^{(q)}(c;a)} w^{(q)}(x;a).$$

PROOF. Let $a, c \in \mathbb{R}$, $a < c$ and $x, b \in [a, c]$. Define the refracted Lévy process $(\tilde{U}_t)_{t\geq 0}$ such that $\tilde{U}_t := U_t - a$ and optional times

$$\tilde{\kappa}_0^- := \inf\{t > 0 : \tilde{U}_t < 0\} \text{ and } \tilde{\kappa}_{c-a}^+ := \inf\{t > 0 : \tilde{U}_t > c - a\}.$$

By using quasi-space homogeneity property for U, we get that

$$\mathbb{E}_x[e^{-q\kappa_c^+}; \kappa_c^+ < \kappa_a^-] = \mathbb{E}_{x-a}[e^{-q\tilde{\kappa}_{c-a}^+}; \tilde{\kappa}_{c-a}^+ < \tilde{\kappa}_0^-].$$

By Theorem 2.11 (i) and a change of variable it follows that

$$\begin{aligned}
&\mathbb{E}_{x-a}[e^{-q\tilde{\kappa}_{c-a}^+}; \tilde{\kappa}_{c-a}^+ < \tilde{\kappa}_0^-] \\
&= \frac{W^{(q)}(x-a) + \alpha \mathbb{1}_{\{x-a\geq b-a\}} \int_{b-a}^{x-a} \mathbb{W}^{(q)}(x-a-y) W^{(q)\prime}(y)\mathrm{d}y}{W^{(q)}(c-a) + \alpha \int_{b-a}^{c-a} \mathbb{W}^{(q)}(c-a-y) W^{(q)\prime}(y)\mathrm{d}y} \\
&= \frac{W^{(q)}(x-a) + \alpha \mathbb{1}_{\{x\geq b\}} \int_{b}^{x} \mathbb{W}^{(q)}(x-y) W^{(q)\prime}(y-a)\mathrm{d}y}{W^{(q)}(c-a) + \alpha \int_{b}^{c} \mathbb{W}^{(q)}(c-y) W^{(q)\prime}(y-a)\mathrm{d}y} \\
&= \frac{w^{(q)}(x;a)}{w^{(q)}(c;a)},
\end{aligned}$$

where for the last equality we used (2.42) and (2.43).

The other part of the proof is similar to the first part. One just needs to use Theorem 2.11 (ii) and does the same steps as in the proof of the first part. □

2.3. Scale function equalities

In this section we present some equalities which we are going to use in the last chapter. Let us begin with the main equality from which we derive the other results.

LEMMA 2.13. *Let $p, q, x \geq 0$ and define α as in Theorem 2.7. Then*

$$\alpha \int_0^x \mathbb{W}^{(p)}(x-y) W^{(q)}(y)\mathrm{d}y + (p-q) \int_0^x \int_0^y \mathbb{W}^{(p)}(y-z) W^{(q)}(z)\mathrm{d}z\mathrm{d}y = \int_0^x \mathbb{W}^{(p)}(y)\mathrm{d}y - \int_0^x W^{(q)}(y)\mathrm{d}y. \tag{2.44}$$

PROOF. Let us first consider the left hand side of (2.44). We notice that by taking the Laplace transform and using the convolution theorem, we get that

$$\int_0^\infty e^{-\lambda x} \alpha \int_0^x \mathbb{W}^{(p)}(x-y) W^{(q)}(y)\mathrm{d}y\mathrm{d}x = \alpha \int_0^\infty e^{-\lambda x} \mathbb{W}^{(p)}(x)\mathrm{d}x \int_0^\infty e^{-\lambda y} W^{(q)}(y)\mathrm{d}y. \tag{2.45}$$

In the same way we get by integration by parts for the integral with respect to dx that

$$
\begin{aligned}
&(p-q)\int_0^\infty e^{-\lambda x}\int_0^x\int_0^y \mathbb{W}^{(p)}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y\mathrm{d}x\\
&=\frac{(p-q)}{\lambda}\int_0^\infty e^{-\lambda x}\int_0^x \mathbb{W}^{(p)}(x-y)W^{(q)}(y)\mathrm{d}y\mathrm{d}x\\
&=\frac{(p-q)}{\lambda}\int_0^\infty e^{-\lambda x}\mathbb{W}^{(p)}(x)\mathrm{d}x\int_0^\infty e^{-\lambda y}W^{(q)}(y)\mathrm{d}y, \qquad (2.46)
\end{aligned}
$$

where the last line can be seen by changing the order of integration and using that $\mathbb{W}^{(p)}=0$ for $x<y$. By Proposition 2.1 we have for (2.45) and (2.46) that

$$\alpha\int_0^\infty e^{-\lambda x}\mathbb{W}^{(p)}(x)\mathrm{d}x\int_0^\infty e^{-\lambda y}W^{(q)}(y)\mathrm{d}y=\alpha\cdot\frac{1}{\Psi(\lambda)-p}\cdot\frac{1}{\psi(\lambda)-q} \qquad (2.47)$$

and

(2.48)
$$\frac{(p-q)}{\lambda}\int_0^\infty e^{-\lambda x}\mathbb{W}^{(p)}(x)\mathrm{d}x\int_0^\infty e^{-\lambda y}W^{(q)}(y)\mathrm{d}y=\frac{(p-q)}{\lambda}\cdot\frac{1}{\Psi(\lambda)-p}\cdot\frac{1}{\psi(\lambda)-q}.$$

Let us consider now the right hand side of (2.44). Again by taking the Laplace transform and integrating by parts we get that

$$
\begin{aligned}
&\int_0^\infty e^{-\lambda x}\int_0^x \mathbb{W}^{(p)}(y)\mathrm{d}y\mathrm{d}x-\int_0^\infty e^{-\lambda x}\int_0^x W^{(q)}(y)\mathrm{d}y\mathrm{d}x\\
&=\frac{1}{\lambda}\int_0^\infty e^{-\lambda x}\mathbb{W}^{(p)}(x)\mathrm{d}x-\frac{1}{\lambda}\int_0^\infty e^{-\lambda x}W^{(q)}(x)\mathrm{d}x\\
&=\frac{1}{\lambda}\cdot\frac{1}{\Psi(\lambda)-p}-\frac{1}{\lambda}\cdot\frac{1}{\psi(\lambda)-q}, \qquad (2.49)
\end{aligned}
$$

where in the end we used again Proposition 2.1. From (2.14) if follows that

$$\Psi(\lambda)-p=\psi(\lambda)-\alpha\lambda-p.$$

We use (2.47) and (2.48) to compute the Laplace transform of the left hand side of (2.44) as follows

$$
\begin{aligned}
&\alpha\cdot\frac{1}{\Psi(\lambda)-p}\cdot\frac{1}{\psi(\lambda)-q}+\frac{(p-q)}{\lambda}\cdot\frac{1}{\Psi(\lambda)-p}\cdot\frac{1}{\psi(\lambda)-q}\\
&=\left(\alpha+\frac{(p-q)}{\lambda}\right)\cdot\frac{1}{\Psi(\lambda)-p}\cdot\frac{1}{\psi(\lambda)-q}\\
&=\left(\frac{\psi(\lambda)-q-\psi(\lambda)+\alpha\lambda+p}{\lambda}\right)\cdot\frac{1}{\Psi(\lambda)-p}\cdot\frac{1}{\psi(\lambda)-q}\\
&=\left(\frac{(\psi(\lambda)-q)-(\Psi(\lambda)-p)}{\lambda}\right)\cdot\frac{1}{\Psi(\lambda)-p}\cdot\frac{1}{\psi(\lambda)-q}\\
&=\frac{1}{\lambda}\cdot\frac{1}{\Psi(\lambda)-p}-\frac{1}{\lambda}\cdot\frac{1}{\psi(\lambda)-q}. \qquad (2.50)
\end{aligned}
$$

Now from (2.49) and (2.50) it follows that the Laplace transforms of the left and right hand side of (2.44) are equal. Since the Laplace transform describes a continuous function uniquely, we get that equality holds in (2.44), and the proof is done. □

From Lemma 2.13 we can derive the following corollaries.

