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DEPARTMENT OF MATHEMATICS
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APPLICATIONS OF THE QUASIHYPERBOLIC METRIC

DEBANJAN NANDI

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Jyväskylä, August 2018

Debanjan Nandi

List of included articles

This dissertation consists of an introductory part and the following articles:

- [A] P. Koskela, D. Nandi, A. Nicolau. Accessible parts of boundary for simply connected domains. Preprint; https://arxiv.org/abs/1709.06802
- [B] P. Goldstein, C. Guo, P. Koskela, D. Nandi. Characterizations of generalized John domains in $\mathbb{R}^n$ via metric duality. Preprint; https://arxiv.org/abs/1710.02050
- [C] D. Nandi, T. Rajala, T. Schultz. A density result for homogeneous Sobolev spaces on planar domains. Preprint; https://arxiv.org/abs/1801.02824
- [D] D. Nandi. A density result for homogeneous Sobolev spaces. Preprint; https://arxiv.org/abs/1803.08715

In the introductory part these articles will be referred to as [A], [B], [C] and [D] whereas other references will be referred as [1], [2],...

The author of this dissertation has actively taken part in the research of the joint articles [A, B, C].

Preface

A basic fact in real analysis says that if every continuous real-valued function defined in $E \subset \mathbb{R}^n$ is bounded, then E is compact and therefore each continuous function in E is uniformly continuous. There seems to be a recurrent and fruitful scheme: studying 'geometric' properties satisfied by suitable classes of functions with given regularity, defined in a set, helps investigating geometric or topological restrictions on the set; on the other hand, suitable geometric properties of sets indicate better regularity for certain classes of functions or mappings defined in the set. Among the numerous instances of this phenomenon, we mention for example the property of admitting a local Poincaré type inequality by Lipschitz functions defined in a set in a suitable metric space, which under appropriate conditions imposes further geometric restrictions on the set (linear local connectedness; see Heinonen-Koskela [19], for example). Such properties prove useful in studying quasiconformal mappings defined in these sets (see [19]).

The present dissertation may be considered to fit roughly into this scheme. For instance, the analytical properties of conformal parametrizations are used in [A] to study boundary accessibility in simply connected domains from which a pointwise weighted Hardy inequality for compactly supported smooth functions can be deduced. The approach in [A] is of course classical and provides quantitative estimates for the size of the accessible boundary. In the other direction, certain hyperbolic type domains are shown to be suitable for the density of functions with bounded derivatives in the (homogeneous) Sobolev classes defined in them (see [D], and also [C]). The article [B] also contributes in this direction where we study domains which have good geometric properties (overlapping with the other articles in the dissertation) from the point of view mentioned above of admitting nice properties for classes of functions and mappings defined in them. Hyperbolicity, or existence of suitable metrics with 'negative curvature' is the key property in many of the domains we consider. We will introduce next some of the concepts being used in this dissertation and then discuss some of the motivation and possible applications of the included articles.

1. Background

1.1. **Quasihyperbolic metric.** The quasihyperbolic metric, a generalization of the hyperbolic metric of simply connected planar domains was introduced in Gehring-Palka [8] in 1976. Given a domain $\Omega \subsetneq \mathbb{R}^n$ and points $x, y \in \Omega$, the quasihyperbolic distance between x and y is defined as

$$k_\Omega(x, y) = \inf_\gamma \int \frac{|dz|}{\mathrm{d}_\Omega(z)},$$

where the infimum is taken over all rectifiable curves joining x and y in Ω, the integral is the line integral of the density $1/\mathrm{d}_\Omega(\cdot)$ over such a curve γ and $\mathrm{d}_\Omega(z)$ is the distance from any $z \in \Omega$ to the boundary $\partial\Omega$. It coincides with the hyperbolic metric when Ω is the half plane and is comparable to it in the general case by an application of the Koebe-distortion theorem (see for example Pommerenke [31]). It follows from the Arzela-Ascoli theorem that (Ω, k_Ω) is a geodesic space.

1.2. **John and Ball separation.**

Definition 1.1 (Ball separation property)**.** Let $\Omega \subsetneq \mathbb{R}^n$ be a domain. A curve $\Gamma \subset \Omega$ satisfies the c-ball separation property, for some $c \geq 1$, if for every point $z \in \Gamma$, the intrinsic ball $B = B_\Omega(z, c\mathrm{d}_\Omega(z))$ separates any $x, y \in \Gamma \setminus B$ in Ω such that $z \in \Gamma(x, y)$, where $\Gamma(x, y)$ denotes the subcurve of Γ between x and y. The domain Ω has the c-ball separation property if every quasihyperbolic geodesic in Ω satisfies the c- ball separation property.

Definition 1.2 (John domains)**.** Let $A \geq 1$. We say that a domain $\Omega \subsetneq \mathbb{R}^n$ is John with center x_0 if $x_0 \in \Omega$ is such that for any point $x \in \Omega$, there exists a curve γ_x joining x and x_0 in Ω such that

$$l(\gamma_x(x, z)) \leq A\mathrm{d}_\Omega(z),$$

for all $z \in \gamma_x$. The curves γ_x are called A-John curves.

The ball separation property was introduced in Koskela-Buckley [5] where it was shown that if a domain has the separation property and admits a Sobolev-Poincaré inequality, then it is a John domain. It is easy to see that John curves satisfy the separation property. It makes sense to ask under what conditions John domains have the property that all quasihyperbolic geodesics are John curves. It was shown in Gehring-Hag-Martio [10] that in planar simply connected domains this is true and they also gave examples of domains where it is false. It turns out that planar simply connected domains satisfy the separation condition which is sufficient for this purpose and the topological condition of simply connectedness is not relevant. In fact the theorem below states that the domain needs to only be checked for the weaker local connectivity property (see Section 1.3 for the definitions of φ-LC-2 and φ-John).

Theorem 1.3 (Proposition 5.5, [B]). *Let $\Omega \subsetneq \mathbb{R}^n$ be a domain with the C-ball separation property. Then Ω is φ-John if and only if Ω is φ-LC-2, quantitatively. Moreover, the quasihyperbolic geodesics are ψ-John curves, where $\psi(t) = \psi(Ct)$.*

1.3. **Generalized John domains.** We recall the quasihyperbolic boundary condition defined in Gehring-Martio [11]. This condition appeared already in Becker-Pommerenke [2] where it was used to study Hölder continuity of conformal mappings. A domain Ω satisfies the quasihyperbolic boundary condition if there exists a point $x_0 \in \Omega$ such that for all $x \in \Omega$,

$$k_\Omega(x_0, x) \leq a \log\left(\frac{\mathrm{d}_\Omega(x_0)}{\mathrm{d}_\Omega(x)}\right) + b,$$

where $a \geq 1$ and $b \geq 0$ are constants. It was shown in [2] that conformal mappings from a disk onto a domain which has the quasihyperbolic boundary condition are Hölder continuous. It was proved in [11] that a quasiconformal mapping from a domain in $\mathbb{R}^n$ is Hölder continuous on all balls contained in the domain (with uniformly bounded constants for Hölder continuity) if and only if the image satisfies the quasihyperbolic boundary condition. Also, certain global Poincaré inequalities hold in domains with the quasihyperbolic boundary conditions. See for example Smith-Steganga [32]. See also Koskela-Onninen-Tyson [26] for validity of (q,p)-Poincaré inequalities in domains satisfying the quasihyperbolic boundary condition leading to solvability conditions for Neumann problems in planar domains. See Koskela-Onninen-Tyson [25] for global Hölder continuity of quasiconformal mappings (in the intrinsic metric, obtained via capacity estimates) when the image satisfies the quasihyperbolic boundary condition.

The quasihyperbolic boundary condition implies for quasihyperbolic geodesics Γ_x joining the fixed point x_0 to a point $x \in \Omega$ the inequality

$$k_\Omega(x_0, z) \leq a \log\left(\frac{\mathrm{d}_\Omega(x_0)}{l(\Gamma_x(x,z))}\right) + b_1,$$

for any $z \in \Gamma_x$, where $b_1 = b_1(a,b)$ (see [32]). It is not difficult to see that John domains satisfy the quasihyperbolic boundary condition with a and b depending only on the John constant of Ω. Domains which satisfy the quasihyperbolic boundary condition, are s-John domains in the sense that the quasihyperbolic geodesics Γ_x joining the center x_0

to an arbitrary point x satisfy

$$l(\Gamma_x(x,z)) \leq A\mathrm{d}_\Omega(z)^s,$$

where $0 < s \leq 1$ depends only on a, b, x_0 and Ω. See Hajłasz-Koskela [16] for Sobolev imbedding theorems in these domains. It makes sense to generalize the definition of John domains in the following way. Given a homeomorphism $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ and $\varphi(t) \geq t$, for $t \in [0,\infty)$, we say Ω is φ-John if there exists a center x_0 such that quasihyperbolic geodesics Γ_x starting from x_0 satisfy

$$l(\Gamma_x(x,z)) \leq \varphi(\mathrm{d}_\Omega(z)),$$

whenever $z \in \Gamma_x$. This generalization was introduced in Guo-Koskela [13] where several properties of these domains were studied. See Guo [12] for modulus of continuity (in terms of φ) of quasiconformal mappings when the image is φ-John; see also Guo-Koskela [14].

Next we state two properties which were introduced by Gehring in [6] for characterizing quasidisks. A set $E \subset \dot{\mathbb{R}}^n$ is called *Linearly Locally Connected* (LLC) if there is a constant $C \geq 1$ such that

- (LLC-1) each pair of points in $B(x,r) \cap E$ can be joined in $B(x,Cr) \cap E$, and
- (LLC-2) each pair of points in $E \backslash B(x,Cr)$ can be joined in $E \setminus B(x,r)$.

The LLC-condition can be generalized to the non-linear case as follows: a set $E \subset \dot{\mathbb{R}}^n$ is (φ,ψ)-*locally connected* ((φ,ψ)-LC) if

- (φ-LC-1) each pair of points in $B(x,r) \cap E$ can be joined in $B(x,\varphi(r)) \cap E$, and
- (ψ-LC-2) each pair of points in $E \backslash B(x,r)$ can be joined in $E \backslash B(x,\psi(r))$,

where $\varphi,\psi : [0,\infty) \to [0,\infty)$ are smooth increasing functions such that $\varphi(0) = \psi(0) = 0$, $\varphi(r) \geq r$ and $\psi(r) \leq r$ for all $r > 0$. Depending on what is meant by joining, one can consider *pathwise* and *continuumwise* versions of φ-LC-1, ψ-LC-2, and (φ,ψ)-LC. It is known that for Jordan curves in the plane that if one of the complementary domains is LLC-1, then the other is LLC-2 (see Näkki-Väisälä [29] and also [13] for the generalized case in the plane). Väisälä [34] generalized this duality to higher dimensions. We have the following generalization of metric duality results of Väisälä [34] to the non linear case.

Theorem 1.4 (Proposition 1.1, [B]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that Ω satisfies the ball separation property and that $H^1(\overline{\Omega}) = 0$. Then the following conditions are quantitatively equivalent:*

1). Ω is φ-diam John;

2). Ω is φ-LC-2;

3). the complementary domain $\Omega' = \mathbb{R}^n \backslash \overline{\Omega}$ is homologically $(n-2,\varphi)$-bounded turning.

In the above theorem, homologically (p,φ)-bounded turning is the generalization of the bounded turning condition for curves (which says that given any pair of points in the domain there exists a continuum joining them of diameter comparably smaller than the distance between the points) to higher dimensions, by using p-cycles and $(p+1)$-chains (see [34], and [B] for the non-linear version). There is also an analog of higher dimensional bounded turning that uses homotopy groups (see Section 7, [B]). It is shown in [B, Theorem 1.4] that homotopically bounded turning and homologically bounded turning are quantitatively not the same.

1.4. **Gehring-Hayman property.**

Definition 1.5 (Gehring-Hayman property)**.** Let $\Omega \subsetneq \mathbb{R}^n$ be a domain. A curve $\Gamma \subset \Omega$ satisfies the c-Gehring-Hayman property, for $c \geq 1$, if

$$l(\Gamma(x,y)) \leq c\lambda_\Omega(x,y),$$

for all $x, y \in \Gamma$, where $\lambda_\Omega(x,y)$ is the infimum of the lengths of all curves joining x and y in Ω. The domain Ω has the c-Gehring-Hayman property if every quasihyperbolic geodesic $\Gamma \subset \Omega$ satisfies the c-Gehring-Hayman property.

There is also the corresponding diameter version (see below) of this condition which is sometimes useful. A curve Γ has the diameter Gehring-Hayman property if for all $x, y \in \Gamma$

$$\operatorname{diam}(\Gamma(x,y)) \leq c\delta_\Omega(x,y),$$

where $\delta_\Omega(x,y)$ is the infimum of the diameters of all curves joining x and y in Ω. The Gehring-Hayman property was first proved for hyperbolic geodesics in simply connected domains in Gehring-Hayman [7]. It was later shown to hold for quasihyperbolic geodesics in quasiconformal images of uniform domains [20] and for conformal metric deformations of unit balls [3]. In [D] we use suitable local versions of the Gehring-Hayman property for quasihyperbolic geodesics.

1.5. **Whitney decomposition.** Given a domain $\Omega \subsetneq \mathbb{R}^n$ the Whitey decomposition $\mathcal{W}$ of Ω is a collection of cubes (edges parallel to coordinate axes) with disjoint interiors which cover Ω and satisfy:

(i) if $Q \in \mathcal{W}$, then

$$\frac{1}{\lambda_1}\sqrt{n}\, l(Q) \leq \mathrm{d}(Q, \partial\Omega) \leq \lambda_2 \sqrt{n}\, l(Q),$$

and

(ii) if $Q, Q' \in \mathcal{W}$ intersect, then

$$\frac{1}{\lambda_1} l(Q) \leq l(Q') \leq \lambda_2 l(Q),$$

where $\lambda_1, \lambda_2 \geq 4$ and $l(Q)$ denotes the edge-length of a cube Q (see for example Whitney [35] or the book of Stein [33]). Given a Whitney cube $Q \in \mathcal{W}$, a simple computation shows that

$$\log\left(\frac{2+\lambda_2}{1+\lambda_2}\right) \leq \operatorname{diam}_k(Q) \leq \lambda_1, \tag{1}$$

where $\operatorname{diam}_k(Q)$ is the diameter of Q in the quasihyperbolic metric k_Ω. In applications, we do not require the sets in the decomposition to be restricted to cubes and allow for more general connected sets Q which satisfy the properties above with edge length of cubes replaced by diameter of the respective sets which are additionally required to satisfy $B(x, c_1\operatorname{diam}(Q)) \subset Q \subset B(x, c_2\operatorname{diam}(Q))$, for some $0 < c_1, c_2$ and $x \in Q$.

1.6. **Gromov hyperbolicity.**

Definition 1.6 (Gromov hyperbolicity)**.** Let $0 \leq \delta < \infty$. A domain $\Omega \subsetneq \mathbb{R}^n$ is δ-Gromov hyperbolic if all quasihyperbolic geodesic triangles are δ-thin in the quasihyperbolic metric. That is, given any three points $x, y, z \in \Omega$ and quasihyperbolic geodesics Γ_{xy}, Γ_{yz} and Γ_{zx} joining them pairwise, it holds for any w lying in any of the three geodesics that the ball of radius δ, in the quasihyperbolic metric, centered at w intersects the union of the remaining two geodesics.

Gromov hyperbolicity of domains was introduced and studied in [4], where more general spaces were considered. The Whitney decomposition of a domain provides another way to identify the Gromov hyperbolicity of a domain. Indeed, let $\Omega \subset \mathbb{R}^n$ be bounded and $\mathcal{W}$ its Whitney decomposition. Enumerate $\mathcal{W}$ as $\{Q_0, Q_1, \ldots\}$ where Q_0 is one of the cubes with the largest diameter. Let x_i be the center of Q_i. Let $G(\mathcal{W})$ be the graph with $\{x_i\}_i$ as the set of vertices and edges consisting of pairs $\{x_j, x_l\}$ where Q_j and Q_l intersect in a face. Equip $G(\mathcal{W})$ with the graph metric d_G, for which $\mathrm{d}_G(x_i, x_j)$ is the smallest number of edges in a path joining x_i to x_j in G. It follows from (1) that

$$\mathrm{d}_G(x_i, x_j) \simeq k_\Omega(x_i, x_j),$$

with constants depending only on λ_1 and λ_2 (see for example [25]). Thus (G, d_G) and (Ω, k) are quasiisometric with constants given by λ_1

and λ_2. It follows that Ω is Gromov hyperbolic, if and only if $G(\mathcal{W})$ is Gromov hyperbolic.

Definition 1.7 (Uniform spaces [4])**.** A non-complete metric space X is A-uniform if there exists a constant $A \geq 1$ such that for each pair of points $x, y \in X$, there exists a curve $\gamma_{xy} \subset \Omega$ joining x and y such that

(*i*) $l(\gamma_{xy}) \leq A|x-y|$,
(*ii*) For any $z \in \gamma_{xy}$, $l(\gamma_{xy}(x,z) \wedge l(\gamma_{xy}(z,y))) \leq A\mathrm{d}_X(z)$,

where $\mathrm{d}_X(z)$ is the distance between z and the boundary ∂X. The curves γ_{xy} are called uniform curves. Curves that satisfy only the second requirement are called doubly-John.

It was shown in [4] that A-uniform domains are $\delta(A)$-Gromov hyperbolic. On the other hand, every Gromov hyperbolic domain can be equipped with a metric, obtained via a conformal deformation of the euclidean metric, which makes it a uniform space.

Definition 1.8 (Conformal modulus)**.** Let $Q > 1$. Let X be a rectifiably connected metric space. Let μ be a Borel measure in X. The Q-modulus of a family $\mathcal{G}$ of curves in X is

$$\mathrm{mod}_Q(\mathcal{G}) = \inf \int_X f^Q d\mu,$$

where the infimum is taken over all Borel functions $f : X \to [0, \infty]$ such that

$$\int_\gamma f ds \geq 1$$

for all $\gamma \in \mathcal{G}$.

Definition 1.9 (Loewner Spaces)**.** Let $Q > 1$. Let X be a rectifiably connected metric space. Let μ be a Borel measure in X. Then X is Loewner space if the function

$$\varphi(t) = \inf\{\mathrm{mod}_Q(E, F; X) : \Delta(E, F) \leq t\}$$

is positive for each $t > 0$, where E and F are any non-degenerate disjoint continua in X with

$$\Delta(E, F) = \frac{\mathrm{dist}(E, F)}{\mathrm{diam}\, E \wedge \mathrm{diam}\, F}$$

and $(E, F; U)$ is the family of all curves in U joining the sets $E, F \subset U$.

The following characterization of Gromov hyperbolic domains is available.

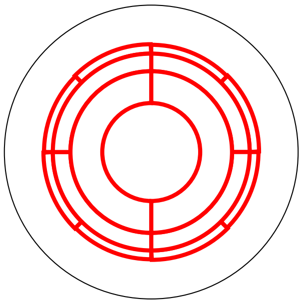

FIGURE 1. First four generations of a Whitney decomposition $\mathcal{W}$ of the unit disc.

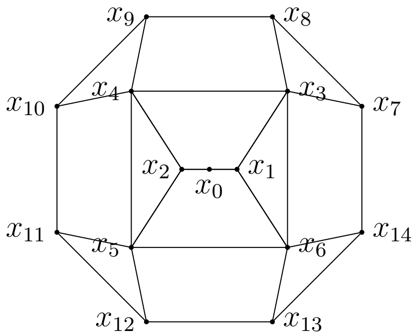

FIGURE 2. The graph $G(\mathcal{W})$.

Theorem 1.10 ([4],[1])**.** *A δ-Gromov hyperbolic domain has both the c_1-ball separation property and the c_2-Gehring-Hayman property, for $c_1 = c_1(\delta, n)$ and $c_2 = c_2(\delta, n)$. Conversely, a domain which has both the c_1-ball separation property and the c_2-Gehring-Hayman property is also δ-Gromov hyperbolic, for $\delta = \delta(c_1, c_2, n)$.*

We show in [D] that the diameter version of the Gehring-Hayman condition is also true.

Theorem 1.11. *Suppose that $\Omega \subset \mathbb{R}^n$ is a δ-Gromov hyperbolic domain. Then there exists a constant $M = M(\delta, n)$ such that for any pair of points $x, y \in \Omega$ we have*

$$diam(\Gamma) \leq M\delta_\Omega(x, y) \tag{2}$$

for any quasihyperbolic geodesic Γ joining x and y in Ω.

For the counterpart of Theorem 1.11 in the special case of hyperbolic geodesics in simply connected domains see Pommerenke [31, page 88]. For the proof of Theorem 1.11 we use the aforementioned method of uniformization developed in [4]. It can be shown that the uniform space thus obtained is a bounded Loewner space (when equipped with corresponding measure) and that conformal modulus of families of curves is preserved under this deformation. Also the boundary of the deformed spaces is quasisymmetrically equivalent to the boundary of (Ω, k) with its quasisymmetric gauge (of visual metrics).

1.7. **A dense subset of Sobolev spaces.** Let $\Omega \subset \mathbb{R}^n$. Let $p \in [1, \infty]$ and $k \in \mathbb{N}$. We write $L^{k,p}(\Omega)$ for the space of Sobolev functions with

p-integrable distributional derivatives of order k;

$$L^{k,p}(\Omega) = \{u \in L^1_{loc}(\Omega) : D^\alpha u \in L^p(\Omega), \text{ if } |\alpha| = k\}.$$

We equip it with the homogeneous Sobolev seminorm $\sum_{|\alpha|=k} \|\nabla^\alpha u\|_{L^p(\Omega)}$. The (non-homogeneous) Sobolev space $W^{k,p}(\Omega)$ is defined as

$$W^{k,p}(\Omega) = \{u \in L^1_{loc}(\Omega) : D^\alpha u \in L^p(\Omega), \text{ if } |\alpha| \leq k\}$$

and is equipped with the norm $\sum_{|\alpha|\leq k} \|\nabla^\alpha u\|_{L^p(\Omega)}$. Here and below an n-multi-index α is an n-vector of non-negative integers and $|\alpha|$ is its ℓ_1-norm.

In articles [C] and [D], we study the question of density of functions with bounded derivatives in the homogeneous Sobolev spaces. Such a density does not hold in general (see for example [23]). It was proved in Koskela-Zhang [22] that in the class of first order Sobolev spaces in planar simply connected domains, $W^{1,\infty}$ functions are dense in both the homogeneous and non-homogeneous Sobolev norms. The same density result was generalized to the case of Gromov hyperbolic domains in all dimensions higher than or equal to two in Koskela-Rajala-Zhang [23]. We generalize their results for higher order Sobolev spaces with the homogeneous norm.

Theorem 1.12 (Theorem 1.1, [C])**.** *Let $k \in \mathbb{N}$, $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then the subspace $W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$ is dense in the space $L^{k,p}(\Omega)$.*

Theorem 1.13 (Theorem 1.1, [D])**.** *Let $0 \leq \delta < \infty$, $k \in \mathbb{N}$ and $p \in [1, \infty)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded δ-Gromov hyperbolic domain. Then $W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$ is dense in $L^{k,p}(\Omega)$.*

For defining an approximating sequence, we use the values of a smooth approximation of the Sobolev function u to be approximated, in an increasing (nested) sequence of precompact subdomains (cores) of Ω to assign values to points in the rest of the domain by a suitable reflection procedure (inner extension). The smooth approximation to u, is approximated by polynomials in Whitney cubes (see for example Jones [21]) and the ball separation property is used to decompose the remaining domain (outside of the core) into components blocked from the core by suitable neighbourhoods of Whitney cubes at the boundary of the core. The Gehring-Hayman inequality is used to find chains of Whitney cubes of bounded length (for example in the metric d_G for the centers) joining cubes whose neighbourhoods intersect. The Poincaré inequality is then used to obtain bounds on the oscillations of the polynomial approximations in cubes whose neighbourhoods intersect. Finally, a suitable partition of unity is used to show that the polynomials can be smoothly summed up to obtain an approximating function with bounded k-derivatives.

1.8. **Boundary accessibility.** We prove the following theorem about the Hausdorff content of the set of points in the boundary of a simply connected planar domain which can be joined by a John curve (with suitable John constant) starting from a given point in the domain and of length comparable to the distance from the given point to the boundary.

Theorem 1.14 (Theorem 1.1, [A]). *Let Ω be a bounded, simply connected domain in the plane. Let $0 < \alpha < 1$ be fixed. Given $z \in \Omega$, there is a John subdomain $\Omega_z \subset \Omega$ with center z and John constant depending only on α such that*

$$\mathcal{H}^{\alpha}_{\infty}(\partial\Omega_z \cap \partial\Omega) \geq c(\alpha)\, d_{\Omega}(z)^{\alpha}.$$

In the above theorem, we used the following definition of Hausdorff content where to each $A \subset \mathbb{R}^2$ and $\alpha \geq 0$ is assigned the number

$$\mathcal{H}^{\alpha}_{\infty}(A) := \inf\{ \sum_{j=1}^{\infty} \operatorname{diam}(E_j)^{\alpha} : \, E_j \subset \mathbb{R}^2, \; A \subset \bigcup_{j\in\mathbb{N}} E_j \}.$$

Theorem 1.14 tells in particular that the Hausdorff dimension of the set of points in the boundary which are the vertices of a twisted interior cone of suitable length is bounded below by one. It may be viewed as the quantitative version of a result of Makarov [28].

1.9. **Weighted Hardy inequalities.** For a domain $\Omega \subset \mathbb{R}^n$ and a compactly supported smooth function, $u \in C_0^{\infty}(\Omega)$ the inequality

$$\int_{\Omega} |u(x)|^p \mathrm{d}_{\Omega}(x)^{\beta - p}\, dx \leq C \int_{\Omega} |\nabla u(x)|^p \mathrm{d}_{\Omega}(x)^{\beta}\, dx$$

is the weighted (classical) Hardy inequality (see [17],[18] for the classical Hardy inequality and [30], [24] for higher dimensional versions of it).

The integral Hardy inequality above follows from the stronger pointwise version:

$$|u(x)| \leq C d_{\Omega}(x)^{1-\frac{\beta}{p}} \sup_{0<r<2\mathrm{d}_{\Omega}(x)} \left(\fint_{B(x,r)\cap\Omega} |\nabla u|^q \mathrm{d}_{\Omega}^{q\beta/p} \right)^{1/q}, \qquad (3)$$

where $1 < q < p$ and $-\infty < \beta < \infty$. This pointwise version of the Hardy inequality is a corollary of Theorem 1.14.

Corollary 1.15. *Let $\Omega \subset \mathbb{C}$ be simply connected. Let $1 < p < \infty$. Then, for each $\beta < p - 1$ there exist $1 < q(\beta, p) < p$ and $C > 0$ for which the weighted pointwise Hardy inequality* (3) *holds for each $u \in C_0^{\infty}(\Omega)$ and $x \in \Omega$.*

The corresponding integral weighted Hardy inequalities were already established in [27] by different methods.

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Accessible parts of boundary for simply connected domains

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ACCESSIBLE PARTS OF BOUNDARY FOR SIMPLY CONNECTED DOMAINS

PEKKA KOSKELA, DEBANJAN NANDI, AND ARTUR NICOLAU

Abstract. For a bounded simply connected domain $\Omega \subset \mathbb{R}^2$, any point $z \in \Omega$ and any $0 < \alpha < 1$, we give a lower bound for the α-dimensional Hausdorff content of the set of points in the boundary of Ω which can be joined to z by a John curve with a suitable John constant depending only on α, in terms of the distance of z to $\partial\Omega$. In fact this set in the boundary contains the intersection $\partial\Omega_z \cap \partial\Omega$ of the boundary of a John sub-domain Ω_z of Ω, centered at z, with the boundary of Ω. This may be understood as a quantitative version of a result of Makarov. This estimate is then applied to obtain the pointwise version of a weighted Hardy inequality.

1. Introduction

Let $\Omega \subset \mathbb{C}$ be a domain. We say that Ω is C-John with center z_0 if for any $z \in \Omega$ there exists a rectifiable curve γ_z joining z and z_0 in Ω such that for any point z' in the image of γ_z, it holds that

$$C\mathrm{d}_\Omega(z') \geq l(\gamma_z(z', z)),$$

where $\mathrm{d}_\Omega(z') := \mathrm{dist}(z', \partial\Omega)$ and $l(\gamma_z(z', z))$ is the length of the sub-curve between z' and z. Given $A \subset \mathbb{C}$, we define the α-Hausdorff content as

$$\mathcal{H}^\alpha_\infty(A) := \inf\{ \sum_{j=1}^{\infty} \mathrm{diam}(E_j)^\alpha : E_j \subset \mathbb{C},\ A \subset \bigcup_{j\in\mathbb{N}} E_j\}.$$

Given a simply connected John domain and $z \in \Omega$ there is a John subdomain Ω_z with center z so that, for the ball in the intrinsic metric (defined by taking the infimum of the lengths of rectifiable paths in the domain joining pairs of points) of radius $2\mathrm{d}_\Omega(z)$, we have $B_\Omega(z, 2\mathrm{d}_\Omega(z)) \subset \Omega_z$; see [5], for example. This statement is quantitative in the sense that the John constant of Ω_z depends only on the John constant of Ω. It is easy to see that this conclusion fails for general simply connected Ω: we may not capture all of $\partial B_\Omega(z, 2\mathrm{d}_\Omega(z)) \cap \partial\Omega$ by $\partial\Omega_z$ for a John subdomain Ω_z for a fixed John constant. The best we can hope for is

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to capture a part of $\partial\Omega$ of $\mathcal{H}^1$-content of the order of $\mathrm{d}_\Omega(z)$. Our main result gives a rather optimal conclusion.

Theorem 1.1. *Let Ω be a bounded, simply connected domain in the plane. Let $0 < \alpha < 1$ be fixed. Given $z \in \Omega$, there is a John subdomain $\Omega_z \subset \Omega$ with center z and John constant depending only on α such that*

$$\mathcal{H}^\alpha_\infty(\partial\Omega_z \cap \partial\Omega) \geq c(\alpha) d_\Omega(z)^\alpha.$$

The motivation for Theorem 1.1 partially arises from the weighted pointwise Hardy inequalities (see [1], [4], [5])

$$|u(x)| \leq C d_\Omega(x)^{1-\frac{\beta}{p}} \sup_{0<r<2\mathrm{d}_\Omega(x)} \left(\fint_{B(x,r)\cap\Omega} |\nabla u|^q \mathrm{d}_\Omega^{q\beta/p} \right)^{1/q} \tag{1}$$

where $u \in C_0^\infty(\Omega)$, $1 < q < p$ and $-\infty < \beta < \infty$. This inequality immediately yields the usual weighted Hardy inequality (see [2],[3] for the classical Hardy inequality and [8], [5] for higher dimensional versions of it)

$$\int_\Omega |u(x)|^p \mathrm{d}_\Omega(x)^{\beta-p}\, dx \leq C \int_\Omega |\nabla u(x)|^p \mathrm{d}_\Omega(x)^\beta\, dx$$

via the boundedness of the Hardy-Littlewood maximal operator on $L^{p/q}$. The pointwise Hardy inequalities were shown in [5] to hold for any simply connected John domain for all $1 < p < \infty$ and every $\beta < p - 1$. This is the optimal range even for Lipschitz domains; see [8]. From Theorem 1.1 together with Theorem 5.1 in [5] we have the following corollary.

Corollary 1.2. *Let $\Omega \subset \mathbb{C}$ be simply connected. Let $1 < p < \infty$. Then, for each $\beta < p - 1$ there exist $1 < q(\beta, p) < p$ and $C > 0$ such that the weighted pointwise Hardy inequality (1) holds for each $x \in \Omega$.*

Above, q and C are independent of Ω. The corresponding weighted Hardy inequalities were already established in [6]. Our proof of Theorem 1.1 is based on the following estimate for conformal maps which we expect to be of independent interest. Let $\mathbb{H}$ be the upper half plane.

Theorem 1.3. *Let $f : \mathbb{H} \to \Omega$ be a conformal map. Let $0 < \alpha < 1$ be fixed. Then there exists $C(\alpha) > 0$ such that the following holds.*

Given $z_0 = x_0 + iy_0 \in \mathbb{H}$, there exists a set $E = E(z_0, \alpha) \subset (x_0 - y_0/2, x_0 + y_0/2)$ such that

(a) $\mathcal{H}^\alpha_\infty(E) \geq \frac{y_0^\alpha}{C(\alpha)}$

(b) $\frac{1}{C(\alpha)} |f'(z_0)| \leq |f'(w)| \leq C(\alpha) |f'(z_0)|$

for any point w in the sawtooth region $S(E) := \{x + iy : x \in (x_0 - y_0/2, x_0 + y_0/2),\ d(x, E) \leq y < y_0\}$.

Theorem 1.3 can be understood as a quantitative version of a result of Makarov; see Theorem 5.1 of [7], see also corollary 1.4 of [11]. Our proof of Theorem 1.3 uses Makarov's idea of approximating Bloch functions by dyadic martingales. Theorem 1.1 then follows from Theorem 1.3 and Lemma 3.3 below; in fact we have that $(\Omega_{z_0}, \mathrm{d}_{\Omega_{z_0}})$ is bilipschitz equivalent to the sawtooth region in Theorem 1.3.

2. Preliminaries

Let $\mathbb{D}$ be the unit disk in the complex plane. A function $g : \mathbb{D} \to \mathbb{C}$ is called a Bloch function if it is analytic and

$$\|g\|_{\mathcal{B}} := \sup_{z\in\mathbb{D}} (1 - |z|^2)|g'(z)| < \infty.$$

This defines a seminorm. The Bloch functions form a complex Banach space $\mathcal{B}$ with the norm $|g(0)| + \|g\|_{\mathcal{B}}$.

Given a univalent analytic function $f : \mathbb{D} \to \mathbb{C}$ we have that the function $\log f'$ is a Bloch function by the Koebe Distortion theorem with $\|\log f'\|_{\mathcal{B}} \leq 6$. Conversely, given a function $g \in \mathcal{B}$ with $\|g\|_{\mathcal{B}} \leq 2$, there exists a univalent function $f : \mathbb{D} \to \mathbb{C}$ such that $g = \log f'$, see (Chapter 4, [10]). Given a conformal map $f : \mathbb{H} \to \mathbb{C}$, it follows by a conformal change of coordinates that

$$\sup_{z\in\mathbb{H}} \operatorname{Im}(z)|g'(z)| < 6$$

where g is the function $\log f'$.

