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**Author(s):** Lahti, Panu

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# A NEW CARTAN-TYPE PROPERTY AND STRICT QUASICOVERINGS WHEN $p = 1$ IN METRIC SPACES

Panu Lahti

University of Jyväskylä, Department of Mathematics and Statistics  
P.O. Box 35, FI-40014 University of Jyväskylä, Finland; panu.k.lahti@jyu.fi

**Abstract.** In a complete metric space that is equipped with a doubling measure and supports a Poincaré inequality, we prove a new Cartan-type property for the fine topology in the case  $p = 1$ . Then we use this property to prove the existence of 1-finely open *strict subsets* and *strict quasicoverings* of 1-finely open sets. As an application, we study fine Newton–Sobolev spaces in the case  $p = 1$ , that is, Newton–Sobolev spaces defined on 1-finely open sets.

## 1. Introduction

Nonlinear fine potential theory in metric spaces has been studied in several papers in recent years, see [7, 8, 6]. Much of nonlinear potential theory, for  $1 < p < \infty$ , deals with  $p$ -harmonic functions, which are local minimizers of the  $L^p$ -norm of  $|\nabla u|$ . Such minimizers can be defined also in metric measure spaces by using *upper gradients*, and the notion can be extended to the case  $p = 1$  by considering functions of least gradient, which are BV functions that minimize the total variation locally; see Section 2 for definitions.

Nonlinear fine potential theory is concerned with studying  $p$ -harmonic functions and related superminimizers by means of the  *$p$ -fine topology*. For nonlinear fine potential theory and its history in the Euclidean setting, for  $1 < p < \infty$ , see especially the monographs [1, 15, 23], as well as the monograph [3] in the metric setting. The typical assumptions of a metric space, which we make also in this paper, are that the space is complete, equipped with a doubling measure, and supports a Poincaré inequality.

A central result in fine potential theory is the (*weak*) *Cartan property* for superminimizer functions. In [21] we proved the following formulation of this property in the case  $p = 1$ .

**Theorem 1.1.** [21, Theorem 1.1] *Let  $A \subset X$  and let  $x \in X \setminus A$  such that  $A$  is 1-thin at  $x$ . Then there exist  $R > 0$  and  $u_1, u_2 \in \text{BV}(X)$  that are 1-superminimizers in  $B(x, R)$  such that  $\max\{u_1^\wedge, u_2^\wedge\} = 1$  in  $A \cap B(x, R)$  and  $u_1^\vee(x) = 0 = u_2^\vee(x)$ .*

In [22] we used this property to prove the so-called Choquet property concerning finely open and *quasiopen* sets in the case  $p = 1$ , similarly as can be done when  $1 < p < \infty$  (see [8]). On the other hand, it is natural to consider an alternative version of the weak Cartan property. In the case  $p > 1$ , superminimizers are Newton–Sobolev functions, but in the case  $p = 1$  they are only BV functions and so the question arises whether the functions  $u_1, u_2$  above can be replaced by a Newton–Sobolev function

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(even though it would no longer be a superminimizer). In Theorem 3.11 we show that such a new Cartan-type property indeed holds.

It is said that a set  $A$  is a  $p$ -strict subset of a set  $D$  if there exists a Newton–Sobolev function  $u \in N^{1,p}(X)$  such that  $u = 1$  on  $A$  and  $u = 0$  on  $X \setminus D$ . In [7] it was shown that if  $U$  is a  $p$ -finely open set ( $1 < p < \infty$ ) and  $x \in U$ , then there exists a  $p$ -finely open strict subset  $V \Subset U$  such that  $x \in V$ . The proof was based on the weak Cartan property. In Theorem 4.3 we show that the analogous result is true in the case  $p = 1$ . Here we need the Cartan-type property involving a Newton–Sobolev function (instead of the BV superminimizer functions).

This result on the existence of 1-strict subsets can be combined with the *quasi-Lindelöf principle* to prove the existence of *strict quasicoverings* of 1-finely open sets, that is, countable coverings by 1-finely open strict subsets. We do this in Proposition 5.4, and it is again analogous to the case  $1 < p < \infty$ , see [7]. Such coverings will be useful in future research when considering partition of unity arguments in finely open sets. In this paper, we apply strict quasicoverings in defining and studying *fine Newton–Sobolev spaces*, that is, Newton–Sobolev spaces defined on finely open or quasiopen sets. In the case  $1 < p < \infty$ , these were studied in [7]. In Section 5 we show that the theory we have developed allows us to prove directly analogous results in the case  $p = 1$ .

## 2. Preliminaries

In this section we introduce the notation, definitions, and assumptions used in the paper. Throughout this paper,  $(X, d, \mu)$  is a complete metric space that is equipped with a metric  $d$  and a Borel regular outer measure  $\mu$  that satisfies a doubling property, meaning that there exists a constant  $C_d \geq 1$  such that

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty$$

for every ball  $B(x, r) := \{y \in X : d(y, x) < r\}$ . We also assume that  $X$  supports a  $(1, 1)$ -Poincaré inequality defined below, and that  $X$  contains at least 2 points. For a ball  $B = B(x, r)$  and  $a > 0$ , we sometimes abbreviate  $aB := B(x, ar)$ ; note that in metric spaces, a ball (as a set) does not necessarily have a unique center and radius, but we will always understand these to be predetermined for the balls that we consider. By iterating the doubling condition, we obtain for any  $x \in X$  and any  $y \in B(x, R)$  with  $0 < r \leq R < \infty$  that

$$(2.1) \quad \frac{\mu(B(y, r))}{\mu(B(x, R))} \geq \frac{1}{C_d^2} \left(\frac{r}{R}\right)^Q,$$

where  $Q > 1$  only depends on the doubling constant  $C_d$ . When we want to state that a constant  $C$  depends on the parameters  $a, b, \dots$ , we write  $C = C(a, b, \dots)$ . When a property holds outside a set of  $\mu$ -measure zero, we say that it holds almost everywhere, abbreviated a.e.

As a complete metric space equipped with a doubling measure,  $X$  is proper, that is, closed and bounded sets are compact. For any  $\mu$ -measurable set  $D \subset X$ , we define  $\text{Lip}_{\text{loc}}(D)$  to be the space of functions  $u$  on  $D$  such that for every  $x \in D$  there exists  $r > 0$  such that  $u \in \text{Lip}(D \cap B(x, r))$ . For an open set  $\Omega \subset X$ , a function  $u \in \text{Lip}_{\text{loc}}(\Omega)$  is then in  $\text{Lip}(\Omega')$  for every open  $\Omega' \Subset \Omega$ ; this notation means that  $\overline{\Omega'}$  is a compact subset of  $\Omega$ . Other local spaces of functions are defined analogously.

For any  $A \subset X$  and  $0 < R < \infty$ , the restricted Hausdorff content of codimension one is defined to be

$$\mathcal{H}_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i \leq R \right\}.$$

The codimension one Hausdorff measure of  $A \subset X$  is then defined to be

$$\mathcal{H}(A) := \lim_{R \rightarrow 0} \mathcal{H}_R(A).$$

All functions defined on  $X$  or its subsets will take values in  $[-\infty, \infty]$ . By a curve we mean a nonconstant rectifiable continuous mapping from a compact interval of the real line into  $X$ . A nonnegative Borel function  $g$  on  $X$  is an upper gradient of a function  $u$  on  $X$  if for all curves  $\gamma$ , we have

$$(2.2) \quad |u(x) - u(y)| \leq \int_{\gamma} g \, ds,$$

where  $x$  and  $y$  are the end points of  $\gamma$  and the curve integral is defined by using an arc-length parametrization, see [16, Section 2] where upper gradients were originally introduced. We interpret  $|u(x) - u(y)| = \infty$  whenever at least one of  $|u(x)|, |u(y)|$  is infinite.