COROLLARY 2.14. *Let $p, q, x \geq 0$ and define α as in Theorem 2.7. Then*

$$(q-p)\int_0^x \mathbb{W}^{(p)}(x-y)W^{(q)}(y)\mathrm{d}y$$
$$= W^{(q)}(x) - \mathbb{W}^{(p)}(x) + \alpha\big[W^{(q)}(0)\mathbb{W}^{(p)}(x) + \int_0^x \mathbb{W}^{(p)}(x-y)W^{(q)\prime}(y)\mathrm{d}y\big].$$

PROOF. The relation follows from Lemma 2.13 by differentiating both sides with respect to x and integration by parts. □

Note that Corollary 2.14 holds for all $p, q, x \geq 0$ and α as in Theorem 2.7. Later on we are going to use this corollary for example in cases like $q = p$ and $\alpha = 0$. In the case $q = p$ we get that

$$W^{(q)}(x) - \mathbb{W}^{(q)}(x) + \alpha\left(W^{(q)}(0)\mathbb{W}^{(q)}(x) + \int_0^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)\mathrm{d}y\right) = 0. \tag{2.51}$$

It is important to notice that from $\alpha = 0$ it follows by (2.14) and uniqueness of the scale functions that $\mathbb{W}^{(q)} = W^{(q)}$. Hence when $\alpha = 0$ Corollary 2.14 gets into the form

$$(q-p)\int_0^x W^{(p)}(x-y)W^{(q)}(y)\mathrm{d}y = W^{(q)}(x) - W^{(p)}(x). \tag{2.52}$$

COROLLARY 2.15. *Let $q, x \geq 0$ and define α as in Theorem 2.7. Then for $x > b$ it holds*

$$w^{(q)}(x;0) = \big[1 - \alpha W^{(q)}(0)\big]\mathbb{W}^{(q)}(x) - \alpha\int_0^b \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)\mathrm{d}y.$$

PROOF. This follows from (2.42) and (2.51). □

CHAPTER 3

Risk model

In this chapter we present a risk model, in which we apply a refracted Lévy process. We begin with the section which presents equations which give us the expected time that a given process spends in the so called red zone. After that we continue with a risk model where we compute the probability of bankruptcy.

3.1. Time spent in the red zone

Now we are interested in how long the given refracted Lévy process stays below the given boundary, which we later consider as a red zone. To get into this problem let us first state the following lemma.

LEMMA 3.1 ([**10**], Lemma 1). *For all $q, p \geq 0$ and $b \leq x \leq c$, one has that*

$$\begin{aligned}
&\mathbb{E}_x[e^{-p\nu_b^-} W^{(q)}(Y_{\nu_b^-}); \nu_b^- < \nu_c^+] \\
&= w^{(q)}(x;0) - (q-p)\int_b^x \mathbb{W}^{(p)}(x-y)w^{(q)}(y;0)\mathrm{d}y \\
&\quad - \frac{\mathbb{W}^{(p)}(x-b)}{\mathbb{W}^{(p)}(c-b)}\left(w^{(q)}(c;0) - (q-p)\int_b^c \mathbb{W}^{(p)}(c-y)w^{(q)}(y;0)\mathrm{d}y\right),
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_x[e^{-p\nu_b^-} Z^{(q)}(Y_{\nu_b^-}); \nu_b^- < \nu_c^+] \\
&= z^{(q)}(x;0) - (q-p)\int_b^x \mathbb{W}^{(p)}(x-y)z^{(q)}(y;0)\mathrm{d}y \\
&\quad - \frac{\mathbb{W}^{(p)}(x-b)}{\mathbb{W}^{(p)}(c-b)}\left(z^{(q)}(c;0) - (q-p)\int_b^c \mathbb{W}^{(p)}(c-y)z^{(q)}(y;0)\mathrm{d}y\right).
\end{aligned}$$

PROOF. See for example the proof of Lemma 1 in [**10**]. □

Next we present the theorem which gives us the representations for the joint Laplace transforms

$$\left(\kappa_a^-, \int_0^{\kappa_a^-} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s\right) \text{ and } \left(\kappa_c^+, \int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s\right). \tag{3.1}$$

Notice that for example $\int_0^{\kappa_a^-} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s$ stands for the time that the process U spent below b before going below a. These representations give us the expected time that the given refracted Lévy process spent below b. Let us now begin with the theorem.

THEOREM 3.2 ([**10**], Theorem 3). *For all $q, p \geq 0$ and $x, b \in [a, c]$ it holds*

(i)

$$\begin{aligned}
&\mathbb{E}_x[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}; \kappa_c^+ < \kappa_a^-] \\
&= \frac{w^{(p+q)}(x;a) - q\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(p)}(x-y)w^{(p+q)}(y;a)\mathrm{d}y}{w^{(p+q)}(c;a) - q\int_b^c \mathbb{W}^{(p)}(c-y)w^{(p+q)}(y;a)\mathrm{d}y}.
\end{aligned}$$

(ii)

$$\begin{aligned}
&\mathbb{E}_x[e^{-p\kappa_a^- - q\int_0^{\kappa_a^-} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}; \kappa_a^- < \kappa_c^+] \\
&= z^{(p+q)}(x;a) - q\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(p)}(x-y)z^{(p+q)}(y;a)\mathrm{d}y \\
&\quad + \frac{z^{(p+q)}(c;a) - q\int_b^c \mathbb{W}^{(p)}(c-y)z^{(p+q)}(y;a)\mathrm{d}y}{w^{(p+q)}(c;a) - q\int_b^c \mathbb{W}^{(p)}(c-y)w^{(p+q)}(y;a)\mathrm{d}y} \\
&\quad \times \left(w^{(p+q)}(x;a) - q\mathbb{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(p)}(x-y)w^{(p+q)}(y;a)\mathrm{d}y \right).
\end{aligned}$$

PROOF. The proof follows the proof of Theorem 3 in [**10**].
Part (i): Let us begin with the case that $a \leq x < b \leq c$. Since process U can not jump up and $x < b \leq c$, we have that

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right] \\
&= \mathbb{E}_x\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\mathbb{1}_{\{\kappa_b^+<\kappa_a^-\}}\right] \\
&= \mathbb{E}_x\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_b^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s - q\int_{\kappa_b^+}^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\mathbb{1}_{\{\kappa_b^+<\kappa_a^-\}}\right] \\
&= \mathbb{E}_x\left[e^{-p\kappa_c^+ - q\kappa_b^+ - q\int_{\kappa_b^+}^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\mathbb{1}_{\{\kappa_b^+<\kappa_a^-\}}\right] \\
&= \mathbb{E}_x\left[e^{-p\kappa_c^+ - q\kappa_b^+ - p\kappa_b^+ + p\kappa_b^+ - q\int_{\kappa_b^+}^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\mathbb{1}_{\{\kappa_b^+<\kappa_a^-\}}\right] \\
&= \mathbb{E}_x\left[e^{-(q+p)\kappa_b^+}\mathbb{1}_{\{\kappa_b^+<\kappa_a^-\}}e^{-p(\kappa_c^+ - \kappa_b^+) - q\int_{\kappa_b^+}^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right].
\end{aligned}$$

By using the tower property, the fact that $e^{-(q+p)\kappa_b^+}$ and $\mathbb{1}_{\{\kappa_b^+<\kappa_a^-\}}$ are $\mathcal{F}_{(\kappa_b^+)+}$-measurable and the strong Markov property of U we get that