Let us introduce some notation. Given a closed interval $I \subset \mathbb{R}$ we denote by x_I the center of I and $z_I := x_I + i|I|$. We denote by $Q(I)$ the square $\{x + iy : x \in I,\ y \in (0, |I|)\}$. The intrinsic metric of a domain $\Omega \subset \mathbb{C}$ is given by $d_\Omega(x, y) := \inf\{l(\gamma_{x,y}) : \gamma_{x,y}$ is a rectifiable curve joining x and y in $\Omega\}$. The euclidean disk with center z and radius r is denoted by $B(z, r)$ and $B_\Omega(z, r)$ is the corresponding intrinsic ball. We denote by $\operatorname{diam}(A)$, the diameter of a set $A \subset \mathbb{C}$. We denote by $\operatorname{diam}_\Omega(A)$ the diameter of a subset $A \subset \Omega$ measured with respect to the intrinsic metric of Ω.

3. Proofs of the theorems

We first sketch the proof of Theorem 1.3. The set E constructed below is a Cantor-type set. One considers the harmonic function $u = \log|f'|$, the real part of the Bloch function $\log f'$, where f is the conformal map from Theorem 1.3. The construction involves selecting "good" parts in the boundary near which the function u remains essentially bounded and estimating the size of the "bad" parts in the boundary where the difference from a fixed value is large and positive or large and negative. The good parts correspond to the points in the boundary, accessible from some interior point of Ω by a John curve. The key observation is that it is possible to recursively choose subsets from the bad parts of the boundary, near which the difference from the fixed value is "up" and "down" at consecutive generations so that the final error in the intersection is not too large. The set E consists of the good parts and an intersection of suitable nested sets of the bad part. The Hausdorff content of E is shown to be large by the mass distribution principle after defining a limit measure supported on E.

We use the following well known lemma in the proof of Theorem 1.3; see Lemma 2.2 of [9].

Lemma 3.1. *Let u be a harmonic function in the upper half plane $\mathbb{H}$ such that*

$$\sup_{z\in\mathbb{H}} Im(z)|\nabla u(z)| \leq A.$$

Let $I \subset \mathbb{R}$ be an interval and let $\{I_j\}$ be a collection of pairwise disjoint dyadic subintervals of I and assume additionally that u is bounded in $Q(I)\setminus \cup_j Q(I_j)$. Then we have

$$u(z_I) = \sum_j u(z_{I_j})\frac{|I_j|}{|I|} + \frac{1}{|I|}\int_{I\setminus\cup_j I_j} u(x)dx + O(A). \tag{2}$$

Proof. Let us write y for the imaginary part of z and fix $0 < \epsilon < 1$. Green's theorem applied to the harmonic functions u and y in the domain $U_\epsilon := (Q(I)\setminus\overline{\cup_j(Q(I_j))}) \cap \{y > \epsilon\}$ gives

$$\int_{U_\epsilon} y\Delta u - \int_{U_\epsilon} u\Delta y = \int_{\partial U_\epsilon} y\nabla u\cdot\nu ds - \int_{\partial U_\epsilon} u\nabla y\cdot\nu ds$$

and thus

$$\int_{\partial U_\epsilon} u\nabla y\cdot\nu ds = \int_{\partial U_\epsilon} y\nabla u\cdot\nu ds \tag{3}$$

where ν is the outward unit normal vector. The absolute value of the latter integral is bounded by $10A|I|$ by assumption. Note that the oscillation of u on the upper edges of $Q(I)$ and $Q(I_j)$ is bounded; indeed

$$|u(x + i|I_j|) - u(x' + i|I_j|)| \leq A|x - x'|/|I_j|$$

for $x, x' \in I_j$. From (3) we have

$$u(z_I) = \sum_{|I_j|>\epsilon} u(z_{I_j})\frac{|I_j|}{|I|} + \frac{1}{|I|}\int_I u(x+i\epsilon)\chi_{I\setminus(\underset{|I_j|>\epsilon}{\cup} I_j)}(x)\,dx + O(A)$$

because the vertical sides of $\partial(Q(I_j))$ do not contribute to the integral. The estimate now follows once we let $\epsilon \to 0$, since the function u has radial limits almost everywhere in $I\setminus\cup_j I_j$. □

Lemma 3.2. *Let u be a harmonic function in the upper half plane $\mathbb{H}$ such that*

$$\sup_{z\in\mathbb{H}} Im(z)|\nabla u(z)| \leq A.$$

Then there is a number $M_0 = M_0(A)$ such that the following holds for any $M > M_0$.

Given any interval $I \subset \mathbb{R}$, define

$$G(I) := \{Re(z) : z \in Q(I), \sup_{0<Im(z)<|I|} |u(z) - u(z_I)| \leq M + A\sqrt{2}\},$$

and assume that $|G(I)| \leq \frac{|I|}{100}$. Consider the family $\mathcal{F}(I)$ of maximal dyadic subintervals $I_j \subset I$ such that $|u(z_{I_j}) - u(z_I)| \geq M$. Then we have

(a) $|u(z) - u(z_I)| \leq M + A\sqrt{2}$ *for any* $z \in Q(I) \setminus \bigcup_j Q(I_j)$. *In particular,* $|u(z_{I_j}) - u(z_I)| \leq M + A\sqrt{2}$.

(b) $|I_j| \leq 2^{-\frac{M}{A\sqrt{2}}}|I|$ *for every* $I_j \in \mathcal{F}(I)$.

(c) *Consider the family $\mathcal{F}^+(I)$ (respectively $\mathcal{F}^-(I)$) of intervals in $\mathcal{F}(I)$ such that $u(z_{I_j}) - u(z_I) \geq M$ (respectively $u(z_{I_j}) - u(z_I) \leq -M$). Then we have*
   (i) $\sum_{I_j \in \mathcal{F}^+(I)} |I_j| \geq |I|/4$
   (ii) $\sum_{I_j \in \mathcal{F}^-(I)} |I_j| \geq |I|/4$

Proof. Given $z = x + iy \in \mathbb{H}$ such that $x \in I$ (respectively I_j) and $|I|/2 < y < |I|$ (respectively $|I_j|/2 < y < |I_j|$) it follows that $|u(z) - u(z_I)| \leq A\sqrt{2}$ (respectively $|u(z) - u(z_{I_j})| \leq A\sqrt{2}$) by our hypothesis.

Part (b) follows by iterating the above inequality and part (a) follows from the maximality of the dyadic intervals.

For part (c) we write the estimate from Lemma 3.1 as

$$\sum_j (u(z_{I_j}) - u(z_I))\frac{|I_j|}{|I|} + \frac{1}{|I|}\int_{I \setminus \bigcup_j I_j} (u(x) - u(z_I))dx = \delta$$

where $\delta = \delta(u, A)$ lies in the interval $[-\delta_A, \delta_A]$, where δ_A is a constant that depends only on A. We observe that $I \setminus \bigcup_j I_j \subset G(I)$. Thus the absolute value of the integral is bounded by $\frac{M+A\sqrt{2}}{100}$, by part (a) and the assumption that $|G(I)| \leq |I|/100$. Hence we have

$$\left| \sum_j (u(z_{I_j}) - u(z_I))\frac{|I_j|}{|I|} \right| \leq \frac{M + A\sqrt{2}}{100} + |\delta|.$$

Next we note that $M \leq |u(z_{I_j}) - u(z_I)| \leq M + A\sqrt{2}$ for any j. Part (c) then follows from this. Indeed, if

$$\sum_{I_j \in \mathcal{F}^+(I)} \frac{|I_j|}{|I|} \leq \frac{1}{4}, \tag{4}$$

then we have

$$\sum_{I_j \in \mathcal{F}^-(I)} \frac{|I_j|}{|I|} \geq \frac{74}{100}$$

and

$$-\frac{M+A\sqrt{2}}{100}-|\delta| \leq \frac{M+A\sqrt{2}}{4}+\sum_{I_j\in\mathcal{F}^-(I)}(u(z_{I_j})-u(z_I))\frac{|I_j|}{|I|}$$

from which we get

$$\sum_{I_j\in\mathcal{F}^-(I)}\frac{|I_j|}{|I|}\leq \frac{26}{100}+\frac{26A\sqrt{2}}{4M}+\frac{\delta_A}{M}.$$

This contradicts (4) if $M > M_0(A)$. The other inequality in part (c) follows similarly. □

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $0<\alpha<1$ be fixed.

We may assume without loss of generality that $z_0 = i$. We construct the set E as follows. Set $u(z) = \log(|f'(z)|)$. Then u is the real part of a Bloch function and thus satisfies the hypothesis of Lemma 3.2 with $A = 6$.

Denote by I_0 the interval $(-\frac{1}{2}, \frac{1}{2})$. Consider the set $Q(I^{(0)})$ and the subset $G(I^{(0)})$ as defined in Lemma 3.2. An interval I is called "good" if $|G(I)| \geq |I|/100$ and "bad" otherwise. If $|G(I^{(0)})| \geq |I^{(0)}|/100$, then set $E = G(I^{(0)})$. Then $\mathcal{H}^\alpha_\infty(E) \gtrsim 1$ and the claim follows.

So we assume that the other case holds and consider the maximal family $\mathcal{F}(I^{(0)})$ of subintervals $I_j \subset I_0$ as chosen in Lemma 3.2, with $M = M(\alpha)$ to be fixed later. Thus I_0 is a bad interval and we may apply Lemma 3.2. We have

$$|I| \leq 2^{-\frac{M}{6\sqrt{2}}}|I^{(0)}| \quad \text{if } I\in\mathcal{F}^+(I^{(0)})$$

and

$$\sum_{I\in\mathcal{F}^+(I^{(0)})}|I| \geq |I^{(0)}|/4.$$

The first generation $\mathcal{G}_1 = \mathcal{G}_1(I^{(0)})$ is formed by the subsets $G(I)$ of the good intervals $I \in \mathcal{F}^+(I^{(0)})$ and by the bad intervals $I \in \mathcal{F}^+(I^{(0)})$. We write $\mathcal{G}_1 = \mathcal{G}_1^g \cup \mathcal{G}_1^b$ where

$$\mathcal{G}_1^g(I^{(0)}) = \{G(I) : I\in\mathcal{F}^+(I^{(0)}) \text{ is good}\}$$

and

$$\mathcal{G}_1^b(I^{(0)}) = \{I\in\mathcal{F}^+(I^{(0)}) : I \text{ is bad}\}.$$

We also have

$$\sum_{I\in\mathcal{G}_1}|I| \geq |I^{(0)}|/400.$$

The construction stops in the sets in the family $\mathcal{G}_1^g$ of good sets. In the sets $I \in \mathcal{G}_1^b$ it continues as follows.

Fix $I^{(1)} \in \mathcal{G}_1^b$. Since $I^{(1)}$ is bad we can apply Lemma 3.2 and consider the collection $\mathcal{F}^-(I^{(1)})$ which satisfies

$$|I| \leq 2^{-\frac{M}{6\sqrt{2}}}|I^{(1)}| \quad \text{if } I \in \mathcal{F}^-(I^{(1)})$$

and

$$\sum_{I \in \mathcal{F}^-(I^{(1)})} |I| \geq |I^{(1)}|/400.$$

The first generation $\mathcal{G}_1(I^{(1)})$ of the interval $I^{(1)} \in \mathcal{G}_1^b$ is written $\mathcal{G}_1^g(I^{(1)}) = \mathcal{G}_1(I^{(1)}) \cup \mathcal{G}_1^b(I^{(1)})$, where

$$\mathcal{G}_1^g(I^{(1)}) = \{G(I) : I \in \mathcal{F}^-(I^{(1)}) \text{ is good}\}$$

and

$$\mathcal{G}_1^b(I^{(1)}) = \{I \in \mathcal{F}^-(I^{(1)}) : I \text{ is bad}\}.$$

We use the first generation as defined above, of members of the collection $\mathcal{G}_1^b(I^{(0)})$, to define the second generation $\mathcal{G}_2(I^{(0)}) = \mathcal{G}_2^g(I^{(0)}) \cup \mathcal{G}_2^b(I^{(0)})$, where

$$\mathcal{G}_2^g(I^{(0)}) = \bigcup_{I^{(1)} \in \mathcal{G}_1^b(I^{(0)})} \mathcal{G}_1^g(I^{(1)})$$

and

$$\mathcal{G}_2^b(I^{(0)}) = \bigcup_{I^{(1)} \in \mathcal{G}_1^b(I^{(0)})} \mathcal{G}_1^b(I^{(1)}).$$

We also have

$$\sum_{\substack{I \subset I^{(1)} \\ I \in \mathcal{G}_2}} |I| \geq |I^{(1)}|/400 \quad \text{for any } I^{(1)} \in \mathcal{G}_1^b.$$

Observe that u oscillates to the right of $u(z_{I_0})$ in the first step of the construction and to the left of $u(z_I)$ in the second. We have

$$|u(z_I) - u(z_{I^{(0)}})| \leq 12\sqrt{2} \quad \text{for any } I \in \mathcal{G}_2^b.$$

Again the construction continues in the intervals of $\mathcal{G}_2^b(I^{(0)})$. Since the errors cancel but do not vanish, we use a slightly different value of M, if needed, for choosing the maximal family $\mathcal{F}(I)$ for the bad intervals $I \in \mathcal{G}_2^b$, so that the errors do not add up. More precisely, given $I^{(2)} \in \mathcal{G}_2^b$, we choose a value M' from the interval $[M - 6\sqrt{2}, M + 6\sqrt{2}]$ such that $u(z_{I^{(2)}}) + M' = u(z_{I^{(0)}}) + M$.

So the construction stops after finitely many steps or continues indefinitely providing new generations $\mathcal{G}_n$. Let $I^{(n)} \in \mathcal{G}_n$. Either $I^{(n)}$ is of the form $G(\tilde{I})$ and the construction stops in $I^{(n)}$ or $I^{(n)} \in \mathcal{G}_n^b(I^{(0)})$ and the construction provides new sets and intervals of $\mathcal{G}_{n+1}$ contained in $I^{(n)}$ which satisfy

$$|I| \leq 2^{-\frac{M}{6\sqrt{2}}}|I^{(n)}| \quad \text{if } I \subset I^{(n)}\ , I \in \mathcal{G}_{n+1}$$

and

$$\sum_{\substack{I\subset I^{(n)}\\ I\in\mathcal{G}_{n+1}}} |I| \geq |I^{(n)}|/400.$$

We define

$$E := (\cup_n \mathcal{G}_n^g) \cup (\cap_n \mathcal{G}_n^b).$$

By construction for any $x \in E$ and any $0 \leq y \leq 1$ we have

$$|\log|f'(x+iy)| - \log|f'(i)|| \leq 2M + 24.$$

Consider the set $S(E)$ as defined in the statement of the theorem. Since u is a Bloch function, the previous estimate gives that

$$e^{-2M-30}|f'(i)| \leq |f'(w)| \leq e^{2M+30}|f'(i)|$$

for any $w \in S(E)$. Thus part (b) of the statement follows.

Part (a) of the statement follows if it is shown that

$$\mathcal{H}_\infty^\alpha(E) \geq c(\alpha).$$

To prove the last estimate, it suffices by the mass distribution principle to construct a positive measure μ with $\mu(E) \geq 1$ such that there exists a constant $c(\alpha) > 0$ with

$$\mu(I) \leq c(\alpha)|I|^\alpha,$$

for any interval $I \subseteq I_0$. The measure μ will be the limit of certain measures μ_n supported in the union $(\underset{k\leq n}{\cup} \mathcal{G}_k^g) \cup \mathcal{G}_n^b$, where $\mathcal{G}_k^g$ are the good parts of the previous generations.

Next we construct the measure μ. Let $\mu_0 = dx \llcorner I^{(0)}$. Consider

$$a(I^{(0)}) = \frac{|I^{(0)}|}{\sum_{I\in\mathcal{G}_1} |I|}$$

which satisfies $a(I^{(0)}) \leq 400$. By defining

$$\mu_1 := a(I^{(0)}) \sum_{I\in\mathcal{G}_1} dx \llcorner I$$

we have $\mu_1(I^{(0)}) = 1$. The measure μ_2 will coincide with μ_1 on $\mathcal{G}_1^g$. On $\mathcal{G}_2$ the measure μ_2 will be defined by redistributing the mass of μ_1. More concretely, if $I^{(1)} \in \mathcal{G}_1^b$ set

$$a(I^{(1)}) = \frac{\mu_1(I^{(1)})}{\sum_{\substack{I\subset I^{(1)}\\ I\in\mathcal{G}_2}} |I|}.$$

Since

$$a(I^{(1)}) = \frac{|I^{(1)}|}{\sum_{\substack{I\subset I^{(1)}\\ I\in\mathcal{G}_2}} |I|} \frac{\mu_1(I^{(1)})}{|I^{(1)}|} = a(I^{(0)}) \frac{|I^{(1)}|}{\sum_{\substack{I\subset I^{(1)}\\ I\in\mathcal{G}_2}} |I|},$$

we deduce that $a(I^{(1)}) \leq 400^2$. Define

$$\mu_2 = \mu_1 \llcorner \mathcal{G}_1^g + \sum_{I^{(1)} \in \mathcal{G}_1^b} a(I^{(1)}) \sum_{\substack{I \subset I^{(1)} \\ I \in \mathcal{G}_2}} dx \llcorner I.$$

The measures $\mu_3, \ldots, \mu_n, \ldots$ are defined recursively. Observe that $\mu_k(I) = \mu_n(I)$ for any $k \geq n$, provided $I \in \mathcal{G}_n$. Moreover, if $I \in \mathcal{G}_n$ we have

$$\frac{\mu_n(I)}{|I|} \leq 400^n.$$

Finally set

$$\mu = \lim_{n \to \infty} \mu_n.$$

It is clear that spt$\mu \subset E$ and $\mu(E) = 1$. We want to check that $\mu(I) \leq c(\alpha)|I|^\alpha$ for any interval $I \subseteq I_0$. Let $J \subset I_0$ be an interval. We may assume that there is a positive integer j such that

$$2^{-\frac{M(j+1)}{6\sqrt{2}}} \leq |J| \leq 2^{-\frac{Mj}{6\sqrt{2}}}.$$

Let $\mathcal{G}_j(J)$ (respectively $\mathcal{G}_k^g(J)$) be the family of sets of generation $\mathcal{G}_j$ (respectively $\mathcal{G}_k^g$) which intersect J. Let $\mathcal{A}_j(J)$ be the family of sets in $\cup_{k=0}^{j-1} \mathcal{G}_k^g(J)$ of diameters smaller than $2^{-\frac{Mj}{6\sqrt{2}}}$. Since the sets in $\mathcal{G}_j(J) \cup \mathcal{A}_j(J)$ intersect J and have diameter smaller than $2|J|$, we have

$$\mathcal{G}_j(J) \cup \mathcal{A}_j(J) \subset 4J.$$

Hence

$$\begin{aligned} \mu(J) &\leq \sum_{I \in \mathcal{G}_j(J)} \mu(I) + \sum_{k=0}^{j-1} \sum_{I \in \mathcal{G}_k^g(J)} \mu(I \cap J) \\ &\leq \sum_{I \in \mathcal{G}_j(J) \cup \mathcal{A}_j(J)} \mu(I) + \sum_{I \in \cup_{k=0}^{j-1} \mathcal{G}_k^g(J) \setminus \mathcal{A}_j(J)} \mu(I \cap J) \\ &=: A + B. \end{aligned}$$

If $I \in \mathcal{G}_j(J) \cup \mathcal{A}_j(J)$, then we have

$$\mu(I) = \mu_j(I) = \frac{\mu_j(I)}{|I|}|I| \leq 400^j |I|.$$

Hence

$$A \leq 400^j \sum_{\substack{I \subset 4J \\ I \in \mathcal{G}_j(J) \cup \mathcal{A}_j(J)}} |I| \leq 4 \cdot 400^j |J|.$$

Since the sets in $\cup_{k=0}^{j-1} \mathcal{G}_k^g(J) \setminus \mathcal{A}_j(J)$ intersect J and are contained in intervals of length larger than $|J|$ which are pairwise disjoint, the collection $\cup_{k=0}^{j-1} \mathcal{G}_k^g(J) \setminus \mathcal{A}_j(J)$ is contained in at most two intervals L_1 and L_2. Now

$$B \leq \sum_{i=1}^{2} \mu(L_i) \leq 400^j \sum_{i=1}^{2} |L_i \cap J| \leq 400^j [J|.$$

Hence $\mu(J) \leq A + B \leq 5 \cdot 400^j |J|$.

Since $|J| \leq 2^{-\frac{Mj}{6\sqrt{2}}}$, we deduce that

$$\mu(J) \leq 5|J|^{1-\frac{6\sqrt{2}\ln_2(400)}{M}}.$$

We choose M large enough so that $1 - \frac{6\sqrt{2}\ln_2(400)}{M} > \alpha$. The theorem follows with this value of M. □

The proof of Theorem 1.1 uses the following auxiliary result.

Lemma 3.3. *Assume that a set $E = E(z_0, \alpha) \subset \mathbb{R}$ exists, for which part (b) of Theorem 1.3 is satisfied. Then we have that $f(S(E))$ is a John domain with John constant depending on α. Moreover,*

$$\mathcal{H}^\alpha_\infty(f(E)) \geq c(\alpha)|f'(z_0)|^\alpha \mathcal{H}^\alpha_\infty(E).$$

Proof. We may assume that E is compact. We have for $f(z) \in f(S(E))$ that

$$\mathrm{d}_{f(S(E))}(f(z), \partial f(S(E))) \gtrsim c(\alpha)|f'(z_0)|\mathrm{d}_{S(E)}(z)$$

from which it follows that $f(S(E))$ is John, with John curves being the images of the John curves in $S(E)$ (which we may take to be the vertical line segment joining the point z to the upper edge of $Q(I)$ followed by a horizontal line segment till z_I). The constant only depends on α.

Let $\{B_i\}_i$ be a countable (possibly finite) collection of disks covering $f(E)$. We may assume that $f(z_0) \notin B_i$ for all indices i. For each point $f(\hat{z}) \in f(E) \cap B_i$ consider the John curve $\gamma_{\hat{z}}$ joining $f(\hat{z})$ to the John center $f(z_0)$. Let $f(z) \in \gamma_{\hat{z}}$ be a point such that $l(\gamma_{\hat{z}}(f(\hat{z}), f(z))) = \mathrm{rad}(B_i) =: R_i$. By the John condition we have that $\mathrm{d}_{f(S(E))}(f(z)) \geq \frac{1}{c(\alpha)} R_i$.

In the following we write $S'(E)$ for the set $f(S(E))$. Consider the collection of intrinsic balls $\{B_{S'(E)}(f(z), c(\alpha)\mathrm{d}_{S'(E)}(f(z)))\}_{f(\hat{z})}$ in the intrinsic metric of $S'(E)$. The balls in this collection cover the set $f(E) \cap B_i$. By the $5r$-covering theorem we find pairwise disjoint balls

$$B_{S'(E)}(f(z^i_j), c(\alpha)\mathrm{d}_{S'(E)}(f(z^i_j))),\ j = 1, 2, \ldots$$

such that $\{B_{S'(E)}(f(z^i_j), 5c(\alpha)\mathrm{d}_{S'(E)}(f(z^i_j)))\}_j$ covers $f(E) \cap B_i$.

We have also an upper bound $N(\alpha)$ for the number N_i of the pairwise disjoint intrinsic balls found above for each $f(E) \cap B_i$, since every ball $B_{S'(E)}(f(z^i_j), c(\alpha)\mathrm{d}_{S'(E)}(f(z^i_j)))$ in the collection contains the euclidean disk $B(f(z^i_j), R_i/c(\alpha))$. Let the collection of finitely many such intrinsic balls chosen for each index i be denoted together $\{B_{ij}\}_{\substack{i\in\mathbb{N}\\ 1\leq j\leq N_i}}$ where for given i and $1 \leq j \leq N_i$, $B_{ij} = B_{S'(E)}(f(z^i_j), 5c(\alpha)\mathrm{d}_{S'(E)}(f(z^i_j)))$. We have

$$\begin{aligned}\sum_i (\operatorname{diam}(B_i))^\alpha &\gtrsim \sum_i \sum_{j=1}^{N_i} (\operatorname{diam}_{S'(E)}(B_{ij}))^\alpha \\ &\geq c(\alpha)|f'(z_0)|^\alpha \sum_{i,j} (\operatorname{diam}_{S(E)}(f^{-1}(B_{ij})))^\alpha \\ &\geq c(\alpha)|f'(z_0)|^\alpha \mathcal{H}^\alpha_\infty(E)\end{aligned}$$

The lemma follows. □

Proof of Theorem 1.1. Let f be the conformal map from $\mathbb{H}$ to Ω. Consider the set $E = E(f^{-1}(z), \alpha)$ obtained using Theorem 1.3. Applying Lemma 3.3 we have

$$\mathcal{H}^\alpha_\infty(f(E)) \gtrsim (e^{-2M}|f'(f^{-1}(z))|)^\alpha \mathcal{H}^\alpha_\infty(E).$$

Theorem 1.1 now follows by combining the above estimate with part (a) of Theorem 1.3 and observing that, by part (b), z can be joined to $\partial\Omega$ by a curve which is the bilipschitz image of a curve in $\mathbb{H}$ of length comparable to $\operatorname{Im} f^{-1}(z)$ joining $f^{-1}(z)$ to $\mathbb{R}$.

□

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[B]

Characterizations of generalized John domains in $\mathbb{R}^n$ via metric duality

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CHARACTERIZATIONS OF GENERALIZED JOHN DOMAINS IN $\mathbb{R}^n$ VIA METRIC DUALITY

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Abstract. Using the metric duality theory developed by Väisälä, we characterize generalized John domains in terms of higher dimensional homological bounded turning for their complements under mild assumptions. Simple examples indicate that our assumptions for such a characterization are optimal. Furthermore, we show that similar results in terms of higher dimensional homotopic bounded turning do not hold in dimension three.

Contents

1. Introduction

Recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is a *John domain* if there is a constant C and a point $x_0 \in \Omega$ so that, for each $x \in \Omega$, one can find a rectifiable curve $\gamma\colon [0,1] \to \Omega$ with $\gamma(0) = x$, $\gamma(1) = x_0$ and with

$$Cd(\gamma(t), \partial\Omega) \geq l(\gamma([0,t])) \tag{1.1}$$

for each $0 < t \leq 1$. Alternatively, one may replace the right-hand side with $|\gamma(t) - x|$ or with $\operatorname{diam}(\gamma([0,t]))$, see [24], and rectifiability of γ is not needed in these two

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modifications to the above definition. Here John refers to F. John, who used this condition in his work on elasticity [21]; the terminology was introduced by Martio and Sarvas [22]. The class of John domains includes all smooth domains, Lipschitz domains and certain fractal domains (for example snowflake-type domains).

In the planar case, Näkki and Väisälä [24] performed a detailed study of the *John disks*, i.e., simply connected planar John domains. In particular, all the different definitions of a John disk were shown to be equivalent, quantitatively (see Section 2 for a precise meaning of that term). Moreover, the following well known properties of John disks were proved there:

i). a bounded simply connected domain is a John disk if and only if it is *inner uniform*,

ii). a bounded simply connected domain is a John disk if and only if it is LLC-2 and if and only if its complement is of *bounded turning*.

Moreover, both statements are quantitative in the sense that the coefficients associated to John disks, inner uniform domains, LLC-2 domains and bounded turning domains depend only on each other. See Section 2 below for the precise definitions of inner uniform domains, LLC-2 sets and bounded turning sets.

Motivated by the recent studies on *generalized quasidisks* [16, 14], Guo and Koskela have introduced in [15] the class of *φ-John domains*, which forms a natural generalization of the class of John domains. Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a continuous, increasing function with $\varphi(0) = 0$ and $\varphi(t) \geq t$ for all $t > 0$. A bounded domain Ω in $\mathbb{R}^n$ is called a *φ-length John domain* when (1.1) is replaced with

$$\varphi(Cd(\gamma(t), \partial\Omega)) \geq l(\gamma([0,t])). \tag{1.2}$$

The concepts of *φ-dist* and *φ-diam John domains* are defined analogously. A corresponding curve γ is called a *φ-dist (diam, length) John curve*.

Unlike the John disk case, being a φ-length John domain is not necessarily quantitatively equivalent with being a φ-diam John domain, despite of the fact that the latter is quantitatively equivalent with being a φ-dist John domain; see [15]. However, the well-known properties i) and ii) above do have natural formulations in the category of generalized John disks [15]. Property i) and the first part of property ii) were further generalized to higher dimensions in [13], where the simply connectedness assumption was replaced by *Gromov-hyperbolicity* with respect to the quasihyperbolic metric (see e.g. [7] for the precise definition).

In this paper, we aim at finding a natural substitute to the bounded turning of the complement in ii) for a certain class of domains in higher dimensions. Instead of more or less standard assumption Ω being Gromov-hyperbolic with respect to the quasihyperbolic metric, we require a weaker ball separation property, introduced in [8] and studied further for instance in [7, 6]. This property has turned out to be useful in many problems, for example in connection with Sobolev-Poincare inequalities [8] and uniform continuity of quasiconformal mappings into irregular domains [13].

A crucial observation towards this generalization was made by Väisälä in [26], where he discovered a general *metric duality* property:

> *quantitative metric properties of open subsets of Euclidean spaces can be derived from corresponding properties of their closed complements, and vice versa.*

This is in spirit similar to Ahlfors' three point characterization of *quasidisks* [2]: intrinsic properties of a Jordan curve provide quantitative geometric information about the two complementary components, and conversely.

A principal difference from the planar case, as discovered by Väisälä (see also [4, 5, 19]), is that one needs *p-dimensional analogues* of the *linear local connectivity* and *bounded turning* properties. These definitions, based on *homology* for open sets $U \subset \dot{\mathbb{R}}^n$ and on *cohomology* for closed sets $X \subset \dot{\mathbb{R}}^n$, are given in Section 3.

We begin by recording the following result that follows rather easily from the techniques in [26]. We will provide a detailed proof in Section 5 below.

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that Ω satisfies the ball separation property and that $H^1(\overline{\Omega}) = 0$. Then the following conditions are quantitatively equivalent:*

1). Ω is φ-diam John;
2). Ω is φ-LC-2;
3). the complementary domain $\Omega' = \mathbb{R}^n \backslash \overline{\Omega}$ is homologically $(n-2, \varphi)$-bounded turning.

Under the assumption of Proposition 1.1, the condition 2) (or equivalently 3)) implies that not only Ω is φ-diam John, but also that quasihyperbolic geodesics starting from the John center are φ-diam John curves; see Proposition 6.1 below.

The boundedness assumption on Ω in Proposition 1.1 is not needed if we extend the notion of φ-diam John domains to the unbounded case, as in Section 5. Thus Proposition 1.1 can be regarded as a bounded version of the following, slightly more general result.

Proposition 1.2. *Let $\Omega \subsetneq \mathbb{R}^n$ be a proper subdomain such that Ω satisfies the ball separation property and that $H^1(\overline{\Omega}) = 0$. Then the following conditions are quantitatively equivalent:*

1). Ω is $(0,\varphi)$-John;
2). Ω is φ-LC-2;
3). the complementary domain $\Omega' = \dot{\mathbb{R}}^n \backslash \overline{\Omega}$ is homologically $(n-2, \varphi)$-bounded turning.

The equivalence of 1) and 3) in Proposition 1.2 in the linear case ($\varphi(t) = t$) actually (under stronger assumptions than ours) can be obtained as a corollary to the main results in [26] . However, some extra (nontrivial) work is necessary because of nonlinearity of φ and our weaker assumption of the ball separation property. Moreover, this additional separation property cannot be dropped from Proposition 1.2 as indicated by the following example.

Example 1.3. *There exists a domain $\Omega \subsetneq \mathbb{R}^3$ with $H^1(\overline{\Omega}) = 0$ such that Ω is LLC-2, but not C-diam John for any $C \geq 1$. In particular, Ω fails to have the ball separation property.*

Construction of Example 1.3. Simply rotate $|y| = (1-x)^2, 0 \leq x \leq 1$ about the y-axis to sweep out a cusp domain in the space (see Figure 1).

□

The duality in Proposition 1.1 is formulated in terms of *homological bounded turning*, which in general looks apparently weaker than the *homotopic bounded turning*. On the other hand, when $n = 2$ and Ω is simply connected, these two concepts are indeed quantitatively equivalent. Thus one could still expect that a stronger form of Theorem 1.2 in terms of homotopic bounded turning could hold, if Ω enjoys a suitable higher dimensional "simply connectedness" assumption. As already mentioned earlier, one natural such kind of assumption would be the Gromov-hyperbolicity with

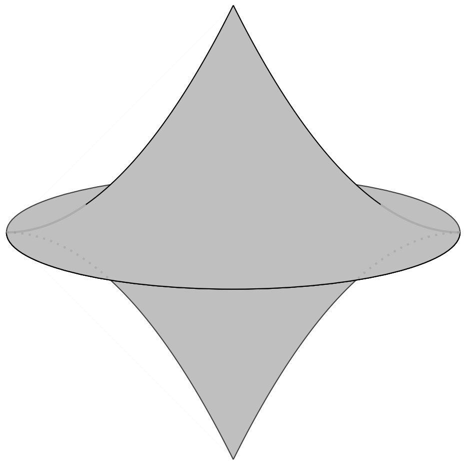

FIGURE 1. Example 1.3

respect to the quasihyperbolic metric. We will refer to these domains as Gromov-hyperbolic domains. Somewhat surprisingly, our main result of this paper shows that this expectation is false when $n = 3$.

Theorem 1.4. *There exists a domain $\Omega \subset \mathbb{R}^3$ with the following properties*

1). Ω is homeomorphic to the open unit ball $\mathbb{B}$;
2). $\overline{\Omega}$ is homeomorphic to the closed unit ball $\overline{\mathbb{B}}$;
3). Ω is a uniform domain with ball separation property;
4). $U = \mathbb{R}^3 \backslash \overline{\Omega}$ is homologically $(1, C)$-bounded turning for some positive constant $C > 1$.
5). U is not homotopically $(1, C)$-bounded turning for any $C > 1$.

The definition of a uniform domain is given in Section 2 below. Bounded uniform domains are John domains and quasiconformal images of uniform domains are typical examples of Gromov-hyperbolic domains, see [7].

Let us shortly comment on the ideas behind the construction of the domain Ω. The starting point for the construction was the question, whether there exists a domain U homeomorphic to a closed ball, which is hlog-$(1, C)$-inner joinable, but not htop-$(1, C)$-inner joinable for any $C > 0$, or, in other words, that, for any $a \in U$ and any ball $B(a, r) \subset \mathbb{R}^n$, every loop contained in $U \setminus B(a, Cr)$ is homologically trivial in $U \setminus B(a, r)$, however, for all C there are $a \in U$, $r > 0$ and a loop γ in $U \setminus B(a, Cr)$ which is not homotopically trivial in $U \setminus B(a, r)$. Such γ is then a commutator of loops in $U \setminus B(a, r)$.