Let  $1 \leq p < \infty$ ; we are going to work solely with  $p = 1$ , but we give definitions that cover all values of  $p$  where it takes no extra work. We say that a family of curves  $\Gamma$  is of zero  $p$ -modulus if there is a nonnegative Borel function  $\rho \in L^p(X)$  such that for all curves  $\gamma \in \Gamma$ , the curve integral  $\int_{\gamma} \rho \, ds$  is infinite. A property is said to hold for  $p$ -almost every curve if it fails only for a curve family with zero  $p$ -modulus. If  $g$  is a nonnegative  $\mu$ -measurable function on  $X$  and (2.2) holds for  $p$ -almost every curve, we say that  $g$  is a  $p$ -weak upper gradient of  $u$ . By only considering curves  $\gamma$  in a set  $D \subset X$ , we can talk about a function  $g$  being a ( $p$ -weak) upper gradient of  $u$  in  $D$ .

Let  $D \subset X$  be a  $\mu$ -measurable set. We define the norm

$$\|u\|_{N^{1,p}(D)} := \|u\|_{L^p(D)} + \inf \|g\|_{L^p(D)},$$

where the infimum is taken over all  $p$ -weak upper gradients  $g$  of  $u$  in  $D$ . The usual Sobolev space  $W^{1,p}$  is replaced in the metric setting by the Newton–Sobolev space

$$N^{1,p}(D) := \{u : \|u\|_{N^{1,p}(D)} < \infty\},$$

which was first introduced in [25]. We understand every Newton–Sobolev function to be defined at every  $x \in D$  (even though  $\|\cdot\|_{N^{1,p}(D)}$  is then only a seminorm). It is known that for any  $u \in N^{1,p}_{\text{loc}}(D)$ , there exists a minimal  $p$ -weak upper gradient of  $u$  in  $D$ , always denoted by  $g_u$ , satisfying  $g_u \leq g$  a.e. on  $D$  for any  $p$ -weak upper gradient  $g \in L^p_{\text{loc}}(D)$  of  $u$  in  $D$ , see [3, Theorem 2.25].

For any  $D \subset X$ , the space of Newton–Sobolev functions with zero boundary values is defined to be

$$N^{1,p}_0(D) := \{u|_D : u \in N^{1,p}(X) \text{ and } u = 0 \text{ on } X \setminus D\}.$$

This is a subspace of  $N^{1,p}(D)$  when  $D$  is  $\mu$ -measurable, and it can always be understood to be a subspace of  $N^{1,p}(X)$ .

The  $p$ -capacity of a set  $A \subset X$  is

$$\text{Cap}_p(A) := \inf \|u\|_{N^{1,p}(X)},$$

where the infimum is taken over all functions  $u \in N^{1,p}(X)$  such that  $u \geq 1$  on  $A$ . If a property holds outside a set  $A \subset X$  with  $\text{Cap}_p(A) = 0$ , we say that it holds  $p$ -quasieverywhere, or  $p$ -q.e. If  $D \subset X$  is  $\mu$ -measurable, then

$$(2.3) \quad \|u\|_{N^{1,p}(D)} = 0 \quad \text{if } u = 0 \text{ } p\text{-q.e. on } D,$$

see [3, Proposition 1.61].

We know that  $\text{Cap}_p$  is an outer capacity, meaning that

$$\text{Cap}_p(A) = \inf_{\substack{W \text{ open} \\ A \subset W}} \text{Cap}_p(W)$$

for any  $A \subset X$ , see e.g. [3, Theorem 5.31]. By [14, Theorem 4.3, Theorem 5.1], for any  $A \subset X$  it holds that

$$(2.4) \quad \text{Cap}_1(A) = 0 \quad \text{if and only if } \mathcal{H}(A) = 0.$$

We say that a set  $U \subset X$  is  $p$ -quasiopen if for every  $\varepsilon > 0$  there is an open set  $G \subset X$  such that  $\text{Cap}_p(G) < \varepsilon$  and  $U \cup G$  is open. We say that a function  $u$  defined on a set  $D \subset X$  is  $p$ -quasicontinuous on  $D$  if for every  $\varepsilon > 0$  there is an open set  $G \subset X$  such that  $\text{Cap}_p(G) < \varepsilon$  and  $u|_{D \setminus G}$  is continuous (as a real-valued function). It is a well-known fact that Newton–Sobolev functions are quasicontinuous; for a proof of the following theorem, see [10, Theorem 1.1] or [3, Theorem 5.29].

**Theorem 2.5.** *Let  $\Omega \subset X$  be open and let  $u \in N^{1,p}(\Omega)$  (with  $1 \leq p < \infty$ ). Then  $u$  is  $p$ -quasicontinuous on  $\Omega$ .*

The variational  $p$ -capacity of a set  $A \subset D$  with respect to  $D \subset X$  is given by

$$\text{cap}_p(A, D) := \inf \int_X g_u^p d\mu,$$

where the infimum is taken over functions  $u \in N_0^{1,p}(D)$  such that  $u \geq 1$  on  $A$ , and  $g_u$  is the minimal  $p$ -weak upper gradient of  $u$  (in  $X$ ). By truncation, we see that we can also assume that  $0 \leq u \leq 1$  on  $X$  (and the same applies to the  $p$ -capacity). For basic properties satisfied by capacities, such as monotonicity and countable subadditivity, see [3, 4].

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, following [24]. See also the monographs [2, 11, 12, 13, 26] for the classical theory in the Euclidean setting. Let  $\Omega \subset X$  be an open set. Given  $u \in L_{\text{loc}}^1(\Omega)$ , the total variation of  $u$  in  $\Omega$  is defined to be

$$\|Du\|(\Omega) := \inf \left\{ \liminf_{i \rightarrow \infty} \int_{\Omega} g_{u_i} d\mu : u_i \in \text{Lip}_{\text{loc}}(\Omega), u_i \rightarrow u \text{ in } L_{\text{loc}}^1(\Omega) \right\},$$

where each  $g_{u_i}$  is the minimal 1-weak upper gradient of  $u_i$  in  $\Omega$ . (In [24], local Lipschitz constants were used instead of upper gradients, but the properties of the total variation can be proved similarly with either definition.) We say that a function  $u \in L^1(\Omega)$  is of bounded variation, and denote  $u \in \text{BV}(\Omega)$ , if  $\|Du\|(\Omega) < \infty$ . For an arbitrary set  $A \subset X$ , we define

$$\|Du\|(A) := \inf_{\substack{W \text{ open} \\ A \subset W}} \|Du\|(W).$$

If  $u \in L_{\text{loc}}^1(\Omega)$  and  $\|Du\|(\Omega) < \infty$ , then  $\|Du\|(\cdot)$  is a Radon measure on  $\Omega$  by [24, Theorem 3.4]. A  $\mu$ -measurable set  $E \subset X$  is said to be of finite perimeter if  $\|D\chi_E\|(X) < \infty$ , where  $\chi_E$  is the characteristic function of  $E$ . The perimeter of  $E$  in  $\Omega$  is also denoted by  $P(E, \Omega) := \|D\chi_E\|(\Omega)$ .

The lower and upper approximate limits of a function  $u$  on  $X$  are defined respectively by

$$u^\wedge(x) := \sup \left\{ t \in \mathbf{R} : \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \{u < t\})}{\mu(B(x, r))} = 0 \right\}$$

and

$$u^\vee(x) := \inf \left\{ t \in \mathbf{R} : \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \{u > t\})}{\mu(B(x, r))} = 0 \right\}.$$

Unlike Newton–Sobolev functions, we understand BV functions to be  $\mu$ -equivalence classes. To consider fine properties, we need to consider the pointwise representatives  $u^\wedge$  and  $u^\vee$ .