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-(q+p)\kappa_b^+}\mathbb{1}_{\{\kappa_b^+<\kappa_a^-\}}e^{-p(\kappa_c^+ - \kappa_b^+) - q\int_{\kappa_b^+}^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right] \\
&= \mathbb{E}_x\left[\mathbb{E}_x\left[e^{-(q+p)\kappa_b^+}\mathbb{1}_{\{\kappa_b^+<\kappa_a^-\}}e^{-p(\kappa_c^+ - \kappa_b^+) - q\int_{\kappa_b^+}^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]\middle|\mathcal{F}_{(\kappa_b^+)+}\right] \\
&= \mathbb{E}_x\left[e^{-(q+p)\kappa_b^+}\mathbb{1}_{\{\kappa_b^+<\kappa_a^-\}}\mathbb{E}_x\left[e^{-p(\kappa_c^+ - \kappa_b^+) - q\int_{\kappa_b^+}^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]\middle|\mathcal{F}_{(\kappa_b^+)+}\right] \\
&= \mathbb{E}_x\left[e^{-(q+p)\kappa_b^+}\mathbb{1}_{\{\kappa_b^+<\kappa_a^-\}}\mathbb{E}_b\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]\right] \\
&= \mathbb{E}_x\left[e^{-(q+p)\kappa_b^+}\mathbb{1}_{\{\kappa_b^+<\kappa_a^-\}}\right]\mathbb{E}_b\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]. \qquad (3.2)
\end{aligned}$$

Let us now consider the case that $b \le x \le c$. First we notice that

$$\begin{aligned} &\mathbb{E}_x\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right] \\ &= \mathbb{E}_x\left[e^{-p\kappa_c^+}\mathbb{1}_{\{\kappa_c^+<\kappa_b^-\}}\right] + \mathbb{E}_x\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]. \end{aligned} \tag{3.3}$$

For the second component in (3.3) we get by using the tower property, the fact that $e^{-p\kappa_b^-}$ and $\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}$ are $\mathcal{F}_{(\kappa_b^-)+}$-measurable and the strong Markov property of U that

$$\begin{aligned} &\mathbb{E}_x\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right] \\ &= \mathbb{E}_x\left[e^{-p\kappa_c^+ - q\int_{\kappa_b^-}^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right] \\ &= \mathbb{E}_x\left[e^{-p\kappa_c^+ + p\kappa_b^- - p\kappa_b^- - q\int_{\kappa_b^-}^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right] \\ &= \mathbb{E}_x\left[\mathbb{E}_x\left[e^{-p\kappa_c^+ + p\kappa_b^- - p\kappa_b^- - q\int_{\kappa_b^-}^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]\middle|\mathcal{F}_{(\kappa_b^-)+}\right] \\ &= \mathbb{E}_x\left[e^{-p\kappa_b^-}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}\mathbb{E}_x\left[e^{-p(\kappa_c^+ - p\kappa_b^-) - q\int_{\kappa_b^-}^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]\middle|\mathcal{F}_{(\kappa_b^-)+}\right] \\ &= \mathbb{E}_x\left[e^{-p\kappa_b^-}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}\mathbb{E}_{U_{\kappa_b^-}}\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]\right], \end{aligned}$$

where in the first equality we used that $b \le x$ and $\mathbb{1}_{\{U_s<b\}} = 0$ before hitting time κ_b^-. Now for the inside expected value we can use the equality (3.2) and hence it follows by using Theorem 2.4 that

$$\begin{aligned} &\mathbb{E}_x\left[e^{-p\kappa_b^-}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}\mathbb{E}_{U_{\kappa_b^-}}\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]\right] \\ &= \mathbb{E}_x\left[e^{-p\kappa_b^-}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}\mathbb{E}_{U_{\kappa_b^-}}\left[e^{-(q+p)\kappa_b^+}\mathbb{1}_{\{\kappa_b^+<\kappa_a^-\}}\right]\mathbb{E}_b\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]\right] \\ &= \mathbb{E}_x\left[e^{-p\kappa_b^-}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}\frac{W^{(q+p)}(U_{\kappa_b^-}-a)}{W^{(q+p)}(b-a)}\mathbb{E}_b\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]\right] \\ &= \frac{\mathbb{E}_b\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]}{W^{(q+p)}(b-a)}\mathbb{E}_x\left[e^{-p\kappa_b^-}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}W^{(q+p)}(U_{\kappa_b^-}-a)\right]. \end{aligned} \tag{3.4}$$

Since $U = Y$ above b, we get from Theorem 2.4 that

$$\mathbb{E}_x\left[e^{-p\kappa_c^+}\mathbb{1}_{\{\kappa_c^+<\kappa_b^-\}}\right] = \mathbb{E}_x\left[e^{-p\nu_c^+}\mathbb{1}_{\{\nu_c^+<\nu_b^-\}}\right] = \frac{\mathbb{W}^{(p)}(x-b)}{\mathbb{W}^{(p)}(c-b)}.$$

Applying above equation and (3.4) in (3.3) it follows that

$$\mathbb{E}_x\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]$$

(3.5)

$$= \frac{\mathbb{W}^{(p)}(x-b)}{\mathbb{W}^{(p)}(c-b)} + \frac{\mathbb{E}_b\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]}{W^{(q+p)}(b-a)}\mathbb{E}_x\left[e^{-p\kappa_b^-}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}W^{(q+p)}(U_{\kappa_b^-}-a)\right].$$

Setting $x=b$ in above equation we get that

$$\mathbb{E}_b\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right] = \frac{\frac{\mathbb{W}^{(p)}(0)}{\mathbb{W}^{(p)}(c-b)}}{1-\left(\frac{\mathbb{E}_b[e^{-p\kappa_b^-}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}W^{(q+p)}(U_{\kappa_b^-}-a)]}{W^{(q+p)}(b-a)}\right)}. \tag{3.6}$$

Since $U=Y$ when $U>b$, we get by using the spatial homogeneity of Y, Lemma 3.1 and the change of variable that

$$\mathbb{E}_x[e^{-p\kappa_b^-}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}W^{(q+p)}(U_{\kappa_b^-}-a)] = \mathbb{E}_x[e^{-p\nu_b^-}\mathbb{1}_{\{\nu_b^-<\nu_c^+\}}W^{(q+p)}(Y_{\nu_b^-}-a)]$$

$$= \mathbb{E}_{x-a}[e^{-p\nu_{b-a}^-}\mathbb{1}_{\{\nu_{b-a}^-<\nu_{c-a}^+\}}W^{(q+p)}(Y_{\nu_{b-a}^-})]$$

$$= w^{(q+p)}(x-a;0) - (q+p-p)\int_{b-a}^{x-a}\mathbb{W}^{(p)}(x-a-y)w^{(q+p)}(y;0)\mathrm{d}y$$

$$- \frac{\mathbb{W}^{(p)}(x-a-b+a)}{\mathbb{W}^{(p)}(c-a-b+a)}\left(w^{(q+p)}(c-a;0) - (q+p-p)\int_{b-a}^{c-a}\mathbb{W}^{(p)}(c-a-y)w^{(q+p)}(y;0)\mathrm{d}y\right)$$

(3.7)

$$= w^{(q+p)}(x;a) - q\int_b^x \mathbb{W}^{(p)}(x-y)w^{(q+p)}(y;a)\mathrm{d}y$$

$$- \frac{\mathbb{W}^{(p)}(x-b)}{\mathbb{W}^{(p)}(c-b)}\left(w^{(q+p)}(c;a) - q\int_b^c \mathbb{W}^{(p)}(c-y)w^{(q+p)}(y;a)\mathrm{d}y\right).$$

Applying (3.7) in (3.6) and setting $x=b$ we get, by using (2.42), after reducing that

$$\mathbb{E}_b\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right] = \frac{W^{(q+p)}(b-a)}{w^{(q+p)}(c;a) - q\int_b^c \mathbb{W}^{(p)}(c-y)w^{(q+p)}(y;a)\mathrm{d}y}. \tag{3.8}$$