In sufficiently high dimension n, one can easily find a domain U with these properties: let M be a manifold with perfect, non-trivial fundamental group (e.g. let M be the Poincaré sphere). Recall that a group is perfect if its abelianization is trivial, i.e. every element is a commutator. Then M embeds in $\mathbb{S}^k$ for k sufficiently large, and we may take V to be a small tubular neighborhood of M in $\mathbb{S}^k$. We construct U by embedding a countable number of copies V_m of V in $S^k = \partial B^{k+1} = B$ and attaching to B along V_m a cylinder $V_m \times [0, m]$, for $m = 1, 2, \ldots$. Then, for any $r > 1$, $U \setminus B(0, r)$ consists of a countable number of cylinders over V, which have perfect fundamental groups, and thus every loop in $U \setminus B(0, r)$ is a commutator. On the other hand, if $r > 1$ and $C > 1$, every non-trivial loop in $U \setminus B(0, Cr)$ is still non-trivial in $U \setminus B(0, r)$.

In dimension $n = 3$, however, it is not obvious how to construct a domain U homeomorphic to a ball and such that $U \setminus B$, for an Euclidean ball B, has perfect fundamental group. A well known example of a domain with perfect π_1 is the complement of the Alexander's horned sphere ([17, Example 2B.2]), but then it is given as a *Euclidean* ball minus a set diffeomorphic to a ball, and not the other way around. In fact, as Ian Agol pointed to us in a discussion on MathOverflow [1], at least one component of $U \setminus B$ has a non-perfect fundamental group – or all have trivial fundamental groups. Thus in dimension 3 we have to proceed differently. The construction of U mimics the construction of an infinite grope (see e.g. [9, Section 38]), but, since we wish to have in the end a domain, we construct *finite gropes* and attach a countable number of them to a half-space, just like we could do with cylinders in high dimensions.

Our construction of Theorem 1.4 relies heavily on three dimensional topology, in particular, the close relation of the fundamental group and the first homology group. Thus, this kind of constructions do not generalize easily to higher dimensions (i.e. $n \geq 4$). In fact, we do not know whether Theorem 1.2 holds in terms of the homotopic bounded turning assumption when $n \geq 4$.

Some of the arguments that we use in the proofs of our results in Section 3 and Section 4 are rather similar to the ones in [24] for the case of $\varphi(t) = t$. For the convenience of the readers we have included full details even in these cases.

The paper is organized as follows. Section 2 fixes the notation and basic definitions. We introduce the joinability conditions in Section 3 and study their basic properties in Section 4. In Section 5, we prove Proposition 1.1 and show its sharpness in terms of given assumptions. Some remarks on quasihyperbolic geodesics in domains with the ball separation property are given in Section 6. In Section 7, we discuss the relation of homotopic bounded turning and homological bounded turning, and present a basic example to indicate their differences, and in the final section, Section 8, we present our construction of Theorem 1.4.

2. Notation and Definitions

Notation. The one-point compactification of $\mathbb{R}^n$ is denoted by $\dot{\mathbb{R}}^n$, that is $\dot{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. The closure of a set $U \subset \mathbb{R}^n$ is denoted $\overline{U}$ and the boundary ∂U. The open ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$ is denoted by $B(x,r)$ and in the case of the unit ball we omit the centre and the radius, writing $\mathbb{B} := B(0,1)$. The boundary of $B(x,r)$ will be denoted by $S(x,r)$ and in the case of the boundary of the unit ball, by $\mathbb{S} := S(0,1)$. The symbol Ω always refers to a domain, i.e. a connected and open subset of $\mathbb{R}^n$. Whenever we write $\gamma(x,y)$ or γ_{xy}, it refers to a curve or an arc from x to y. For an open or closed set X in $\dot{\mathbb{R}}^n$, we denote by $H_p(X)$ the reduced singular p-homology group of X and by $H^p(X)$ the Alexander-Spanier p-cohomology group, both with coefficients in $\mathbb{Z}$. Occasionally, we shall need the unreduced integral zero-homology and cohomology groups of X, which, following Väisälä ([26]), we denote by $\overline{H_0}(X)$ and $\overline{H^0(X)}$, recall that if $X \neq \varnothing$, $\overline{H_0}(X) = H_0(X) \oplus \mathbb{Z}$ and $\overline{H^0}(X) = H^0(X) \oplus \mathbb{Z}$.

If a condition P with data v implies a condition P' with data v' so that v' depends only on v, then we say that P implies P' quantitatively, and we say that P and P' are quantitatively equivalent, if in addition P implies P quantitatively, as well.

Local connectivity. A set $E \subset \dot{\mathbb{R}}^n$ is called *Linearly Locally Connected* (LLC) if there is a constant $C \geq 1$ such that

(LLC-1) each pair of points in $B(x,r) \cap E$ can be joined in $B(x,Cr) \cap E$, and

(LLC-2) each pair of points in $E \setminus B(x,Cr)$ can be joined in $E \setminus B(x,r)$.

The LLC-condition can be generalized to the non-linear case as follows: a set $E \subset \dot{\mathbb{R}}^n$ is (φ, ψ)-*locally connected* ((φ, ψ)-LC) if

(φ-LC-1) each pair of points in $B(x, r) \cap E$ can be joined in $B(x, \varphi(r)) \cap E$, and

(ψ-LC-2) each pair of points in $E \backslash B(x, r)$ can be joined in $E \setminus B(x, \psi(r))$,

where $\varphi, \psi : [0, \infty) \to [0, \infty)$ are smooth increasing functions such that $\varphi(0) = \psi(0) = 0$, $\varphi(r) \geq r$ and $\psi(r) \leq r$ for all $r > 0$. Depending on what is meant by joining, one can consider *pathwise* and *continuumwise* versions of φ-LC-1, ψ-LC-2, and (φ, ψ)-LC. If E is locally compact, and locally path-connected, then pathwise connectivity is quantitatively equivalent to continuumwise connectivity (see e.g. [20]).

Bounded turning. A subset $E \subset \dot{\mathbb{R}}^n$ is φ-*bounded turning* if there exists a continuous function φ such that each pair of points $x, y \in E$ can be joined by a continuum γ in E satisfying

$$\operatorname{diam}(\gamma) \leq \varphi(|x - y|). \tag{2.1}$$

When $\varphi(t) = Ct$, we recover the so-called C-*bounded turning* or simply *bounded turning sets*.

Uniformity. A domain $\Omega \subset \mathbb{R}^n$ is called a *uniform domain*, if there exists a constant $A \geq 1$ such that each pair of points $x_1, x_2 \in \Omega$ can be joined by a rectifiable curve γ in Ω for which

$$\min_{i=1,2} l(\gamma([x_j, \gamma(t)])) \leq A d(\gamma(t), \partial\Omega) \tag{2.2}$$

and

$$l(\gamma) \leq A d(x_1, x_2). \tag{2.3}$$

A curve γ as above is called an A-*uniform curve*.

Inner uniformity. A domain $\Omega \subset \mathbb{R}^n$ is φ-*dist (diam, length) inner uniform*, if there exists a constant $C > 0$ such that each pair of points $x_1, x_2 \in \Omega$ can be joined by a curve γ in Ω for which

$$\min_{i=1,2} S(\gamma([x_j, \gamma(t)])) \leq \varphi(C d(\gamma(t), \partial\Omega)) \tag{2.4}$$

and

$$S(\gamma) \leq C d_I(x_1, x_2) \tag{2.5}$$

with $S(\gamma)$ equal to $|\gamma(1) - \gamma(0)|$, $\operatorname{diam}(\gamma)$ and $l(\gamma)$, respectively. When $\varphi(t) = t$, we recover the definition of an *inner uniform domain*.

Quasihyperbolic metric and quasihyperbolic geodesics. The *quasihyperbolic metric* k_Ω in a domain $\Omega \subsetneq \mathbb{R}^n$ is defined to be

$$k_\Omega(x, y) = \inf_\gamma k_\Omega\text{-length}(\gamma),$$

where the infimum is taken over all rectifiable curves γ in Ω that join x to y and

$$k_\Omega\text{-length}(\gamma) = \int_\gamma \frac{ds}{d(x, \partial\Omega)}$$

denotes the *quasihyperbolic length* of γ in Ω. This metric was introduced by Gehring and Palka in [12]. A curve γ joining x to y for which k_Ω-length$(\gamma) = k_\Omega(x, y)$ is called a *quasihyperbolic geodesic*. Quasihyperbolic geodesics joining any two points of a proper subdomain of $\mathbb{R}^n$ always exist but they need not be unique; see [11, Lemma 1]. Given a pair of points $x, y \in \Omega$, we denote by $[x, y]$ a quasihyperbolic geodesic that joins x and y.

Ball separation property. Let $\Omega \subset \mathbb{R}^n$ be a proper domain. We say that Ω satisfies the *ball separation property* if there exists a constant $C > 0$ such that for each pair of points $x, y \in \Omega$, for each $z \in [x, y]$, and for every curve γ in Ω joining x to y it holds that

$$B_\Omega(z, Cd(z, \partial\Omega)) \cap \gamma \neq \varnothing, \tag{2.6}$$

where $B_\Omega(z, r)$ is the ball of radius $r > 0$ in the intrinsic length metric of Ω, defined as the infimum of the lengths of curves in Ω joining any pair of points.

3. Basic algebraic topology concepts

In this section, we define a nonlinear variant of the joinability conditions introduced by Väisälä [26].

Let

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \tag{3.1}$$

be a short sequence of *groups* and *homomorphisms*. We say that the sequence is *fast* if $ker(\beta\alpha) = ker\alpha$ or, equivalently, $ker(\beta\alpha) \subset ker\alpha$. Dually, the sequence is *slow* if $im(\beta\alpha) = im\beta$ or, equivalently, $im\beta \subset im(\beta\alpha)$. In particular, the short sequence in (3.1) is fast if $\alpha = 0$ and slow if $\beta = 0$.

Let $A \subset \dot{\mathbb{R}}^n$, $a \in A\backslash\{\infty\}$, $r > 0$ and $\varphi : [0, \infty) \to [0, \infty)$ be a homeomorphism such that $\varphi(r) \geq r$ and $\frac{\varphi(r)}{r}$ is non-decreasing. For each integer p, inclusions induce four sequences

(a): $H_p(A \cap B(a, r)) \to H_p(A \cap B(a, \varphi(r))) \to H_p(A)$;
(b): $H_p(A\backslash\overline{B}(a, \varphi(r))) \to H_p(A\backslash\overline{B}(a, r)) \to H_p(A)$;
(c): $H^p(A) \to H^p(A \cap \overline{B}(a, \varphi(r))) \to H^p(A \cap \overline{B}(a, r))$;
(d): $H^p(A) \to H^p(A\backslash B(a, r)) \to H^p(A\backslash B(a, \varphi(r)))$.

If the sequence (a) is fast for every $a \in A\backslash\{\infty\}$ and $r > 0$, we say that A is *homologically outer* (p, φ)*-joinable*. If (b) is fast for all such a, r, then A is *homologically inner* (p, φ)*-joinable*. If (c) is slow for all such a, r, then A is *cohomologically outer* (p, φ)*-joinable*. If (d) is slow for all such a, r, then A is *cohomologically inner* (p, φ)*-joinable*.

We shall abbreviate the words "homologically" and "cohomologically" by *hlog* and *cohlog*, respectively. We say that A is *hlog* (p, φ)*-joinable* if A is both hlog outer (p, φ)-joinable and hlog inner (p, φ)-joinable. The concept of *cohlog* (p, φ)*-joinability* is defined in an analogous way.

The homological joinability properties can be defined more explicitly in terms of cycles and chains. For example, an open set $U \subset \mathbb{R}^n$ is homologically outer (p, φ)-joinable if and only if, given $a \in U$ and $r > 0$, a p-cycle in $U \cap B(a, r)$ bounds in $U \cap B(a, \varphi(r))$ whenever it bounds in U.

Note that in the definitions of four (p, φ)-joinability properties, one requires the corresponding sequences $(a), (b), (c)$ and (d) to be fast or slow for all $a \in A\backslash\{\infty\}$. Hence, these properties are intrinsic properties of A (referred as absolute joinability). It is often convenient to consider these conditions also for points a outside A and we say that A has *one of the four properties in* $\mathbb{R}^n$ (refereed as relative joinability) if the corresponding condition holds for all $a \in \mathbb{R}^n$. The next lemma shows that the relative joinability is in fact quantitatively equivalent to the absolute joinability; compare with [26, Lemma 2.5].

Lemma 3.1. *If $p \geq 0$ and if $A \subset \dot{\mathbb{R}}^n$ is hlog outer (p, φ)-joinable for all $a \in A\backslash\{\infty\}$ and $r > 0$, then A is hlog outer $(p, 2\varphi + \mathrm{id})$-joinable in $\mathbb{R}^n$. The corresponding statement is valid for the other three joinability properties as well.*

Proof. Let $a \in \mathbb{R}^n$, $r > 0$. Writing $\varphi'(r) = 2\varphi(r) + r$, we must show that the sequence

$$H_p(A \cap B(a,r)) \to H_p(A \cap B(a,\varphi'(r))) \to H_p(A)$$

is fast. If $A \cap B(a,r) = \emptyset$, the first group is trivial, and, consequently, the sequence is fast. If $A \cap B(a,r) \neq \emptyset$, choose a point $x \in A \cap B(a,r)$. Now

$$B(a,r) \subset B(x,2r) \subset B(x,2\varphi(r)) \subset B(a,\varphi'(r)),$$

and we obtain the commutative diagram

$$\begin{array}{ccccc} H_p(A \cap B(a,r)) & \longrightarrow & H_p(A \cap B(a,\varphi'(r))) & \longrightarrow & H_p(A) \\ \downarrow & & \uparrow & & \\ H_p(A \cap B(x,2r)) & \longrightarrow & H_p(A \cap B(x,2\varphi(r))) & \longrightarrow & H_p(A). \end{array}$$

Since the lower row is fast, so is the upper row.

Next assume that A is hlog inner (p,φ)-joinable and that $a \in M$, $r > 0$. We must show that the sequence

$$H_p(A \backslash \overline{B}(a,\varphi'(r))) \to H_p(A \backslash \overline{B}(a,r)) \to H_p(A)$$

is fast. If $A \cap \overline{B}(a,r) = \emptyset$, the sequence is trivially fast since then the second map is the identity. If $A \cap \overline{B}(a,r) \neq \emptyset$, choose $x \in A \cap \overline{B}(a,r)$- Now

$$\overline{B}(a,r) \subset \overline{B}(x,2r) \subset \overline{B}(x,2\varphi(r)) \subset \overline{B}(a,\varphi'(r)),$$

and we can proceed essentially as in the first case.

The cohlog cases are treated by analogous arguments.

□

The following Metric Duality Theorem is due to Väisälä [26, Theorem 2.7]. (The statement in [26, Theorem 2.7] is written in the case $\varphi(t) = ct$, but the proof there works for general φ without change.)

Theorem 3.2 (Metric Duality Theorem)**.** *Suppose that U is an open set in $\dot{\mathbb{R}}^n$ and p is an integer with $0 \le p \le n-2$. Set $X = \dot{\mathbb{R}}^n \backslash U$ and $q = n-2-p$. Then*

(1) U is hlog outer (p,φ)-joinable in $\mathbb{R}^n$ if and only if X is cohlog inner (q,φ)-joinable in $\mathbb{R}^n$;

(2) U is hlog inner (p,φ)-joinable in $\mathbb{R}^n$ if and only if X is cohlog outer (q,φ)-joinable in $\mathbb{R}^n$;

(3) U is hlog (p,φ)-joinable in $\mathbb{R}^n$ if and only if X is cohlog (q,φ)-joinable in $\mathbb{R}^n$.

Note that in the formulation of Theorem 3.2, one uses relative joinability. With the aid of Lemma 3.1, one obtains also the corresponding absolute version of the duality theorem: if U or X is outer or inner (p,φ)-joinable, then X or U is inner or outer $(q, 2\varphi + \mathrm{id})$-joinable.

4. Basic properties of joinability

In this section, we consider some useful consequences of the non-linear joinability that we have introduced in the previous section. The presentation here is parallel to [26, Section 3] for the linear case.

As in [26], to simplify terminology, we omit the word "hlog" if A is open in $\dot{\mathbb{R}}^n$ and the word "cohlog" if A is closed in $\dot{\mathbb{R}}^n$.

The following theorem was proved in [26, Theorem 3.2] for the linear case. The proof in [26, Theorem 3.2] also works in our more general situation.

Theorem 4.1. *Suppose that $A \subset \dot{\mathbb{R}}^n$ is open or closed. Then A is outer or inner (p, φ)-joinable if and only if each component of A is outer or inner (p, φ)-joinable, respectively.*

Lemma 4.2 ([26], Lemma 3.4)**.** *Let $X \subset \dot{\mathbb{R}}^n$ be a measurable subset and let $A \subset B \subset X$. Then the following conditions are equivalent:*

(1) The sequence $\overline{H_0}(A) \to \overline{H_0}(B) \to \overline{H_0}(X)$ is fast;
(2) The sequence $H_0(A) \to H_0(B) \to H_0(X)$ is fast;
(3) Points $x, y \in A$ can be joined by a path in B whenever they can be joined by a path in X.

A direct application of Lemma 4.2 gives us the following result.

Theorem 4.3. *Let $A \subset \dot{\mathbb{R}}^n$. Then the following two conditions are equivalent:*

- *A is hlog outer $(0, \varphi)$-joinable in $\mathbb{R}^n$;*
- *Every path component of A is pathwise φ-LC-1.*

The following two conditions are also equivalent:

- *A is hlog inner $(0, \varphi)$-joinable in $\mathbb{R}^n$;*
- *Every path component of A is pathwise φ-LC-2.*

Recall that two points x and $y \in A$ are separated in a topological space T, if x and y belong to different quasi-components of A, that is, A can be written as a disjoint union of two closed set E and F with x in E and y in F. Equivalently, there is a continuous map $\alpha : A \to \{0, 1\}$ with $\alpha(x) = 1$ and $\alpha(y) = 1$.

Lemma 4.4. [26, Lemma 3.7] *Let $X \subset \dot{\mathbb{R}}^n$ be a measurable subset and assume that $A \subset B \subset X$. Then the following two conditions are equivalent:*

(1) The sequence $\overline{H^0}(X) \to \overline{H^0}(B) \to \overline{H^0}(A)$ is slow;
(2) The sequence $H^0(X) \to H^0(B) \to H^0(A)$ is slow.

Moreover, they imply the condition

"If points $x, y \in A$ are separated in B, then they are separated in X."

If X is compact metrizable and if A is closed in X, then all three conditions are equivalent.

Theorem 4.5. *Let $A \subset \dot{\mathbb{R}}^n$ be compact. Then the following two conditions are equivalent:*

- *A is cohlog outer $(0, \varphi)$-joinable in $\mathbb{R}^n$;*
- *Every path component of A is continuumwise φ-LC-1.*

The following two conditions are also equivalent:

- *A is cohlog inner $(0, \varphi)$-joinable in $\mathbb{R}^n$;*
- *Every path component of A is continuumwise φ-LC-2.*

Proof. We only prove the first part, since the proof of the second part is similar. Suppose that A is cohlog outer (p, φ)-joinable, that C is a component of A and that $a \in \mathbb{R}^n$, $r > 0$. Let $x, y \in C \cap \overline{B}(a, r)$. Then, the sequence

$$H^0(A) \to H^0(A \cap \overline{B}(a, \varphi(r))) \to H^0(A \cap \overline{B}(a, r))$$

is slow. Since x and y are not separated in A, it follows from Lemma 4.4 that they are not separated in $A \cap \overline{B}(a, \varphi(r))$. Since this set is compact and since the quasi-components of a compact set are components, there is a component of $A \cap \overline{B}(a, \varphi(r))$ containing x and y. Hence every component of A is continuumwise φ-LC-1.

Conversely, assume that every component of A is continuumwise φ-LC-1. Let $a \in \mathbb{R}^n$ and $r > 0$. It suffices to show that the sequence

$$H^0(A) \to H^0(A \cap \overline{B}(a, \varphi(r))) \to H^0(A \cap \overline{B}(a, r))$$

is slow. Let $x, y \in A \cap \overline{B}(a, r)$ be points which are not separated in A. Since A is compact, these points belong to a component C of A. By the φ-LC-1 condition, there is a continuum α with $\{x, y\} \subset \alpha \subset C \cap \overline{B}(a, \varphi(r))$. Hence x and y are not separated in $A \cap \overline{B}(a, \varphi(r))$. It follows again from Lemma 4.4 that the above sequence is slow and hence the theorem follows. □

The following duality theorem in the plane generalizes the corresponding results in the linear case [26, Theorem 3.11].

Theorem 4.6 (Duality in the plane). *Let U be open in $\dot{\mathbb{R}}^2$ and let $X = \dot{\mathbb{R}}^2 \setminus U$. Then*

(1) The components of U are pathwise φ-LC-1 if and only if the components of X are continuumwise φ-LC-2;

(2) The components of U are pathwise ψ-LC-2 if and only if the components of X are continuumwise ψ-LC-1;

(3) The components of U are pathwise (φ, ψ)-LC if and only if the components of X are continuumwise (ψ, φ)-LC.

Proof. The claim follows from Theorem 3.2, Theorem 4.3 and Theorem 4.5. □

Remark 4.7. *1. If X is assumed to be connected, then the components of U are simply connected domains in $\dot{\mathbb{R}}^2$. For such domains, the property φ-LC-2 is known to be quantitatively equivalent to φ-diam John property, see [15]. Recall that φ-LC-1 is equivalent to φ-bounded turning. Hence we obtain that: a continuum $X \subset \dot{\mathbb{R}}^2$ is of φ-bounded turning if and only if, quantitatively, all components of $\dot{\mathbb{R}}^2 \setminus X$ are φ-diam John domains.*

2. If $X \subset \dot{\mathbb{R}}^2$ is compact and locally path-connected, one can replace the 'continuumwise (ψ, φ)-LC", quantitatively, by "(ψ, φ)-LC".

5. Joinability, bounded turning and John condition

Let Ω be a domain in $\dot{\mathbb{R}}^n$. Following [26], we say that Ω is (p, φ)-John if $H_p(\Omega) = 0$ and for every p-cycle z bounding in Ω there is a $(p+1)$-chain $g \subset \Omega$ such that $\partial g = z$ and

$$d(x, |z|) \le \varphi(d(x, \partial\Omega)) \quad \text{for all } x \in |g|. \tag{5.1}$$

As in the classical case, we call the above condition (5.1) the *lens condition*. It is straightforward to check that bounded $(0, \varphi)$-John domains are φ-diam John, quantitatively.

Let $\Omega \subset \dot{\mathbb{R}}^n$ be a set. If each p-cycle z in Ω bounds a chain g with $\operatorname{diam}(|g|) \le \varphi(\operatorname{diam}(|z|))$, then Ω is said to be *hlog (p, φ)-bounded turning*, or briefly *hlog (p, φ)-BT*. Note that when $\varphi(r) = cr$, we recover the definitions of hlog (p, c)-bounded turning. For $p = 0$, it is easy to see that the definition is equivalent to the usual φ-bounded turning.

The next lemma is an easy consequence of the definition of hlog bounded turning.

Lemma 5.1. *Let $p \in \mathbb{N}$ and $\Omega \subset \dot{\mathbb{R}}^n$ be a set such that $H_p(\Omega) = 0$. Then the following two conditions are quantitatively equivalent:*

- *Ω is hlog (p, φ)-BT;*
- *Ω is hlog outer (p, φ)-joinable.*

Proof. Suppose Ω is hlog (p,φ)-BT. Then every p-cycle z in Ω bounds a chain $g \subset \Omega$ such that $\operatorname{diam}(|g|) \le \varphi(\operatorname{diam}(|f|))$. This implies that for $a \in \Omega$ and $r > 0$, the sequence

$$H_p(\Omega \cap B(a,r)) \to H_p(\Omega \cap B(a, 2\varphi(r))) \to H_p(\Omega) \tag{5.2}$$

is fast. Hence by Lemma 3.1, Ω is hlog (p,φ')-joinable. Conversely, if (5.2) is fast for all $a \in \Omega$ and $r > 0$, then by the assumption $H_p(\Omega) = 0$, we deduce that the mapping $H_p(\Omega \cap B(a,r)) \to H_p(\Omega \cap B(a, 2\varphi(r)))$ is zero as a homomorphism. Therefore, Ω is (p,φ')-BT. □

Lemma 5.2. [26, Lemma 5.3] *Suppose that $0 \le p \le n-2$ and that $A, V \subset \dot{\mathbb{R}}^n$ are such that V is open, $H_p(V) = 0$, and $V^c \subset \operatorname{int} A$. Then the map $H_p(A \cap V) \to H_p(A)$ is an isomorphism.*

The next result can be regarded as a p-dimensional version of the fact that φ-diam John domains are φ-LC-2, quantitatively.

Theorem 5.3. *Let $U \subset \dot{\mathbb{R}}^n$ be a (p,φ)-John domain with $0 \le p \le n-2$. Then U is hlog inner (p,φ)-joinable, quantitatively.*

Proof. Let $a \in \mathbb{R}^n$ and $r > 0$. Let z be a p-cycle in $U \backslash \overline{B}(a, 2\varphi(r)+r)$ bounding in U. We need to show that z bounds in $U \backslash \overline{B}(a,r)$. Since U is (p,φ)-John, $z = \partial g$ for some $(p+1)$-chain g satisfying the lens condition in U. If $|g| \cap \overline{B}(a,r) = \emptyset$, there is nothing to prove. In the opposite case, we fix a point $x \in |g| \cap \overline{B}(a,r)$. Now

$$2\varphi(r) = 2\varphi(r) + r - r < d(x, |z|) \le \varphi(d(x, \partial U)),$$

and hence $\overline{B}(a,r) \subset \overline{B}(a, \varphi^{-1}(2\varphi(r))) \subset U$. Applying Lemma 5.2 with $A = U$ and $V = \overline{B}(a,r)^c$ we see that z bounds in $U \backslash \overline{B}(a,r)$. □

As a particular consequence of Theorem 5.3, we infer that φ-diam John domains U in $\mathbb{R}^n$ are φ-LC-2, quantitatively.

Proposition 5.4. *Let $\Omega \subset \dot{\mathbb{R}}^n$ be a set such that $H^1(\overline{\Omega}) = 0$. Then $\overline{\Omega}$ is φ-LC-2 if and only if, $\Omega' = \dot{\mathbb{R}}^n \backslash \overline{\Omega}$ is $(n-2,\varphi)$-bounded turning, quantitatively.*

Proof. By the Alexander duality (see e.g. [23, Theorem 74.1]), $H_{n-2}(\Omega') = 0$. Now,

$$\begin{aligned} \Omega' \text{ is } (n-2,\varphi)\text{-bounded turning} &\overset{\text{Lemma 5.1}}{\Longleftrightarrow} \Omega' \text{ is hlog outer } (n-2,\varphi)\text{-joinable} \\ &\overset{\text{Theorem 3.2}}{\Longleftrightarrow} \overline{\Omega} \text{ is cohlog inner } (0,\varphi)\text{-joinable} \\ &\overset{\text{Theorem 4.5}}{\Longleftrightarrow} \overline{\Omega} \text{ is cohlog inner } (0,\varphi)\text{-joinable.} \end{aligned}$$

□

Proposition 5.5. *Let $\Omega \subset \mathbb{R}^n$ be a proper domain with the ball separation property. Then Ω is (0,φ)-John if and only if Ω is φ-LC-2, quantitatively.*

Proof. Based on Theorem 5.3, it suffices to show that if Ω is φ-LC-2, then it is $(0,\psi)$-John for some ψ that is quantitatively equivalent to φ, i.e., there exists some positive constant C such that $\psi(t) = \varphi(Ct)$ for all $t > 0$. We first claim that for each pair of points $x, y \in \Omega$, there exists a quasihyperbolic geodesic γ that joins x and y with the property that

$$\text{either } \gamma_{zx} \text{ or } \gamma_{zy} \text{ is contained in } B\Big(z, \psi\big(d(z,\partial\Omega)\big)\Big) \tag{5.3}$$

for all $z \in \gamma$ with $\psi = \varphi(Ct)$, where C is the constant from the ball separation property. Indeed, if (5.3) fails for some $z \in \gamma$, then there exists $x_0 \in \gamma_{zx}$ and $y \in \gamma_{zy}$

such that x and y are outside the ball $B\Big(z,\psi\big(d(z,\partial\Omega)\big)\Big)$. Then, by the φ-LC-2 condition, they can be joined outside $B(z, Cd(z,\partial\Omega))$, which contradicts the fact that Ω satisfies the ball separation property with constant C. Thus (5.3) holds for all $z \in \gamma$, and consequently, Ω is $(0,\psi)$-John. □

Proof of Proposition 1.2. Proposition 1.2 follows from Proposition 5.4 and Proposition 5.5 upon noticing that for a domain in $\mathbb{R}^n$, Ω is φ-LC-2 if and only if, quantitatively, $\overline{\Omega}$ is φ-LC-2. □

We also point out the following characterization of John domains in three dimension, which is essentially due to Väisälä [26].

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^3$ be a domain with trivial homology groups, such that $\Omega' = \dot{\mathbb{R}}^3 \backslash \Omega$ is LLC-1. Then Ω is John if and only if Ω' is hlog $(1,c)$-bounded turning, quantitatively.*

Proof. The necessity is a direct consequence of Theorem 5.3, Lemma 5.1 and Theorem 3.2.

For the sufficiency, note that by [26, Theorem 5.21], if Ω satisfies $H_1(\Omega) = 0 = H_2(\Omega)$ and if Ω is hlog inner $(0,c_0)$-joinable and hlog $(1,c)$-inner joinable, then Ω is c'-John, quantitatively. In our situation, by the metric duality Theorem 3.2, Ω being hlog $(1,c)$-inner joinable is quantitatively equivalent to the condition that Ω' is LLC-1 and Ω being LLC-2 is quantitatively equivalent to being outer $(1,c)$-joinable, which is further equivalent to being hlog $(1,c)$-bounded turning by Lemma 5.1. Thus the claim follows. □

6. Quasihyperbolic geodesics in domains with ball separation property

Recall that from Proposition 5.5, we know that if $\Omega \subset \mathbb{R}^n$ is a bounded domain with the ball separation property and if Ω is φ-LC-2, then Ω is a φ-diam John domain with a center point x_0. Moreover, the proof implies that quasihyperbolic geodesics starting from x_0 are φ-diam John curves. We formulate this result as a separate proposition below.

Proposition 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the ball separation property. If Ω is φ-LC-2, then Ω is a φ-diam John domain with a center point x_0 and quasihyperbolic geodesics starting from x_0 are φ-diam John curves.*

Note that Proposition 6.1 was previously known to hold for bounded Gromov hyperbolic φ-diam John domains [13, Proposition 3.8]. As a consequence of Proposition 6.1 and [13, Proof of Theorem 3.1 (2)], we obtain the following result, that generalizes [13, Theorem 3.1 (2)] by removing the a priori Gromov-hyperbolicity assumption.

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the ball separation property. If Ω is φ-LC-2, then Ω is η-length John for*

$$\eta(t) = C\int_0^{\varphi(Ct)} \left(\frac{s}{\varphi^{-1}(s)}\right)^{n-1} ds, \tag{6.1}$$

provided this integral converges. The statement is essentially sharp in the sense that η defined in (6.1) is best possible.

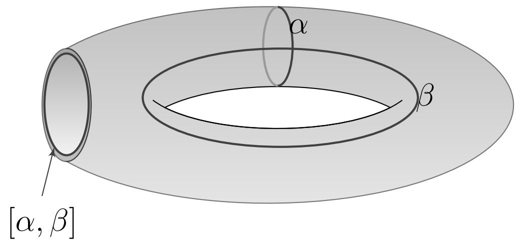

Figure 2. A pinched torus – the loop at the cut is a commutator of the loops α and β.

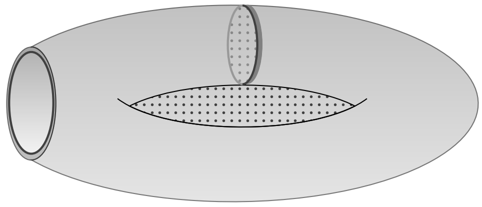

Figure 3. Filling α and β with disks.

7. The bounded example

Let $p \in \mathbb{N}$, $\varphi : [0,\infty) \to [0,\infty)$ be a homeomorphism, and $\Omega \subset \dot{\mathbb{R}}^n$ be a domain. If every map $f : S^p \to \Omega$ has an extension $g : \overline{B}^{p+1} \to \Omega$ such that

$$\operatorname{diam}(|g|) \le \varphi(\operatorname{diam}(|f|)), \tag{7.1}$$

then Ω is said to be htop (p,φ)-bounded turning, or briefly htop (p,φ)-BT. Equivalently, Ω is htop (p,φ)-bounded turning if $\pi_p(\Omega) = 0$ and the sequence

$$\pi_p(\Omega \cap B(x,r)) \to \pi_p(\Omega \cap B(x,\varphi(r))) \to \pi_p(\Omega)$$

is fast. Here π_p denotes the p-th order homotopy group (see e.g. [17, 25] for the definition). As always, Ω is htop (p,C)-bounded turning if it is htop (p,φ)-bounded turning with $\varphi(r) = Cr$.

Similarly, we can define htop inner joinability as in the homology case (simply replacing H_p with π_p).

Next, we construct a bounded domain $\Omega \subset \mathbb{R}^3$, topologically equivalent to an open ball, that is hlog $(1,C)$-bounded turning for some $C > 1$, but not htop $(1,C)$-bounded turning for any $C > 1$.

7.1. **An infinite mushroom – construction.** The starting point of our construction is a pinched torus, i.e., a torus with a disk cut out.

Since we are interested in constructing a domain, not a surface, we use a 'thickened' torus, i.e. a tubular neighborhood of the torus surface. The thickness of the torus is uniform, and equal to 10^{-3} of the diameter of the whole torus.

In the next step, we glue (thickened) disks along the loops α and β, see Figure 3. The resulting set has trivial fundamental group and it is homeomorphic (and diffemorphic, if we keep the boundary smooth) to a unit ball. This can be easily visualized if we (diffeomorphically) thicken the disks plugging the loops α and β so that they almost fill the torus, see Figure 4. Note also that if we plugged only alpha with a thickened disk, we would obtain a set diffeomorphic to a filled torus.

Note that if the tentacles are cut off at some level, the resulting set is again (diffeomorphic to) a pinched torus from Figure 2.

We extend the tentacles further down, cut off the one representing the β loop, and attach a pinched torus to it. In the resulting space, the loop β, seen on Figure 6 as the cut in the larger tentacle, becomes a commutator of the two loops in the attached pinched torus.