We will assume throughout the paper that  $X$  supports a  $(1, 1)$ -Poincaré inequality, meaning that there exist constants  $C_P > 0$  and  $\lambda \geq 1$  such that for every ball  $B(x, r)$ , every  $u \in L^1_{\text{loc}}(X)$ , and every upper gradient  $g$  of  $u$ , we have

$$(2.6) \quad \int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq C_P r \int_{B(x, \lambda r)} g \, d\mu,$$

where

$$u_{B(x, r)} := \int_{B(x, r)} u \, d\mu := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u \, d\mu.$$

The  $(1, 1)$ -Poincaré inequality implies the following Sobolev inequality: if  $x \in X$ ,  $0 < r < \frac{1}{4} \text{diam } X$ , and  $u \in N_0^{1,1}(B(x, r))$ , then

$$(2.7) \quad \int_{B(x, r)} |u| \, d\mu \leq C_S r \int_{B(x, r)} g_u \, d\mu$$

for some constant  $C_S = C_S(C_d, C_P) \geq 1$ , see [3, Theorem 5.51]. By applying this to approximating functions in the definition of the total variation, we obtain for any  $x \in X$ ,  $0 < r < \frac{1}{4} \text{diam } X$ , and any  $\mu$ -measurable set  $E \subset B(x, r)$

$$(2.8) \quad \mu(E) \leq C_S r P(E, X).$$

Next we define the fine topology in the case  $p = 1$ .

**Definition 2.9.** We say that  $A \subset X$  is 1-thin at the point  $x \in X$  if

$$\lim_{r \rightarrow 0} r \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} = 0.$$

We say that a set  $U \subset X$  is 1-finely open if  $X \setminus U$  is 1-thin at every  $x \in U$ . Then we define the 1-fine topology as the collection of 1-finely open sets on  $X$  (see [20, Lemma 4.2] for a proof of the fact that this is indeed a topology).

We denote the 1-fine interior of a set  $H \subset X$ , i.e. the largest 1-finely open set contained in  $H$ , by  $\text{fine-int } H$ . We denote the 1-fine closure of  $H \subset X$ , i.e. the smallest 1-finely closed set containing  $H$ , by  $\overline{H}^1$ . We define the 1-base  $b_1 H$  of  $H \subset X$  to be the set of points in  $X$  where  $H$  is not 1-thin. We say that a function  $u$  defined on a set  $U \subset X$  is 1-finely continuous at  $x \in U$  if it is continuous at  $x$  when  $U$  is equipped with the induced 1-fine topology on  $U$  and  $[-\infty, \infty]$  is equipped with the usual topology.

By [3, Proposition 6.16], for all  $x \in X$  and  $0 < r < \frac{1}{8} \text{diam } X$  (in fact, the second inequality holds for all  $r > 0$ )

$$(2.10) \quad \frac{\mu(B(x, r))}{2C_S r} \leq \text{cap}_1(B(x, r), B(x, 2r)) \leq \frac{C_d \mu(B(x, r))}{r},$$

and so obviously  $W \subset b_1W$  for any open set  $W \subset X$ .

The following fact is given in [19, Proposition 3.3]:

$$(2.11) \quad \text{Cap}_1(\overline{A}^1) = \text{Cap}_1(A) \quad \text{for any } A \subset X.$$

The following result describes the close relationship between finely open and quasiopen sets.

**Theorem 2.12.** [22, Corollary 6.12] *A set  $U \subset X$  is 1-quasiopen if and only if it is the union of a 1-finely open set and a  $\mathcal{H}$ -negligible set.*

For an open set  $\Omega \subset X$ , we denote by  $\text{BV}_c(\Omega)$  the class of functions  $\varphi \in \text{BV}(\Omega)$  with compact support in  $\Omega$ , that is,  $\text{spt } \varphi \Subset \Omega$ .

**Definition 2.13.** We say that  $u \in \text{BV}_{\text{loc}}(\Omega)$  is a 1-minimizer in  $\Omega$  if for all  $\varphi \in \text{BV}_c(\Omega)$ ,

$$(2.14) \quad \|Du\|(\text{spt } \varphi) \leq \|D(u + \varphi)\|(\text{spt } \varphi).$$

We say that  $u \in \text{BV}_{\text{loc}}(\Omega)$  is a 1-superminimizer in  $\Omega$  if (2.14) holds for all nonnegative  $\varphi \in \text{BV}_c(\Omega)$ .

More precisely, we should talk about  $\text{spt } |\varphi|^\vee$ , since  $\varphi$  is only a.e. defined. In the literature, 1-minimizers are usually called functions of least gradient.

### 3. A new Cartan-type property

In this section we prove the new Cartan-type property, given in Theorem 3.11. First we take note of a few results that we will need in the proofs; the following is given in [3, Lemma 11.22].

**Lemma 3.1.** *Let  $x \in X$ ,  $r > 0$ , and  $A \subset B(x, r)$ . Then for every  $1 < s < t$  with  $tr < \frac{1}{4} \text{diam } X$ , we have*

$$\text{cap}_1(A, B(x, tr)) \leq \text{cap}_1(A, B(x, sr)) \leq C_S \left(1 + \frac{t}{s-1}\right) \text{cap}_1(A, B(x, tr)),$$

where  $C_S$  is the constant from the Sobolev inequality (2.7).

**Theorem 3.2.** [21, Theorem 3.11] *Let  $u$  be a 1-superminimizer in an open set  $\Omega \subset X$ . Then  $u^\wedge : \Omega \rightarrow (-\infty, \infty]$  is lower semicontinuous.*

As mentioned in the introduction, in [21] we proved a weak Cartan property for  $p = 1$ , more precisely in the following form.

**Theorem 3.3.** [21, Theorem 5.2] *Let  $A \subset X$  and let  $x \in X \setminus A$  be such that  $A$  is 1-thin at  $x$ . Then there exist  $R > 0$  and  $E_0, E_1 \subset X$  such that  $\chi_{E_0}, \chi_{E_1} \in \text{BV}(X)$ ,  $\chi_{E_0}$  and  $\chi_{E_1}$  are 1-superminimizers in  $B(x, R)$ ,  $\max\{\chi_{E_0}^\wedge, \chi_{E_1}^\wedge\} = 1$  in  $A \cap B(x, R)$ ,  $\chi_{E_0}^\vee(x) = 0 = \chi_{E_1}^\vee(x)$ ,  $\{\max\{\chi_{E_0}^\vee, \chi_{E_1}^\vee\} > 0\}$  is 1-thin at  $x$ , and*

$$\lim_{r \rightarrow 0} r \frac{P(E_0, B(x, r))}{\mu(B(x, r))} = 0, \quad \lim_{r \rightarrow 0} r \frac{P(E_1, B(x, r))}{\mu(B(x, r))} = 0.$$

Now we collect a few facts that are not included in the above statement, but are given in the proof in [21]. Defining  $B_j := B(x, 2^{-j}R)$  and  $H_j := B_j \setminus \frac{9}{10}\overline{B_{j+1}}$  for  $j = 0, 1, \dots$ , there exists an open set  $W \supset A$  that is 1-thin at  $x$ ,

$$(3.4) \quad W \cap \bigcup_{j=0,2,\dots} H_j \subset E_0 \quad \text{and} \quad W \cap \bigcup_{j=1,3,\dots} H_j \subset E_1,$$

and (see [21, Eq. (5.4)])

$$(3.5) \quad \begin{aligned} E_0 &\subset \left(\frac{3}{2}B_0 \setminus \frac{4}{5}B_1\right) \cup \bigcup_{j=2,4,\dots}^{\infty} \frac{5}{4}B_j \setminus \frac{4}{5}B_{j+1} \quad \text{and} \\ E_1 &\subset \left(\frac{3}{2}B_1 \setminus \frac{4}{5}B_2\right) \cup \bigcup_{j=3,5,\dots}^{\infty} \frac{5}{4}B_j \setminus \frac{4}{5}B_{j+1}. \end{aligned}$$

Moreover, by [21, Eq. (5.5)], for all  $i = 2, 4, 6, \dots$  we have

$$(3.6) \quad P(E_0 \cap \frac{5}{4}B_i, X) \leq 5C_S \operatorname{cap}_1(W \cap B_i, 2B_i),$$

and similarly for all  $i = 3, 5, 7, \dots$ ,

$$(3.7) \quad P(E_1 \cap \frac{5}{4}B_i, X) \leq 5C_S \operatorname{cap}_1(W \cap B_i, 2B_i).$$

From the proof it can also be seen that if  $R > 0$  is chosen to be smaller, all of the above results still hold. The same will then apply to the conclusion of the next lemma. Let  $B_j$  and  $H_j$  be defined as above.