Inserting (3.8) into (3.5) and using (3.7) leads us to following expression

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-p\kappa_c^+ - q\int_0^{\kappa_c^+}\mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\mathbb{1}_{\{\kappa_c^+<\kappa_a^-\}}\right]\\
&= \frac{\mathbb{W}^{(p)}(x-b)}{\mathbb{W}^{(p)}(c-b)} + \frac{\mathbb{E}_x\left[e^{-p\kappa_b^-}\mathbb{1}_{\{\kappa_b^-<\kappa_c^+\}}W^{(q+p)}(U_{\kappa_b^-}-a)\right]}{w^{(q+p)}(c;a) - q\int_b^c \mathbb{W}^{(p)}(c-y)w^{(q+p)}(y;a)\mathrm{d}y}\\
&= \frac{\mathbb{W}^{(p)}(x-b)}{\mathbb{W}^{(p)}(c-b)} + \frac{w^{(q+p)}(x;a) - q\int_b^x \mathbb{W}^{(p)}(x-y)w^{(q+p)}(y;a)\mathrm{d}y}{w^{(q+p)}(c;a) - q\int_b^c \mathbb{W}^{(p)}(c-y)w^{(q+p)}(y;a)\mathrm{d}y}\\
&\quad - \frac{\mathbb{W}^{(p)}(x-b)[w^{(q+p)}(c;a) - q\int_b^c \mathbb{W}^{(p)}(c-y)w^{(q+p)}(y;a)\mathrm{d}y]}{\mathbb{W}^{(p)}(c-b)[w^{(q+p)}(c;a) - q\int_b^c \mathbb{W}^{(p)}(c-y)w^{(q+p)}(y;a)\mathrm{d}y]}\\
&= \frac{w^{(q+p)}(x;a) - q\int_b^x \mathbb{W}^{(p)}(x-y)w^{(q+p)}(y;a)\mathrm{d}y}{w^{(q+p)}(c;a) - q\int_b^c \mathbb{W}^{(p)}(c-y)w^{(q+p)}(y;a)\mathrm{d}y},
\end{aligned}$$

which is what we wanted to show.

The proof of part (ii) can be derived in the same way as the part (i). □

With Theorem 3.2 we can compute the expected time that the given process, which can describe for example some insurance company's financial situation, spent below the given boundary b but above a before crossing c or a.

In risk theory the boundary b is in many cases chosen to be the level below which our company is in financial distress. The boundaries a and c are usually chosen in the following way. If the process U goes below a, then ruin occurs, and if U crosses c, the company has financial solvency.

3.2. Probability of bankruptcy

In this section we apply the knowledge about scale functions and exit problems and give the probability of bankruptcy. We start by giving an example of a situation where one can apply refracted Lévy processes. Let us assume that we want to describe the financial situation of some insurance company. Assume that the spectrally negative Lévy process $(Y_t)_{t\geq 0}$ describes the surplus of the company. Now the jumps can be taught to be the consequence from the claims that the company gets. For small claim sizes it is usually assumed that jumps are exponentially distributed and that is what we assume here too. If we assume that the company becomes bankrupt when the surplus goes below 0 it is natural to think that we want the company's surplus to be above some artificial level b such that for example with high probability one claim can not cause bankruptcy. In this case we have that the level b depends on the claim size distribution. If the company's surplus is between 0 and b we say that the company is in financial distress. When the company is in financial distress it is natural to think that some restructuring should happen. One solution for restructuring, so the company has better chances to overcome financial distress, would be to increase premiums, which can be modelled with linear drift. Now we have the new process which describes the surplus of our company which is same as process $(Y_t)_{t\geq 0}$ above b and below b it is process $(Y_t)_{t\geq 0}$ increased by a linear drift. This process is actually the refracted Lévy process $(U_t)_{t\geq 0}$ from Definition 2.6 if we define $X_t := Y_t + \alpha t$. Let us now present the model with which we can compute the probability of bankruptcy, when the surplus is described by refracted Lévy process. For more information about the model see for example [**10**, Remark 4].

Recall the processes $(U_t)_{t\geq 0}$, $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$, the scale functions and the corresponding optional times from the beginning of Section 2.2. The definition of bankruptcy is borrowed from [**10**] and is as follows: bankruptcy happens, if the process U spends too much time in the interval $(-\infty, b)$ or drops too deep. We clarify the definition with the following functions. For $q > 0$ define the function $\omega : \mathbb{R} \longrightarrow [0, \infty]$ such that

$$\omega(x) = \begin{cases} 0, & \text{if } x \geq b, \\ q, & \text{if } 0 \leq x < b, \\ \infty, & \text{if } x < 0. \end{cases}$$

For the function ω define the corresponding bankruptcy time ρ_ω by

$$\rho_\omega = \inf\{t \geq 0; \int_0^t \omega(U_s)\mathrm{d}s > e_1\}, \tag{3.9}$$

where e_1 is an exponentially distributed random variable with rate 1 that is independent of U.

We assume that the condition, which in literature is called the net profit condition, $\mathbb{E}[X_1] > \alpha$ holds, in which case the probability of bankruptcy is less than 1 and

$$\lim_{c\to\infty} \mathbb{W}(c) = \frac{1}{\psi'(0+) - \alpha} = \frac{1}{\mathbb{E}[X_1] - \alpha}. \tag{3.10}$$

In our model, if bankruptcy occurs, it occurs when the process U is in set $[0, b)$ or drops below 0. By using the law of total probability we get that

$$\mathbb{P}_x(0 \leq U_{\rho_\omega} < b, \rho_\omega < \infty) + \mathbb{P}_x(U_{\rho_\omega} < 0, \rho_\omega < \infty) + \mathbb{P}_x(\rho_\omega = \infty) = 1, \tag{3.11}$$

where for first two probabilities we get that

$$\begin{aligned} \mathbb{P}_x(0 \leq U_{\rho_\omega} < b, \rho_\omega < \infty) &= \mathbb{P}_x\left(\int_0^{\kappa_0^-} \omega(U_s)\mathrm{d}s > e_1\right) \\ &= 1 - \mathbb{E}_x\left[e^{-q\int_0^{\kappa_0^-} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}\right] \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \mathbb{P}_x(U_{\rho_\omega} < 0, \rho_\omega < \infty) &= \mathbb{P}_x\left(\int_0^{\kappa_0^-} \omega(U_s)\mathrm{d}s \leq e_1, \kappa_0^- < \infty\right) \\ &= \mathbb{E}_x\left[e^{-q\int_0^{\kappa_0^-} \mathbb{1}_{\{U_s<b\}}\mathrm{d}s}; \kappa_0^- < \infty\right]. \end{aligned} \tag{3.13}$$

The equality (3.12) describes the probability of bankruptcy while surplus is between 0 and b and the equality (3.13) gives the probability of bankruptcy when surplus drops below 0 with the initial capital $x \geq 0$. The probability $\mathbb{P}_x(\rho_\omega = \infty)$ in (3.11) gives us the so called survival probability, that is the probability that bankruptcy does not occur with the initial capital $x \geq 0$. The probabilities in (3.11) can be calculated with help of the following corollary.