Next, we extend another two tentacles from the disks glued into the smaller pinched torus (see Figure 8), then, we iterate the construction. Of the two loops generating the fundamental group of the pinched torus attached at stage k, one (α_k) is filled with a disk, extended to a tentacle, the other (β_k) - with a pinched torus. If

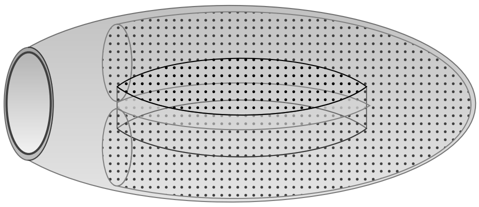

Figure 4. The piched torus with α and β filled with thickened disks is diffeomorphic to a 3-ball.

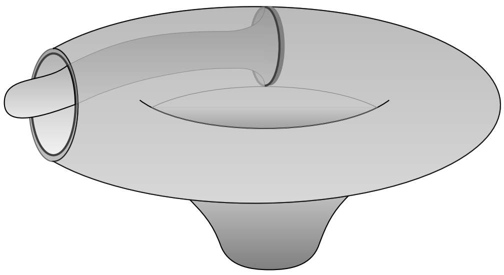

Figure 5. Extending 'tentacles' out of plugging disks.

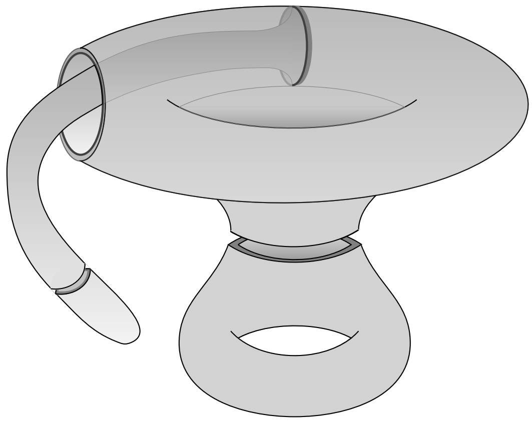

Figure 6. Cutting the tentacle representing β loop and attaching a pinched torus to it. Cutting off both tentacles reproduces a pinched torus.

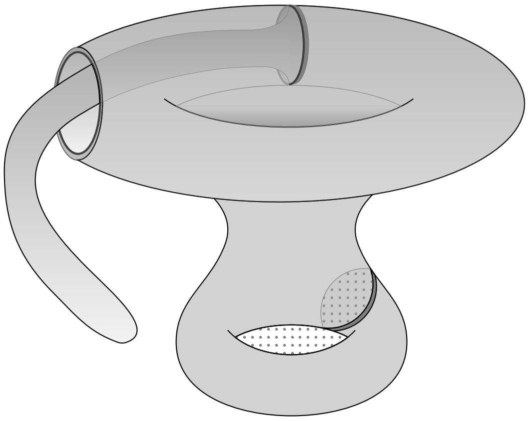

Figure 7. Filling the loops in the smaller pinched torus with thickened disks.

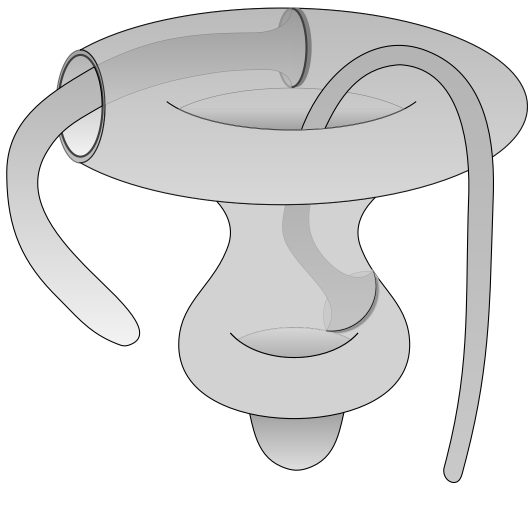

Figure 8. Extending tentacles from the attached pinched torus.

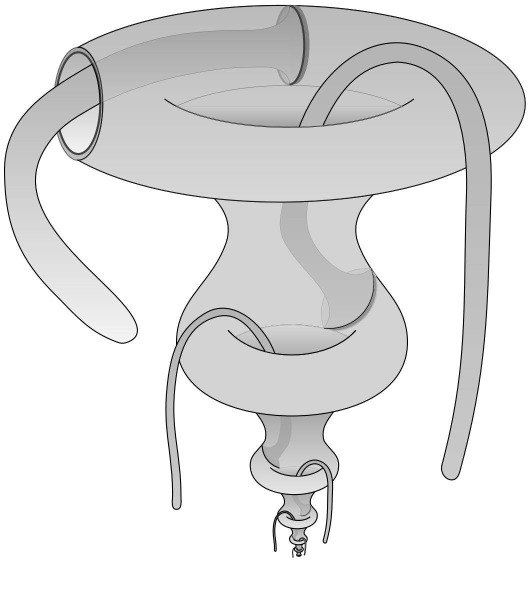

Figure 9. The infinite mushroom.

we cut off the tentacles constructed at stage $(k+2)$, α_k becomes contractible (the tentacle extended from α_k ends in stage $(k+1)$) and β_k represents the commutator $[\alpha_{k+1}, \beta_{k+1}]$. Of course, we cannot keep the thickness of the tubular neighborhood

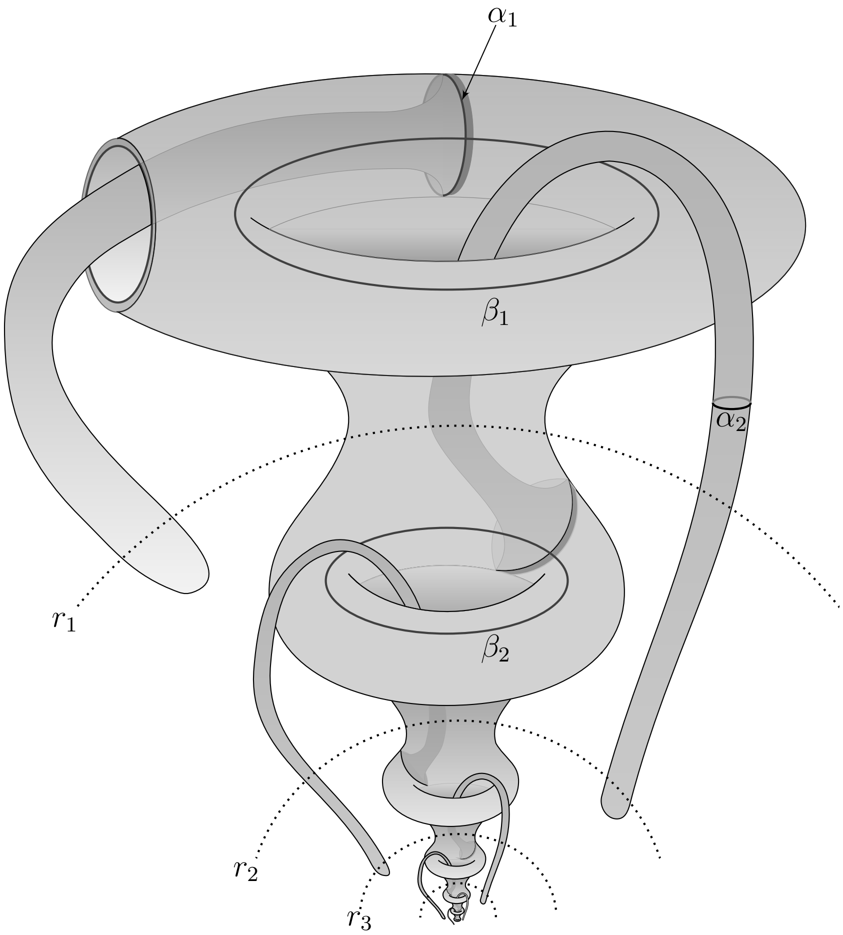

FIGURE 10. The loop β_1 is a commutator of β_2 and α_2.

fixed – it has to decrease with subsequent stages of the construction, but we keep the thickness in parts modified at stage k comparable to 10^{-3} of the diameter of the pinched torus attached at stage k.

Note that at each stage we attach along β_k a pinched torus with both loops filled with disks, i.e. a space diffeomorphic to a ball, or, equivalently, to a thickened disk, which shows that the $k+1$-st stage of our construction is diffeomorphic to the k-th stage, and, by induction, to the first stage – to a pinched torus with both loops plugged with thickened disks, i.e. to a space diffeomorphic to a ball.

The resulting set M_∞, to which we shall refer as *the infinite mushroom*, is presented in Figure 9.

Denote the limit point of M_∞ by z. At each point $y \in M_\infty$ the thickness of the mushroom, the size of the stage of construction and the distance to z are comparable. Assume we have a ball $B = B(x,r)$ centered at $x \neq z$ such that $B \cap M_\infty \neq \varnothing$. We either have $B \subset\subset intM_\infty$ (and then the fundamental group of $M_\infty \setminus B$ is the same as that of M_∞), or the diameter of B is greater than the thickness of M_∞ at points where B cuts M_∞. Then, if k-th stage of M_∞ is the earliest stage cut by B, the inflated ball $10^4 B = B(x, 10^4 r)$ contains all the stages of M_∞ from $(k-2)$-th up.

Let (r_i) be the sequence of radii such that $\frac{r_k}{r_{k+2}} = c_0 << 100$ for all k and that the sphere $S(z, r_k)$ separates stage k of M from stages $(k+1), (k+2), \ldots$ (see Figure 10).

The set $S_1 = M_\infty \setminus B(z, r_1)$ is homeomorphic to a pinched torus plus a thick cylinder obtained from a dissected tentacle. Therefore, any loop in S_1 is generated by

α_1, β_1 and α_2. However, in $S_3 = M_\infty \setminus B(z, r_3)$, the loops α_1 and α_2 are contractible, and the loop β_1 binds a pinched torus attached at stage 2 of our construction, thus it becomes a commutator (up to orientation, $\beta_1 = [\beta_2, \alpha_2]$). Therefore, the inclusion map $S_1 \overset{i_1}{\hookrightarrow} S_3$ induces a zero mapping on the first homology group.

The same phenomenon holds for any inclusion $M_\infty \setminus B(z, r_k) \overset{i_k}{\hookrightarrow} M_\infty \setminus B(z, r_{k+2})$.

Note also that the infinite mushroom M_∞ is contractible (it is homeomorphic to a thick cylinder that thins to a single point at the center, or, more precisely, to a closed ball minus a double open cone). Thus the sequence

$$H_1(M_\infty \setminus B(z, r_k)) \xrightarrow{(i_k)_*} H_1(M_\infty \setminus B(z, r_{k+2})) \to H_1(M_\infty) = 0$$

is fast.

Since the ratio r_k/r_{k+2} is constant and much less than 100, we have that for any $r > 0$ the sequence

$$H_1(M_\infty \setminus B(z, 100r)) \xrightarrow{(i_k)_*} H_1(M_\infty \setminus B(z, r)) \to H_1(M_\infty) = 0 \tag{7.2}$$

is fast.

By earlier remarks, the sequence

$$H_1(M_\infty \setminus B(x, 10^4 r)) \xrightarrow{(i_k)_*} H_1(M_\infty \setminus B(x, r)) \to H_1(M_\infty) = 0, \tag{7.3}$$

for any $x \in \mathbb{R}^n$ and $r > 0$, is fast, as well.

On the other hand, the sequence analogous to (7.2) in the first homotopy groups

$$\pi_1(M_\infty \setminus B(z, r_k)) \xrightarrow{(i_k)_*} \pi_1(M_\infty \setminus B(z, r_{k+2})) \to \pi_1(M_\infty) = 0 \tag{7.4}$$

is not fast at all – the loop β_1 is not contractible in $M_\infty \setminus B(z, r_k)$ for any k.

7.2. **Finite mushrooms.** The problem with M_∞ is that it is not a domain, and its interior does not have the above property, since it is not contractible – and neither the homology sequence (7.2), nor the homology sequence (7.4) is fast with intM_∞ in place of M_∞.

To overcome this difficulty, we observe that if, instead of constructing an infinite mushroom M_∞, we stop at some finite stage M_k (e.g. M_2 depicted in Figure 8), we obtain, as we observed before, a set diffeomorphic to a ball, with closure diffeomorphic to a closed ball, which is both hlog and htop inner $(1, C)$-joinable, but the higher k, the higher is the constant C in htop inner $(1, C)$-joinability. This comes from the fact that for balls $B(z, r)$ centered at the limit point of the infinite mushroom M_∞, there is no difference between $M_\infty \setminus B(z, r)$ and $M_k \setminus B(z, r)$, unless r is sufficiently small. Thus the sequence

$$\pi_1(M_k \setminus B(z, r_1)) \xrightarrow{(i_k)_*} \pi_1(M_k \setminus B(z, r)) \to \pi_1(M_k) = 0$$

is fast only if $r < r_k$ (and the ratio r_1/r_k tends to infinity). At the same time the analogous sequences for homology grups are fast, because, as before, the inclusion maps induce zero maps on homology.

These observations are now valid also for the interiors of the mushrooms M_k and are independent of scaling.

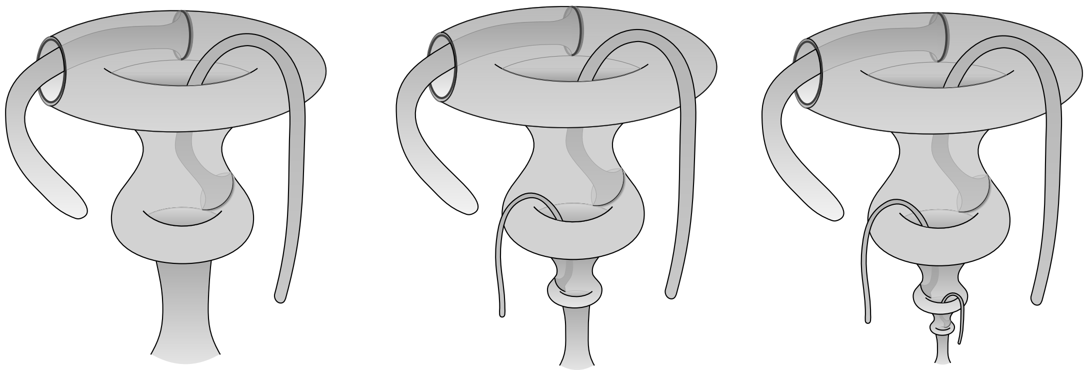

FIGURE 11. Finite mushrooms, to be planted on a ball

7.3. **Mushrooms on a ball.** Our example Ω of a bounded domain in $\mathbb{R}^3$, which is hlog inner $(1, C)$-joinable for $C > 10^4$, but not htop inner $(1, C)$-joinable for any $C > 0$ will consist of an open ball B, with an infinite countable family of finite mushrooms M_k attached to it. We attach the mushrooms sufficiently far away from each other: the distance between any two mushrooms is more than twice the size of the larger of these two. This can be obtained by attaching not the original mushrooms M_k, but their sufficiently small copies: M_k rescaled by a factor λ_k (we shall refer to this smaller copy of M_k as to $\lambda_k M_k$. This is to ensure that if any ball $B(x, r)$ cuts two different mushrooms, then $B(x, 2r)$ contains both mushrooms – and thus any loop in $\Omega \setminus B(x, 2r)$ lies outside these two mushrooms and thus it is contractible outside $B(x, r)$).

Then, there is no uniform constant C such that Ω is htop inner $(1, C)$-joinable, since for any C we can find k such that $r_1/r_k > C$ – and then, for the ball $B = B(z, r)$ of radius $r = \lambda_{k+1} r_k$, centered at the limit point z of the mushroom $\lambda_{k+1} M_{k+1}$ the sequence

$$\pi_1(\Omega \setminus B(z, Cr)) \xrightarrow{(i)_*} \pi_1(\Omega \setminus B(z, r)) \to \pi_1(\Omega) = 0$$

is not fast.

At the same time, by the same arguments as for the infinite mushroom M_∞, Ω is hlog inner $(1, C)$-joinable for $C > 10^4$. We can also deduce this fact via the metric duality theorem as follows: by Theorem 4.6 and Theorem 4.5, Ω is hlog inner $(1, C)$-joinable if and only if quantitatively its complementary is C_1-LLC-1. In our case, it is clear that $\dot{\mathbb{R}}^n \setminus \Omega$ is 10^4-LLC-1.

Note that each of the finite mushrooms M_k is diffeomorphic to a ball, and by taking that ball sufficiently small, we may assure that the diffeomorphism has small derivative. Therefore, Ω is homeomorphic to an open ball, and the homeomorphism is differentiable (in particular, it is Lipschitz). The inverse homeomorphism, however, is not differentiable.

8. The unbounded example

In this section, we give the explicit construction of the example in Theorem 1.4. The basic idea behind that example is quite similar to the bounded example in the previous section, except that this time, we break the htop outer joinability.

8.1. **Trumpets.** Geometrically, a k-trumpet is a k-mushroom that, if extended to ∞-mushroom, would have a limit point at infinity. In particular, every stage of construction is of the same size and thickness. A 3-trumpet is depicted on Figure

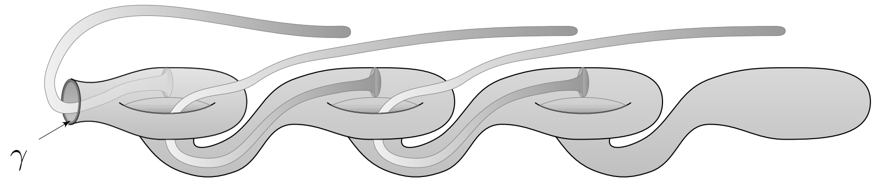

FIGURE 12. A 3-trumpet and its γ loop.

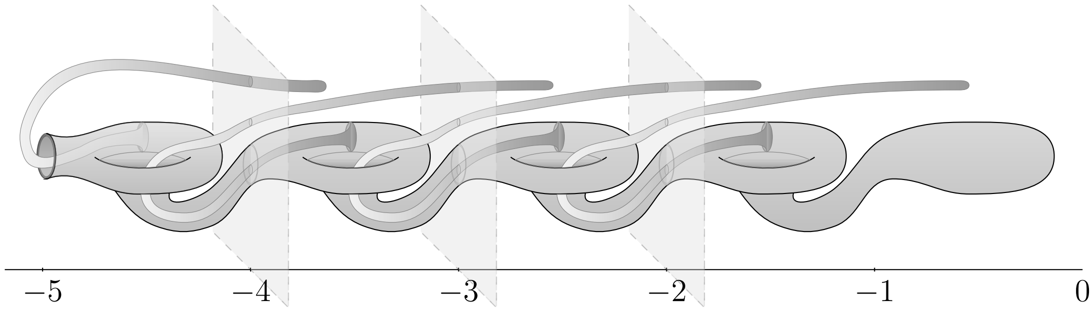

FIGURE 13. Vertical planes intersecting a trumpet.

12 (as in the case of mushrooms, the trumpet is an open tubular neighborhood of the contractible surface that one sees on the picture).

Every k-trumpet is diffeomorphic to the mushroom M_k, and thus is diffeomorphic to an open ball, and its closure is diffeomophic to a closed ball. It also shares the crucial property of the mushroom: any loop that is contained in the first ℓ stages of construction becomes homologically trivial (i.e. either contractible, or a commutator) when considered as a loop in the first $\ell + 2$ stages.

Assume the last, trivial stage of the k-trumpet T_k (the "mouthpiece") is contained in $\{-1 < x_1 < 0\}$. Then, denoting by L_ℓ the half-space $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, x_1 \leq \ell\}$ we have that the inclusion mapping $T_k \cap L_\ell \xhookrightarrow{i} T_k \cap L_{\ell+2}$ induces a zero mapping in the first homology group, thus the sequence

$$H_1(T_k \cap L_\ell) \xrightarrow{i_*} H_1(T_k \cap L_{\ell+2}) \to H_1(T_k) = 0$$

is fast. At the same time, the sequence

$$\pi_1(T_k \cap L_\ell) \xrightarrow{i_*} \pi_1(T_k \cap L_{\ell+2}) \to \pi_1(T_k) = 0$$

is not fast as long as $\ell \in [-k, -3]$ – this is exactly the same phenomenon as in the case of a finite mushroom. Indeed, to each k-trumpet we can associate the loop γ of the pinched torus in the first stage of the construction (c.f. loop $[\alpha, \beta]$ in Figure 2). Then γ is not contractible in $T_k \cap L_\ell$ for any $\ell \leq -1$.

8.2. **Trumpet wall.** To construct our infinite example U, we attach an infinite countable family of trumpets T_k to a vertical wall (i.e. the half-space $\{(x_1, \ldots, x_n) \,:\, x_1 > 0\}$), assuming again that the trumpets are far apart from each other: the distance between any two is at least twenty times the size of the larger one. It is clear that the complementary domain $\Omega = \dot{\mathbb{R}}^n \backslash \overline{U}$ is a $(0, C)$-John domain and it satisfies the ball separation property (recall definition from Section 5). Every k-trumpet is diffeomorphic to a ball and its closure is diffeomorphic to a closed ball, thus, Ω satisfies the conditions 1) and 2) in Example 1.4, since it is, essentially, a half-space with some (sets diffeomorphic to) balls nicely attached, each away from the other ones.

We next show that the trumpet wall U is an example of a domain which is homologically $(1, C)$-bounded turning for some positive constant $C > 1$, but not homotopically $(1, C)$-bounded turning for any $C > 1$.

To see that it is not htop $(1, C)$-BT for any given $C > 1$, it suffices to consider the γ loops of different trumpets in the trumpet wall. Each of these loops is contractible (i.e. it binds a disk in U), but such a disk has to reach the wall, and thus its diameter is as large as the diameter of the whole trumpet. Since we have trumpets of arbitrary length, U is not htop $(1, C)$-BT for any given constant $C > 1$.

To see that it is hlog $(1, C)$-bounded turning for some constant $C > 1$, let us assume γ is a loop in U.

If γ passes through more than one trumpet, then the ball B centered at some point of it, of radius equal to $\mathrm{diam}(\gamma)$, contains all the trumpets that γ intersects (thanks to the fact that the distances between trumpets are at least twice their sizes) and the loop γ is contractible in $U \cap B$ (and thus it bounds a disk of diameter comparable with $\mathrm{diam}(\gamma)$).

Assume now that γ is contained in a single trumpet. Then we have two possible cases:

- γ has very small diameter – less than the thickness of the trumpet. Then it binds a small disk of comparable diameter (recall that a trumpet is diffeomorphic and bi-Lipschitz equivalent to a ball, or if one prefers, thick cylinder plugged at one end, along which end it is attached to the half-space).
- γ has diameter comparable or much larger than the size of a single stage of the trumpet. Let T denote the sum of stages of the trumpet intersected by γ. Form $\tilde{T}$ by adding to T two more stages of the trumpet towards the wall (add one or none if T already contains last-but-one or the last stage of the trumpet towards the wall). Then $\tilde{T}$ has diameter comparable with T, thus comparable with $\mathrm{diam}(\gamma)$ and γ is homologicaly trivial in $\tilde{T}$ (it is a boundary in $\tilde{T}$).

Thus we conclude that U is hlog $(1, C)$-bounded turning with $C > 100$. Alternatively, we can also conclude U being hlog $(1, C)$-bounded turning by noticing that $\dot{\mathbb{R}}^n \backslash U$ is C-LLC-2 with some absolute constant $C > 100$.

8.3. **Uniformity.** Finally, we show that Ω is an A-uniform domain for some constant $A \geq 1$. For this, it suffices to show that for each pair of points $x, y \in \Omega$, we may find an A-uniform curve γ_{xy} in Ω joining x and y.

Before turning to the rigorous proof, let us briefly point out the difficulty in finding a uniform curve between the points x and y. If the points are sufficiently close to each other (say very close in a small neighborhood of the same trumpet), then by our construction of trumpet, we know that (away from the hyperplane) it consists of self-similar "mushrooms", each of which is a Lipschitz domain, in particular an A-uniform domain, and so we may connect them by a A-uniform curve. If the points are very far away from each other (say stay in neighborhoods of different trumpets), then it is not difficult to connect them by curves with property (2.2). Moreover, since the distance between each two trumpets is at least twenty times the size of the larger one, it is easy to adjust the corresponding curve so that it also satisfies (2.3). Thus, the essential difficulty lies in the case when these two points stay in the same neighborhood of some trumpet but are not so close to each other. This would correspond to Case 2 below.

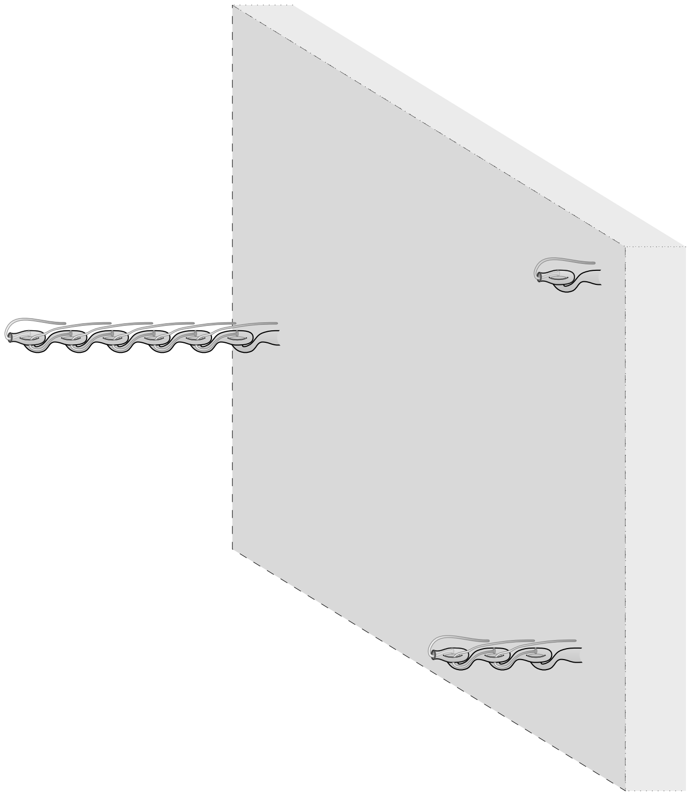

FIGURE 14. A piece of the trumpet wall.

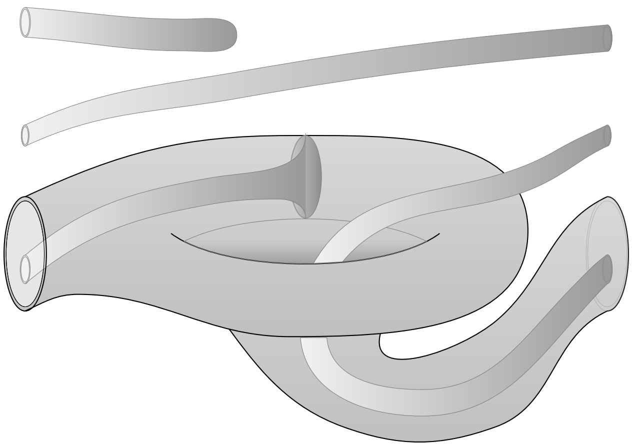

FIGURE 15. T_k^i has 4 components.

Let $k \in \mathbb{N}, k > 2$. Let C_k^i be the cubical neighbourhood with one of its faces intersecting (perpendicularly) the mouth of the i-th level of the trumpet T_k such that the opposite face intersects the mouth of the $(i+2)$-th level of the trumpet. Let B_k^i be the cubical neighbourhood contained in C_k^i with its edge parallel to e_3 and e_2 of the same length as edge parallel to e_3 and respectively e_2 of C_k^i and the edge normal to the e_1 direction of length half as edge parallel to e_1 of C_k^i; see Figures 15 and 16.

We will call the two intersecting faces horizontal faces and the four other faces vertical faces of C_k^i. For each k there are $i_k = k-2$ such neighbourhoods which cover the "middle portion" of the trumpet T_k. We also define a cubical neighbourhood C_k^0 which has one of its faces lying in $\partial\mathbb{H}^3$ and the opposite face intersecting the mouth

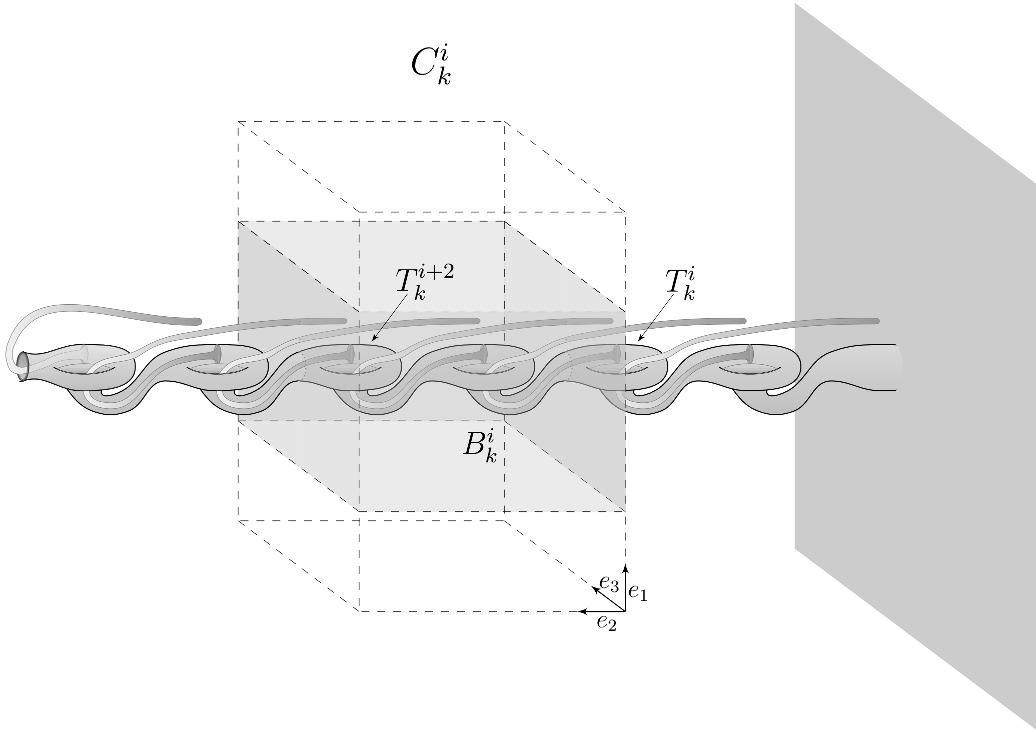

FIGURE 16. The rectangular parallelepiped C_k^i contains T_k^{i+1} and T_k^{i+2}.

of the 2-nd level and a neighbourhood $C_k^{i_k+1}$ with one of its faces intersecting the mouth of the (i_k-1)-th level as shown in Figure 16. Recall that the thickness of the trumpet is 10^{-3} and the diameter of each level T_k^i and the distance between consecutive levels are 1 (in particular comparable to 10^{-3}). The distance between each of the vertical faces of C_k^i to the i-th level T_k^i is at least 10 and comparably smaller than 1. Note also that the distance between the non-intersecting horizontal surface of $C_k^{i_k+1}$ to T_k is at least 1 and comparably smaller than 1. Set

$$U_k = \bigcup_{i=0}^{i_k+1} C_k^i$$

and notice that $T_k \subset U_k$. We consider the following cases.

Case 1: $x, y \in U_k$ for some $k \in \mathbb{N}$ and $|x-y| < 10^{-3}$.

In this case, we assign to the pair (x, y) any one of the C_k^i such that $x, y \in C_k^i$ and obtain an A-uniform curve γ_{xy} joining x and y in $C_k^i \cap \Omega$.

Case 2: $x, y \in U_k$ and $|x-y| \geq 10^{-3}$.

In this case, we may assume that $\{x, y\} \not\subset C_k^l$ for any l. Let C_k^i and C_k^j be the cubical neighbourhoods containing x and y, respectively. If $x \in B_k^i$ then let γ_x be a quasihyperbolic geodesic joining x to a point x' in the vertical face J^i of C_k^i that has the standard coordinate vector e_1 as the outward normal and such that

$$\delta_{C_k^i}(x, x') \leq 2d(x, J^i),$$

where $\delta_{C_k^i}(x, x')$ is the diameter distance in C_k^i obtained by taking the infimum over the diameter of all curves joining x and x' in C_k^i.

Let x'' be the point where γ_x meets ∂B_k^i. With a slightly abuse of notation, from now on, we also denote by γ_x the subcurve of γ_x lying between x and x''. Similarly, if $y \in B_k^j$, we may find a point $y'' \in B_k^j$ and a quasihyperbolic geodesic γ_y joining y and y''. We let $\gamma_{x''y''}$ denote the half-circle not intersecting B_k^i with centre as the

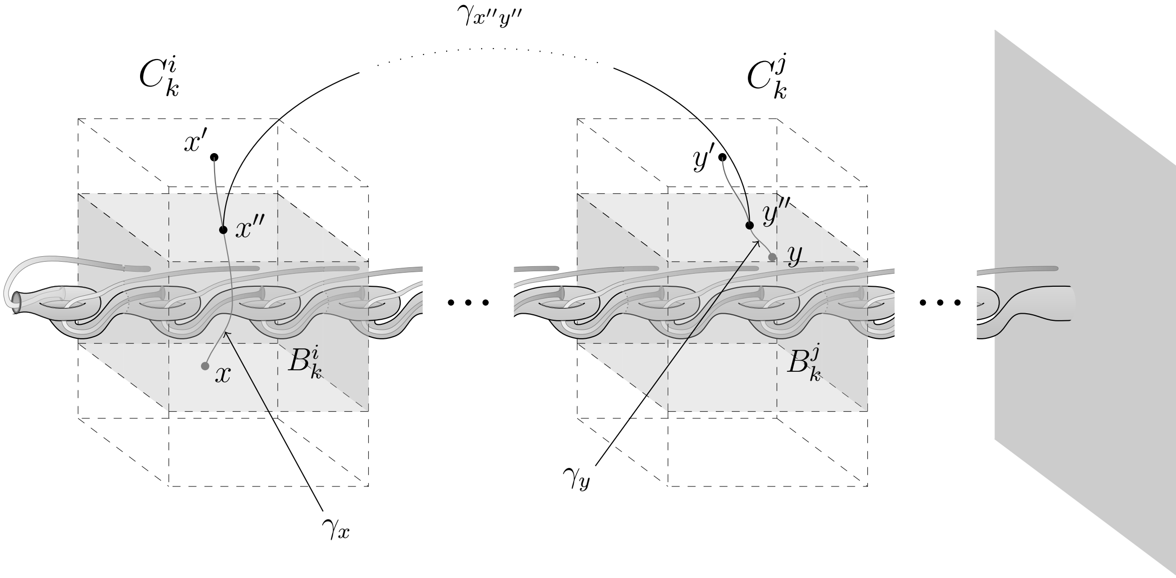

FIGURE 17. Case 2.

midpoint of the line segment joining x'' and y''; of diameter $|x''-y''|$ and lying in the plane that is normal to the plane coinciding with J^i and intersecting it along the line containing x'' and y''. We denote by γ_{xy} the curve joining x and y obtained by concatenation of the curves γ_x, $\gamma_{x''y''}$ and γ_y; see Figure 17. It is clear that

$$l(\gamma_{xy}) \leq A|x-y|$$

for some absolute constant $A \geq 1$. Furthermore, it also follows from our construction that γ_{xy} satisfies (2.2) for a suitable constant $A \geq 1$ and thus it is an A-uniform curve joining x and y.