**Lemma 3.8.** *Let  $A \subset X$  and let  $x \in X \setminus A$  be such that  $A$  is 1-thin at  $x$ . Then there exists a number  $R > 0$ , an open set  $W \supset A$  that is 1-thin at  $x$ , and open sets  $F_j \supset W \cap H_j$  such that  $F_j \subset \frac{5}{4}B_j \setminus \frac{3}{4}B_{j+1}$  for all  $j = 0, 1, \dots$ , and*

$$(3.9) \quad \sum_{j=i}^{\infty} P(F_j, X) \leq 50C_S^2 \operatorname{cap}_1(W \cap B_i, 2B_i)$$

for all  $i = 0, 1, \dots$

*Proof.* By using the weak Cartan property (Theorem 3.3), choose  $R > 0$  and  $E_0, E_1 \subset X$  such that  $\chi_{E_0}, \chi_{E_1} \in \operatorname{BV}(X)$  and  $\chi_{E_0}$  and  $\chi_{E_1}$  are 1-superminimizers in  $B(x, R)$ . We can assume that  $R < \frac{1}{2} \operatorname{diam} X$ . Also let  $W \supset A$  be an open set that is 1-thin at  $x$ , as described above. Define

$$F_j := \{\chi_{E_0}^\wedge > 0\} \cap \frac{5}{4}B_j \setminus \frac{3}{4}\overline{B_{j+1}} \quad \text{for } j = 2, 4, \dots,$$

and

$$F_j := \{\chi_{E_1}^\wedge > 0\} \cap \frac{5}{4}B_j \setminus \frac{3}{4}\overline{B_{j+1}} \quad \text{for } j = 3, 5, \dots$$

By (3.4), we have  $F_j \supset W \cap H_j$  for all  $j = 2, 3, \dots$  as desired. The sets  $F_j$  are open by Theorem 3.2. By Lebesgue’s differentiation theorem, the sets  $\{\chi_{E_0}^\wedge > 0\}$  and  $\{\chi_{E_1}^\wedge > 0\}$  differ from  $E_0$  and  $E_1$ , respectively, only by a set of  $\mu$ -measure zero. Thus by (3.5) and the fact that the sets  $F_j$  are at a positive distance from each other, we find that for all  $i = 2, 4, \dots$ ,

$$P(E_0 \cap \frac{5}{4}B_i, X) = P\left(\bigcup_{j=i,i+2,\dots} F_j, X\right) = \sum_{j=i,i+2,\dots} P(F_j, X),$$

and similarly for all  $i = 3, 5, \dots$ ,

$$P(E_1 \cap \frac{5}{4}B_i, X) = \sum_{j=i,i+2,\dots} P(F_j, X).$$



Combining these with (3.6) and (3.7), and using Lemma 3.1, we have for all  $i = 2, 3, \dots$

$$\begin{aligned} \sum_{j=i}^{\infty} P(F_j, X) &\leq 5C_S(\text{cap}_1(W \cap B_i, 2B_i) + \text{cap}_1(W \cap B_{i+1}, 2B_{i+1})) \\ &\leq 5C_S(\text{cap}_1(W \cap B_i, 2B_i) + 5C_S \text{cap}_1(W \cap B_{i+1}, 4B_{i+1})) \\ &\leq 25C_S^2(\text{cap}_1(W \cap B_i, 2B_i) + \text{cap}_1(W \cap B_i, 4B_{i+1})) \\ &= 50C_S^2 \text{cap}_1(W \cap B_i, 2B_i). \end{aligned}$$

Then by replacing  $R$  with  $R/4$ , we have the result. □

Recall the constant  $\lambda \geq 1$  from the Poincaré inequality (2.6). We have the following boxing inequality from [18, Theorem 3.1]. Note that in [18] it is assumed that  $\mu(X) = \infty$ , but the proof reveals that we can alternatively assume  $\mu(F) < \mu(X)/2$ .

**Theorem 3.10.** *Let  $F \subset X$  be an open set of finite perimeter with  $\mu(F) < \mu(X)/2$  (in particular,  $\mu(F)$  is finite). Then there exists a collection of balls  $\{B_k = B(x_k, r_k)\}_{k \in \mathbb{N}}$  such that the balls  $\lambda B_k$  are disjoint,  $F \subset \bigcup_{k=1}^{\infty} 5\lambda B_k$ ,*

$$\frac{1}{2C_d} \leq \frac{\mu(B_k \cap F)}{\mu(B_k)} \leq \frac{1}{2}$$

for all  $k \in \mathbb{N}$ , and

$$\sum_{k=1}^{\infty} \frac{\mu(5\lambda B_k)}{5\lambda r_k} \leq C_B P(F, X)$$

for some constant  $C_B = C_B(C_d, C_P, \lambda)$ .

Now we can show the following Cartan-type property.

**Theorem 3.11.** *Let  $A \subset X$  and let  $x \in X \setminus A$  be such that  $A$  is 1-thin at  $x$ . Then there exists a number  $R > 0$ , open sets  $G \subset V \subset X$ , and a function  $\eta \in N_0^{1,1}(V)$  such that  $A \cap B(x, R) \subset G$ ,  $V$  is 1-thin at  $x$ ,  $0 \leq \eta \leq 1$  on  $X$ ,  $\eta = 1$  on  $G$ , and*

$$(3.12) \quad \lim_{r \rightarrow 0} \frac{r}{\mu(B(x, r))} \|\eta\|_{N^{1,1}(B(x, r))} = 0.$$

*Proof.* Take  $R > 0$ , an open set  $W \supset A$ , and open sets  $F_j \subset \frac{5}{4}B_j \setminus \frac{3}{4}B_{j+1}$  as given by Lemma 3.8. Let

$$\delta := \frac{1}{2^8(680\lambda)^Q C_d^3 C_S^3},$$

where  $Q > 1$  is the exponent in (2.1). We can assume that  $R \leq \min\{1, \frac{1}{8} \text{diam } X\}$ . Since  $\mu(\{x\}) = 0$  (see [3, Corollary 4.3]), we can also assume  $R$  to be so small that  $\mu(\frac{5}{4}B_0) < \frac{1}{2}\mu(X)$ , and so also  $\mu(F_j) < \frac{1}{2}\mu(X)$  for all  $j = 0, 1, \dots$ . Since  $W$  is 1-thin at  $x$ , we can further assume that  $R$  is so small that

$$(3.13) \quad 2^{-j} R \frac{\text{cap}_1(W \cap B_j, 2B_j)}{\mu(B_j)} < \delta$$

for all  $j = 0, 1, \dots$ . Fix  $j$ . By the boxing inequality (Theorem 3.10) we find a collection of balls  $\{B_k^j = B(x_k^j, r_k^j)\}_{k=1}^{\infty}$  such that the balls  $\lambda B_k^j$  are disjoint,  $F_j \subset \bigcup_{k=1}^{\infty} 5\lambda B_k^j$ ,

$$(3.14) \quad \frac{1}{2C_d} \leq \frac{\mu(B_k^j \cap F_j)}{\mu(B_k^j)} \leq \frac{1}{2}$$

for all  $k \in \mathbf{N}$ , and

$$(3.15) \quad \sum_{k=1}^{\infty} \frac{\mu(5\lambda B_k^j)}{5\lambda r_k^j} \leq C_B P(F_j, X).$$

Thus we have

$$\begin{aligned} \mu(B_k^j) &\leq 2C_d \mu(B_k^j \cap F_j) \leq 2C_d \mu(F_j) \\ &\leq 2^{2-j} RC_d C_S P(F_j, X) \quad \text{by (2.8)} \\ &\leq 2^{8-j} RC_d C_S^3 \text{cap}_1(W \cap B_j, 2B_j) \quad \text{by (3.9)}. \end{aligned}$$