COROLLARY 3.3 ([**10**], Corollary 1). *Assume the net profit condition* $\mathbb{E}[X_1] > \alpha$ *is verified. Then for* $q > 0$ *and* $x, b \geq 0$ *the following assertions are true:*

(i)

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-q\int_0^{\kappa_0^-}\mathbb{1}_{\{U_s<b\}}\mathrm{d}s}; \kappa_0^- < \infty\right] \\
&= z^{(q)}(x;0) - q\int_b^x \mathbb{W}(x-y)z^{(q)}(y;0)\mathrm{d}y \\
&\quad + \frac{(\mathbb{E}[X_1]-\alpha) + q\int_0^b(\mathbb{Z}^{(q)}(y) - \alpha W^{(q)}(y)\mathbb{Z}^{(q)}(b-y))\mathrm{d}y}{Z^{(q)}(b) - \alpha W^{(q)}(b)} \\
&\quad \times \left(w^{(q)}(x;0) - q\int_b^x \mathbb{W}(x-y)w^{(q)}(y;0)\mathrm{d}y\right).
\end{aligned}$$

(ii)

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-q\int_0^{\kappa_0^-}\mathbb{1}_{\{U_s<b\}}\mathrm{d}s}; \kappa_0^- = \infty\right] \\
&= (\mathbb{E}[X_1]-\alpha)\frac{w^{(q)}(x;0) - q\int_b^x \mathbb{W}(x-y)w^{(q)}(y;0)\mathrm{d}y}{Z^{(q)}(b) - \alpha W^{(q)}(b)}.
\end{aligned}$$

PROOF. Part (i): First we notice that

$$\mathbb{E}_x\left[e^{-q\int_0^{\kappa_0^-}\mathbb{1}_{\{U_s<b\}}\mathrm{d}s}; \kappa_0^- < \infty\right] = \lim_{c\to\infty}\mathbb{E}_x\left[e^{-q\int_0^{\kappa_0^-}\mathbb{1}_{\{U_s<b\}}\mathrm{d}s}; \kappa_0^- < \kappa_c^+\right]. \tag{3.14}$$

Since from Theorem 3.2 (ii) it follows that

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-q\int_0^{\kappa_0^-}\mathbb{1}_{\{U_s<b\}}\mathrm{d}s}; \kappa_0^- < \kappa_c^+\right] \\
&= z^{(q)}(x;0) - q\int_b^x \mathbb{W}(x-y)z^{(q)}(y;0)\mathrm{d}y \\
&\quad + \frac{z^{(q)}(c;0) - q\int_b^c \mathbb{W}(c-y)z^{(q)}(y;0)\mathrm{d}y}{w^{(q)}(c;0) - q\int_b^c \mathbb{W}(c-y)w^{(q)}(y;0)\mathrm{d}y} \\
&\quad \times \left(w^{(q)}(x;0) - q\int_b^x \mathbb{W}(x-y)w^{(q)}(y;0)\mathrm{d}y\right),
\end{aligned} \tag{3.15}$$

the only thing we need to show is that

$$\begin{aligned}
&\lim_{c\to\infty}\frac{z^{(q)}(c;0) - q\int_b^c \mathbb{W}(c-y)z^{(q)}(y;0)\mathrm{d}y}{w^{(q)}(c;0) - q\int_b^c \mathbb{W}(c-y)w^{(q)}(y;0)\mathrm{d}y} \\
&= \frac{(\mathbb{E}[X_1]-\alpha) + q\int_0^b(\mathbb{Z}^{(q)}(y) - \alpha W^{(q)}(y)\mathbb{Z}^{(q)}(b-y))\mathrm{d}y}{Z^{(q)}(b) - \alpha W^{(q)}(b)}.
\end{aligned} \tag{3.16}$$

For $c > b$, by using Corollary 2.15, we have that

$$\begin{aligned}
&w^{(q)}(c;0) - q\int_b^c \mathbb{W}(c-y)w^{(q)}(y;0)\mathrm{d}y \\
&= (1-\alpha W^{(q)}(0))\mathbb{W}^{(q)}(c) - \alpha\int_0^b \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y \\
&\quad - q\int_b^c \mathbb{W}(c-y)\left[(1-\alpha W^{(q)}(0))\mathbb{W}^{(q)}(y) - \alpha\int_0^b \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\right]\mathrm{d}y \\
&= (1-\alpha W^{(q)}(0))\left[\mathbb{W}^{(q)}(c) - q\int_b^c \mathbb{W}(c-y)\mathbb{W}^{(q)}(y)\mathrm{d}y\right] \\
&\quad - \alpha\left[\int_0^b \mathbb{W}^{(q)}(c-y)W^{(q)\prime}(y)\mathrm{d}y - \int_0^b W^{(q)\prime}(z)q\int_b^c \mathbb{W}(c-y)\mathbb{W}^{(q)}(y-z)\mathrm{d}y\mathrm{d}z\right].
\end{aligned} \tag{3.17}$$

From (2.52) it follows that

$$\begin{aligned}
&q\int_b^c \mathbb{W}(c-y)\mathbb{W}^{(q)}(y)\mathrm{d}y \\
&= q\int_0^c \mathbb{W}(c-y)\mathbb{W}^{(q)}(y)\mathrm{d}y - q\int_0^b \mathbb{W}(c-y)\mathbb{W}^{(q)}(y)\mathrm{d}y \\
&= \mathbb{W}^{(q)}(c) - \mathbb{W}(c) - q\int_0^b \mathbb{W}(c-y)\mathbb{W}^{(q)}(y)\mathrm{d}y
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
q\int_b^c \mathbb{W}(c-y)\mathbb{W}^{(q)}(y-z)\mathrm{d}y &= q\int_{b-z}^{c-z} \mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}y \\
&= q\int_0^{c-z} \mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}y - q\int_0^{b-z} \mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}y \\
&= \mathbb{W}^{(q)}(c-z) - \mathbb{W}(c-z) - q\int_0^{b-z} \mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}y.
\end{aligned} \tag{3.19}$$

Applying (3.18) and (3.19) in (3.17) gives that

$$\begin{aligned}
&(1-\alpha W^{(q)}(0))\left[\mathbb{W}^{(q)}(c) - q\int_b^c \mathbb{W}(c-y)\mathbb{W}^{(q)}(y)\mathrm{d}y\right] \\
&\quad - \alpha\left[\int_0^b \mathbb{W}^{(q)}(c-y)W^{(q)\prime}(y)\mathrm{d}y - \int_0^b W^{(q)\prime}(z)q\int_b^c \mathbb{W}(c-y)\mathbb{W}^{(q)}(y-z)\mathrm{d}y\mathrm{d}z\right] \\
&= (1-\alpha W^{(q)}(0))\left[\mathbb{W}(c) + q\int_0^b \mathbb{W}(c-y)\mathbb{W}^{(q)}(y)\mathrm{d}y\right] \\
&\quad - \alpha\int_0^b W^{(q)\prime}(z)\left[\mathbb{W}(c-z) + q\int_0^{b-z} \mathbb{W}(c-z-y)\mathbb{W}^{(q)}(y)\mathrm{d}y\right]\mathrm{d}z.
\end{aligned}$$

Since the function $\mathbb{W}$ is non-decreasing, we can use the monotone convergence theorem together with the definition

$$\mathbb{Z}^{(q)}(x) = 1 + q\int_0^x \mathbb{W}^{(q)}(y)\mathrm{d}y \tag{3.20}$$

and $\lim_{c\to\infty}\frac{\mathbb{W}(c-y)}{\mathbb{W}(c)} = 1$, which follows from the fact that scale functions are continuous, non-degreasing and $\mathbb{W}(x) < \infty$ for $x \geq 0$, and get that

$$\begin{aligned}
&\lim_{c\to\infty}\frac{w^{(q)}(c;0) - q\int_b^c \mathbb{W}(c-y)w^{(q)}(y;0)\mathrm{d}y}{\mathbb{W}(c)}\\
&= \lim_{c\to\infty}\left\{\frac{(1-\alpha W^{(q)}(0))\left[\mathbb{W}(c) + q\int_0^b \mathbb{W}(c-y)\mathbb{W}^{(q)}(y)\mathrm{d}y\right]}{\mathbb{W}(c)}\right.\\
&\quad\left. - \frac{\alpha\int_0^b W^{(q)\prime}(z)\left[\mathbb{W}(c-z) + q\int_0^{b-z}\mathbb{W}(c-z-y)\mathbb{W}^{(q)}(y)\mathrm{d}y\right]\mathrm{d}z}{\mathbb{W}(c)}\right\}\\
&= (1-\alpha W^{(q)}(0))\mathbb{Z}^{(q)}(b) - \alpha\int_0^b W^{(q)\prime}(z)\mathbb{Z}^{(q)}(b-z)\mathrm{d}z.
\end{aligned}$$