In the case when either $x \notin B_k^i$ or $y \notin B_k^j$, we denote by x'' (resepectively y'') the normal projection of x (respectively y) in the plane coinciding with J^i and choose a similar half-circle passing through x'' and y'' and then obtain a curve γ_{xy} by concatenation of the corresponding curves. It is straightforward to check as in the previous case that γ_{xy} is an A-uniform curve for some absolute constant $A \geq 1$.

Case 3: $x, y \in \Omega \setminus \cup_k U_k$.

In this case, observe that $\Omega\setminus\cup_k U_k$ is an A-uniform domain (indeed even a Lipschitz domain) for some large enough constant A and that an A-uniform curve joining x and y in $\Omega\setminus \cup_k U_k$ is also an A-uniform curve joining x and y in Ω.

Case 4: $x \in U_i$ and $y \in U_j$, $i \neq j$, or $x \in U_i$ for some i and $y \notin U_j$ for any j.

In this case, the required A-uniform curve joining x and y can be obtained by arguing as in Case 2 and concatenating the corresponding uniform curves. We leave the details to the interested readers.

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A density result for homogeneous Sobolev spaces on planar domains

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A DENSITY RESULT FOR HOMOGENEOUS SOBOLEV SPACES ON PLANAR DOMAINS

DEBANJAN NANDI, TAPIO RAJALA, AND TIMO SCHULTZ

Abstract. We show that in a bounded simply connected planar domain Ω the smooth Sobolev functions $W^{k,\infty}(\Omega)\cap C^\infty(\Omega)$ are dense in the homogeneous Sobolev spaces $L^{k,p}(\Omega)$.

1. Introduction

By the result of Meyers-Serrin [16] it is known that $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$ for every open set Ω in $\mathbb{R}^d$. The space $C^\infty(\mathbb{R}^d)$ is not always dense in $W^{k,p}(\Omega)$, for example when Ω is a slit disk. However, a slit disk is not a very appealing example as it is not the interior of its closure. Counterexamples for the density satisfying $\Omega = \operatorname{int}(\overline{\Omega})$ were given by Amick [1] and Kolsrud [9]. In fact, in these examples even $C(\overline{\Omega})$ is not dense in $W^{k,p}(\Omega)$. Going further in counterexamples, O'Farrell [18] constructed a domain satisfying $\Omega = \operatorname{int}(\overline{\Omega})$ where $W^{k,\infty}(\Omega)$ is not dense in $W^{k,p}(\Omega)$ for any k and p. The domain constructed by O'Farrell was infinitely connected. From the recent results of Koskela-Zhang [14] and Koskela-Rajala-Zhang [13] we can conclude that this is necessary for such constructions in the plane, since $W^{1,\infty}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for all finitely connected bounded planar domains (see also the earlier work by Giacomini-Trebeschi [4]). Further examples of domains where $W^{1,p}(\Omega)$ is not dense in $W^{1,q}(\Omega)$ were constructed by Koskela [11] and Koskela-Rajala-Zhang [13].

In this note we continue the study of density of $W^{k,\infty}(\Omega)$ in $W^{k,p}(\Omega)$. Let us remark that such density clearly holds in the case where the Sobolev functions in $W^{k,p}(\Omega)$ can be extended to Sobolev functions defined on the whole $\mathbb{R}^2$. By work of Jones [8], this is true when $\partial\Omega$ is a quasi-circle. (See also the works [6, 5, 7].) Geometric characterizations of Sobolev extension domains are known, especially in the planar simply connected domains when $k = 1$, see [3, 10, 19, 12].

Being an extension domain is only a sufficient condition for the density. For example, there are Jordan domains Ω and functions $f \in W^{1,p}(\Omega)$ that cannot be extended to a function in $W^{1,p}(\mathbb{R}^2)$. However, global smooth functions are dense in $W^{1,p}(\Omega)$ for any Jordan domain and any $p \in [1,\infty]$, see Lewis [15] and Koskela-Zhang [14]. For $W^{k,p}(\Omega)$ with $k \geq 2$ this is still unknown.

In [20] Smith-Stanoyevitch-Stegenga studied the density of $C^\infty(\mathbb{R}^2)$ as well as the density of functions in $C^\infty(\Omega)$ with bounded derivatives, in $W^{k,p}(\Omega)$. For the latter class they

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obtained a density result assuming Ω to be starshaped or to satisfy an interior segment condition. For the smaller class of functions $C^\infty(\mathbb{R}^2)$ they also required an extra assumption on the boundary points to be m_2-limit points. (See also Bishop [2] for a counterexample on a related question.)

The result of Koskela-Zhang [14] showing that $W^{1,\infty}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for every bounded simply connected planar domain was generalized to higher dimensions by Koskela-Rajala-Zhang [13]. They showed that simply connectedness is not sufficient to give such a density result, but Gromov hyperbolicity in the hyperbolic distance is. In this paper we provide another generalization to the Koskela-Zhang result by going to higher order Sobolev spaces. We show that if we restrict attention to the homogenous norm, then being simply connected is sufficient for domains in the plane.

For a domain $\Omega \subset \mathbb{R}^2$ and $p \in [1,\infty)$, by homogenous Sobolev space $L^{k,p}(\Omega)$ we mean functions with p-integrable distributional derivatives of order k;

$$L^{k,p}(\Omega) = \{u \in L^1_{loc}(\Omega) : \nabla^\alpha u \in L^p(\Omega), \text{ if } |\alpha| = k\},$$

with semi-norm $\sum_{|\alpha|=k} \|\nabla^\alpha u\|_{L^p(\Omega)}$, where α is any 2-vector of non-negative integers and $|\alpha|$ is its ℓ_1-norm. The (non-homogenous) Sobolev space $W^{k,p}(\Omega)$ is defined as

$$W^{k,p}(\Omega) = \{u \in L^1_{loc}(\Omega) : \nabla^\alpha u \in L^p(\Omega), \text{ if } |\alpha| \le k\},$$

with norm $\sum_{|\alpha|\le k} \|\nabla^\alpha u\|_{L^p(\Omega)}$.

Theorem 1.1. *Let $k \in \mathbb{N}$, $p \in [1,\infty)$ and $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then the subspace $W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$ is dense in the space $L^{k,p}(\Omega)$.*

The approach in [13] differs from ours in that there the approximating functions are defined via shifting matters to the disk via the Riemann mapping. Instead, we directly make a Whitney decomposition of the domain and a rough reflection to define our approximating sequence. We achieve this via an elementary use of simply connectedness in the plane. In both of these approaches the values of the function in a suitable compact set are used to define a smooth function in the entire domain which approximates the original function in Sobolev norm. For this we employ similar tools as used by Jones in [8].

In p-Poincaré domains, that is domains Ω where a p-Poincaré inequality

$$\int_\Omega |u - u_D|^p \,\mathrm{d}x \le C \int_\Omega |\nabla u|^p \,\mathrm{d}x$$

holds, we can bound the integrals of the lower order derivatives by the integrals of the higher order ones and thus we obtain the following corollary to our Theorem 1.1.

Corollary 1.2. *Let $k \in \mathbb{N}$, $p \in [1,\infty)$ and $\Omega \subset \mathbb{R}^2$ be a bounded simply connected p-Poincaré domain. Then $W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$ is dense in the space $W^{k,p}(\Omega)$.*

For instance Hölder-domains are p-Poincaré domains for $p \ge 2$, see Smith-Stegenga [21]. It still remains an open question whether Corollary 1.2 holds if one drops the assumption of being a p-Poincaré domain.

Next we come to the question of density of $C^\infty(\mathbb{R}^2)$ functions in $L^{k,p}(\Omega)$ in our setting of bounded simply connected domains. We have the following corollary which is analogous to

[14, Corollary 1.2], where it is shown that Ω being Jordan is sufficient. A small modification of the argument there applies to our situation as well. See the end of Section 4 for the proof.

Corollary 1.3. *Let $k \in \mathbb{N}$, $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^2$ be a Jordan domain. Then $C^\infty(\mathbb{R}^2)$ is dense in the space $L^{k,p}(\Omega)$.*

In Section 2, we collect the necessary ingredients which will be used for defining the approximating sequence; these include a suitable Whitney-type decomposition of a simply connected domain and a local polynomial approximation of Sobolev functions. In Section 3 we describe a partition of the domain using the Whitney-type decomposition of Section 2, which is needed for obtaining a suitable partition of unity. Then in Section 4, we define the approximating sequence and present the necessary estimates for proving Theorem 1.1 and Corollary 1.3.

2. Preliminaries

For sets $A, B \subset \mathbb{R}^2$ we denote the diameter of A by $\mathrm{diam}(A)$ and the distance between A and B by $\mathrm{dist}(A, B)$. We denote by $B(x, r)$ the open ball with center $x \in \mathbb{R}^2$ and radius $r > 0$ and more generally, by $B(A, r)$ the open r-neighbourhood of a set $A \subset \mathbb{R}^2$. Given a connected set $E \subset \mathbb{R}^2$ and points $x, y \in E$, we define the inner distance $\mathrm{d}_E(x, y)$ between x and y in E to be the infimum of lengths of curves in E joining x to y. (Notice that in general the infimum might have value ∞.) We write the inner distance in E between sets $A, B \subset E$ as $\mathrm{dist}_E(A, B)$.

With a slight abuse of notation, by a curve γ we refer to both, a continuous mapping $\gamma\colon [0, 1] \to \mathbb{R}^2$ and its image $\gamma([0, 1])$. Given two curves $\gamma_1, \gamma_2\colon [0, 1] \to \mathbb{R}^2$ such that $\gamma_1(1) = \gamma_2(0)$, we denote by $\gamma_1 * \gamma_2\colon [0, 1] \to \mathbb{R}^2$ the concatenated curve $\gamma_1 * \gamma_2(t) = \gamma_1(2t)$ for $t \leq 1/2$ and $\gamma_1 * \gamma_2(t) = 2t - 1$ for $t \geq 1/2$. We denote the length of a curve γ by $L(\gamma)$.

We will use the following facts in plane topology whose proofs can be found in the book of Newman [17, Chapter VI, Theorem 5.1 and Chapter V, Theorem 11.8].

Lemma 2.1. *Let Ω be a simply connected domain in $\mathbb{R}^2$ and $\gamma\colon [0, 1] \to \mathbb{R}^2$ a continuous curve that is injective on $(0, 1)$, whose endpoints $\gamma(0)$ and $\gamma(1)$ are in $\partial\Omega$ and interior $\gamma((0, 1))$ in Ω. Then $\Omega \setminus \gamma$ has two connected components, both of which are simply connected.*

In the case where Ω is Jordan and γ is homeomorphic to a closed interval, the two connected components of $\Omega \setminus \gamma$ have boundaries $\gamma \cup J_1$ and $\gamma \cup J_2$, where J_1 and J_2 are the two connected components of $\partial\Omega \setminus \gamma$.

2.1. **A dyadic decomposition.**

Although it is standard to consider a Whitney decomposition of a domain in $\mathbb{R}^d$ (see for instance Whitney [23] or the book of Stein [22, Chapter VI]), we will use a precise construction of such a decomposition. We present this construction below. Here and later on we denote the sidelength of a square Q by $l(Q)$.

For notational convenience we start the Whitney decomposition below from squares with sidelength 2^{-1}. Formally, by rescaling, we may consider all bounded domains $\Omega \subset \mathbb{R}^2$ to have $\operatorname{diam}(\Omega) \le 1$ in which case no Whitney decomposition would have squares larger than the ones used below regardless of the starting scale.

Definition 2.2 (Whitney decomposition)**.** Let $\Omega \subset \mathbb{R}^2$ be a bounded (simply connected) open set. Let $\mathcal{Q}_n$ be the collection of all closed dyadic squares of sidelength 2^{-n}. Define a Whitney decomposition as $\tilde{\mathcal{F}} := \bigcup_{n\in\mathbb{N}} \tilde{\mathcal{F}}_n$ where the sets $\tilde{\mathcal{F}}_n$ are defined recursively as follows. Define

$$\tilde{\mathcal{F}}_1 := \left\{ Q \in \mathcal{Q}_1 : \bigcup_{\substack{Q'\in\mathcal{Q}_1 \\ Q'\cap Q\neq\emptyset}} Q' \subset \Omega \right\}$$

and

$$\tilde{\mathcal{F}}_{n+1} := \left\{ Q \in \mathcal{Q}_{n+1} : Q \not\subset \tilde{F}_n \text{ and } \bigcup_{\substack{Q'\in\mathcal{Q}_{n+1} \\ Q'\cap Q\neq\emptyset}} Q' \subset \Omega \right\},$$

where $\tilde{F}_n = \bigcup_{j\le n} \bigcup_{Q\in\tilde{\mathcal{F}}_j} Q$.

Lemma 2.3. *A Whitney decomposition given by Definition 2.2 has the following properties.*

(W1) $\Omega = \bigcup_{Q\in\tilde{\mathcal{F}}} Q$
(W2) $l(Q) < \operatorname{dist}(Q,\Omega^c) \le 3\sqrt{2}l(Q) = 3\operatorname{diam}(Q)$ *for all* $Q \in \tilde{\mathcal{F}}$
(W3) $\operatorname{int} Q_1 \cap \operatorname{int} Q_2 = \emptyset$ *for all* $Q_1, Q_2 \in \tilde{\mathcal{F}}, Q_1 \neq Q_2$
(W4) If $Q_1, Q_2 \in \tilde{\mathcal{F}}$ *and* $Q_1 \cap Q_2 \neq \emptyset$*, then* $\frac{l(Q_1)}{l(Q_2)} \le 2$.

Proof. Although the proof is very elementary, we give it here for completeness.

For (W1), take any $x \in \Omega$ and $n \in \mathbb{N}$ such that $x \in Q \in \mathcal{Q}_n$, where $2^{-n+2}\sqrt{2} < \operatorname{dist}(x,\Omega^c) \le 2^{-n+3}\sqrt{2}$. Then for any $Q' \in \mathcal{Q}_n$ with $Q' \cap Q \neq \emptyset$ we have $Q' \subset \Omega$. Hence by definition either $Q \in \tilde{F}_n$ or $x \in Q \subset Q'' \in \tilde{\mathcal{F}}_i$ for some $i < n$.

In order to see (W2), let $Q \in \tilde{\mathcal{F}}_n$. Then all $Q' \subset \Omega$ for all $Q' \in \mathcal{Q}_n$ with $Q' \cap Q \neq \emptyset$. Consequently, $\operatorname{dist}(Q,\Omega^c) > 2^{-n} = l(Q)$. For the upper bound, suppose $\operatorname{dist} Q, \Omega^c > 3\sqrt{2}2^{-n}$. Let $Q_2 \in \mathcal{Q}_{n-1}$ be such that $Q \subset Q_2$. Then $\operatorname{dist}(Q_2,\Omega^c) > \sqrt{2}2^{-n+1}$ and so $Q_3 \subset \Omega$ for all $Q_3 \in \mathcal{Q}_{n-1}$ for which $Q_2 \cap Q_3 \neq \emptyset$. Thus $Q_2 \in \tilde{F}_{n-1}$ or $Q_2 \subset Q_4 \in \tilde{F}_i$ for some $i < n-1$. In either case, $Q \notin \tilde{\mathcal{F}}_n$ giving a contradiction.

Property (W3) holds by the recursion in the definition and the fact that the dyadic squares are nested.

Suppose (W4) is not true. Then there exist $Q_1 \in \tilde{\mathcal{F}}_n$ and $Q_2 \in \tilde{\mathcal{F}}_m$ with $n < m-1$ and $Q_1 \cap Q_2 \neq \emptyset$. Let $Q_3 \in \tilde{F}_{n+1}$ be such that $Q_2 \subset Q_3$. Then

$$\bigcup_{\substack{Q'\in\mathcal{Q}_{n+1} \\ Q'\cap Q_3\neq\emptyset}} Q' \subset \bigcup_{\substack{Q'\in\mathcal{Q}_n \\ Q'\cap Q_1\neq\emptyset}} Q' \subset \Omega$$

and so either $Q_3 \in \tilde{\mathcal{F}}_{n+1}$ or $Q_3 \subset \tilde{F}_n$. In both cases $Q_2 \subset \tilde{F}_{n+1}$ and so $Q_2 \notin \tilde{\mathcal{F}}_m$. $\square$

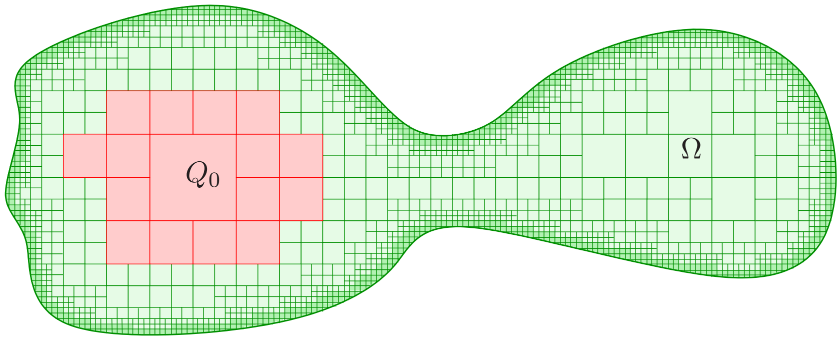

FIGURE 1. A core part D_n is selected from the Whitney decomposition of Ω by taking the connected component containing Q_0 of the interior of the union of Whitney squares with sidelength at least 2^{-n}.

By a chain of dyadic squares $\{Q_i\}_{i=1}^m$ we mean a collection of sets $Q_i \in \tilde{\mathcal{F}}$ such that $Q_i \cap Q_{i+1}$ is a non-degenerate line segment for all $i \in \{1, \dots, m-1\}$. We say that the chain connects Q_1 and Q_m.

2.2. **Approximating polynomials.** We record here the following two Lemmas from [8] which will be used when estimating the approximation in Section 4.

Lemma 2.4 (Lemma 2.1, [8])**.** *Let Q be any square in $\mathbb{R}^2$ and P be a polynomial of degree k defined in $\mathbb{R}^2$. Let $E, F \subset Q$ be such that $|E|, |F| > \eta|Q|$ where $\eta > 0$. Then*

$$\|P\|_{L^p(E)} \le C(\eta, k)\|P\|_{L^p(F)}.$$

Given a function $u \in C^\infty(\Omega)$ and a bounded set $E \subset \Omega$, we define (see [8]) the polynomial approximation of u in E , $P_k(u, E)$ to be the polynomial of order $k-1$ which satisfies

$$\int_E \nabla^\alpha (u - P_k(u, E)) = 0$$

for each $\alpha = (\alpha_1, \alpha_2)$ such that $|\alpha| = \alpha_1 + \alpha_2 \le k-1$. Once k is fixed, we denote the polynomial approximation of u in a dyadic square Q as P_Q

The next lemma is a consequence of Poincaré inequality for Lipschitz domains.

Lemma 2.5 (Lemma 3.1, [8])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain and $\tilde{\mathcal{F}}$ a Whitney decomposition of Ω. Fix α such that $|\alpha| \le k$. Let $\{Q_i\}_{i=1}^m$ in $\tilde{\mathcal{F}}$ be a chain of dyadic squares in $\tilde{\mathcal{F}}$.Then we have*

$$\|\nabla^\alpha (P_{Q_1} - P_{Q_m})\|_{L^p(Q_1)} \le Cl(Q_1)^{k-|\alpha|}\|\nabla^k u\|_{L^p(\bigcup_{i=1}^m Q_i)},$$

where $\nabla^k u$ is the vector $(\nabla^\alpha u)_{|\alpha|=k}$ normed by the ℓ_2-norm and $C = C(m)$.

In what follows, given $\beta = (\beta_1, \beta_2)$ and $\alpha = (\alpha_1, \alpha_2)$, we write $\beta \le \alpha$ if the inequality holds coordinate-wise.

3. Decomposition of the domain

From now on we fix a bounded simply connected domain $\Omega \subset \mathbb{R}^2$ and a Whitney decomposition $\tilde{\mathcal{F}}$ of Ω given by Definition 2.2. For our purposes we need to choose at each level a nice enough subcollection of $\tilde{\mathcal{F}}_n$, namely we take connected components of the Whitney decomposition (see Figure 1). More precisely we fix $Q_0 \in \tilde{\mathcal{F}}_1$ and for each $n \in \mathbb{N}$ let C_n be the connected component of the interior of $\tilde{F}_n$ that has $\operatorname{int} Q_0$ as a subset. We define

$$\mathcal{F}_{n,j} := \{Q \in \tilde{\mathcal{F}}_j : \operatorname{int} Q \subset C_n\}$$

and using this the families of squares

$$\mathcal{F}_n := \mathcal{F}_{n,n}, \qquad \mathcal{D}_n := \bigcup_{j \le n} \mathcal{F}_{n,j}$$

and the corresponding sets for two of the above collections by

$$F_n := \bigcup_{Q \in \mathcal{F}_n} Q \quad \text{ and } \quad D_n := \bigcup_{Q \in \mathcal{D}_n} Q = \overline{C}_n.$$

The collection of boundary layer squares in $\mathcal{D}_n$ is denoted by

$$\partial \mathcal{D}_n := \left\{ Q \in \mathcal{D}_n : Q \cap \overline{(\Omega \setminus D_n)} \neq \emptyset \right\}.$$

With this notation we have the following lemma.

Lemma 3.1. *The above collections have the properties:*

(i) $\mathcal{D}_n \subset \mathcal{D}_{n+1}$ *for all* $k \in \mathbb{N}$.
(ii) $\Omega = \bigcup_{n \in \mathbb{N}} D_n$.
(iii) If $Q_1, Q_2 \in \mathcal{F}_n$ *and* $Q_1 \cap Q_2$ *is a singleton, then there exists* $Q_3 \in \mathcal{D}_n$ *for which* $Q_1 \cap Q_2 \cap Q_3 \neq \emptyset$.
(iv) If $Q \in \partial \mathcal{D}_n$, *then* $Q \in \mathcal{F}_n$.
(v) If $Q \in \partial \mathcal{D}_n$, *then* $Q \cap \overline{(\Omega \setminus \tilde{F}_n)} \neq \emptyset$.
(vi) The set C_n *is simply connected.*

Proof. The property (i) is obvious by the definitions of $\mathcal{F}_{n,j}$ and $\mathcal{D}_n$ since $C_n \subset C_{n+1}$.

For (ii) it suffices to prove that for every $Q \in \tilde{\mathcal{F}}$ there exists $n \in \mathbb{N}$ so that $Q \in \mathcal{D}_n$. Let $Q \in \mathcal{F}_n$. Since Ω is connected and open, there exists a path γ in Ω joining Q to Q_0. By the fact that $\tilde{F}_j \subset \operatorname{int} \tilde{F}_{j+1}$ and the property (W1) of the decomposition $\tilde{\mathcal{F}}$ we have that $\Omega = \cup_{j \in \mathbb{N}} \operatorname{int} \tilde{F}_j$. Then by the compactness of γ there exists $m \geq n$ so that $\gamma \subset \operatorname{int} \tilde{F}_m$. Hence $Q \in \mathcal{D}_m$.

For (iii) let $Q_1, Q_2 \in \mathcal{F}_n$ be so that $Q_1 \cap Q_2$ is a singleton $\{q\}$. Assume that the claim is false. Then for the two squares $Q \in \mathcal{Q}_n$ that intersect both Q_1 and Q_2 it is true that $Q \notin \tilde{\mathcal{F}}_n$ and $Q \not\subset Q'$ for all $Q' \in \tilde{\mathcal{F}}_{n-1}$. Let q_1 and q_2 be the centres of the squares Q_1 and Q_2 respectively. Consider a curve $\gamma' \colon [0,1] \to \Omega$ for which $\gamma'_0 = q_1$, $\gamma'_1 = q_1$ and $\gamma' \subset C_n$. Such a curve exists by the definition of C_n. We may also assume that γ' is an injective

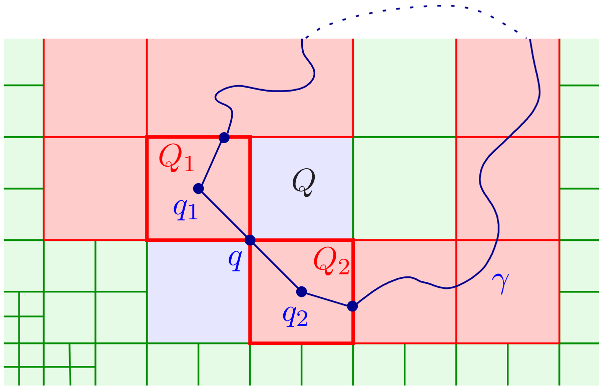

FIGURE 2. The constructed Jordan curve γ in the proof of Lemma 3.1 ((iii)) has in its interior domain a dyadic square Q that also has to be an element of $\mathcal{D}_n$.

curve. Let $t_0 \coloneqq \sup\{t : \gamma'(t) \in Q_1\}$ and $t_1 \coloneqq \inf\{t \geq t_0 : \gamma'(t) \in Q_2\}$. Define a Jordan curve

$$\gamma \coloneqq \gamma^1 * \gamma^2 * \gamma'|_{[t_0,t_1]} * \gamma^3 * \gamma^4,$$

where γ^1, γ^2, γ^3 and γ^4 correspond to the line segments $[q, q_1]$, $[q_1, \gamma'_{t_0}]$, $[\gamma'_{t_1}, q_2]$ and $[q_2, q]$ respectively. By Jordan curve theorem γ divides $\mathbb{R}^2$ into two components, one of which is precompact (see Figure 2). Denote the precompact component by A.

For small enough ball B around q we have by the definition of γ that $B \setminus \gamma$ has exactly two components. Since γ is a Jordan curve one of those components has to contain an interior point of A and thus the whole component lies inside A. On the other hand that component has to intersect with one of the dyadic squares in $\mathcal{Q}_n$ touching both Q_1 and Q_2 (but being different from Q_1 and Q_2). Let $Q \in \mathcal{Q}_n$ be that square. Now for all the neighbouring squares $\tilde{Q} \in \mathcal{Q}_n$ (except the opposite one) of Q either $\tilde{Q} \cap \gamma([0,1]) \neq \emptyset$ implying that $\tilde{Q} \in \mathcal{D}_n$ or $\tilde{Q}$ is in the precompact component of $\mathbb{R}^2 \setminus \gamma([0,1])$ and thus by simply connectedness $\tilde{Q} \subset \Omega$. Since $Q_1 \in \mathcal{F}_n$, also the opposite square of Q is a subset of Ω. Hence $Q \in \tilde{\mathcal{F}}_n$ or $Q \subset Q' \in \mathcal{F}_{n-1}$ which is a contradiction. Thus we have proven (iii).

In order to see (iv), suppose that there exists $Q \in \partial\mathcal{D}_n$ such that $Q \notin \mathcal{F}_n$. Then $Q \in \mathcal{F}_{n,i} \subset \tilde{\mathcal{F}}_i$ for some $i < n$. By Property (W4), for all the $Q' \in \tilde{\mathcal{F}}$ with $Q' \cap Q \neq \emptyset$ we have $Q' \in \tilde{\mathcal{F}}_j$ for $j \leq i + 1 \leq n$. Thus, $Q' \subset D_n$ and $Q \notin \partial\mathcal{D}_n$ giving a contradiction.

If property (v) fails for some $Q \in \partial\mathcal{D}_n$, then for every $Q' \in \tilde{\mathcal{F}}$ with $Q' \cap Q \neq \emptyset$ we have $Q' \in \tilde{\mathcal{F}}_i$ for some $i \leq n$. Thus, again $Q' \subset D_n$ and $Q \notin \partial\mathcal{D}_n$ giving a contradiction.

Finally, we prove property (vi). Since C_n is open it suffices to prove that every Jordan curve is loop homotopic to a constant loop. Suppose this is not the case. Then there exists a Jordan curve γ that is not homotopic to a constant loop, and a point $x \in \Omega \setminus C_n$ that lies inside γ. In particular there exists $Q \in \mathcal{Q}_n$ such that $Q \not\subset D_n$ which lies inside γ and for which $Q \cap D_n$ is an edge of a square. Now by similar argument as in (iii) we conclude that $Q \in \mathcal{D}_n$, which is a contradiction. $\square$

The next lemma shows that we can connect the boundary of D_n to the boundary of Ω with a short curve in the complement of D_n.

Lemma 3.2. *For each point $x \in \partial D_n$, there exists an injective curve $\gamma\colon [0,1] \to \mathbb{R}^2$ so that $\gamma(0) = x$, $\gamma(1) \in \partial\Omega$, $\gamma(0,1) \subset \Omega \setminus \operatorname{int} D_n$ and $L(\gamma) \leq 2\sqrt{2}l(Q)$.*

Proof. Let $Q \in \partial\mathcal{D}_n$ be such that $x \in Q \cap \partial D_n$. By Lemma 3.1 (v) we have that there exists a square $Q' \in \mathcal{Q}_n$ touching Q at x so that $Q' \notin \tilde{\mathcal{F}}_n$ and $Q' \not\subset \tilde{Q}$ for every $\tilde{Q} \in \tilde{F}_j$, $j < n$. Thus, there exists a neighbouring square $Q'' \in \mathcal{Q}_n$ of Q' and a point $y \in \partial\Omega \cap Q''$. Let γ^1 be a curve corresponding to a line segment connecting x to a point $z \in Q' \cap Q''$ and let γ^2 be a curve corresponding to a line segment connecting z to y. Moreover, let $t_0 := \inf\{t : \gamma^2_t \in \partial\Omega\}$. Since $\partial\Omega$ is closed, we have that $\gamma^2_{t_0} \in \partial\Omega$. Define a curve $\gamma := \gamma^1 * \gamma^2|_{[0,t_0]}$. For γ we have that $\gamma(0,1) \subset (\Omega \setminus \operatorname{int} D_n) \cap (Q' \cup Q'')$, $\gamma(0) = x$, $\gamma(1) \in \partial\Omega$ and $L(\gamma) \leq d(Q') + d(Q'') = 2\sqrt{2}l(Q)$. □

Observe that by Lemma 3.1 (iv) we have $\partial D_n = \bigcup_{Q \in \partial\mathcal{D}_n}(Q \cap \partial D_n)$. Thus, by Lemma 3.1 (iii) we have that ∂D_n is locally homeomorphic to the real line. Since by Lemma 3.1 (vi) C_n is simply connected, we have that $\partial D_n = \partial C_n$ is connected. Hence, ∂D_n is a Jordan curve. Thus, we may write

$$\partial D_n = \bigcup_{i=1}^{L_n} I_i, \tag{3.1}$$

where $I_i = [y_i, y_{i+1}]$ is an edge of a square in $\mathcal{F}_n$ with vertices y_i and y_{i+1}, and $y_1 = y_{L_n+1}$.

For the rest of the paper we fix a constant $M > (4\sqrt{2} + 2)$. However, the following lemma is true for any $M > 0$ and with C depending on M.

Lemma 3.3. *There exists $C \in \mathbb{N}$ so that for any $n \in \mathbb{N}$ and $x, y \in \partial D_n$ with $d_{\partial D_n}(x,y) \geq 2^{-n}C$, and for any γ in $\Omega \setminus \operatorname{int} D_n$ connecting x to y we have that $\gamma \cap (\Omega \setminus B(x, M2^{-n})) \neq \emptyset$. In particular, $L(\gamma) \geq M2^{-n}$.*

Proof. By taking a slightly larger C, namely $C + 2$, we may assume that $x = y_i$ and $y = y_j$ for some i and j, where y_i, y_j are two endpoints of intervals from the collection $\{I_i\}$ forming the boundary as noted above. Moreover, by symmetry we may assume that $i < j$ and $j - i \leq n + 1 - j$. Since each I_i is a side for two squares in $\mathcal{Q}_n$, by taking C large enough, we obtain

$$\mathcal{H}^2(B(x, 2(M+1)2^{-n})) = \pi(2(M+1)2^{-n})^2 < \frac{1}{2}C(2^{-n})^2 \leq \mathcal{H}^2(\bigcup Q),$$

where the union is taken over all $Q \in \mathcal{Q}_n$ having I_m as one of it sides for some $i < m \leq j-1$. Therefore, one of the intervals I_{m_1}, for $i < m_1 \leq j-1$, has to intersect with the complement of the ball $B(x, 2M2^{-n})$. Let $Q'_1 \in \partial\mathcal{D}_n$ be the boundary square corresponding to that interval and let $q_1 \in I_{m_1} \setminus B(x, 2M2^{-n})$. By symmetry, there also exists $Q'_2 \in \partial\mathcal{D}_n$ whose side is some I_{m_2} with $m_2 \notin \{i+1, i+1, \ldots, j-1\}$ such that there is $q_2 \in I_{m_2} \setminus B(x, 2M2^{-n})$.

Suppose now that there exists a curve γ in $\Omega \setminus \operatorname{int} D_n$ joining x to y with $\gamma \subset B(x, M2^{-n})$. We may assume that γ is injective, and by compactness that $\gamma(t) \in \Omega \setminus D_n$ for every $t \in (0,1)$. Then, for $i = 1, 2$ we have that $B(Q'_i, 2\sqrt{2}l(Q'_i)) \subset B(q, M2^{-n})$ and hence

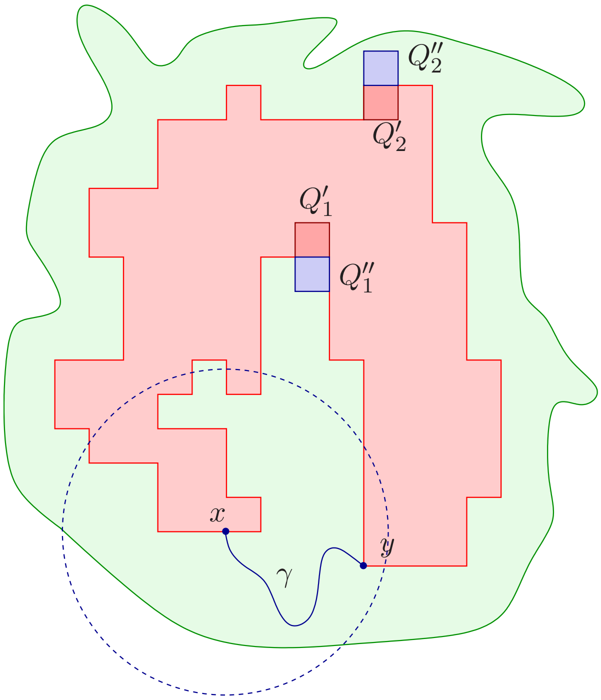

FIGURE 3. In the proof of Lemma 3.3 we assume towards a contradiction that x and y can be connected by a short curve γ in $\Omega \setminus D_n$. This will imply that one more square in $\mathcal{Q}_n$ (here Q_1'') will be a subset of D_n.