Thus for all  $k \in \mathbf{N}$ ,

$$(3.16) \quad \frac{\mu(B_k^j)}{\mu(B_j)} \leq 2^8 C_d C_S^3 2^{-j} R \frac{\text{cap}_1(W \cap B_j, 2B_j)}{\mu(B_j)} \leq 2^8 C_d C_S^3 \delta$$

by (3.13). By (3.14) we necessarily have  $F_j \cap B_k^j \neq \emptyset$  for all  $k \in \mathbf{N}$ , and so  $\frac{5}{4}B_j \cap B_k^j \neq \emptyset$ . Now if  $r_k^j \geq 2^{-j}R$  for some  $k \in \mathbf{N}$ , then  $B_j \subset 4B_k^j$  and so

$$\frac{\mu(B_k^j)}{\mu(B_j)} \geq \frac{1}{C_d^2},$$

contradicting (3.16) by our choice of  $\delta$ . Thus  $r_k^j \leq 2^{-j}R$  for all  $k \in \mathbf{N}$ , so that  $x_k^j \in 3B_j$ , and thus by (2.1),

$$\left( \frac{r_k^j}{2^{-j+2}R} \right)^Q \leq C_d^2 \frac{\mu(B_k^j)}{\mu(4B_j)} \leq C_d^2 \frac{\mu(B_k^j)}{\mu(B_j)} \leq 2^8 C_d^3 C_S^3 \delta$$

by (3.16), so that by our choice of  $\delta$ ,

$$(3.17) \quad r_k^j \leq (2^8 C_d^3 C_S^3 \delta)^{1/Q} 2^{-j+2}R = \frac{2^{-j}R}{170\lambda}.$$

Thus recalling that  $F_j \cap B_k^j \neq \emptyset$ , so that  $(\frac{5}{4}B_j \setminus \frac{3}{4}B_{j+1}) \cap B_k^j \neq \emptyset$ , we conclude that  $20\lambda B_k^j \subset B_{j-1} \setminus B_{j+2}$  (let  $B_{-1} := B(x, 2R)$ ). Now, define Lipschitz functions

$$\xi_k^j := \max \left\{ 0, 1 - \frac{\text{dist}(\cdot, 10\lambda B_k^j)}{10\lambda r_k^j} \right\}, \quad k \in \mathbf{N},$$

so that  $\xi_k^j = 1$  on  $10\lambda B_k^j$  and  $\xi_k^j = 0$  on  $X \setminus 20\lambda B_k^j$ . Using the basic properties of 1-weak upper gradients, see [3, Corollary 2.21], we obtain

$$\int_X g_{\xi_k^j} d\mu \leq \frac{\mu(20\lambda B_k^j)}{10\lambda r_k^j}.$$

Define  $V := \bigcup_{j=0}^\infty \bigcup_{k=1}^\infty 10\lambda B_k^j$ . Now for every  $i = 1, 2, \dots$ ,

$$\begin{aligned}
 \text{cap}_1(V \cap B_i, 4B_i) &\leq \text{cap}_1\left(\bigcup_{j=i-1}^\infty \bigcup_{k=1}^\infty 10\lambda B_k^j, 4B_i\right) \\
 &\leq \sum_{j=i-1}^\infty \sum_{k=1}^\infty \text{cap}_1(10\lambda B_k^j, 4B_i) \\
 (3.18) \quad &\leq \sum_{j=i-1}^\infty \sum_{k=1}^\infty \int_X g_{\xi_k^j} d\mu \leq \sum_{j=i-1}^\infty \sum_{k=1}^\infty \frac{\mu(20\lambda B_k^j)}{10\lambda r_k^j} \\
 &\leq C_d^2 C_B \sum_{j=i-1}^\infty P(F_j, X) \quad \text{by (3.15)} \\
 &\leq 50C_d^2 C_B C_S^2 \text{cap}_1(W \cap B_{i-1}, 2B_{i-1}) \quad \text{by (3.9)}.
 \end{aligned}$$

Thus

$$2^{-i} R \frac{\text{cap}_1(V \cap B_i, 4B_i)}{\mu(B_i)} \leq 50C_d^3 C_B C_S^2 2^{-i+1} R \frac{\text{cap}_1(W \cap B_{i-1}, 2B_{i-1})}{\mu(B_{i-1})} \rightarrow 0$$

as  $i \rightarrow \infty$ , since  $W$  is 1-thin at  $x$ . By Lemma 3.1 it is then straightforward to show that  $V$  is also 1-thin at  $x$ . Let us also define the Lipschitz functions

$$\eta_k^j := \max\left\{0, 1 - \frac{\text{dist}(\cdot, 5\lambda B_k^j)}{5\lambda r_k^j}\right\}, \quad j = 0, 1, \dots, \quad k = 1, 2, \dots,$$

so that  $\eta_k^j = 1$  on  $5\lambda B_k^j$  and  $\eta_k^j = 0$  on  $X \setminus 10\lambda B_k^j$ , and then let

$$\eta := \sup_{j=0,1,\dots, k=1,2,\dots} \eta_k^j.$$

Recall from Lemma 3.8 that  $\bigcup_{j=0}^\infty F_j \supset W \cap B(x, R)$ ; thus  $\eta \geq 1$  on

$$G := \bigcup_{j=0}^\infty \bigcup_{k=1}^\infty 5\lambda B_k^j \supset \bigcup_{j=0}^\infty F_j \supset W \cap B(x, R) \supset A \cap B(x, R).$$

By [3, Lemma 1.52] we know that  $g_\eta \leq \sum_{j=0}^\infty \sum_{k=1}^\infty g_{\eta_k^j}$ . Thus for any  $i = 1, 2, \dots$ ,

$$\begin{aligned}
 \int_{B_i} g_\eta d\mu &\leq \sum_{j=0}^\infty \sum_{k=1}^\infty \int_{B_i} g_{\eta_k^j} d\mu \leq \sum_{j=i-1}^\infty \sum_{k=1}^\infty \int_X g_{\eta_k^j} d\mu \\
 &\leq 50C_d C_B C_S^2 \text{cap}_1(W \cap B_{i-1}, 2B_{i-1}),
 \end{aligned}$$

where the last inequality follows just as in the last four lines of (3.18). Since we assumed  $R \leq 1$  and so  $5\lambda r_k^j \leq 1$  by (3.17), we similarly get

$$\|\eta\|_{L^1(B_i)} \leq 50C_d C_B C_S^2 \text{cap}_1(W \cap B_{i-1}, 2B_{i-1}).$$

Using the fact that  $W$  is 1-thin at  $x$  and the doubling property of  $\mu$ , we get (3.12). Estimating just as in the last four lines of (3.18), now with  $i = 1$ , we get

$$\int_X g_\eta d\mu \leq \sum_{j=0}^\infty \sum_{k=1}^\infty \int_X g_{\eta_k^j} d\mu \leq 50C_d C_B C_S^2 \text{cap}_1(W \cap B_0, 2B_0) < \infty.$$

Thus  $\eta \in N^{1,1}(X)$ . Clearly  $\eta = 0$  on  $X \setminus V$ , and so  $\eta \in N_0^{1,1}(V)$ . □

### 4. 1-strict subsets

In this section we study 1-strict subsets which are defined as follows.

**Definition 4.1.** A set  $A \subset D$  is a 1-strict subset of  $D \subset X$  if there is a function  $u \in N_0^{1,1}(D)$  such that  $u = 1$  on  $A$ .

Equivalently,  $A$  is a 1-strict subset of  $D$  if  $\text{cap}_1(A, D) < \infty$ . In [22, Proposition 6.7] we proved the following result by using the weak Cartan property (Theorem 3.3).

**Proposition 4.2.** Let  $U \subset X$  be 1-finely open and let  $x \in U$ . Then there exists a 1-finely open set  $W$  such that  $x \in W \subset U$ , and a function  $w \in \text{BV}(X)$  such that  $0 \leq w \leq 1$  on  $X$ ,  $w^\wedge = 1$  on  $W$ , and  $\text{spt } w \Subset U$ .