By using integration by parts, relation (3.20), Lemma 2.13 for case $p = q$ and relation (2.3) we get that

$$\begin{aligned}
&(1-\alpha W^{(q)}(0))\mathbb{Z}^{(q)}(x) - \alpha\int_0^x W^{(q)\prime}(y)\mathbb{Z}^{(q)}(x-y)\mathrm{d}y\\
&= (1-\alpha W^{(q)}(0))\mathbb{Z}^{(q)}(x) + \alpha W^{(q)}(0)\mathbb{Z}^{(q)}(x) - \alpha W^{(q)}(x) - \alpha\int_0^x W^{(q)}(y)q\mathbb{W}^{(q)}(x-y)\mathrm{d}y\\
&= \mathbb{Z}^{(q)}(x) - \alpha W^{(q)}(x) - q\alpha\int_0^x \mathbb{W}^{(q)}(x-y)W^{(q)}(y)\mathrm{d}y\\
&= 1 + q\int_0^x \mathbb{W}^{(q)}(y)\mathrm{d}y - \alpha W^{(q)}(x) - q\alpha\int_0^x \mathbb{W}^{(q)}(x-y)W^{(q)}(y)\mathrm{d}y\\
&= 1 + q\int_0^x \mathbb{W}^{(q)}(y)\mathrm{d}y - \alpha W^{(q)}(x) - q\int_0^x \mathbb{W}^{(q)}(y)\mathrm{d}y + q\int_0^x W^{(q)}(y)\mathrm{d}y\\
&= 1 - \alpha W^{(q)}(x) + q\int_0^x W^{(q)}(y)\mathrm{d}y\\
&= Z^{(q)}(x) - \alpha W^{(q)}(x).
\end{aligned}$$

Now we have that

$$\lim_{c\to\infty}\frac{w^{(q)}(c;0) - q\int_b^c \mathbb{W}(c-y)w^{(q)}(y;0)\mathrm{d}y}{\mathbb{W}(c)} = Z^{(q)}(x) - \alpha W^{(q)}(x). \tag{3.21}$$

Also for $c > b$, by using Corollary 2.15 and the definitions of z and Z, which we presented in (2.43) and (2.3), we have that

$$\begin{aligned}
&z^{(q)}(c;0) - q\int_b^c \mathbb{W}(c-y)z^{(q)}(y;0)\mathrm{d}y \\
&= Z^{(q)}(c) + \alpha q\int_b^c \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y \\
&\quad - q\int_b^c \mathbb{W}(c-y)\left[Z^{(q)}(y) + \alpha q\int_b^y \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\right]\mathrm{d}y \\
&= 1 + q\int_0^c W^{(q)}(y)\mathrm{d}y + \alpha q\int_b^c \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y \\
&\quad - q\int_b^c \mathbb{W}(c-y)\left[1 + q\int_0^y W^{(q)}(z)\mathrm{d}z + \alpha q\int_b^y \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\right]\mathrm{d}y \\
&= 1 + q\int_0^c W^{(q)}(y)\mathrm{d}y + \alpha q\int_0^c \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y - \alpha q\int_0^b \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y \\
&\quad - q\int_b^c \mathbb{W}(c-y)\mathrm{d}y - q\int_b^c \mathbb{W}(c-y)\left[q\int_0^y W^{(q)}(z)\mathrm{d}z + \alpha q\int_b^y \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\right]\mathrm{d}y.
\end{aligned}$$

By using (3.20) we get that

$$\begin{aligned}
&1 + q\int_0^c W^{(q)}(y)\mathrm{d}y + \alpha q\int_0^c \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y - \alpha q\int_0^b \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y \\
&\quad - q\int_b^c \mathbb{W}(c-y)\mathrm{d}y - q\int_b^c \mathbb{W}(c-y)\left[q\int_0^y W^{(q)}(z)\mathrm{d}z + \alpha q\int_b^y \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\right]\mathrm{d}y \\
&= 1 + q\int_0^b \mathbb{W}(c-y)\mathbb{Z}^{(q)}(y)\mathrm{d}y - q\int_0^b \mathbb{W}(c-y)\mathrm{d}y - q\int_0^b \mathbb{W}(c-y)q\int_0^y \mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&\quad + q\int_0^c W^{(q)}(y)\mathrm{d}y + \alpha q\int_0^c \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y - \alpha q\int_0^b \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y \\
&\quad - q\int_b^c \mathbb{W}(c-y)\mathrm{d}y - q\int_b^c \mathbb{W}(c-y)q\int_0^y W^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&\quad - q\int_b^c \mathbb{W}(c-y)\alpha q\int_b^y \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y.
\end{aligned}$$

By setting $p = 0$ and $\alpha = 0$, which means that $\mathbb{W}$ and W are equal, we get from Corollary 2.14 that

$$q\int_0^x \mathbb{W}(x-z)\mathbb{W}^{(q)}(z)\mathrm{d}z = \mathbb{W}^{(q)}(x) - \mathbb{W}(x)$$

so that

(3.22)

$$\begin{aligned}
&-\alpha q\int_0^b \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y \\
&= -\alpha q\int_0^b \mathbb{W}(c-y)W^{(q)}(y)\mathrm{d}y - \alpha q\int_0^b q\int_0^{c-y} \mathbb{W}(c-y-z)\mathbb{W}^{(q)}(z)\mathrm{d}zW^{(q)}(y)\mathrm{d}y
\end{aligned}$$

By using the relation (3.22) and the fact that $\int_0^c \mathbb{W}(c-y)\mathrm{d}y = \int_0^c \mathbb{W}(z)\mathrm{d}z$, by change of variable, we get that

$$
\begin{aligned}
&1+q\int_0^b \mathbb{W}(c-y)\mathbb{Z}^{(q)}(y)\mathrm{d}y - q\int_0^b \mathbb{W}(c-y)\mathrm{d}y - q\int_0^b \mathbb{W}(c-y)q\int_0^y \mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y\\
&\quad +q\int_0^c W^{(q)}(y)\mathrm{d}y + \alpha q\int_0^c \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y - \alpha q\int_0^b \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y\\
&\quad -q\int_b^c \mathbb{W}(c-y)\mathrm{d}y - q\int_b^c \mathbb{W}(c-y)q\int_0^y W^{(q)}(z)\mathrm{d}z\mathrm{d}y\\
&\quad -q\int_b^c \mathbb{W}(c-y)\alpha q\int_b^y \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y\\
&= 1+q\int_0^b \mathbb{W}(c-y)\mathbb{Z}^{(q)}(y)\mathrm{d}y - \alpha q\int_0^b \mathbb{W}(c-y)W^{(q)}(y)\mathrm{d}y\\
&\quad +q\int_0^c W^{(q)}(y)\mathrm{d}y - q\int_0^c \mathbb{W}(y)\mathrm{d}y + \alpha q\int_0^c \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y\\
&\quad -q\int_0^c \mathbb{W}(c-y)q\int_0^y \mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y - \alpha q\int_0^b q\int_0^{c-z} \mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}yW^{(q)}(z)\mathrm{d}z\\
&\quad -q\int_b^c \mathbb{W}(c-y)\alpha q\int_b^y \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y.
\end{aligned}
$$