$B(Q_i', 2\sqrt{2}l(Q')) \cap \gamma = \emptyset$. Now by definition of Q_i' there is a neighbouring square $Q_i'' \in \mathcal{Q}_n$ of Q_i' which is not a subset of D_n, see Figure 3. We claim that either Q_1'' or Q_2'' lies inside the Jordan curve γ' obtained by concatenating the curve γ and the part of the boundary, denoted by γ'', obtained from the intervals $\{I_h\}_{h=i}^{j-1}$, or by concatenating γ and $\partial D_n \setminus \gamma''$.

This can be seen in the following way. Consider $\Omega \xrightarrow{h} \mathbb{R}^2 \hookrightarrow \mathbb{S}^2$, where h is a homeomorphism and the inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{S}^2$ is the inverse of the stereographic projection. Under this composite map $\mathbb{S}^2 \setminus D_n$ is a simply connected domain. Hence, by Lemma 2.1 $(\mathbb{S}^2 \setminus D_n) \setminus \gamma$ has exactly two components whose boundaries are the two connected components of $\partial D_n \setminus \gamma$ together with γ. Thus, $(\Omega \setminus D_n) \setminus \gamma = (\mathbb{S}^2 \setminus D_n) \setminus \gamma$ has exactly two components. Since $\partial Q_1'' \cap \partial D_n$ and $\partial Q_2'' \cap \partial D_n$ are in two different connected components of $\partial D_n \setminus \gamma$, we conclude that Q_1'' and Q_2'' are in different components of $(\Omega \setminus D_n) \setminus \gamma$. We denote the Q_i'' that lies inside the Jordan curve by Q''.

Since $Q'' \subset B(Q', \sqrt{2}l(Q'))$, we have that every neighbouring square of Q'' either lies inside γ' or is an element of $\partial \mathcal{D}_n$. In particular, by the simply connectedness of Ω they all are subsets of Ω. Hence, $Q'' \subset D_n$ which is a contradiction. Thus, we have proven that $\gamma \cap (\Omega \setminus B(x, M2^{-n})) \neq \emptyset$. □

Let us now partition $\Omega \setminus D_n$ in the following way. Recall (3.1). Notice that for large enough n we have that $L_n \geq 2C$. Define $x_1 := y_1$ and then $x_m := y_{(m-1)C}$ until $L_n + 1 - (m-$

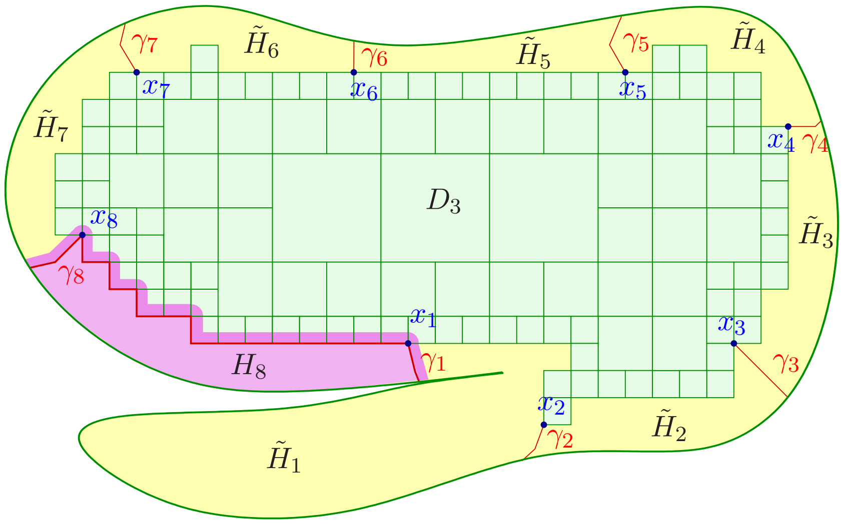

FIGURE 4. Here the domain Ω is decomposed into the core part D_3 and eight boundary parts $\tilde{H}_i$. A neighbourhood H_8 of $\tilde{H}_8$ is also illustrated.

$1)C < 2C$. Notice that for every $i \neq j$ we have $d_{\partial D_n}(x_i, x_j) \geq 2^{-n}C$. We now partition the set $\Omega \setminus D_n$ up to Lebesgue measure zero into connected sets $\{\tilde{H}_j\}_{j=1}^m$ where $\tilde{H}_j$ is the open set bounded by γ_j, γ_{j+1} given by Lemma 3.2 for points x_j and x_{j+1}, and $J_j := \bigcup_{i=Cj}^{C(j+1)} I_i$ (with interior in $\Omega \setminus D_n$). This partition is well defined by Lemma 2.1. Notice that since $L(\gamma_i) \leq M$ for all i, we have that $\gamma_i \cap \gamma_j = \emptyset$ for all $i \neq j$. Let us define H_j as the connected component containing $\tilde{H}_j$ of the set $\Omega \cap \left(\tilde{H}_j \cup B_{\mathbb{R}^2}(\gamma_j \cup \gamma_{j+1} \cup J_j, \delta)\right)$, where $\delta = 2^{-n-3}$. See Figure 4 for an illustration of the decomposition. Although the decomposition depends on n, for simplicity we do not display the dependence in the notation. A crucial property of our decomposition is the following lemma.

Lemma 3.4. *We have $H_j \cap H_i \neq \emptyset$, if and only if $|i - j| \leq 1$ in a cyclic manner.*

Proof. Trivially $\gamma_{i+1} \in H_i \cap H_{i+1}$. Thus, we only need to show that $H_j \cap H_i \neq \emptyset$ implies $|i - j| \leq 1$. We may assume that $i \neq j$. Let $x \in H_i \cap H_j$.

Suppose first that $x \in \tilde{H}_i$. Then, by (path) connectedness of H_j there exists a path γ in H_j from x to $\tilde{H}_j$. Let

$$t_0 := \inf\{t \in [0,1] : \gamma(t) \notin \tilde{H}_i\}.$$

Then, $\gamma(t_0) \notin \tilde{H}_i$ but $\gamma(t) \in H_i \cap H_j$. Thus it suffices to consider the case when $x \notin \tilde{H}_i \cup \tilde{H}_j$.

Suppose now that $x \in D_n$. Since $\delta < \frac{2^{-n}}{2}$, we have that $x \in Q$ for some $Q \in \partial\mathcal{D}_n$. Then, there are neighbouring squares $Q_i, Q_j \in \mathcal{Q}_n$ of Q for which $Q_i \cap \tilde{H}_i \neq \emptyset$ and $Q_j \cap \tilde{H}_j \neq \emptyset$. Since δ is small, we may choose the Q_i, Q_j so that $Q_i \cap Q_j \neq \emptyset$. If $Q_i = Q_j$ or if Q_i and Q_j have a common edge, then there is a curve γ' in $Q_i \cup Q_j$ from $\tilde{H}_i$ to $\tilde{H}_j$ with $L(\gamma') < 2\delta$. If $Q_i \cap Q_j$ is a singleton, then by Lemma 3.1 (iii) the neighbouring square $Q' \neq Q$ of both

Q_i and Q_j lies in $\Omega \setminus \operatorname{int} D_n$. Indeed, if this were not the case, then $Q', Q \in \mathcal{F}_n$ and $Q' \cap Q$ is a singleton, implying that $Q_i \in \mathcal{D}_n$ or $Q_j \in \mathcal{D}_n$. Thus, there exists a curve γ' in $\Omega \setminus D_n$ joining Q_i and Q_j with $L(\gamma') < 4\delta$.

Now, we have $Q_i \cap J_i \neq \emptyset$ or $Q_i \cap (\gamma_i \cup \gamma_{i+1}) \neq \emptyset$. Notice that $\gamma_i \cap J_i \neq \emptyset \neq \gamma_{i+1} \cap J_i$. By Lemma 3.2 we have $\max(l(\gamma_i), l(\gamma_{i+1})) \leq 2\sqrt{2} \cdot 2^{-n}$. Combining these observations with the analogous ones for Q_j, we have that J_i and J_j can be connected by a curve in $\Omega \setminus D_k$ with length less than $4\delta + 4\sqrt{2} \cdot 2^{-n} < 2^{-n}M$. Hence, we have by Lemma 3.3 that $\operatorname{dist}_{\partial D_n}(J_i, J_j) \leq C$. Thus, $|i - j| \leq 1$ in cyclical manner.

We are left with the case where $x \in \Omega \setminus (D_n \cup \tilde{H}_i \cup \tilde{H}_j)$. By definition we have that $B(D_n, 2\delta) \subset \Omega$. Thus, if $\operatorname{dist}(x, J_i) < \delta$, we may join x to J_i by a curve in $\Omega \setminus \operatorname{int} D_n$ with length less than δ. If $\operatorname{dist}(x, J_i) \geq \delta$, then $x \in B(\gamma_m, \delta)$, where $m \in \{i, i+1\}$. By path connectedness of H_i there is a curve γ in H_i connecting x to $\gamma_i \cup \gamma_{i+1} \cup J_i$. We want to prove that x can be joined to γ_m in δ-neighbourhood of γ_m. If (a subcurve of) γ is not such a curve, then we may define

$$t_0 := \inf\{t \in [0,1] : \gamma(t) \in B(D_n, \delta)\}.$$

Then, $\gamma|_{[0,t_0]} \subset B(\gamma_m, \delta)$. Therefore, there exists a point $y \in \gamma_m$ with $d(\gamma(t_0), y) < \delta$. In particular, the line segment $[\gamma(t_0), y]$ lies in $(\Omega \setminus D_n) \cap B(\gamma_m, \delta)$ and thus we have proven that there exists a curve γ' in $(\Omega \setminus D_n) \cap B(\gamma_m, \delta)$ connecting x to γ_m. By the definition of γ_m we have that $\gamma' \subset B(\gamma_m(0), 2\sqrt{2} \cdot 2^{-n} + \delta)$. By the same argument for j we conclude that J_i and J_j can actually be connected by a curve γ in $(\Omega \setminus \operatorname{int} D_n) \cap B(\gamma(0), 4\sqrt{2} \cdot 2^{-n} + 2\delta)$. Hence, by Lemma 3.3 $\operatorname{dist}_{\partial D_n}(J_i, J_j) < C$, and thus $|i - j| \leq 1$ in cyclical manner. □

4. Approximation

In this section we finish the proof of Theorem 1.1 by making a partition of unity using the decomposition of Ω constructed in Section 3 and by approximating a given function by polynomials in this decomposition. Recall that our aim is to show that for any $u \in L^{k,p}(\Omega)$ and $\epsilon > 0$ there exists a function $u_\epsilon \in W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$ with $\|\nabla^k u - \nabla^k u_\epsilon\|_{L^p(\Omega)} \lesssim \epsilon$. By noting that $L^{k,p}(\Omega) \cap C^\infty(\Omega)$ is dense in $L^{k,p}(\Omega)$ we may assume that function $u \in L^{k,p}(\Omega) \cap C^\infty(\Omega)$. From now on, let u and $\epsilon > 0$ be fixed.

Using the notation from Section 3, we write the domain Ω as the union of the core part D_n and the boundary regions $\{H_i\}_{i=1}^l$. For each $\tilde{H}_i$ we let $\mathcal{I}_i$ be the collection of squares Q in $\partial\mathcal{D}_n$ such that $Q \cap \tilde{H}_i \neq \emptyset$, which are bounded in number independently of n. We need to decide what polynomial to attach to each set H_i. For this purpose, for each $1 \leq i \leq l$ we assign a square $Q_i \in \mathcal{I}_i$. We call Q_i the associated square of H_i.

Given $Q \in \mathcal{I}_i$ we set $\mathcal{P}_Q := \bigcup_{j=i-1}^{i+1} \{Q' \in \mathcal{I}_j\}$, which is a collection of squares from a suitable neighbourhood of Q.

Recall the approximating polynomials P_Q introduced in Section 2.2. We abbreviate $P_i = P_{Q_i}$ for the associated squares Q_i.

We make a smooth partition of unity by using a Euclidean mollification. (Compare to [14] where the inner distance in Ω was used for the mollification.) Let ρ_r denote a standard Euclidean mollifier supported in $B(0, r)$. We start with a collection of functions $\{\tilde{\psi}_i\}_{i=0}^l$, where $\tilde{\psi}_0 = \chi_{D_n} * \rho_{2^{-n-5}}$ and $\tilde{\psi}_i = \left(\chi_{\tilde{H}_i} * \rho_{2^{-n-5}}\right)|_{H_i}$ for $i \geq 1$. Using this we obtain a partition of unity $\{\psi_i\}_{i=0}^{l}$ by setting $\psi_i = \tilde{\psi}_i / \sum_{j=0}^{l} \tilde{\psi}_j$.

Now the partition of unity $\{\psi_i\}_{i=0}^{l}$ satisfies the following.

(1) The function ψ_0 is supported in $B(D_n, \frac{2^{-n}}{10})$.
(2) For $i \geq 1$ the function ψ_i is supported in H_i.
(3) For all i, $0 \leq \psi_i \leq 1$.
(4) $\sum \psi_i \equiv 1$ on Ω.
(5) For all i, $|\nabla^\alpha \psi_i| \leq C_\alpha 2^{-n|\alpha|}$ for all multi-indeces α.

We will fix n later such that for the function u_ϵ defined as

$$u_\epsilon(x) := u(x)\psi_0(x) + \sum_{i=1}^{l} \psi_i(x) P_i(x)$$

for $x \in \Omega$, we have

$$\|\nabla^k u - \nabla^k u_\epsilon\|_{L^p(\Omega)} < C\epsilon.$$

Note that $u_\epsilon = u$ on D_{n-1}; indeed $D_{n-1} \cap \psi_i = \emptyset$ for $i \geq 1$, see Lemma 3.1 (iv).

First of all, we consider only n large enough so that

$$\|\nabla^k u\|_{L^p(\Omega \setminus D_{n-1})} \leq \epsilon. \tag{4.1}$$

Now, we need to show that n can actually be chosen large enough so that also

$$\|\nabla^k u_\epsilon\|_{L^p(\Omega \setminus D_{n-1})} \leq C\epsilon.$$

So, we compute for $Q \in \mathcal{I}_i$ and $|\alpha| = k$

$$\begin{aligned} \|\nabla^\alpha u_\epsilon\|_{L^p(Q)} &\leq \sum_{\beta \leq \alpha} \left(\int_Q |\nabla^\beta u - \nabla^\beta P_i(x)|^p |\nabla^{\alpha-\beta} \psi_0(x)|^p \,\mathrm{d}x \right)^{1/p} \\ &\quad + \sum_{\beta \leq \alpha} \sum_j \left(\int_Q |\nabla^\beta P_j(x) - \nabla^\beta P_i(x)|^p |\nabla^{\alpha-\beta} \psi_j(x)|^p \,\mathrm{d}x \right)^{1/p} \\ &=: A_1 + A_2, \end{aligned} \tag{4.2}$$

where A_1 and A_2 are the first and second terms on the right hand side of the inequality and we used that for $\beta < \alpha$, $\sum_j \nabla^{\alpha-\beta}\psi_j = 0$ and order of P_i is at most $k-1$. We first estimate A_1 as

$$\begin{aligned} A_1 &\lesssim \sum_{\beta \leq \alpha} 2^{n(|\alpha|-|\beta|)} \|\nabla^\beta u - \nabla^\beta P_i\|_{L^p(Q)} \\ &\lesssim \sum_{\beta \leq \alpha} 2^{n(|\alpha|-|\beta|)} (\|\nabla^\beta P_i - \nabla^\beta P_Q\|_{L^p(Q)} + \|\nabla^\beta u - \nabla^\beta P_Q\|_{L^p(Q)}) \\ &\lesssim \sum_{\beta \leq \alpha} 2^{n(|\alpha|-|\beta|)} 2^{n(|\beta|-k)} \|\nabla^k u\|_{L^p(\cup \tilde{Q})} \\ &\lesssim \|\nabla^k u\|_{L^p(\cup \tilde{Q})}, \end{aligned} \tag{4.3}$$

where in the third inequality we used that Q_i (associated square of $\tilde{H}_i$) and Q may be joined by a chain of bounded number of squares from $\mathcal{I}_i$ by our construction, and therefore we may apply Lemma 2.5. Similarly we estimate A_2 as

$$\begin{aligned} A_2 &\lesssim \sum_{\beta\le\alpha}\sum_{j=i-1}^{i+1}\left(\int_Q |\nabla^\beta P_j(x)-\nabla^\beta P_i(x)|^p|\nabla^{\alpha-\beta}\psi_j(x)|^p\,\mathrm{d}x\right)^{1/p} \\ &\lesssim \sum_{\beta\le\alpha} 2^{n(|\alpha|-|\beta|)}\sum_{j=i-1}^{i+1}(\|\nabla^\beta P_j-\nabla^\beta P_Q\|_{L^p(Q)}+\|\nabla^\beta P_i-\nabla^\beta P_Q\|_{L^p(Q)}) \\ &\lesssim \sum_{\beta\le\alpha}2^{n(|\alpha|-|\beta|)}2^{n(|\beta|-k)}\|\nabla^k u\|_{L^p(\cup\tilde{Q})} \\ &\lesssim \|\nabla^k u\|_{L^p(\cup\tilde{Q})}, \end{aligned} \tag{4.4}$$

where again in the second inequality we used that if $\psi_j(x)\neq 0$ for $x\in Q\in\mathcal{I}_i$ then by our construction Q_j and Q can be joined by a chain of bounded number of squares as j is either $i-1, i$ or $i+1$ (cyclically); and therefore we can apply Lemma 2.5.

For $Q\in\tilde{\mathcal{F}}\setminus\mathcal{D}_n$ such that $Q\cap\operatorname{spt}(\psi_0)\neq\emptyset$, we assign to Q a square $Q'\in\mathcal{I}_i$, such that $Q\cap Q'\neq\emptyset$. Note that such a square Q' exists by our construction. Then Q and Q' can be joined by a chain of bounded (by an absolute constant) number of squares from $\mathcal{D}_{n+1}$. We choose such a chain for Q and denote it by $\mathcal{B}_Q$. We also set

$$\mathcal{J}_n := \{Q\in\tilde{\mathcal{F}}\setminus\mathcal{D}_n : Q\cap\operatorname{spt}(\psi_0)\neq\emptyset\}.$$

We estimate using Lemma 2.5 exactly as above (see (4.2)) to obtain for $|\alpha|=k$

$$\begin{aligned} \|\nabla^\alpha u_\epsilon\|_{L^p(Q)} &\le \sum_{\beta\le\alpha}\left(\int_Q|\nabla^\beta u-\nabla^\beta P_Q(x)|^p|\nabla^{\alpha-\beta}\psi_0(x)|^p\,\mathrm{d}x\right)^{1/p} \\ &\quad+\sum_{\beta\le\alpha}\sum_j\left(\int_Q|\nabla^\beta P_j(x)-\nabla^\beta P_Q(x)|^p|\nabla^{\alpha-\beta}\psi_j(x)|^p\,\mathrm{d}x\right)^{1/p} \\ &=: B_1+B_2. \end{aligned} \tag{4.5}$$

Again, we estimate separately,

$$\begin{aligned} B_1 &\lesssim \sum_{\beta\le\alpha}2^{n(|\alpha|-|\beta|)}\|\nabla^\beta u-\nabla^\beta P_Q\|_{L^p(Q)} \\ &\lesssim \|\nabla^k u\|_{L^p(Q)} \end{aligned}$$

and

$$\begin{aligned}
B_2 &\lesssim \sum_{\beta\leq\alpha}\sum_{j=i-1}^{i+1}\left(\int_Q |\nabla^\beta P_j(x)-\nabla^\beta P_Q(x)|^p|\nabla^{\alpha-\beta}\psi_j(x)|^p\,\mathrm{d}x\right)^{1/p}\\
&\lesssim \sum_{\beta\leq\alpha} 2^{n(|\alpha|-|\beta|)}\sum_{j=i-1}^{i+1}(\|\nabla^\beta P_j-\nabla^\beta P_{Q'}\|_{L^p(Q)}+\|\nabla^\beta P_{Q'}-\nabla^\beta P_Q\|_{L^p(Q)})\\
&\lesssim \sum_{\beta\leq\alpha} 2^{n(|\alpha|-|\beta|)}2^{n(|\beta|-k)}(\|\nabla^k u\|_{L^p(\bigcup_{Q''\in\mathcal{P}_{Q'}}Q'')}+\|\nabla^k u\|_{L^p(\bigcup_{Q''\in\mathcal{B}_{Q'}}Q'')})\\
&\lesssim \|\nabla^k u\|_{L^p(\bigcup_{Q''\in\mathcal{P}_{Q'}\cup\mathcal{B}_{Q'}}Q'')}.
\end{aligned}$$

Next we note that $\nabla^k u_\epsilon\equiv 0$ in $\tilde{H}_i\setminus\cup_{j\neq i}\operatorname{spt}(\psi_j)$ and we compute for $|\alpha|=k$

$$\begin{aligned}
\|\nabla^\alpha u_\epsilon\|_{L^p(H_i)} \leq& \sum_{Q\in\tilde{Q}_i}\|\nabla^\alpha u_\epsilon\|_{L^p(Q)}+\sum_{Q\in\mathcal{J}_n,Q\cap H_i\neq\emptyset}\|\nabla^\alpha u_\epsilon\|_{L^p(Q)}\\
&+\sum_{j=i-1}^{i+1}\|\nabla^\alpha u_\epsilon\|_{L^p((\operatorname{spt}(\psi_j)\cap\operatorname{spt}(\psi_i))\setminus\bigcup_{Q''\in\partial\mathcal{D}_n\bigcap\mathcal{J}_n}Q'')}
\end{aligned}\tag{4.6}$$

The terms in the first and second summands have been estimated earlier. Denoting $H_i':=(\cup_{j=i-1}^{i+1}\operatorname{spt}(\psi_j)\cap\operatorname{spt}(\psi_i))\setminus\bigcup_{Q''\in\partial\mathcal{D}_n\bigcap\mathcal{J}_n}Q''$, we estimate now the third one;

$$\begin{aligned}
\|\nabla^\alpha u_\epsilon\|_{L^p(H_i')} &\lesssim \sum_{\beta\leq\alpha}2^{n(|\alpha|-|\beta|)}\sum_{j=i-1}^{i+1}\|\nabla^\beta P_j-\nabla^\beta P_i\|_{L^p(H_i')}\\
&\lesssim \sum_{\beta\leq\alpha}2^{n(|\alpha|-|\beta|)}(\|\nabla^\beta P_j-\nabla^\beta P_i\|_{L^p(Q_i)}\\
&\lesssim \sum_{\beta\leq\alpha}2^{n(|\alpha|-|\beta|)}2^{n(|\beta|-k)}\|\nabla^k u\|_{L^p(\bigcup_{Q'\in\mathcal{P}_{Q_i}}Q')}\\
&\lesssim \|\nabla^k u\|_{L^p(\bigcup_{Q'\in\mathcal{P}_{Q_i}}Q')},
\end{aligned}\tag{4.7}$$

where we used the facts that for $\beta<\alpha$, $\nabla^{\alpha-\beta}\sum_j(\psi_j)=0$ and $\psi_0\equiv 0$ in H_i' in the first inequality, Lemma 2.4 in the second inequality since $H_i'\subset CQ_i$ for some absolute constant C coming from Lemma 3.2 and in the third inequality we used Lemma 2.5.

Remark 4.1. Note that for each $Q\in\mathcal{I}_i$ we have $\mathcal{P}_Q=\mathcal{P}_{Q_i}$ where Q_i is the associated square of $\tilde{H}_i$. We note that any $Q'\in\partial\mathcal{D}_n$ occurs in at most three distinct collections $\mathcal{P}_{Q_i}$. Moreover any $Q\in\mathcal{D}_{n+1}$ appears in only a bounded number of the collections $\mathcal{B}_{Q''}$, where $Q''\in\mathcal{J}_n$. In particular, any $Q'\in\partial\mathcal{D}_n$ appears in only a bounded number of the collections $\mathcal{B}_{Q''}$, where $Q''\in\mathcal{J}_n$. The bounds are provided by absolute constants coming from volume comparison.

Now it follows from equations (4.3), (4.4), (4.5), (4.6) and (4.7) that

$$\begin{aligned}\|\nabla^\alpha u_\epsilon\|_{L^p(\Omega\setminus C_n)} &\lesssim \sum_i \|\nabla^k u\|_{L^p(H_i)} + \|\nabla^k u\|_{L^p(\bigcup_{Q\in\partial\mathcal{D}_n} Q)} \\ &\lesssim \|\nabla^k u\|_{L^p(\bigcup_{Q\in\partial\mathcal{D}_n} Q)} + \|\nabla^k u\|_{L^p(\bigcup_{Q\in\mathcal{J}_n} Q)} + \|\nabla^k u\|_{L^p(\bigcup_{\substack{Q\in\mathcal{J}_n\\ Q'\in\mathcal{B}_Q}} Q')}\end{aligned} \tag{4.8}$$

when $|\alpha| = k$. By Remark 4.1 we may choose n such that

$$\|\nabla^k u\|_{L^p(\bigcup_{Q\in\partial\mathcal{D}_n} Q)} + \|\nabla^k u\|_{L^p(\bigcup_{Q\in\mathcal{J}_n} Q)} + \|\nabla^k u\|_{L^p(\bigcup_{\substack{Q\in\mathcal{J}_n\\ Q'\in\mathcal{B}_Q}} Q')} < \epsilon.$$

Then, the claim follows from (4.1) and (4.8).

Remark 4.2. We note that when $k = 1$ we may take the function to be smooth as well as bounded for showing the density of $W^{1,\infty}(\Omega)$ in $W^{1,p}(\Omega)$. This is because truncations approximate the functions in $W^{1,p}(\Omega)$. This allows us to also approximate the L^p norm of u. Indeed let $u \in W^{1,p}(\Omega)\cap C^\infty(\Omega)\cap L^\infty(\Omega)$ such that $\|u\|_{L^\infty} \le M$. Decompose the domain as in the above construction; then choose n large enough such that $\|u\|_{W^{1,p}(\Omega\setminus D_{n-1})} \le \epsilon$ and $M|\Omega \setminus D_{n-1}| < \epsilon$. Then it follows from estimates in the proof that the function u_ϵ defined as above approximates u in $W^{1,p}(\Omega)$ with error given by ϵ. This conclusion is the content of [14].

Finally, let us show how the smooth approximation in Jordan domains is done.

Proof of Corollary 1.3. The argument we need follows the one used to prove [14, Corollary 1.2]. As in [14], given a bounded Jordan domain we approximate it from outside by a nested sequence of Lipschitz and simply connected domains G_s which are obtained for example by taking the complement of the unbounded connected component of the union Whitney squares larger than 2^{-s} from the Whitney decomposition of the complementary Jordan domain of Ω.

Then, we note that for given n, taking s_n large enough, we have that the squares in $\partial\mathcal{D}_n$ are Whitney type sets in G_{s_n}, meaning they have diameters comparable to the distance from the boundary of G_{s_n}.

Note that $G_{s_n} \subset B(\Omega, 2^{-s_n+5})$ are simply connected. Now the set $G_{s_n} \setminus \bar{C}_n$ (recall that C_n is a suitable connected component of the interior of the union of the Whitney squares of scale less than 2^{-n}) can be decomposed in the same way as $\Omega \setminus \bar{C}_n$ was decomposed into the sets $\tilde{H}_i$ in Section 3.

We may then follow the argument used in the proof of Theorem 1.1 to obtain an approximating sequence of functions u_n in G_{s_n} which are in the space $W^{k,\infty}(G_{s_n}) \cap L^{k,p}(G_{s_n}) \cap C^\infty(G_{s_n})$. By multiplying with a smooth cut-off function that is 1 on Ω and compactly supported in G_{s_n}, we obtain a sequence of global smooth functions having the desired properties. □

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[D]

A density result for homogeneous Sobolev spaces

Debanjan Nandi

A DENSITY RESULT FOR HOMOGENEOUS SOBOLEV SPACES

DEBANJAN NANDI

Abstract. We show that in a bounded Gromov hyperbolic domain Ω smooth functions with bounded derivatives $C^\infty(\Omega) \cap W^{k,\infty}(\Omega)$ are dense in the homogeneous Sobolev spaces $L^{k,p}(\Omega)$.

1. Introduction

We continue the study of density of functions with bounded derivatives in the space of Sobolev functions in a domain in $\mathbb{R}^n$. It was shown by Koskela-Zhang [13] that for a simply connected planar domain $\Omega \subset \mathbb{R}^2$, $W^{1,\infty}$ is dense in $W^{1,p}$ and in the special case of Jordan domains also $C^\infty(\mathbb{R}^2) \cap W^{1,\infty}(\Omega)$ is dense. The above result of Koskela-Zhang has been generalized to have the density of $W^{k,\infty}$ in the homogeneous Sobolev space $L^{k,p}$ for $k \in \mathbb{N}$ in planar simply connected domains by Nandi-Rajala-Schultz [19]. In dimensions higher than two however simply connectedness is not sufficient (see for example [14]). Recall that simply connected planar domains are negatively curved in the (quasi) hyperbolic metric. A useful metric generalization of negatively curved spaces was introduced by Gromov [8], in the context of group theory. Following Bonk-Heinonen-Koskela [4], we call a domain Gromov hyperbolic if, when equipped with the quasihyperbolic metric, it is δ-hyperbolic in the sense of Gromov, for some $\delta \geq 0$ (see Section 3 for definitions). Gromov hyperbolicity has turned out to be a sufficient condition for the density of $W^{1,\infty}$ in $W^{1,p}$, as shown by Koskela-Rajala-Zhang [14], and these are primarily the domains we consider in this paper.

A simply connected domain, equipped with the quasihyperbolic metric (equivalent to the hyperbolic metric by the Koebe distortion theorem) in the plane, as mentioned before, is Gromov hyperbolic. Conversely, a Gromov hyperbolic domain with uniqueness of quasihyperbolic geodesics is simply connected. The latter is of course true in higher dimensions as well (indeed, in this case, for any pair of points, any curve γ joining the points is homotopic to the unique quasihyperbolic geodesic Γ joining the given points, with the homotopy given by quasihyperbolic geodesics joining the points $\gamma(t)$ and $\Gamma(t)$ once Γ is parametrized suitably). Gromov hyperbolic domains are often seen as a topological generalization of planar simply connected domains; for it was shown by Bonk-Heinonen-Koskela [4] that Gromov hyperbolic domains are conformally equivalent to suitable uniform metric spaces equipped with the quasihyperbolic metric and corresponding suitable measures. Finally, we note that in higher dimensions, simply connectedness alone does not imply Gromov hyperbolicity; consider for example the unit ball in $\mathbb{R}^3$ deformed to have a cuspidal-wedge along an equator.

We denote by $L^{k,p}$ the space of functions with finite homogeneous Sobolev norm (see Section 3 for definition). We obtain the following extension of the result of Koskela-Rajala-Zhang [14] to Sobolev spaces of higher order.

Key words and phrases. Sobolev space, Gromov hyperbolic, density, quasihyperbolic.

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Theorem 1.1. *Let $0 \leq \delta < \infty$, $k \geq 1$ and $1 \leq p < \infty$. If $\Omega \subset \mathbb{R}^n$ is a bounded δ-Gromov hyperbolic domain, then $C^\infty(\Omega) \cap W^{k,\infty}(\Omega)$ is dense in $L^{k,p}(\Omega)$.*

We have the following corollary to Theorem 1.1.

Corollary 1.2. *Let $0 \leq \delta < \infty$, $k \geq 1$ and $1 \leq p < \infty$. If Ω is a bounded δ-Gromov hyperbolic domain with C^0-boundary, then $C^\infty(\mathbb{R}^n) \cap W^{k,\infty}(\Omega)$ is dense in $L^{k,p}(\Omega)$.*

In the statement of Corollary 1.2, for a domain with C^0-boundary, we require that given $x \in \partial\Omega$, there is a neighbourhood $U_x \subset \mathbb{R}^n$ of x such that $U_x \cap \Omega$ has the representation $y_n < f(y_1, \dots, y_{n-1})$ in suitable coordinates, with a continuous function f (see Maz'ya [18]).

The approximating functions are obtained by an inner extension of a smooth approximation to the function to be approximated, from an increasing sequence of suitable compact subsets, to the domain in question. This method of inner extension for this purpose has been used in [13], [14] and [19]. In [13] this extension is done using the conformal parametrization of simply connected domains. In [19], the topology of the plane is utilized directly to construct an approximating sequence although the hyperbolic structure of planar simply connected domains plays an implicit role in the construction. Our approach in this paper is along the lines of [14].

The information coming from the conformal invariance may be read off in terms of any suitable Whitney decomposition of our domain and towards that end we use as tools the uniformization of Gromov hyperbolic domains developed in [4] along with two of its geometric implications also from [4], employed already in [14], namely the ball separation condition and the Gehring-Hayman condition (see Section 3 below for defintions). Using uniformization we prove a diameter counterpart of the usual Gehring-Hayman condition (see Theorem 5.6) which in its original form relates the lengths of quasihyperbolic geodesics with the intrinsic distance between the points they join (see Pommerenke [20] for the property in simply connected domains). We use also the polynomial approximations of smooth functions in precompact subsets of the domain (see Section 3.3) which were utilized by Jones [11] for showing the extendibility of Sobolev functions defined in uniform domains and also in [19] for the inner extension. We actually prove the density under weaker assumptions than Gromov hyperbolicity (see Theorem 4.1).

One may want to know under what other conditions such density results could hold. Recall that a domain Ω is called *c-John with center x_0* if for each $x \in \Omega$ there exists a curve γ_x joining x to x_0 such that at each point $z \in \gamma_x$ of the curve, c times the distance to the boundary is larger than the length of the subcurve joining x to z. The curve γ_x is called a *c-John curve.* The domain Ω is c-quasiconvex if for each pair of points $x, y \in \Omega$, $c\,\mathrm{d}(x, y) \geq \lambda_\Omega(x, y)$. Here λ_Ω is the intrinsic distance; see Section 2. There are domains that are simultaneously John and quasiconvex for which the density of $C^\infty \cap W^{k,\infty}$-functions in the Sobolev classes $W^{k,p}$ in the homogeneous $L^{k,p}$-norm fails to hold. As an example, we recall the following class of domains which appear in a paper of Koskela [12].