This kind of formulation is sufficient for some purposes, but now we are able to improve it by replacing  $w \in \text{BV}(X)$  with  $w \in N^{1,1}(X)$ . The following is our main result on the existence of 1-strict subsets.

**Theorem 4.3.** Let  $U \subset X$  be 1-finely open and let  $x \in U$ . Then there exists a 1-finely open set  $W$  such that  $x \in W \subset U$ , and a function  $w \in N_0^{1,1}(U)$  such that  $0 \leq w \leq 1$  on  $X$ ,  $w = 1$  on  $W$ , and  $\text{spt } w \Subset U$ . Moreover, if  $\text{Cap}_1(\{x\}) = 0$ , then  $\|w\|_{N^{1,1}(X)}$  can be made arbitrarily small.

*Proof.* Applying Theorem 3.11 with the choice  $A = X \setminus U$ , we find a number  $R > 0$ , open sets  $G \subset V \subset X$ , and a function  $\eta \in N_0^{1,1}(V)$  such that  $B(x, R) \subset G \cup U$ ,  $V$  is 1-thin at  $x$ ,  $0 \leq \eta \leq 1$  on  $X$ ,  $\eta = 1$  on  $G$ , and

$$\lim_{r \rightarrow 0} \frac{r}{\mu(B(x, r))} \|\eta\|_{N^{1,1}(B(x, r))} = 0.$$

Choose  $0 < r \leq R$  such that  $r \|\eta\|_{N^{1,1}(B(x, r))} / \mu(B(x, r)) \leq 1$  and let

$$\rho := \max \left\{ 0, 1 - \frac{4 \text{dist}(\cdot, B(x, r/2))}{r} \right\} \in \text{Lip}(X),$$

so that  $0 \leq \rho \leq 1$  on  $X$ ,  $\rho = 1$  on  $B(x, r/2)$ , and  $\text{spt } \rho \Subset B(x, r)$ . Then let  $w := (1 - \eta)\rho$ . By the Leibniz rule (see [3, Theorem 2.15]), we have  $w \in N^{1,1}(X)$  and

$$\int_X g_w d\mu = \int_{B(x, r)} g_w d\mu \leq \int_{B(x, r)} g_\eta d\mu + \int_{B(x, r)} g_\rho d\mu \leq \frac{\mu(B(x, r))}{r} + \frac{4\mu(B(x, r))}{r}.$$

Thus  $\|w\|_{N^{1,1}(X)} \leq (5/r + 1)\mu(B(x, r))$ . If  $\text{Cap}_1(\{x\}) = 0$ , then also  $\mathcal{H}(\{x\}) = 0$  by (2.4), and so we can make  $\mu(B(x, r))/r$  as small as we like by choosing suitable  $r$ . Then we can also make  $\|w\|_{N^{1,1}(X)}$  arbitrarily small.

Regardless of the value of  $\text{Cap}_1(\{x\})$ , the set  $V$  is 1-thin at  $x$ , that is,  $x \notin b_1V$ . Since  $V$  is open we have  $V \subset b_1V$ ; recall (2.10) and the comment after it. We know that  $\overline{V}^1 = V \cup b_1V$  by [19, Corollary 3.5], so in conclusion  $x \notin \overline{V}^1$ . Thus

$$W := B(x, r/2) \setminus \overline{V}^1 \subset \{w = 1\}$$

is a 1-finely open neighborhood of  $x$ . Finally,  $\text{spt } w$  is compact and

$$\text{spt } w \subset \text{spt } \rho \setminus G \subset (U \cup G) \setminus G \subset U,$$

so that  $\text{spt } w \Subset U$ . Clearly now  $w \in N_0^{1,1}(U)$ . □

Let us make a few more observations concerning 1-strict subsets. In general it is not clear which subsets  $A$  of a set  $D$  are 1-strict subsets. If  $A$  is a compact subset of

an open set  $D$ , we obviously have  $\text{cap}_1(A, D) < \infty$ , and the test function can even be chosen to be Lipschitz. When  $A$  is a compact subset of a 1-quasiopen set  $D$ , we cannot necessarily choose a Lipschitz test function but one might nonetheless suspect that  $\text{cap}_1(A, D) < \infty$ . However, this is not always the case.

**Example 4.4.** Let  $X = \mathbf{R}^2$  (unweighted), denote the origin by 0, and let

$$A := \bigcup_{j=1}^{\infty} A_j \cup \{0\} \quad \text{with} \quad A_j := \{2^{-j}\} \times [-1/(2j), 1/(2j)].$$

Denoting  $A_j^\varepsilon := \{x \in X : \text{dist}(x, A_j) < \varepsilon\}$ , with  $\varepsilon > 0$ , let

$$D := \bigcup_{j=1}^{\infty} D_j \cup \{0\} \quad \text{with} \quad D_j := A_j^{2^{-3j}}.$$

Since all the sets  $D_j$  are disjoint, it is straightforward to check that

$$\text{cap}_1(A, D) = \sum_{j=1}^{\infty} \text{cap}_1(A_j, D_j) = \sum_{j=1}^{\infty} \frac{1}{j} = \infty.$$

Now  $A$  is clearly a compact set, and  $D$  is 1-quasiopen since  $D \cup B(0, r)$  is an open set for every  $r > 0$ .

One can also make the sets  $A, D$  connected by adding the line  $(0, 1/2] \times \{0\}$  to  $A$ , and by adding e.g. the sets  $(2^{-j-1}, 2^{-j}) \times (-2^{-j-1}, 2^{-j-1})$  to  $D$ ; then we still have  $\text{cap}_1(A, D) = \infty$  but the calculation is somewhat more complicated.

The variational 1-capacity is an outer capacity in the following weak sense.

**Proposition 4.5.** *Let  $A \subset D \subset X$ . Then*

$$\text{cap}_1(A, D) = \inf_{\substack{V \text{ 1-quasiopen} \\ A \subset V \subset D}} \text{cap}_1(V, D).$$

*Proof.* We can assume that  $\text{cap}_1(A, D) < \infty$ . Fix  $0 < \varepsilon < 1$ . Take  $u \in N_0^{1,1}(D)$  such that  $u = 1$  on  $A$  and  $\int_X g_u d\mu < \text{cap}_1(A, D) + \varepsilon$ . The set  $V := \{u > 1 - \varepsilon\}$  is 1-quasiopen by Theorem 2.5, and

$$\text{cap}_1(V, D) \leq \int_X g_{u/(1-\varepsilon)} d\mu = \frac{\int_X g_u d\mu}{1-\varepsilon} \leq \frac{\text{cap}_1(A, D) + \varepsilon}{1-\varepsilon}.$$

Since  $0 < \varepsilon < 1$  was arbitrary, we have the result. □

Even though 1-quasiopen sets and 1-finely open sets are very closely related (recall Theorem 2.12), it is not clear whether the following holds.

**Open Problem.** If  $D \subset X$  and  $A \subset \text{fine-int } D$ , do we have

$$\text{cap}_1(A, D) = \inf_{\substack{V \text{ 1-finely open} \\ A \subset V \subset D}} \text{cap}_1(V, D)?$$

Note that according to Theorem 4.3, the above property does hold in the very special case when  $A$  is a point with 1-capacity zero.

Let us say that a set  $K \subset X$  is 1-quasiclosed if  $X \setminus K$  is 1-quasiopen. Now we can show that 1-strict subsets have the following continuity.

**Proposition 4.6.** *Let  $D \subset X$  and let  $K_1 \supset K_2 \supset \dots$  be bounded 1-quasiclosed subsets of  $D$  such that  $\text{cap}_1(K_1, D) < \infty$ . Then for  $K := \bigcap_{i=1}^{\infty} K_i$  we have*

$$\text{cap}_1(K, D) = \lim_{i \rightarrow \infty} \text{cap}_1(K_i, D).$$

We will show in Example 4.7 below that the assumption  $\text{cap}_1(K_1, D) < \infty$  is needed.