(3.23)

$$
\begin{aligned}
&= 1+q\int_0^b \mathbb{W}(c-y)\mathbb{Z}^{(q)}(y)\mathrm{d}y - \alpha q\int_0^b \mathbb{W}(c-y)W^{(q)}(y)\mathrm{d}y\\
&\quad -\alpha q^2\int_0^b W^{(q)}(z)\int_0^{b-z} \mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}y\mathrm{d}z\\
&\quad +\Bigg[\alpha q^2\int_0^b W^{(q)}(z)\int_0^{b-z} \mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}y\mathrm{d}z\\
&\quad +q\int_0^c \left[W^{(q)}(y)-\mathbb{W}(y)\right]\mathrm{d}y + \alpha q\int_0^c \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y\\
&\quad -q\int_0^c \mathbb{W}(c-y)q\int_0^y \mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y - \alpha q^2\int_0^b W^{(q)}(z)\int_0^{c-z} \mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}y\mathrm{d}z\\
&\quad -q\int_b^c \mathbb{W}(c-y)\alpha q\int_b^y \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y\Bigg].
\end{aligned}
$$

From (3.23) it follows that

$$
\begin{aligned}
(3.24)\qquad & z^{(q)}(c;0) - q\int_b^c \mathbb{W}(c-y)z^{(q)}(y;0)\mathrm{d}y\\
&= 1+q\int_0^b \mathbb{W}(c-y)\mathbb{Z}^{(q)}(y)\mathrm{d}y - \alpha q\int_0^b \mathbb{W}(c-y)W^{(q)}(y)\mathrm{d}y\\
&\quad -\alpha q^2\int_0^b W^{(q)}(z)\int_0^{b-z} \mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}y\mathrm{d}z,
\end{aligned}
$$

if we only can show that the expression in $\big[\quad\big]$ is equal to 0. This is what we do next. Putting in Corollary 2.14 $p = 0$ and integrating the equation, then using Fubini and integration by parts with respect to z, we get that

$$
\begin{aligned}
&q\int_0^c \big[W^{(q)}(y) - \mathbb{W}(y)\big]\mathrm{d}y \\
&= q\int_0^c q\int_0^y \mathbb{W}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y - q\int_0^c \alpha\left[W^{(q)}(0)\mathbb{W}(y) + \int_0^y \mathbb{W}(y-z)W^{(q)\prime}(z)\mathrm{d}z\right]\mathrm{d}y \\
&= q\int_0^c q\int_0^y \mathbb{W}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&\quad - q\alpha\int_0^c W^{(q)}(0)\mathbb{W}(y)\mathrm{d}y - q\alpha\int_0^c\int_0^y \mathbb{W}(y-z)W^{(q)\prime}(z)\mathrm{d}z\mathrm{d}y \\
&= q\int_0^c q\int_0^y \mathbb{W}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&\quad - q\alpha\int_0^c W^{(q)}(0)\mathbb{W}(y)\mathrm{d}y - q\alpha\int_0^c\int_z^c \mathbb{W}(y-z)W^{(q)\prime}(z)\mathrm{d}y\mathrm{d}z \\
&= q\int_0^c q\int_0^y \mathbb{W}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&\quad - q\alpha\left[\int_0^c W^{(q)}(0)\mathbb{W}(y)\mathrm{d}y + \int_0^c W^{(q)\prime}(z)\int_0^{c-z}\mathbb{W}(y)\mathrm{d}y\mathrm{d}z\right] \\
&= q\int_0^c q\int_0^y \mathbb{W}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y - \alpha q\int_0^c \mathbb{W}(c-z)W^{(q)}(z)\mathrm{d}z.
\end{aligned}
$$

With the help of above equality proceed with $\big[\quad\big]$ from (3.23) as follows:

(3.25)

$$
\begin{aligned}
A :=& \alpha q^2\int_0^b W^{(q)}(z)\int_0^{b-z}\mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}y\mathrm{d}z \\
&+ q\int_0^c \big[W^{(q)}(y) - \mathbb{W}(y)\big]\mathrm{d}y + \alpha q\int_0^c \mathbb{W}^{(q)}(c-y)W^{(q)}(y)\mathrm{d}y \\
&- q\int_0^c \mathbb{W}(c-y)q\int_0^y \mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y - \alpha q^2\int_0^b W^{(q)}(z)\int_0^{c-z}\mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}y\mathrm{d}z \\
&- q\int_b^c \mathbb{W}(c-y)\alpha q\int_b^y \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
=& \alpha q^2\int_0^b W^{(q)}(z)\int_0^{b-z}\mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}y\mathrm{d}z \\
&+ q\int_0^c q\int_0^y \mathbb{W}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&- q\int_0^c \mathbb{W}(c-y)q\int_0^y \mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y - \alpha q^2\int_0^b W^{(q)}(z)\int_0^{c-z}\mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}y\mathrm{d}z \\
&- q\int_b^c \mathbb{W}(c-y)\alpha q\int_b^y \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&= -\alpha q^2 \int_0^b W^{(q)}(z) \int_{b-z}^{c-z} \mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}y\mathrm{d}z \\
&\quad - \alpha q^2 \int_b^c \mathbb{W}(c-y) \int_b^y \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&\quad + q^2 \int_0^c \int_0^y \mathbb{W}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y - q^2 \int_0^c \mathbb{W}(c-y) \int_0^y \mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y.
\end{aligned}
$$

We substitute $\overline{y} = y - z$ and then use Fubini and change $y = \overline{y}$ again to get

$$
\begin{aligned}
&= -\alpha q^2 \int_0^b W^{(q)}(z) \int_{b-z}^{c-z} \mathbb{W}(c-y-z)\mathbb{W}^{(q)}(y)\mathrm{d}y\mathrm{d}z \\
&= -\alpha q^2 \int_0^b W^{(q)}(z) \int_b^c \mathbb{W}(c-\overline{y})\mathbb{W}^{(q)}(\overline{y}-z)\mathrm{d}\overline{y}\mathrm{d}z \\
&= -\alpha q^2 \int_b^c \mathbb{W}(c-y) \int_0^b \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y,
\end{aligned}
$$

so that we get

$$
\begin{aligned}
A &= -\alpha q^2 \int_b^c \mathbb{W}(c-y) \int_0^y \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&\quad + q^2 \int_0^c \int_0^y \mathbb{W}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y - q^2 \int_0^c \mathbb{W}(c-y) \int_0^y \mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y.
\end{aligned} \tag{3.26}
$$

By using Corollary 2.14 for $p = q$, Fubini and integration by parts we get that

$$
\begin{aligned}
&q^2 \int_0^c \mathbb{W}(c-y) \int_0^y \mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&= q^2 \int_0^c \mathbb{W}(c-y) \int_0^y \left[W^{(q)}(z) + \alpha\left(W^{(q)}(0)\mathbb{W}^{(q)}(z) + \int_0^z \mathbb{W}^{(q)}(z-r)W^{(q)\prime}(r)\mathrm{d}r \right) \right] \mathrm{d}z\mathrm{d}y \\
&= q^2 \int_0^c \mathbb{W}(c-y) \int_0^y W^{(q)}(z)\mathrm{d}z\mathrm{d}y + q^2 \int_0^c \mathbb{W}(c-y) \int_0^y \alpha W^{(q)}(0)\mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&\quad + q^2 \int_0^c \mathbb{W}(c-y) \int_0^y \alpha \int_0^z \mathbb{W}^{(q)}(z-r)W^{(q)\prime}(r)\mathrm{d}r\mathrm{d}z\mathrm{d}y \\
&= q^2 \int_0^c \mathbb{W}(c-y) \int_0^y W^{(q)}(z)\mathrm{d}z\mathrm{d}y + q^2 \int_0^c \mathbb{W}(c-y) \int_0^y \alpha W^{(q)}(0)\mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&\quad + q^2 \int_0^c \mathbb{W}(c-y) \int_0^y \alpha W^{(q)\prime}(r) \int_r^y \mathbb{W}^{(q)}(z-r)\mathrm{d}z\mathrm{d}r\mathrm{d}y \\
&= q^2 \int_0^c \mathbb{W}(c-y) \int_0^y W^{(q)}(z)\mathrm{d}z\mathrm{d}y + q^2 \int_0^c \mathbb{W}(c-y) \int_0^y \alpha W^{(q)}(0)\mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&\quad + q^2 \int_0^c \mathbb{W}(c-y) \int_0^y \alpha W^{(q)\prime}(r) \int_0^{y-r} \mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}r\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&= q^2 \int_0^c \mathbb{W}(c-y) \int_0^y W^{(q)}(z)\mathrm{d}z\mathrm{d}y + q^2 \int_0^c \mathbb{W}(c-y) \int_0^y \alpha W^{(q)}(0)\mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&\quad - q^2 \int_0^c \mathbb{W}(c-y)\alpha W^{(q)}(0) \int_0^y \mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y - q^2 \int_0^c \alpha\mathbb{W}(c-y) \int_0^y W^{(q)}(r)\mathbb{W}^{(q)}(y-r)\mathrm{d}r\mathrm{d}y \\
&= q^2 \int_0^c \mathbb{W}(c-y) \int_0^y W^{(q)}(z)\mathrm{d}z\mathrm{d}y - \alpha q^2 \int_0^c \mathbb{W}(c-y) \int_0^y \mathbb{W}^{(q)}(y-r)W^{(q)}(r)\mathrm{d}r\mathrm{d}y.
\end{aligned} \tag{3.27}
$$