Example 1.3. Let $E \subset \mathbb{R}^{n-1}$ be a p-porous set for $1 < p \leq n$. Then E is removable for $W^{1,p}$ in $\mathbb{R}^n$. Moreover, for each $1 < p \leq n$, there exists a p-porous set $E \subset \mathbb{R}^{n-1}$ that is non-removable for $W^{1,q}$ for any $q < p$.

Here a set E is called removable for the class $W^{1,p}(\mathbb{R}^n)$ if we have that the norm preserving restriction operator $W^{1,p}(\mathbb{R}^n) \hookrightarrow W^{1,p}(\mathbb{R}^n \backslash E)$ is an isomorphism. If E is removable for $W^{1,p}$ then it is removable for $W^{1,q}$ for $q > p$ by the Hölder inequality and since a set being removable is a local condition. The complements of sets removable for $W^{1,q}$ for $q > n$ are quasiconvex;

see for example Koskela-Reitich [15]. Therefore for $p \leq n-1$, the unit ball with a suitable $(p+1)$-porous set as in Example 1.3 removed from the intersection of a $(n-1)$-hyperplane with the concentric ball of radius half, is a John and quasiconvex domain, for which the $C^\infty \cap W^{1,q}$ functions, for $q > p+1$, can not be dense in the class $W^{1,p}$. The Sobolev-Poincare inequality holds in John domains (see Bojarski [2]) from which it follows that $L^{1,p} = W^{1,p}$ in John domains and in particular that $C^\infty \cap W^{1,q}$ is not dense in $L^{1,p}$ for $q > p+1$.

We note however that domains where the John curves may be chosen to be quasihyperbolic geodesics and a suitable quasiconvexity condition involving the lengths of the quasihyperbolic geodesics holds, are Gromov hyperbolic (see Section 3.2) and the required density therefore follows.

We say that a domain Ω admits a global p-Poincaré inequality if for functions $u \in L^1(\Omega)$ with (weak) first order p-integrable derivatives it holds

$$\int_\Omega |u - u_\Omega|^p \leq C \int_\Omega |\nabla u|^p,$$

for $C = C(\Omega)$, where $u_\Omega = \fint_\Omega u$. We have the following corollary to Theorem 1.1.

Corollary 1.4. *Let $0 \leq \delta < \infty$, $k \geq 1$ and $1 \leq p < \infty$. If $\Omega \subsetneq \mathbb{R}^n$ is a δ-hyperbolic domain which admits a global p-Poincaré inequality, then $W^{k,\infty}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

The paper is arranged as follows. In Section 2, we set some of the notation to be used below. In Section 3, we introduce definitions of relevant function spaces and geometric conditions on domains being used in this paper. In Section 4, we prove the density result under the (weaker) geometric conditions from Section 3. In Section 5, we verify that Gromov hyperbolic domains satisfy the geometric assumptions sufficient for the density result which proves Theorem 1.1 and then prove some technical facts required for the proof of Theorem 4.1 in Section 4.

2. Notation

We write $\#A$ for the cardinality of a set A and the Lebesgue $\mathcal{L}^n$-measure of sets $A \subset \mathbb{R}^n$ is denoted $|A|$. For an open cube Q with edges parallel to the coordinate axes, $l(Q)$ denotes its edge length. For a point $x \in \Omega$ we write $\mathrm{d}_\Omega(x)$ for $\mathrm{d}(x, \partial\Omega)$. For a set $A \subset \mathbb{R}^n$ and $\epsilon > 0$, $B(A, \epsilon)$ is the ϵ-neighbourhood of A in $\mathbb{R}^n$. When $A = \{x\}$, we just write $B(x, r)$.

The intrinsic length metric of a domain Ω, denoted $\lambda_\Omega(x, y)$, is the infimum of the euclidean lengths of paths in Ω joining a given pair of points $x, y \in \Omega$. The intrinsic diameter metric is the infimum of euclidean diameters of paths in Ω joining a pair of points and is denoted $\delta_\Omega(x, y)$. For a curve $\gamma \subset \Omega$ (with injective parametrization) and $x, y \in \gamma$ we write $\gamma(x, y)$ for the subcurve of γ between the points x and y. We write $l(\gamma)$ for the length of a rectifiable curve γ.

For $x \in \Omega$, we write $B_\Omega(x, \epsilon)$, for the set of points y in Ω such that $\lambda_\Omega(x, y) < \epsilon$, that is the ϵ ball in the intrinsic metric of Ω. Similarly for sets $A \subset \Omega$ we write $B_\Omega(A, \epsilon)$ for the intrinsic ϵ-neighbourhoods. We use $\mathrm{dist}(\cdot, \cdot)$ to denote the distance between sets obtained by taking infimum of pairwise distances of points lying in the respective sets.

For a set $A \subset \Omega$, the domain Ω in question being fixed, we will write $\bar{A}$ for the closure of A in the relative topology of Ω.

3. Preliminaries

Let $\Omega \subset \mathbb{R}^n$. Let $p \in [1,\infty]$ and $k \in \mathbb{N}$. We write $L^{k,p}(\Omega)$ for the space of Sobolev functions with p-integrable distributional derivatives of order k;

$$L^{k,p}(\Omega) = \{u \in L^1_{loc}(\Omega) : D^\alpha u \in L^p(\Omega),\ \text{if } |\alpha| = k\}.$$

We equip it with the homogeneous Sobolev seminorm $\sum_{|\alpha|=k} \|\nabla^\alpha u\|_{L^p(\Omega)}$. The (non-homogeneous) Sobolev space $W^{k,p}(\Omega)$ is defined as

$$W^{k,p}(\Omega) = \{u \in L^1_{loc}(\Omega) : D^\alpha u \in L^p(\Omega),\ \text{if } |\alpha| \le k\}$$

and is equipped with the norm $\sum_{|\alpha|\le k} \|\nabla^\alpha u\|_{L^p(\Omega)}$. Here and below an n-multi-index α is an n-vector of non-negative integers and $|\alpha|$ is its ℓ_1-norm.

3.1. **Whitney decomposition.** We use the standard Whitney decomposition of a domain $\Omega \subset \mathbb{R}^n$ (see for instance Whitney [23] or the book of Stein [21, Chapter 6]).

3.2. **Some geometric conditions.**

Definition 3.1 (Uniform domain)**.** A domain $\Omega \subsetneq \mathbb{R}^n$ is A-uniform if there exists a constant $A \ge 1$ such that for each pair of points $x, y \in \Omega$, there exists a curve $\gamma_{xy} \subset \Omega$ joining x and y such that

(*i*) $l(\gamma_{xy}) \le A|x-y|$,
(*ii*) For any $z \in \gamma_{xy}$, $l(\gamma_{xy}(x,z) \wedge l(\gamma_{xy}(z,y))) \le A\mathrm{d}_\Omega(z)$.

The curves γ_{xy} are called uniform curves. Curves that satisfy only the second requirement are called doubly-John.

Uniform domains, introduced by Martio-Sarvas [17] are more general than Lipschitz domains but nice enough for being Sobolev extension domains (see Jones [11]). Therefore Lipschitz functions are dense both in the homogeneous and non-homogeneous Sobolev spaces defined in these domains. They come up naturally in the theory of quasiconformal mappings, for example as a characterization for the quasisymmetric images of disks. Note that the definition requires only a non-complete metric space. We may similarly define uniform spaces as metric spaces with non empty topological boundary that satisfy the conditions above. We will require the notion of uniform spaces in Section 5.

Definition 3.2. (Quasihyperbolic metric) Let $\Omega \subsetneq \mathbb{R}^n$ be a domain. Given $x, y \in \Omega$ the quasihyperbolic distance is defined as

$$k_\Omega(x,y) := \inf_{\gamma_{xy}} \int \frac{|dz|}{d_\Omega(z)}$$

where the infimum is taken over all rectifiable curves joining x and y in Ω.

It is a consequence of the Arzela-Ascoli theorem that a quasihyperbolic geodesic (for which the infimum is achieved) exists for each pair of points; see for example the paper of Gehring-Osgood [6] where several facts about the quashyperbolic metric and its relation with quasiconformal mappings had been proved. It was also shown in [6] that in uniform domains

$$k_\Omega(x,y) \simeq \log\left(1 + \frac{|x-y|}{\mathrm{d}_\Omega(x) \wedge \mathrm{d}_\Omega(y)}\right) \tag{1}$$

for $x, y \in \Omega$. It may be noted that quasihyperbolic distance dominates the logarithmic term in (1) in general, that is

$$k_\Omega(x,y) \geq \log\left(1 + \frac{\lambda_\Omega(x,y)}{\mathrm{d}_\Omega(x) \wedge \mathrm{d}_\Omega(y)}\right) \tag{2}$$

for $x, y \in \Omega$, where Ω need not be uniform. We use this and its consequence

$$k_\Omega(x,y) \geq \left|\log\left(\frac{d_\Omega(x)}{d_\Omega(y)}\right)\right|, \tag{3}$$

later in Section 5. We note that the equivalence in (1) characterises uniformity (see [6]). We define here two more quantities (which are equivalent to the logarithmic term in (1) in uniform domains but not in general), which we will require later. Set

$$\Delta_\Omega(x,y) = \log\left(1 + \frac{\delta_\Omega(x,y)}{\mathrm{d}_\Omega(x) \wedge \mathrm{d}_\Omega(y)}\right)$$

and

$$\Lambda_\Omega(x,y) = \log\left(1 + \frac{\lambda_\Omega(x,y)}{\mathrm{d}_\Omega(x) \wedge \mathrm{d}_\Omega(y)}\right),$$

for $x, y \in \Omega \subsetneq \mathbb{R}^n$. These quantities are quasi-metrics in uniform domains.

Definition 3.3 (Ball separation property)**.** Let $\Omega \subsetneq \mathbb{R}^n$ be a domain. A curve $\Gamma \subset \Omega$ satisfies the c-ball separation property, for some $c \geq 1$, if for every point $z \in \Gamma$, the intrinsic ball $B = B_\Omega(z, c\mathrm{d}_\Omega(z))$ separates any $x, y \in \Gamma \setminus B$ in Ω such that $z \in \Gamma(x,y)$. The domain Ω has the c-ball separation property if every quasihyperbolic geodesic in Ω satisfies the c-ball separation property.

The above separation property was perhaps first introduced by Buckley-Koskela [5] who showed that in the class of domains satisfying the above property, if the domain admits a global $(np/(n-p), p)$-Sobolev-Poincaré inequality for $1 \leq p < n$, then the separation condition of geodesics improves to geodesics being John. If a domain Ω is the quasiconformal image of a uniform domain, then the quasihyperbolic geodesics of Ω are the images of diameter-uniform curves (that is properties (i) and (ii) in Definition 3.1 hold with length replaced by diameter; existence of such curves for each pair of points is quantitatively equivalent to the definition of uniformity given here, see Martio-Sarvas [17] for this) under the quasiconformal mapping, see Heinonen-Näkki [9]. An argument using the conformal modulus (see [5]) then shows that quasiconformal images of diameter uniform curves (and thus the quasihyperbolic geodesics in quasiconformal images of uniform domains in particular) have the ball separation property.

Definition 3.4 (Gehring-Hayman property)**.** Let $\Omega \subsetneq \mathbb{R}^n$ be a domain. A curve $\Gamma \subset \Omega$ satisfies the c-Gehring-Hayman property, for $c \geq 1$, if

$$l(\Gamma(x,y)) \leq c\lambda_\Omega(x,y),$$

for all $x, y \in \Gamma$. The domain Ω has the c-Gehring-Hayman property if every quasihyperbolic geodesic $\Gamma \subset \Omega$ satisfies the c-Gehring-Hayman property.

The Gehring-Hayman property appears first perhaps in the paper [7] where it was shown to hold for hyperbolic geodesics in simply connected domains in the plane. Subsequently, it was shown to hold for quasiconformal images of uniform domains by Heinonen-Rohde [10] and for conformal metric deformations of the euclidean unit ball by Bonk-Koskela-Rohde [3]). We will define below a local version of this property which will suffice for our purpose.

By a Gromov hyperbolic domain we mean a domain Ω such that the space (Ω, k_Ω) is Gromov hyperbolic.

Definition 3.5 (Gromov hyperbolicity)**.** Let $0 \le \delta < \infty$. A domain $\Omega \subsetneq \mathbb{R}^n$ is δ-Gromov hyperbolic if all quasihyperbolic geodesic triangles are δ-thin in the quasihyperbolic metric. That is, given any three points $x, y, z \in \Omega$ and quasihyperbolic geodesics Γ_{xy}, Γ_{yz} and Γ_{zx} joining them pairwise, it holds for any w lying in any of the three geodesics that the ball of radius δ, in the quasihyperbolic metric, centered at w intersects the union of the remaining two geodesics.

Quasiconformal images of uniform domains are Gromov hyperbolic (for example by the results in [10] and [5] combined with Theorem 3.6 below). A strictly weaker version of uniformity obtained by using the intrinsic metric of the domain instead of the euclidean metric, for the quasiconvexity of the uniform curve, is often useful. Domains satisfying this latter property are called inner uniform. Inner uniform domains are Gromov hyperbolic (see [4]), and thus functions with bounded derivatives are dense in the homogeneous Sobolev spaces in these domains by Theorem 4.1. In fact, the density in this case holds also in the non-homogeneous Sobolev norm as the hypotheses of Corollary 1.4 are satisfied (see for example [2]). The next theorem is a known characterization of Gromov hyperbolic domains. The geometric implications were proved in [4] using uniformization. It was later shown by Balogh-Buckley [1] that they characterize Gromov hyperbolicity.

Theorem 3.6 ([4],[1])**.** *A δ-Gromov hyperbolic domain has both the c_1-ball separation property and the c_2-Gehring-Hayman property, for $c_1 = c_1(\delta, n)$ and $c_2 = c_2(\delta, n)$. Conversely, a domain which has both the c_1-ball separation property and the c_2-Gehring-Hayman property is also δ-Gromov hyperbolic, for $\delta = \delta(c_1, c_2, n)$.*

We define the following local versions of the Gehring-Hayman condition defined above which are suitable to our purpose. Recall the definitions of $\Lambda_\Omega(\cdot,\cdot)$ and $\Delta_\Omega(\cdot,\cdot)$ from the discussion following Definition 3.2.

Definition 3.7 (Local length/diameter Gehring-Hayman)**.** Let $\Omega \subsetneq \mathbb{R}^n$ be a domain and $c \ge 1, R \ge 0$ be fixed. Let $E \subset \Omega$ be given. We say that E has the (c, R)-length Gehring-Hayman property if for any pair of points $x, y \in E$ for which

$$\Lambda_\Omega(x, y) \le R,$$

each quasihyperbolic geodesic Γ_{xy} joining x and y in Ω satisfies

$$l(\Gamma_{xy}) \le c\, \lambda_\Omega(x, y).$$

We say that E has the (c, R)-diameter Gehring-Hayman property if for any pair of points $x, y \in E$ for which

$$\Delta_\Omega(x, y) \le R,$$

each quasihyperbolic geodesic Γ_{xy} joining x and y in Ω satisfies

$$\operatorname{diam}(\Gamma_{xy}) \le c\, \delta_\Omega(x, y).$$

Remark 3.8. The above definition may be equivalently reformulated by saying $E \subset \Omega$ has the above (c, R)-length Gehring-Hayman property if for all $x, y \in E$, for which

$$\frac{1}{R}\, \mathrm{d}_\Omega(x) \le \mathrm{d}_\Omega(y) \le R\, \mathrm{d}_\Omega(x) \tag{4}$$

and

$$\lambda_\Omega(x,y) \le R\,(\mathrm{d}_\Omega(x) \wedge \mathrm{d}_\Omega(y)), \tag{5}$$

each quasihyperbolic geodesic Γ_{xy} joining x and y in Ω satisfies

$$l(\Gamma_{xy}) \le c\,\lambda_\Omega(x,y).$$

Similarly, we may say that E has the (c,R)-diameter Gehring-Hayman property if

$$\operatorname{diam}(\Gamma_{xy}) \le c\,\delta_\Omega(x,y)$$

holds for any pair of points $x,y \in E$ and quasihyperbolic geodesics Γ_{xy} joining them, where x and y satisfy the condition (4) of being comparably distant from the boundary as above and

$$\delta_\Omega(x,y) \le R\,(\mathrm{d}_\Omega(x) \wedge \mathrm{d}_\Omega(y)).$$

Note that the value of R in this reformulation is related to but different from the R in the previous defintion.

The first condition (4) says for example that quasihyperbolic unit balls centred at x and y have comparable sizes in the euclidean geometry. It is not known if the (c,R)-length and diameter Gehring-Hayman conditions defined above are quantitatively equivalent for domains which have the ball separation property. However, in the case of domains where quasihyperbolic geodesics are doubly-John curves (see Definition 3.1) the equivalence holds in the global sense. A domain is (c,∞)-diameter Gehring-Hayman, if it has the (c,R)-diameter Gehring-Hayman property for all $R > 0$.

Lemma 3.9. *Let $\Omega \subset \mathbb{R}^n$ be a domain such that the quasihyperbolic geodesics are A-doubly-John curves. Suppose Ω has the (c_0,∞)-diameter Gehring-Hayman property. Then Ω also has the c-Gehring-Hayman property, where $c = c(A, c_0, n)$, that is, Ω is inner uniform. The converse is also true quantitatively.*

Proof. Let Γ be a quasihyperbolic geodesic joining $x,y \in \Omega$. Let $z \in \Gamma$ be the midpoint, that is $l(\Gamma(x,z)) = l(\Gamma(z,y)) = l(\Gamma)/2$. We may assume that $\delta_\Omega(x,y) \ge \mathrm{d}_\Omega(z)/2$. Then by the John property of Γ, we have

$$3A\delta_\Omega(x,y) \ge A\mathrm{d}_\Omega(z) \ge l(\Gamma(x,z)).$$

The converse follows from Lemma 5.6. □

The converse utilizes conformal uniformization (see Section 5) and the proof can not be applied if the global inequalities are replaced by the local ones. Lemma 3.11 below says that if there exists any curve that joins x and y and lies uniformly away from the boundary then the local diameter Gehring-Hayman property implies the local length Gehring-Hayman property for the pair $\{x,y\}$. We need to state the following simple lemma before Lemma 3.11.

Lemma 3.10. *Let $M > 0$ be a given number. Suppose Ω has the (c,R)-diameter Gehring-Hayman property and the c_0-ball separation property. Let $x,y \in \Omega$ be such that*

$$\frac{1}{R} d_\Omega(x) \le d_\Omega(y) \le R d_\Omega(x)$$

and

$$\delta_\Omega(x,y) \le R(d_\Omega(x) \wedge d_\Omega(y)).$$

Suppose there exists a curve $\gamma \subset \Omega$ joining x and y such that for each $z \in \gamma$, $d_\Omega(z) \ge (d_\Omega(x) \wedge d_\Omega(y))/M$. Then

$$k_\Omega(x,y) \le c'(c,R,M,n).$$

Proof. Assume that $\delta_\Omega(x,y) \le R(\mathrm{d}_\Omega(x) \wedge \mathrm{d}_\Omega(y))$. Then we have by the (c,R)-diameter Gehring-Hayman property that $\operatorname{diam}(\Gamma) \le cR\mathrm{d}_\Omega(x)$ for any quasihyperbolic geodesic Γ joining x and y. Fix such a geodesic Γ. The ball separation condition implies that, for any $z \in \Gamma$, there exists a $z' \in \gamma$ such that $\lambda_\Omega(z,z') \le c_0\mathrm{d}_\Omega(z)$. Along with our assumption for the curve γ, it follows that

$$\mathrm{d}_\Omega(z) \ge \frac{1}{M(c_0+1)}\mathrm{d}_\Omega(x),$$

and a similar estimate then follows for the edge-lengths of the Whitney cubes intersecting Γ. Since all these cubes lie in a ball of radius at most $2R\mathrm{d}_\Omega(x)$, by the upper bound on the diameter of Γ, there are at most $c'(c,R,M,n)$ of these. This gives an upper bound for $k_\Omega(x,y)$. □

Lemma 3.11. *Let $M > 0$ be a given number. Suppose Ω has the (c,R)-diameter Gehring-Hayman property and the c_0-ball separation property. Let $x, y \in \Omega$ be such that*

$$\frac{1}{R}d_\Omega(x) \le d_\Omega(y) \le R d_\Omega(x)$$

and

$$\lambda_\Omega(x,y) \le R(d_\Omega(x) \wedge d_\Omega(y)).$$

Suppose there exists a curve $\gamma \subset \Omega$ joining x and y such that for each $z \in \gamma$, $d_\Omega(z) \ge (d_\Omega(x) \wedge d_\Omega(y))/M$. Then,

$$l(\Gamma_{xy}) \le c(c,R,M,n)\lambda_\Omega(x,y)$$

where Γ_{xy} is any quasihyperbolic geodesic joining x and y in Ω.

Proof. Since $\delta_\Omega(x,y) \le \lambda_\Omega(x,y)$, we have by the previous lemma that $k_\Omega(x,y) \le c'(c,R,M,n)$.

We may assume that $\lambda_\Omega(x,y) \ge \frac{1}{2}\mathrm{d}_\Omega(x)$. Fix a quasihyperbolic geodesic Γ joining x and y. Note that $l(\Gamma \cap Q) \le 5l(Q)$, for each Whitney cube Q; see [10] for example. Thus, we get that

$$l(\Gamma) \le c' \cdot 5 \cdot 2R\mathrm{d}_\Omega(x) \le 20c'R\lambda_\Omega(x,y)$$

since the number of Whitney cubes intersecting Γ is comparable with $k_\Omega(x,y)$ and their sizes are comparably smaller that $\mathrm{d}_\Omega(x)$. □

Thus the local diameter Gehring-Hayman is *a priori* a stronger condition than the length counterpart, in the sense of the above lemma when the points x and y are chosen correctly. In the next definition we introduce the class of domains for which we show the $W^{k,\infty}$-density to hold.

Definition 3.12 ((c_0,c,R)-radially hyperbolic)**.** Let $c_0, c \ge 1,\ R \ge 0$. We say that a domain $\Omega \subsetneq \mathbb{R}^n$ is (c_0,c,R)-radially hyperbolic with center x_0, if it has the c_0-ball separation property and there exists $x_0 \in \Omega$ so that for any $x_0 \ne x \in \Omega$ and any quasihyperbolic geodesic Γ_x joining x_0 to x, the following are true:

(i) Γ_x has the (c,R)-diameter Gehring-Hayman property (from Definition 3.7).

(ii) Whenever $x_0 \ne y \in \Omega$ and Γ_y is a quasihyperbolic geodesic from x_0 to y such that $(\Gamma_x \cap \Gamma_y) \setminus \{x_0\} \ne \emptyset$, the set $\Gamma_x \cup \Gamma_y$ satisfies either of the two:
  (a) the (c,R)-length Gehring-Hayman property,
  (b) the (c,R)-diameter Gehring-Hayman property.

We will refer to the curves Γ_x for $x_0 \ne x \in \Omega$ as radial geodesics. We will say Ω is (c_0,c,R)-radially hyperbolic when reference to the center x_0 is not required or when x_0 is fixed and understood.

Remark 3.13. We note the following.

(*i*) It is clear that a (c_0, c, R)-radially hyperbolic domain is also (c_0, c, R')-radially hyperbolic whenever $R > R'$.

(*ii*) When the geodesics are unique the second requirement of above definition becomes vacuous. An example of this class would have to be simply connected as noted previously; a planar simply connected domain for example.

(*iii*) If we have that the (c, R)-diameter Gehring-Hayman property holds for all relevant pairs of points in the second requirement, then we get the first requirement as an implication of it. This need not be the case in general, and therefore we specify the first requirement separately.

(*iv*) It follows from Theorem 3.6 and Lemma 5.6 below that a δ-Gromov Hyperbolic domain is a (c_0, c, ∞)-radially hyperbolic domain (for any choice of center), where $c = c(\delta, n)$ and $c_0 = c_0(\delta, n)$. We do not know if the converse is true as the Gehring-Hayman conditions above are required to hold only radially. Also note that when R is close enough to zero, say $R < 1/2$, the domains defined above may only have the ball separation property, as the local Gehring-Hayman conditions above become vacuous. We would like to know if for some $R \in (0, \infty]$, (c_0, c, R)-radially hyperbolic domains are Gromov hyperbolic.

3.3. **Approximating polynomial.** Let $v \in W^{k,p}_{loc}(\Omega)$. For each measurable set $E \Subset \Omega$, let us denote by $P_E = P(v; k, E)$ the unique polynomial of order $k - 1$ such that

$$\int_E \nabla^\alpha (v - P_E) = 0,$$

for all multi-indices α such that $|\alpha| \leq k$ (see Jones [11, page 79]).

We note the following general result.

Lemma 3.14 (Lemma 2.1, [11])**.** *Let Q be any cube in $\mathbb{R}^n$ and P be a polynomial of degree k defined in $\mathbb{R}^n$. Let $E, F \subset Q$ be such that $|E|, |F| > \eta|Q|$ where $\eta > 0$. Then*

$$\|P\|_{L^p(E)} \leq C(\eta, k)\|P\|_{L^p(F)}.$$

Let $Q, Q' \in \mathcal{W}$ be Whitney cubes where $\mathcal{W}$ is the Whitney decomposition of a domain $\Omega \subset \mathbb{R}^n$, such that there is a chain of N_0 Whitney cubes $\{Q = Q_1, Q_2, \ldots, Q_{N_0} = Q'\} \subset \mathcal{W}$ forming a continuum joining Q and Q' such that consecutive cubes from the collection intersect in faces. Then we have the following estimate from [11], which follows via chaining, using the Poincaré inequality.

Lemma 3.15 (Lemma 3.1, [11])**.** *Fix α such that $|\alpha| \leq k$ and let $v \in W^{k,p}_{loc}(\Omega)$. Then for any pair of Whitney cubes as above, we have*

$$\|\nabla^\alpha (P_Q - P_{Q'})\|_{L^p(Q)} \leq C(n, N_0) l(Q)^{k-|\alpha|} \|\nabla^k v\|_{L^p(\cup_{i=1}^{N_0} Q_i)}$$

where $|\nabla^k v|$ is the ℓ_2-norm of the vector $\{\nabla^\alpha v\}_{|\alpha|=k}$.

4. Density in the $L^{k,p}$-norm

In this section we prove the following density result.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded c_0-ball separation domain. Then, given $c \geq 1$, there exists $R_0 = R_0(c_0, c, n)$ such that, if Ω is (c_0, c, R)-radially hyperbolic for any $R \geq R_0$, we have that $C^\infty(\Omega) \cap W^{k,\infty}(\Omega)$ is dense in $L^{k,p}(\Omega)$.*

Let $\epsilon > 0$ be fixed. Let $u \in L^{k,p}(\Omega)$ be given. We will define a function $u_\epsilon \in W^{k,\infty}(\Omega) \cap L^{k,p}(\Omega) \cap C^\infty(\Omega)$ such that $\|u - u_\epsilon\|_{L^{k,p}(\Omega)} \lesssim \epsilon$. We begin by constructing a suitable decomposition of Ω into subsets with finite overlap.

4.1. **A decomposition of Ω.** In this section let $\Omega \subsetneq \mathbb{R}^n$ be a domain with the $\frac{c_0}{10}$-ball separation property for $c_0 \geq 10$. We denote by $\mathcal{W}$ the Whitney decomposition of the domain Ω.

We briefly describe the role of the ball separation condition in the decomposition of Ω. Recall that we want to extend the function from a connected compact 'core' (where a smooth aproximation of the function u to be approximated has bounded derivatives of order up to k) formed by large Whitney cubes to the rest of the domain in such a way that the derivatives of the extension are still bounded. Suitable neighbourhoods (to be determined by the separation property) of cubes in the boundary of the core block parts of the remaining domain which appear as 'tentacles' separated from other tentacles and the core (see Figure 1). The points in a tentacle should then receive their values for the extension of the given function from the cube whose neighbourhood separates it from the core and the other tentacles. However, (it may be seen below that) we need to deal with the case when there are more than one possible choices of cubes which block a given such tentacle. In particular, we need the polynomial approximations (Section 3.3) to our function, in cubes whose neighbourhoods intersect, to oscillate in a controlled way with respect to each other, so that the derivatives are not very large. The construction in [14] takes care of this by considering intrinsic neighbourhoods of cubes for blocking the tentacles. The Gehring-Hayman condition in that case ensures that if suitable intrinsic neighbourhoods of two Whitney cubes intersect, for example with a tentacle, then the euclidean length of the quasihyperbolic geodesic joining their centres is at most, say a constant multiple of the diameter of the larger cube. This is then used to obtain suitable estimates for the oscillations. We will however need to work with euclidean neighbourhoods to define a euclidean partition of unity. This is because the intrinsic distance is only (locally) Lipschitz and we need a partition of unity with k-derivatives. Thus we modify the construction in [14], allowing us also to work under slightly weaker hypothesis. We do this below.

Let $\Omega_m^{(1)}$ denote the interior of the connected component containing x_0 of the set $\cup\{Q : l(Q) \geq 2^{-m}\}$, where m is large enough so that $\Omega_m^{(1)}$ is well defined (contains the point x_0). Define

$$\mathcal{W}_m^{(1)} := \{Q \in \mathcal{W} : Q \subset \Omega_m^{(1)}\}$$

and

$$\mathcal{P}_m^{(1)} := \{Q \in \mathcal{W}_m^{(1)} : 2^{-m} \leq l(Q) < 2^{-(m-2)}\}.$$

Given $Q \in \mathcal{W}$, $\lambda > 0$, λQ is the concentric cube with edge length λ times that of Q. We write $(\lambda Q)_c$ for the component of $\lambda Q \cap \Omega$ that contains Q. Set

$$B_Q = \left(\frac{11}{10} c_0 Q\right)_c.$$

Given a Whitney cube Q and a set A in Ω, we say that Q *blocks* A if x_0 and A lie in different components of $\Omega \backslash \overline{(c_0 Q)_c}$. Given another Whitney cube Q', we write $Q|Q'$, if Q blocks the set $B_{Q'}$. Set

$$\mathcal{P}_m^{(-1)} := \left\{Q \in \mathcal{P}_m^{(1)} : \exists Q' \in \mathcal{P}_m^{(1)} \text{ such that } Q'|Q\right\}$$

and subtract this collection from the original collection to define

$$\mathcal{P}_m = \mathcal{P}_m^{(1)} \backslash \mathcal{P}_m^{(-1)}.$$

Let us denote the components of $\Omega \backslash \bigcup_{Q \in \mathcal{P}_m} \overline{(c_0 Q)_c}$ by V_i' for $i = 0, 1, 2 \ldots$ where V_0' is the component containing x_0. Set

$$\mathcal{V}_i' := \{Q \in \mathcal{P}_m : \overline{V}_i' \cap \overline{(c_0 Q)_c} \neq \emptyset\}$$

for $i = 0, 1, 2, \ldots$

The member cubes of $\mathcal{V}_i'$ together separate the set V_i' from x_0. We want the sets V_i' for $i > 0$ to correspond to the tentacles mentioned above while V_0' corresponds to the core. After that, we would like to find bounds on oscillations of the polynomial approximations to our functions across the cubes in $\mathcal{V}_i$ for each $i > 0$. Some of the sets V_i' however may be 'pockets' composed of large cubes bounded by neighbourhoods of the cubes in $\mathcal{P}_m$. We need to relabel the sets V_i' in order to exclude such 'thick' components and consider them as part of the core instead.

Lemma 4.2. *We have a decomposition*

$$\Omega \backslash \bigcup_{Q \in \mathcal{P}_m} \overline{(c_0 Q)_c} = \bigcup_{i=0}^{l_m} U_i \ \cup \ \bigcup_{i \geq 1} V_i,$$

where the sets V_i' have been relabeled as U_i or V_i so that the following hold.

(*i*) *There are finitely many V_i', denoted and enumerated as $\{U_0 = V_0', \ldots, U_{l_m}\}$ having the property that for any Whitney cube Q such that $Q \cap U_j \neq \emptyset$ for $j = 1, 2, \ldots, l_m$ it holds that $l(Q) \geq 2^{-(m-2)}$. We write $\mathcal{U}_i := \{Q \in \mathcal{P}_m : \overline{U}_i \cap \overline{(c_0 Q)_c} \neq \emptyset\}$ for the collection of boundary cubes whose neighbourhoods bound U_i for $i = 0, \ldots, l_m$.*

(*ii*) *The components $V_i' \notin \{U_0, \ldots, U_{l_m}\}$ are relabeled as $\{V_i\}_i$. For each i, set $\mathcal{V}_i := \{Q \in \mathcal{P}_m : \overline{V}_i \cap \overline{(c_0 Q)_c} \neq \emptyset\}$ as the boundary cubes whose neighbourhoods bound the set V_i. We have that $\# \mathcal{V}_i \leq M$, where $M = M(c_0, n)$.*

Proof. We begin with the component V_0', containing the point x_0. Suppose that there exists a point $z_0 \in Q \subset V_0'$ such that $l(Q) < 2^{-(m-2)}$. Then there exists $Q_0 \in \mathcal{P}_m^{(1)}$ for which $\Gamma_{z_0} \cap Q_0 \neq \emptyset$. By the ball separation property, either x_0 and z_0 lie in different components of $\Omega \setminus \overline{(c_0 Q_0)_c}$ or $z_0 \in \overline{(c_0 Q_0)_c}$. Suppose first that $Q_0 \in \mathcal{P}_m$. This gives a contradiction to our assumption that z_0 and x_0 lie in the same component of $\Omega \setminus \bigcup_{Q \in \mathcal{P}_m} \overline{(c_0 Q)_c}$. If $Q_0 \notin \mathcal{P}_m$ then there exists $Q_0^0 \in \mathcal{P}_m$ such that $Q_0^0 | Q_0$. Since by the ball separtion property either $\overline{(c_0 Q_0)_c}$ separates x_0 and z_0 or $z_0 \in \overline{(c_0 Q_0)_c}$ and since $z_0 \notin \overline{(c_0 Q_0^0)_c}$, we have that z_0 and x_0 are in different components of $\Omega \setminus \bigcup_{Q \in \mathcal{P}_m} \overline{(c_0 Q)_c}$ which is the same contradiction again. Therefore, there is no point $z_0 \in V_0'$ as above. From this we conclude that for every $Q \in \mathcal{W}$ such that $Q \cap V_0' \neq \emptyset$ we have $l(Q) \geq 2^{-(m-2)}$. We set $U_0 = V_0'$ and $\mathcal{U}_0 = \mathcal{V}_0'$.

We continue by induction. Suppose the sets V_j', where $0 \leq j \leq i-1$, have been relabelled. Consider now the component V_i' of $\Omega \setminus \bigcup_{Q \in \mathcal{P}_m} \overline{(c_0 Q)_c}$. We have the following cases.