*Proof.* Fix  $\varepsilon > 0$ . By Proposition 4.5 we find a 1-quasiopen set  $V$  such that  $K \subset V \subset D$  and  $\text{cap}_1(V, D) < \text{cap}_1(K, D) + \varepsilon$ . For each  $j \in \mathbf{N}$  we find an open set  $\tilde{G}_j \subset X$  such that  $V \cup \tilde{G}_j$  is open and  $\text{Cap}_1(\tilde{G}_j) \rightarrow 0$  as  $j \rightarrow \infty$ . For each  $i, j \in \mathbf{N}$ , we find an open set  $G_{i,j} \subset X$  such that  $K_i \setminus G_{i,j}$  is compact and  $\text{Cap}_1(G_{i,j}) < 2^{-i-j}$ . Letting  $G_j := \tilde{G}_j \cup \bigcup_{i=1}^\infty G_{i,j}$  for each  $j \in \mathbf{N}$ , we have that each  $V \cup G_j$  is open, each  $K_i \setminus G_j$  is compact, and  $\text{Cap}_1(G_j) \rightarrow 0$  as  $j \rightarrow \infty$ . Then for each  $j \in \mathbf{N}$  we find a function  $w_j \in N^{1,1}(X)$  such that  $0 \leq w_j \leq 1$  on  $X$ ,  $w_j = 1$  on  $G_j$ , and  $\|w_j\|_{N^{1,1}(X)} \rightarrow 0$  as  $j \rightarrow \infty$ . Passing to a subsequence (not relabeled), we can assume that  $w_j \rightarrow 0$  a.e.

Since  $\text{cap}_1(K_1, D) < \infty$ , we find  $v \in N_0^{1,1}(D)$  such that  $0 \leq v \leq 1$  on  $X$  and  $v = 1$  on  $K_1$ . For each  $j \in \mathbf{N}$ , let  $\rho_j := vw_j$ . Then  $\|\rho_j\|_{L^1(X)} \rightarrow 0$  as  $j \rightarrow \infty$ , and by the Leibniz rule (see [3, Theorem 2.15]),

$$\int_X g_{\rho_j} \, d\mu \leq \int_X g_{w_j} \, d\mu + \int_X w_j g_v \, d\mu \rightarrow 0$$

as  $j \rightarrow \infty$ ; for the second term this follows from Lebesgue’s dominated convergence theorem. Thus  $\text{cap}_1(G_j \cap K_1, D) \rightarrow 0$ . Fix  $j \in \mathbf{N}$  such that  $\text{cap}_1(G_j \cap K_1, D) < \varepsilon$ . Since every  $K_i \setminus G_j$  is compact and  $V \cup G_j$  is open, for some  $i \in \mathbf{N}$  we have  $K_i \setminus G_j \subset V \cup G_j$ . Thus  $K_i \subset V \cup G_j$ . Then

$$\begin{aligned} \text{cap}_1(K_i, D) &\leq \text{cap}_1(V \cup (G_j \cap K_1), D) \leq \text{cap}_1(V, D) + \text{cap}_1(G_j \cap K_1, D) \\ &\leq \text{cap}_1(K, D) + \varepsilon + \text{cap}_1(G_j \cap K_1, D) \leq \text{cap}_1(K, D) + 2\varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary, the proof is concluded. □

**Example 4.7.** In the notation of Example 4.4, let  $K_i := \bigcup_{j=i}^\infty A_j \cup \{0\}$  for each  $i \in \mathbf{N}$ . These are compact sets and similarly as in Example 4.4 we find that  $\text{cap}_1(K_i, D) = \infty$  for every  $i \in \mathbf{N}$ . However,  $\text{cap}_1(K, D) = 0$  for  $K := \bigcap_{i=1}^\infty K_i = \{0\}$ , by the fact that a point has 1-capacity zero and by using (2.3).

### 5. Application to fine Sobolev spaces

Björn–Björn–Latvala [7] have studied different definitions of Newton–Sobolev spaces on quasiopen sets in metric spaces in the case  $1 < p < \infty$ . As an application of the theory we have developed, we show that the analogous results hold for  $p = 1$ .

First we prove the following fact in a very similar way as it is proved in the case  $1 < p < \infty$ , see [8, Theorem 1.4(b)] and [6, Theorem 4.9(b)]. Recall that a function  $u$  defined on a set  $U \subset X$  is 1-quasicontinuous on  $U$  if for every  $\varepsilon > 0$  there is an open set  $G \subset X$  such that  $\text{Cap}_1(G) < \varepsilon$  and  $u|_{U \setminus G}$  is continuous (as a real-valued function).

**Theorem 5.1.** *A function  $u$  on a 1-quasiopen set  $U$  is 1-quasicontinuous on  $U$  if and only if it is finite 1-q.e. and 1-finely continuous 1-q.e. on  $U$ .*

*Proof.* To prove one direction, suppose there is a set  $N \subset U$  such that  $\text{Cap}_1(N) = 0$  and  $u$  is finite and 1-finely continuous at every point in  $V := U \setminus N$ . By Theorem 2.12, we can assume that  $V$  is 1-finely open. Let  $\{(a_j, b_j)\}_{j=1}^\infty$  be an enumeration of all intervals in  $\mathbf{R}$  with rational endpoints and let

$$V_j := \{x \in V : a_j < u(x) < b_j\}.$$

By the 1-fine continuity of  $u$ , the sets  $V_j$  are 1-finely open. Hence by Theorem 2.12, they are also 1-quasiopen. Fix  $\varepsilon > 0$ . There are open sets  $G_j \subset X$  such that  $\text{Cap}_1(G_j) < 2^{-j-1}\varepsilon$  and each  $V_j \cup G_j$  is open. Since  $\text{Cap}_1$  is an outer capacity, there is also an open set  $G_N \supset N$  such that  $\text{Cap}_1(G_N) < \varepsilon/2$ . Now

$$G := G_N \cup \bigcup_{j=1}^{\infty} G_j$$

is an open set such that  $\text{Cap}_1(G) < \varepsilon$ , and  $u|_{U \setminus G}$  is continuous since  $V_j \cup G$  are open sets.

To prove the converse direction, by Theorem 2.12 we know that  $U = V \cup N$ , where  $V$  is 1-finely open and  $\mathcal{H}(N) = 0$ , and then also  $\text{Cap}_1(N) = 0$  by (2.4). By the quasicontinuity of  $u$ , for each  $j \in \mathbf{N}$  we find an open set  $G_j \subset X$  such that  $\text{Cap}_1(G_j) < 1/j$  and  $u|_{V \setminus G_j}$  is continuous. By (2.11), we have  $\text{Cap}_1(\overline{G_j}^{-1}) = \text{Cap}_1(G_j)$  for each  $j \in \mathbf{N}$ , and so the set

$$A := N \cup \bigcap_{j=1}^{\infty} \overline{G_j}^{-1}$$

satisfies  $\text{Cap}_1(A) = 0$ . If  $x \in U \setminus A$ , then  $x \in V \setminus \overline{G_j}^{-1}$  for some  $j \in \mathbf{N}$ . Since  $V \setminus \overline{G_j}^{-1}$  is a 1-finely open set and  $u|_{V \setminus \overline{G_j}^{-1}}$  is continuous, it follows that  $u$  is finite and 1-finely continuous at  $x$ .  $\square$

We will need the following quasi-Lindelöf principle from [22].

**Theorem 5.2.** [22, Theorem 5.2] *For every family  $\mathcal{V}$  of 1-finely open sets there is a countable subfamily  $\mathcal{V}'$  such that*

$$\text{Cap}_1 \left( \bigcup_{V \in \mathcal{V}} V \setminus \bigcup_{V' \in \mathcal{V}'} V' \right) = 0.$$

From now on,  $U \subset X$  is always a 1-quasiopen set. Note that 1-quasiopen sets are  $\mu$ -measurable by [5, Lemma 9.3].