Inserting (3.27) in to (3.26) and by using Fubini and integration by parts it follows that

$$
\begin{aligned}
&- \alpha q^2 \int_b^c \mathbb{W}(c-y) \int_0^y \mathbb{W}^{(q)}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&\quad + q^2 \int_0^c \int_0^y \mathbb{W}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y - q^2 \int_0^c \mathbb{W}(c-y) \int_0^y \mathbb{W}^{(q)}(z)\mathrm{d}z\mathrm{d}y. \\
&= q^2 \int_0^c \int_0^y \mathbb{W}(y-z)W^{(q)}(z)\mathrm{d}z\mathrm{d}y - q^2 \int_0^c \mathbb{W}(c-y) \int_0^y W^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&= q^2 \int_0^c W^{(q)}(z) \int_z^c \mathbb{W}(y-z)\mathrm{d}y\mathrm{d}z - q^2 \int_0^c \mathbb{W}(c-y) \int_0^y W^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&= q^2 \int_0^c W^{(q)}(z) \int_0^{c-z} \mathbb{W}(y)\mathrm{d}y\mathrm{d}z - q^2 \int_0^c \mathbb{W}(c-y) \int_0^y W^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&= q^2 \int_0^c \mathbb{W}(c-y) \int_0^y W^{(q)}(z)\mathrm{d}z\mathrm{d}y - q^2 \int_0^c \mathbb{W}(c-y) \int_0^y W^{(q)}(z)\mathrm{d}z\mathrm{d}y \\
&= 0,
\end{aligned}
$$

which is what we wanted. Now we have that the equality (3.24) holds. By using monotone convergence and the definition of $\mathbb{Z}$ it follows that

$$
\begin{aligned}
&\lim_{c\to\infty} \frac{z^{(q)}(c;0) - q\int_b^c \mathbb{W}(c-y)z^{(q)}(y;0)\mathrm{d}y}{\mathbb{W}(c)} \\
&= \lim_{c\to\infty} \frac{1 + q\int_0^b \mathbb{W}(c-y)\mathbb{Z}^{(q)}(y)\mathrm{d}y}{\mathbb{W}(c)} \\
&\quad + \lim_{c\to\infty} \frac{\alpha q\int_0^b W^{(q)}(z)\big[\mathbb{W}(c-z) + q\int_0^{b-z}\mathbb{W}(c-z-y)\mathbb{W}^{(q)}(y)\mathrm{d}y\big]\mathrm{d}z}{\mathbb{W}(c)} \\
&= \lim_{c\to\infty} \frac{1}{\mathbb{W}(c)} + q\int_0^b \mathbb{Z}^{(q)}(y)\mathrm{d}y - \alpha q \int_0^b W^{(q)}(z)\big[1 + q\int_0^{b-z}\mathbb{W}^{(q)}(y)\mathrm{d}y\big]\mathrm{d}z \\
&= \big(\mathbb{E}[X_1] - \alpha\big) + q\int_0^b \big[\mathbb{Z}(y) - \alpha W^{(q)}(y)\mathbb{Z}^{(q)}(b-y)\big]\mathrm{d}y,
\end{aligned} \tag{3.28}
$$

where in the last equality we also used the equality (3.10). From (3.21) and (3.28) it follows that the equality (3.16) holds and hence the proof is done.

Part (ii): First we notice that

$$
\mathbb{E}_x\Big[e^{-q\int_0^{\kappa_0^-}\mathbb{1}_{\{U_s<b\}}\mathrm{d}s}; \kappa_0^- = \infty\Big] = \lim_{c\to\infty}\mathbb{E}_x\Big[e^{-q\int_0^{\kappa_0^-}\mathbb{1}_{\{U_s<b\}}\mathrm{d}s}; \kappa_0^- > \kappa_c^+\Big].
$$

From Theorem 3.2 (i) it follows that

$$\mathbb{E}_x\left[e^{-q\int_0^{\kappa_0^-}\mathbb{1}_{\{U_s<b\}}\mathrm{d}s};\kappa_0^->\kappa_c^+\right]=\frac{w^{(q)}(x;0)-q\int_b^x\mathbb{W}(x-y)w^{(q)}(y;0)\mathrm{d}y}{w^{(q)}(c;0)-q\int_b^c\mathbb{W}(c-y)w^{(q)}(y;0)\mathrm{d}y}.$$

In part (i) we have shown that

$$\lim_{c\to\infty}\frac{w^{(q)}(c;0)-q\int_b^c\mathbb{W}(c-y)w^{(q)}(y;0)\mathrm{d}y}{\mathbb{W}(c)}=Z^{(q)}(x)-\alpha W^{(q)}(x).$$

Hence we get by using the equality (3.10) that

$$\begin{aligned}&\lim_{c\to\infty}\mathbb{E}_x\left[e^{-q\int_0^{\kappa_0^-}\mathbb{1}_{\{U_s<b\}}\mathrm{d}s};\kappa_0^->\kappa_c^+\right]\\&=\lim_{c\to\infty}\frac{1}{\mathbb{W}(c)}\frac{\mathbb{W}(c)\big[w^{(q)}(x;0)-q\int_b^x\mathbb{W}(x-y)w^{(q)}(y;0)\mathrm{d}y\big]}{w^{(q)}(c;0)-q\int_b^c\mathbb{W}(c-y)w^{(q)}(y;0)\mathrm{d}y}\\&=(\mathbb{E}[X_1]-\alpha)\frac{w^{(q)}(x;0)-q\int_b^x\mathbb{W}(x-y)w^{(q)}(y;0)\mathrm{d}y}{Z^{(q)}(x)-\alpha W^{(q)}(x)},\end{aligned}$$

which is what we needed to show. □

In the model given in the beginning of this section we have that the probability of bankruptcy with the initial capital $x\geq 0$ is

$$\mathbb{P}_x(\rho_\omega<\infty)=\mathbb{P}_x(0\leq U_{\rho_\omega}<b,\rho_\omega<\infty)+\mathbb{P}_x(U_{\rho_\omega}<0,\rho_\omega<\infty), \tag{3.29}$$

where for the probabilities on the right hand side we have the equalities (3.12) and (3.13) which we can now compute with Corollary 3.3. Consequently we can also derive the survival probability of the risk process U, which is

$$\mathbb{P}_x(\rho_\omega=\infty)=1-\mathbb{P}_x(\rho_\omega<\infty), \tag{3.30}$$

where in the right hand side we can apply (3.29).

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