**Definition 5.3.** A family  $\mathcal{B}$  of 1-quasiopen sets is a *1-quasicovering* of  $U$  if it is countable,  $\bigcup_{V \in \mathcal{B}} V \subset U$ , and  $\text{Cap}_1(U \setminus \bigcup_{V \in \mathcal{B}} V) = 0$ . If every  $V \in \mathcal{B}$  is a 1-finely open 1-strict subset of  $U$  with  $V \Subset U$ , then  $\mathcal{B}$  is a *1-strict quasicovering* of  $U$ . Moreover, we say that

1.  $u \in N_{\text{fine-loc}}^{1,1}(U)$  if  $u \in N^{1,1}(V)$  for every 1-finely open 1-strict subset  $V \Subset U$ ;
2.  $u \in N_{\text{quasi-loc}}^{1,1}(U)$  (respectively  $L_{\text{quasi-loc}}^1(U)$ ) if there is a 1-quasicovering  $\mathcal{B}$  of  $U$  such that  $u \in N^{1,1}(V)$  (respectively  $L^1(V)$ ) for every  $V \in \mathcal{B}$ .

**Proposition 5.4.** *There exists a 1-strict quasicovering  $\mathcal{B}$  of  $U$ . Moreover, the associated Newton–Sobolev functions can be chosen compactly supported in  $U$ .*

*Proof.* By Theorem 2.12, we have  $U = V \cup N$ , where  $V$  is 1-finely open and  $\mathcal{H}(N) = 0$ , and then also  $\text{Cap}_1(N) = 0$  by (2.4). For every  $x \in V$ , by Theorem 4.3 we find a 1-finely open set  $V_x \ni x$  such that  $V_x \Subset V$  and an associated function  $v_x \in N_0^{1,1}(V)$  such that  $0 \leq v_x \leq 1$  on  $X$ ,  $v_x = 1$  on  $V_x$ , and  $\text{spt } v_x \Subset V$ . The collection  $\mathcal{B}' := \{V_x\}_{x \in V}$  covers  $V$ , and by the quasi-Lindelöf principle (Theorem 5.2) and the fact that  $\text{Cap}_1(U \setminus V) = 0$ , there exists a countable subcollection  $\mathcal{B} \subset \mathcal{B}'$  such that  $\text{Cap}_1(U \setminus \bigcup_{V_x \in \mathcal{B}} V_x) = 0$ .  $\square$

It follows that  $N_{\text{fine-loc}}^{1,1}(U) \subset N_{\text{quasi-loc}}^{1,1}(U)$ . From now on, since the proofs given in [7] in the case  $1 < p < \infty$  apply almost verbatim also in our setting, we will only point out the differences with [7].

**Theorem 5.5.** *Let  $u \in N_{\text{quasi-loc}}^{1,1}(U)$ . Then  $u$  is finite 1-q.e. and 1-finely continuous 1-q.e. on  $U$ . Thus  $u$  is also 1-quasicontinuous on  $U$ .*

*Proof.* Follow verbatim the proof of [7, Theorem 4.4], except that replace the reference to [7, Proposition 4.2] by Proposition 5.4, and the references to [6, Theorem 4.9(b)] and [8, Theorem 1.4(b)] by Theorem 5.1. □

**Definition 5.6.** A nonnegative function  $\tilde{g}_u$  on  $U$  is a 1-fine upper gradient of  $u \in N_{\text{quasi-loc}}^{1,1}(U)$  if there is a quasicovering  $\mathcal{B}$  of  $U$  such that for every  $V \in \mathcal{B}$ ,  $u \in N^{1,1}(V)$  and  $\tilde{g}_u = g_{u,V}$  a.e. on  $V$ , where  $g_{u,V}$  is the minimal 1-weak upper gradient of  $u$  in  $V$ .

**Lemma 5.7.** *If  $u \in N_{\text{quasi-loc}}^{1,1}(U)$ , then it has a unique (in the a.e. sense) 1-fine upper gradient  $\tilde{g}_u$ .*

*Proof.* Follow verbatim the proof of [7, Lemma 5.2]. □

**Theorem 5.8.** *If  $u \in N_{\text{quasi-loc}}^{1,1}(U)$  and  $\tilde{g}_u$  is a 1-fine upper gradient of  $u$ , then  $\tilde{g}_u$  is a 1-weak upper gradient of  $u$  in  $U$ .*

*Proof.* Follow verbatim the proof of [7, Theorem 5.3]. □

**Proposition 5.9.** *If  $u \in N_{\text{quasi-loc}}^{1,1}(U)$ , then there is a 1-strict quasicovering  $\mathcal{B}$  of  $U$  such that for every  $V \in \mathcal{B}$ , there exists  $u_V \in N^{1,1}(X)$  with  $u = u_V$  on  $V$ .*

*Proof.* Follow verbatim the proof of [7, Proposition 5.5], except that replace the reference to [7, Theorem 4.4] by Theorem 5.5, and [7, Proposition 4.2] by Proposition 5.4. □

The following definition is originally from Kilpeläinen–Malý [17].

**Definition 5.10.** Let  $U \subset \mathbf{R}^n$ . A function  $u \in L^1(U)$  is in  $W^{1,1}(U)$  if

1. there is a quasicovering  $\mathcal{B}$  of  $U$  such that for every  $V \in \mathcal{B}$  there is an open set  $G_V \supset V$  and  $u_V \in W^{1,1}(G_V)$  such that  $u = u_V$  on  $V$ , and
2. the fine gradient  $\nabla u$ , defined by  $\nabla u = \nabla u_V$  a.e. on each  $V \in \mathcal{B}$ , also belongs to  $L^1(U)$ .

Moreover, let

$$\|u\|_{W^{1,1}(U)} := \int_U (|u| + |\nabla u|) \, dx.$$

Recall that we constantly assume  $U$  to be a 1-quasiopen set.

**Theorem 5.11.** *Let  $U \subset \mathbf{R}^n$ . Then  $u \in W^{1,1}(U)$  if and only if there exists  $v \in N^{1,1}(U)$  such that  $v = u$  a.e. on  $U$ . Moreover,  $g_v = |\nabla u|$  a.e. on  $U$  and  $\|v\|_{N^{1,1}(U)} = \|u\|_{W^{1,1}(U)}$ .*

Here  $g_v$  is the minimal 1-weak upper gradient of  $v$  in  $U$ .

*Proof.* Follow verbatim the proof of [7, Theorem 5.7], except that replace the reference to [7, Proposition 5.5] by Proposition 5.9, [3, Proposition A.12] by [3, Corollary A.4], and [7, Theorem 5.4] by Theorem 5.8. □



Returning momentarily to the metric space setting, define the space

$$\widehat{N}^{1,1}(U) := \{u: u = v \text{ a.e. for some } v \in N^{1,1}(U)\}.$$

**Theorem 5.12.** *Let  $u \in \widehat{N}^{1,1}(U)$ . Then  $u \in N^{1,1}(U)$  if and only if  $u$  is 1-quasicontinuous on  $U$ .*

*Proof.* Assume that  $u$  is 1-quasicontinuous on  $U$ . There is  $v \in N^{1,1}(U)$  such that  $u = v$  a.e. on  $U$ . By Theorem 5.5,  $v$  is 1-quasicontinuous on  $U$ . By [3, Proposition 5.23] and [9, Proposition 4.2],  $u = v$  1-q.e. on  $U$ , and thus  $u \in N^{1,1}(U)$  by (2.3).

The converse follows from Theorem 5.5. □

**Theorem 5.13.** *Let  $U \subset \mathbf{R}^n$ , and let  $u$  be an everywhere defined function on  $U$ . Then  $u \in N^{1,1}(U)$  if and only if  $u \in W^{1,1}(U)$  and  $u$  is 1-quasicontinuous. Moreover, then  $\|u\|_{N^{1,1}(U)} = \|u\|_{W^{1,1}(U)}$ .*

*Proof.* This follows from Theorems 5.11 and 5.12. □

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