Case 1:

Suppose first that there exists $z_i \in V_i'$ and $Q_i \in \mathcal{P}_m^{(1)}$ such that $\Gamma_{z_i} \cap Q_i \neq \emptyset$. It follows from the ball separation property that x_0 and z_i are not in the same component of $\Omega \setminus \overline{(c_0Q_i)_c}$. We consider the following subcases.

Case 1.1 $\overline{(c_0Q_i)_c} \cap V_i' = \emptyset$.

Given $Q \in \mathcal{V}_i'$, such that $\overline{(c_0Q_i)_c} \cap \overline{(c_0Q)_c} \neq \emptyset$, we have that x_0 and $\overline{(c_0Q)_c}$ are in separate components of $\Omega \setminus \overline{(c_0Q_i)_c}$. By the definition of $\mathcal{P}_m$ it is impossible that $Q_i|Q$. We conclude that $\overline{(c_0Q_i)_c} \cap B_Q \neq \emptyset$.

Case 1.2: $\overline{(c_0Q_i)_c} \cap V_i' \neq \emptyset$.

In this case $Q_i \notin \mathcal{P}_m$. So let $Q_i^i \in \mathcal{P}_m$ be such that $Q_i^i|Q_i$. Then we are back to the previous case. We conclude again that if $Q \in \mathcal{V}_i'$, then $\overline{(c_0Q_i^i)_c} \cap B_Q \neq \emptyset$.

Since the cubes in $\mathcal{V}_i'$ have comparable sizes, are disjoint and by the above reasoning are contained in a set of measure bounded from above by a constant times the measure of the smallest cube, we get $\#\mathcal{V}_i' \leq M$, where $M = M(c_o, n)$. We set $V_{j_i+1} = V_i'$ and $\mathcal{V}_{j_i+1} = \mathcal{V}_i'$ where j_i is the smallest index such that V_{j_i} has been already defined, (if no such j_i exists, take $j_i = 0$).

Case 2:

Suppose next that it is not possible to find a point z_i as above. Then we have that for any Whitney cube Q such that $Q \cap V_i \neq \emptyset$, $l(Q) \geq 2^{-(m-2)}$, since if $2^{-(m-2)} > l(Q)$ then we get a contradiction with the assumption that there is no point z_i as above. In this case we set $U_{j_i+1} := V_i'$ and $\mathcal{U}_{j_i+1} := \mathcal{V}_i'$, where j_i is the largest index for which U_{j_i} has already been defined.

This concludes the relabelling. □

The sets $\overline{(c_0Q)_c}$ for $Q \in \mathcal{P}_m$, $\overline{V}_i$ and $\overline{U}_j$ provide a decomposition of Ω. Before we may proceed however, we are faced with the issue that uniformly enlarged neighbourhoods of the tentacles $\{V_i\}_i$ might not have bounded overlap which we require if we have them as sets being used to define a partition of unity. Therefore, in what follows we make some modifications to the decomposition of Ω in order to take care of this problem. Namely, we group together some of the sets V_i and the neighbourhoods B_Q of the cubes that block them. This provides a decomposition where suitably uniformly enlarged neighbourhoods of the sets in the decomposition have bounded overlap. We need a lemma first.

Write $\mathcal{V}^{(m)}$ for the union $\bigcup_{i\geq 1} \mathcal{V}_i$ (see part (ii) of the definition of the relabeled sets in Lemma 4.2). Fix an enumeration $\{Q_1, \ldots, Q_{j_m}\}$ of $\mathcal{P}_m$.

Lemma 4.3. *For each $Q_j \in \mathcal{P}_m$, there exists a collection $\{\mathcal{V}_{j_1}, \ldots, \mathcal{V}_{j_l}\}$ so that $Q_j \in \mathcal{V}_{j_s}$ for $1 \leq s \leq l$, which is maximal in the sense that $\mathcal{V}_{j_s} \not\subset \cup_{i\neq s}\mathcal{V}_{j_i}$ for $1 \leq s \leq l$ and $Q_j \in \mathcal{V}_k$ implies $\mathcal{V}_k \subset \bigcup_{1\leq i\leq s} \mathcal{V}_{j_i}$. Moreover, l is bounded above by a constant depending only on c_0 and n.*

Proof. If $Q_j \notin \mathcal{V}^{(m)}$, then set $l = 0$. If $Q_j \in \bigcap_{1\leq s\leq l} \mathcal{V}_{j_s}$, where $\{\mathcal{V}_{j_s}\}_{1\leq s\leq l}$ is a maximal distinct such collection, then for each $Q \in \bigcup_{1\leq s\leq l} \mathcal{V}_{j_s}$, $Q \subset (CQ_j)_c$, where $C = C(c_o, n)$, by part (ii) of our decomposition in Lemma 4.2. Thus, there are an absolutely bounded number of cubes in $\cup_{1\leq s\leq l} \mathcal{V}_{j_s}$. So we get an upper bound for l in terms of c_0 and n. To see the existence of such a maximal collection one observes that there are finitely many distinct collections $\mathcal{V}_i$. □

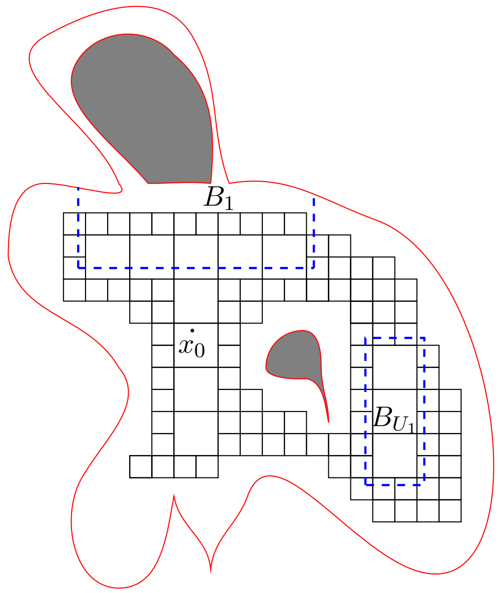

FIGURE 1. Illustrating the sets B_i.

For each $Q_i \in \mathcal{P}_m$, set

$$\tilde{\mathcal{Q}}_i = \bigcup_{1\le j\le l} \mathcal{V}_{i_j},$$

where $\{\mathcal{V}_{i_j}\}_j$ is a chosen collection for Q_i, maximal in the sense of the Lemma 4.3. Note that the definition of $\tilde{\mathcal{Q}}_i$ does not depend on the maximal collection $\mathcal{V}_{i_1}, \dots, \mathcal{V}_{i_l}$ used. Choose a maximal subcollection $\{\mathcal{Q}_i\}_i$ from $\{\tilde{\mathcal{Q}}_i\}_i$ such that collections in $\{\mathcal{Q}_i\}_i$ are distinct in the sense that

$$\bigcup_{Q\in\mathcal{Q}_i} Q \not\subset \bigcup_{i\neq s}\bigcup_{Q\in\mathcal{Q}_s} Q$$

and satisfy

$$\bigcup_i \mathcal{Q}_i = \bigcup_i \tilde{\mathcal{Q}}_i = \mathcal{V}^{(m)}.$$

Denote this collection by $\mathcal{Q}^{(m)} := \{\mathcal{Q}_i\}_i$.

Note that given V_i, there exists $j_i \in \mathbb{N}$ such that V_i and x_0 are in separate components of $\Omega \setminus \bigcup_{Q\in\mathcal{Q}_{j_i}} \overline{(c_0Q)}_c$ (since $\mathcal{V}_i$ is contained in one of these collections). There may be more than one $\mathcal{Q}_{j_i}$ satisfying the above condition, so we assign to V_i the smallest possible index j_i so that $\mathcal{Q}_{j_i}$ works.

Therefore, we may classify the collection of components $\{V_i\}_i$ based on who blocks them as follows. Recall that $B_Q = (\frac{11}{10}c_0Q)_c$. For each $\mathcal{Q}_i \in \mathcal{Q}^{(m)}$, let $\mathcal{B}_i$ be the collection $\{V_{i_j}\}_j$ of components that have been assigned as above to $\mathcal{Q}_i$. Set

$$B_i := \bigcup_{V_{i_j}\in\mathcal{B}_i} \overline{V}_{i_j} \;\cup\; \bigcup_{Q\in\mathcal{Q}_i} B_Q.$$

The sets B_i may be now interpreted as the tentacles mentioned at the beginning of this section (see Figure 1). Fix a cube $Q_i \in \mathcal{Q}_i$, which we will below refer to as the 'assigned' cube of $\mathcal{Q}_i$ (and B_i). We write $\mathcal{B}^{(m)}$ for the collection $\{\mathcal{B}_i\}_i$.

Set $\mathcal{U}^{(m)} = \mathcal{P}_m \setminus \bigcup\{\mathcal{Q}_i \in \mathcal{Q}^{(m)}\}$ as the collection of cubes whose neighbourhoods are not needed for separating the components V_i. Next, we describe the decomposition of Ω which we use for our partition of unity. The sets are

(i) $B_{U_i} := B(U_i, 2^{-m}/100)$, $0 \le i \le l_m$,
(ii) B_i, where $\mathcal{Q}_i \in \mathcal{Q}^{(m)}$ and
(iii) B_Q, where $Q \in \mathcal{U}^{(m)}$.

Then we have the following estimate.

Lemma 4.4. *It holds that*
$$1 \le \sum_{Q\in\mathcal{U}^{(m)}} \chi_{B_Q}(x) + \sum_{0\le i\le l_m} \chi_{B_{U_i}}(x) + \sum_{\mathcal{Q}_i\in\mathcal{Q}^{(m)}} \chi_{B_i}(x) \le c'(c_0, n), \tag{6}$$
for $x \in \Omega$.

Proof. The lower bound follows since $\Omega = \bigcup_{0\le i\le l_m} B_{U_i} \cup \bigcup_{Q_i\in\mathcal{Q}_i} B_i \cup \bigcup_{Q\in\mathcal{U}^{(m)}} B_Q$. For the upper bound, it is clear that
$$\sum_{i=0}^{l_m} \chi_{B_{U_i}} \le c'(n) \text{ and } \sum_{Q\in\mathcal{U}^{(m)}} \chi_{B_Q} \le c'(c_0, n).$$
Let $x \in \Omega$. Suppose $x \in B_{i_i} \cap \ldots \cap B_{i_l}$, where $\{B_{i_j}\}_{j=1}^{l}$ is the collection of all such sets B_i. We want an upper bound for l. Then, since $\{V_j\}_j$ are pairwise disjoint, for any $Q \in \bigcup_{i_1\le s\le i_l} \mathcal{Q}_s$ we have by Lemma 4.3 and the construction of the collections $\mathcal{V}_j$ that
$$\operatorname{dist}(x, Q) \le C2^{-m},$$
where $C = C(c_0)$. This gives an upper bound in terms of c_0 and n for the number of cubes $\# \bigcup_{i_1\le s\le i_l} \mathcal{Q}_s$. So, we get an upper bound A for l, such that $A = A(c_0, n)$. □

Define
$$\Omega_m = \bigcup_{j=0}^{l_m} U_j.$$
We note the following.

(∗) We have that $\Omega_M^{(1)} \subset \Omega_m$ for $M(m, c_0)$ chosen small enough; precisely we require that, $2^{-M} > 10\, c_0\, 2^{-m}$. This follows from our definitions. We therefore have that $|\Omega_m| \to |\Omega|$ as $m \to \infty$.

We will need the following definitions for the ensuing lemmas.

Definition 4.5 (*Trail* and *cover* of Q)**.** Given $Q \in \mathcal{W}$, define the *trail of* Q, denoted $\mathcal{T}(Q)$ to be the set of all points $y \in \Omega$ such that $\Gamma_y(x_0, y) \cap Q \ne \emptyset$. In particular $Q \subset \mathcal{T}(Q)$.

Next, given $Q \in \mathcal{P}_m$ define the *cover of* Q, denoted by $\mathcal{A}(Q)$ to be the collection of those Whitney cubes $Q' \in \mathcal{W}_m^{(1)}$ such that there exists $z \in B_Q$ for which either

(a) $z \in Q'$, or
(b) $z \in \mathcal{T}(Q')$, $Q' \in \mathcal{P}_m^{(1)}$ and $\Gamma_z \cap Q' \ne \emptyset$ for a radial geodesic Γ_z.

The following lemmas are just direct consequences of our geometric assumptions on Ω and the definition of our decomposition.

Lemma 4.6. *Given $Q \in \mathcal{P}_m$, we have that $B_Q \subset \cup\{\mathcal{T}(Q') : Q' \in \mathcal{A}(Q)\}$ and $\#\mathcal{A}(Q) \leq c(c_0, n)$.*

Proof. Let $z \in B_Q$. We have two cases. Let us first assume that $z \in \mathcal{T}(Q')$, where $Q' \in \mathcal{P}_m^{(1)}$ satisfies $\Gamma_z \cap Q' \neq \emptyset$. Then by the ball separation applied to the curve Γ_z, we have that if $\overline{(c_0Q)_c} \cap \overline{(c_0Q')_c} \neq \emptyset$, then x_0 and $(c_0Q)_c$ are in different components of $\Omega \setminus \overline{(c_0Q')_c}$. Since $Q'|Q$ is false, we must have $\overline{(c_0Q')_c} \cap \overline{B}_Q \neq \emptyset$.

Next note that if z is not of above type, then $z \in \Omega_m^{(1)}$ and thus case (a) in the definition of $\mathcal{A}(\cdot)$ must hold. Thus $B_Q \subset \cup\{\mathcal{T}(Q') : Q' \in \mathcal{A}(Q)\}$.

Again, since $\overline{(c_0Q')_c} \cap \overline{B}_Q \neq \emptyset$ for $Q' \in \mathcal{A}(Q)$, by volume comparison we get the required bound on the cardinality of $\mathcal{A}(Q)$. Indeed, the members of $\mathcal{A}(Q)$ are comparably large with respect to Q and are all contained in the set $100c_0Q$. □

So far we have only used the ball separation condition. The next lemma specifies the role of the Gehring-Hayman conditions. Denote by $\text{dist}_k(A, B)$ the distance measured in the quasihyperbolic metric between sets $A, B \subset \Omega$.

Lemma 4.7. *Given $c \geq 1$, there exists $R = R(c_0, c, n)$ such that, if Ω is (c_0, c, R')-radially hyperbolic for $R' \geq R$, then given $Q \in \mathcal{P}_m$, and Q_1 and Q_2 in $\mathcal{A}(Q)$ such that $\mathcal{T}(Q_1) \cap \mathcal{T}(Q_2) \neq \emptyset$, we have that $\text{dist}_k(Q_1, Q_2) \leq M$, where $M = M(c_0, c, n)$.*

The proof of Lemma 4.7 is rather technical and we postpone it to Section 5.

Remark 4.8. We note that if a global diameter Gehring-Hayman property or a suitable local quantitative version of it holds then the proof is simpler. In fact Lemma 4.9 below then follows directly from hypothesis without Lemma 4.7 being necessary. Indeed, if $Q_1, Q_2 \in \mathcal{P}_m$, then pairs of points from Q_1 and Q_2 can be joined by a curve which is uniformly away from the boundary, namely contained in $\Omega_m^{(1)}$; and pairs of such points are comparably distant from the boundary. Lemma 4.9 then follows from Lemma 3.10.

Lemma 4.9. *With the above definitions the following hold:*

- (i) *For any Q and Q' in $\mathcal{P}_m$ such that $B_Q \cap B_{Q'} \neq \emptyset$, we have that $\text{dist}_k(Q, Q') \leq c'(c_0, c, n)$.*
- (ii) *For any $Q \in \mathcal{P}_m$ and $Q' \in \mathcal{A}(Q)$ such that $B_Q \cap Q' \neq \emptyset$, we have that $\text{dist}_k(Q, Q') \leq c'(c_0, c, n)$.*

Proof. By Lemma 4.6, we have that $\#(\mathcal{A}(Q) \cup \mathcal{A}(Q'))$ is finite and bounded above by $c' = c'(c_0, n)$. Also note that $B_Q \cup B_{Q'}$ is connected and $\mathcal{T}(Q'')$ is relatively closed (by Arzela-Ascoli) and connected in Ω for each $Q'' \in \mathcal{A}(Q) \cup \mathcal{A}(Q')$. Moreover, $\{\mathcal{T}(Q'') : Q'' \in \mathcal{A}(Q) \cup \mathcal{A}(Q')\}$ covers $B_Q \cup B_{Q'}$ efficiently (that is, each $\mathcal{T}(Q'')$ necessarily intersects $B_Q \cup B_{Q'}$). Thus we may find an enumeration of $\mathcal{A}(Q) \cup \mathcal{A}(Q')$ say $Q_1, \ldots, Q_s$; $s \leq c'$ such that $\mathcal{T}(Q_i) \cap \mathcal{T}(Q_{i+1}) \neq \emptyset$ for $1 \leq i < s$. Part (i) of the Lemma then follows from Lemma 4.7.

Part (ii) follows arguing similarly as in part (i), since in this case also Q and Q' can again be connected by a chain of trails of cubes in $\mathcal{A}(Q)$ as before, consecutive members of which have intersecting trails when enumerated suitably, so that Lemma 4.7 applies. □

Recall that we fixed a cube $Q_i \in \mathcal{Q}_i$, called the assigned cube of $\mathcal{Q}_i$ and B_i, when $\mathcal{Q}_i \in \mathcal{Q}^{(m)}$. When defining the approximating function, the values of u in Q_i will be used to assign values to the larger set B_i.

Lemma 4.10. *If $Q \in \mathcal{Q}_i$, for $\mathcal{Q}_i \in \mathcal{Q}^{(m)}$, then $dist_k(Q, Q_i) \leq c'$, where $c' = c'(c_0, c, n)$.*

Proof. Note that $\underset{Q\in\mathcal{V}_s}{\cup} \overline{(c_0Q)}_c$ are connected and $\#\mathcal{V}_s$ are uniformly bounded for the sets V_s (recall from property (iii) of the relabeled sets in the decomposition). Thus, $\underset{Q\in\mathcal{Q}_i}{\cup} B_Q$ is connected, since $\{V_s\}_s$ are disjoint. Then the claim follows by induction and proof of Lemma 4.9 after noting that by Lemma 4.3, $\#\mathcal{Q}_i$ is bounded by a constant. □

For each $Q, Q' \in \mathcal{W}_m^{(1)}$ such that $\text{dist}_k(Q, Q') \leq c$, fix a chain of Whitney cubes $\{Q = Q_1, \ldots, Q_s = Q'\}$ with $s \leq c' = c'(c)$ forming a continuum joining Q and Q' and enumerated such that consecutive cubes intersect in faces. Denote by $\mathcal{F}(Q, Q')$ this chain. The selection may be done so that $\mathcal{F}(Q, Q') = \mathcal{F}(Q', Q)$ for any pair of cubes $Q, Q' \in \mathcal{W}_m^{(1)}$. We will need the following lemma in the estimates below.

Given $Q'' \in \mathcal{W}_m^{(1)}$, denote by $\mathcal{F}(Q'')$ the set of pairs (Q, Q') where $Q'' \in \mathcal{F}(Q, Q')$ and one of the following holds:

(i) $Q \in \mathcal{P}_m$ and $Q' \in \mathcal{A}(Q)$ such that $Q' \cap B_Q \neq \emptyset$ or
(ii) $Q, Q' \in \mathcal{P}_m$ such that $B_Q \cap B_{Q'} \neq \emptyset$ or
(iii) $Q = Q_i \in \mathcal{Q}_i$ and $Q' = Q_j \in \mathcal{Q}_j$, where Q_i and Q_j are the assigned cubes of the tentacles B_i and B_j respectively such that $B_i \cap B_j \neq \emptyset$ or
(iv) $Q \in \mathcal{P}_m$ and $Q' = Q_i \in \mathcal{Q}_i$, where Q_i is the assigned cube of the tentacle B_i such that $B_Q \cap B_i \neq \emptyset$.

Then we have the following.

Lemma 4.11. *Given $Q'' \in \mathcal{W}_m^{(1)}$, $\#\mathcal{F}(Q'') \leq c'(c_0, c, n)$.*

Proof. By Lemmas 4.9 and 4.10 we have that for any $(Q, Q') \in \mathcal{F}(Q'')$
$$\text{dist}_k(Q, Q'') \vee \text{dist}_k(Q', Q'') \leq c'(c_0, c, n).$$
Then the claim follows by volume comparison. □

4.2. **Proof of Theorem 4.1.** We begin by defining a partition of unity. Recall the decomposition of Ω obtained in the previous section (Figure 1). Given $Q \in \mathcal{U}^{(m)}$, let $\hat{\psi}_Q$ be a smooth function such that
$$\hat{\psi}_Q|_{(c_0Q)_c} \equiv 1, \ \text{spt}(\hat{\psi}_Q) \subset \frac{11}{10}c_0Q$$
and $|\nabla^\alpha \hat{\psi}_Q(x)| \leq C(\alpha)2^{m|\alpha|}$ for all $x \in \Omega$ and $0 \leq |\alpha| \leq k$.

For $\mathcal{Q}_i \in \mathcal{Q}^{(m)}$, let and let $\hat{\varphi}_i$ be a smooth function such that
$$\hat{\varphi}_i|_{\underset{V_j\in\mathcal{B}_i}{\cup} \overline{V}_j \cup \underset{Q\in\mathcal{Q}_i}{\cup} \overline{(c_0Q)}_c} \equiv 1, \ \text{spt}(\hat{\varphi}_i) \subset \bigcup_{V_j\in\mathcal{B}_i} B(V_j, 2^{-m}/100) \ \cup \ \bigcup_{Q\in\mathcal{Q}_i} (\frac{11}{10}c_0Q)$$
and $|\nabla^\alpha \hat{\varphi}_i(x)| \leq C(\alpha)2^{m|\alpha|}$ for all $x \in \Omega$ and $0 \leq |\alpha| \leq k$.

For U_i, $0 \leq i \leq l_m$, let $\hat{\xi}_i$ be a smooth function such that
$$\hat{\xi}_i|_{U_i} \equiv 1, \ \text{spt}(\hat{\xi}_i) \subset B_{U_i}$$

and $|\nabla^\alpha \hat{\xi}_i| \le C(\alpha)2^{m|\alpha|}$ for all $x \in \Omega$ and $0 \le |\alpha| \le k$. The functions above are obtained by standard mollification of suitable indicator functions related to the sets involved.

We define the partition of unity by setting

$$\varphi_i(x) = \frac{\hat{\varphi}_i(x)}{\sum\limits_{Q \in \mathcal{U}^{(m)}} \hat{\psi}_Q(x) + \sum_{i=0}^{l_m} \hat{\xi}_i(x) + \sum\limits_{\mathcal{Q}_i \in \mathcal{Q}^{(m)}} \hat{\varphi}_i(x)}$$

and similarly defining ψ_Q and ξ_i by dividing by the sum as above. As a consequence of (6) we have

(*i*) $\sum\limits_{Q \in \mathcal{U}^{(m)}} \psi_Q(x) + \sum_{i=0}^{l_m} \xi_i(x) + \sum\limits_{\mathcal{Q}_i \in \mathcal{Q}^{(m)}} \varphi_i(x) = 1$, for all $x \in \Omega$.

(*ii*) $0 \le \psi_Q, \varphi_i, \xi_i \le 1$.

(*iii*) $\max\{\|\nabla^\alpha \psi_Q\|_{L^\infty(\Omega)}, \|\nabla^\alpha \varphi_i\|_{L^\infty(\Omega)}, \|\nabla^\alpha \xi_i\|_{L^\infty(\Omega)}\} \le C(\alpha)2^{m|\alpha|}$, whenever $0 \le |\alpha| \le k$.

(*iv*) $\mathrm{spt}(\psi_Q) \subset \frac{11}{10}c_0 Q$ for $Q \in \mathcal{U}^{(m)}$,
$\mathrm{spt}(\varphi_i) \subset \bigcup\limits_{V_j \in \mathcal{B}_i} B(V_j, 2^{-m}/100) \cup \bigcup\limits_{Q \in \mathcal{Q}_i} (\frac{11}{10}c_0 Q)$ for $\mathcal{Q}_i \in \mathcal{Q}^{(m)}$ and
$\mathrm{spt}(\xi_i) \subset B_{U_i}$ for $i = 1, \ldots, l_m$.

Proof of Theorem 4.1. By the density of smooth functions in $L^{k,p}(\Omega)$ (see Maz'ya's book [18] for example), we may assume without loss of generality that $u \in C^\infty(\Omega)$. Recall the definition of approximating polynomials from Section 3.3. For any Whitney cube Q we write P_Q for the approximating polynomial of u in Q. For $\mathcal{Q}_i \in \mathcal{Q}^{(m)}$, recall $Q_i \in \mathcal{Q}_i$ as the assigned cube for $\mathcal{Q}_i$ (appearing also in Lemma 4.10). We write P_i for the polynomial P_{Q_i}. We define an approximating function by setting

$$u_m(x) := \sum_{Q \in \mathcal{U}^{(m)}} \psi_Q(x) P_Q(x) + \sum_{\mathcal{Q}_i \in \mathcal{Q}^{(m)}} \varphi_i(x) P_i(x) + \sum_{i=0}^{l_m} \xi_i(x) u(x),$$

for $m \in \mathbb{N}$ and for all $x \in \Omega$. We note that $u_m \in W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$. By (6) and property (*iv*) of the partition of unity it suffices to show that $m \in \mathbb{N}$ can be chosen large enough so that

$$\|u_m\|_{L^{k,p}(\bigcup\limits_{Q \in \mathcal{U}^{(m)}} B_Q \cup \bigcup\limits_{\mathcal{Q}_i \in \mathcal{Q}^{(m)}} B_i)} \le \sum_{Q \in \mathcal{U}^{(m)}} \|u_m\|_{L^{k,p}(B_Q)} + \sum_{\mathcal{Q}_i \in \mathcal{Q}^{(m)}} \|u_m\|_{L^{k,p}(B_i)} \lesssim \epsilon.$$

Fix a multi-index α such that $|\alpha| = k$. We note that

$$\nabla^{\alpha-\beta}(\sum_{Q \in \mathcal{U}^{(m)}} \psi_Q + \sum_{\mathcal{Q}_i \in \mathcal{Q}^{(m)}} \varphi_i + \sum_{i=0}^{l_m} \xi_i) = 0 \text{ for } \beta < \alpha \text{ and } \nabla^\alpha P_Q = 0,$$

where $\beta \leq \alpha$ means $0 \leq \beta_i \leq \alpha_i$ for all $0 \leq i \leq n$ where $\alpha = (\alpha_i)_i, \beta = (\beta_i)_i$. Let $Q \in \mathcal{U}^{(m)}$. Then we have

$$\begin{aligned}
\nabla^\alpha u_m(x) &= \sum_{\beta \leq \alpha} \Big(\sum_{Q' \in \mathcal{U}^{(m)}} \nabla^\beta P_{Q'}(x) \nabla^{\alpha-\beta} \psi_{Q'}(x) \\
&+ \sum_{\mathcal{Q}_i \in \mathcal{Q}^{(m)}} \nabla^\beta P_i(x) \nabla^{\alpha-\beta} \varphi_i(x) + \sum_{i=0}^{l_m} \nabla^\beta u(x) \nabla^{\alpha-\beta} \xi_i(x) \Big) \\
&= \sum_{\beta < \alpha} \Big(\sum_{Q' \in \mathcal{U}^{(m)}} \nabla^\beta (P_{Q'}(x) - P_Q(x)) \nabla^{\alpha-\beta} \psi_{Q'}(x) \\
&+ \sum_{\mathcal{Q}_i \in \mathcal{Q}^{(m)}} \nabla^\beta (P_i(x) - P_Q(x)) \nabla^{\alpha-\beta} \varphi_i(x) + \sum_{i=0}^{l_m} \nabla^\beta (u(x) - P_Q(x)) \nabla^{\alpha-\beta} \xi_i(x) \Big) \\
&+ \nabla^\alpha u(x) \sum_{i=0}^{l_m} \xi_i(x).
\end{aligned} \tag{7}$$

Using (7) and property (iii) of the partition of unity, for $Q \in \mathcal{U}^{(m)}$ we get

$$\begin{aligned}
\|\nabla^\alpha u_m\|_{L^p(B_Q)} &\leq \sum_{\beta < \alpha} 2^{m(k-|\beta|)} \sum_{i=0}^{l_m} \|\nabla^\beta (u - P_Q)\|_{L^p(B_Q \cap B_{U_i})} \\
&+ \sum_{\beta < \alpha} 2^{m(k-|\beta|)} \sum_{\substack{Q' \in \mathcal{U}^{(m)} \\ B_Q \cap B_{Q'} \neq \emptyset}} \|\nabla^\beta (P_{Q'} - P_Q)\|_{L^p(B_Q \cap B_{Q'})} \\
&+ \sum_{\beta < \alpha} 2^{m(k-|\beta|)} \sum_{\substack{\mathcal{Q}_i \in \mathcal{Q}^{(m)} \\ B_i \cap B_{Q'} \neq \emptyset}} \|\nabla^\beta (P_i - P_Q)\|_{L^p(B_Q \cap B_i)} + \|(\sum_{i=0}^{l_m} \xi_i) \nabla^\alpha u\|_{L^p(B_Q)} \\
&=: A_1 + A_2 + A_3 + A_4,
\end{aligned} \tag{8}$$

where we write A_i for the summands in their order of appearance.

We estimate them separately. First of all,

$$\begin{aligned}
A_1 &\lesssim \sum_{\beta < \alpha} 2^{m(k-|\beta|)} \sum_{\substack{Q' \in \mathcal{A}(Q) \\ Q' \cap B_Q \neq \emptyset}} \| \nabla^\beta u - \nabla^\beta P_Q\|_{L^p(Q')} \\
&\lesssim \sum_{\beta < \alpha} 2^{m(k-|\beta|)} \sum_{\substack{Q' \in \mathcal{A}(Q) \\ Q' \cap B_Q \neq \emptyset}} \| \nabla^\beta u - \nabla^\beta P_{Q'}\|_{L^p(Q')} \\
&+ \sum_{\beta < \alpha} 2^{m(k-|\beta|)} \sum_{\substack{Q' \in \mathcal{A}(Q) \\ Q' \cap B_Q \neq \emptyset}} \| \nabla^\beta P_{Q'} - \nabla^\beta P_Q\|_{L^p(Q')} \\
&\lesssim \sum_{\substack{Q' \in \mathcal{A}(Q) \\ Q' \cap B_Q \neq \emptyset}} \sum_{Q'' \in \mathcal{F}(Q, Q')} \|\nabla^k u\|_{L^p(Q'')},
\end{aligned} \tag{9}$$

where in the first inequality we observed that $\{Q' \in \mathcal{A}(Q) \mid Q' \cap B_Q \neq \emptyset\}$ covers $B_Q \cap \bigcup_{i=0}^{l_m} \mathrm{spt}(\xi_i)$. The second inequality is triangle inequality whereas in the last inequality we use the Poincaré inequality and Lemma 3.15 respectively.

We estimate

$$\begin{aligned} A_2 &= \sum_{\beta<\alpha} 2^{m(k-\beta|)} \sum_{\substack{Q'\in\mathcal{U}^{(m)} \\ B_Q\cap B_{Q'}\neq\emptyset}} \|\nabla^\beta P_{Q'} - \nabla^\beta P_Q\|_{L^p(B_Q\cap B_{Q'})} \\ &\lesssim \sum_{\beta<\alpha} 2^{m(k-|\beta|)} \sum_{\substack{Q'\in\mathcal{U}^{(m)} \\ B_Q\cap B_{Q'}\neq\emptyset}} \|\nabla^\beta P_{Q'} - \nabla^\beta P_Q\|_{L^p(Q)} \\ &\lesssim \sum_{\substack{Q'\in\mathcal{U}^{(m)} \\ B_Q\cap B_{Q'}\neq\emptyset}} \sum_{Q''\in\mathcal{F}(Q,Q')} \|\nabla^k u\|_{L^p(Q'')}, \end{aligned} \tag{10}$$

where in the first inequality we applied Lemma 3.14 and for the second inequality we used Lemma 3.15. Similarly,

$$\begin{aligned} A_3 &\lesssim \sum_{\beta<\alpha} 2^{m(k-\beta|)} \sum_{\substack{\mathcal{Q}_i\in\mathcal{Q}^{(m)} \\ B_Q\cap B_i\neq\emptyset}} \|\nabla^\beta P_i - \nabla^\beta P_Q\|_{L^p(B_Q\cap B_i)} \\ &\lesssim \sum_{\beta<\alpha} 2^{m(k-|\beta|)} \sum_{\substack{\mathcal{Q}_i\in\mathcal{Q}^{(m)} \\ B_Q\cap B_i\neq\emptyset}} \|\nabla^\beta P_i - \nabla^\beta P_Q\|_{L^p(Q)} \\ &\lesssim \sum_{\substack{\mathcal{Q}_i\in\mathcal{Q}^{(m)} \\ B_Q\cap B_i\neq\emptyset}} \sum_{Q'\in\mathcal{F}(Q,Q_i)} \|\nabla^k u\|_{L^p(Q')}. \end{aligned} \tag{11}$$

Finally,

$$A_4 \leq \sum_{\substack{Q'\in\mathcal{A}(Q) \\ Q'\cap B_Q\neq\emptyset}} \|\nabla^k u\|_{L^p(Q')}. \tag{12}$$

Similarly we estimate $\|\nabla^\alpha u_m\|_{L^p(B_i)}$ for $\mathcal{Q}_i \in \mathcal{Q}^{(m)}$ as follows;

$$\begin{aligned} \|\nabla^\alpha u_m\|_{L^p(B_i)} &\leq \sum_{\beta<\alpha} 2^{m(k-|\beta|)} \sum_{j=0}^{l_m} \|\nabla^\beta(u-P_i)\|_{L^p(B_i\cap B_{U_j})} \\ &+ \sum_{\beta<\alpha} 2^{m(k-|\beta|)} \sum_{\substack{Q\in\mathcal{U}^{(m)} \\ B_i\cap B_Q\neq\emptyset}} \|\nabla^\beta(P_Q - P_i)\|_{L^p(B_i\cap B_Q)} \\ &+ \sum_{\beta<\alpha} 2^{m(k-|\beta|)} \sum_{\substack{Q_j\in\mathcal{Q}^{(m)} \\ B_i\cap B_j\neq\emptyset}} \|\nabla^\beta(P_j - P_i)\|_{L^p(B_j\cap B_i)} + \|(\sum_{j=0}^{l_m}\xi_j)\nabla^\alpha u\|_{L^p(B_i)} \\ &=: A_1' + A_2' + A_3' + A_4'. \end{aligned} \tag{13}$$

Now,

$$
\begin{aligned}
A_1' &\lesssim \sum_{\beta<\alpha} 2^{m(k-|\beta|)} \sum_{Q\in\mathcal{Q}_i} \sum_{\substack{Q'\in\mathcal{A}(Q)\\ Q'\cap B_Q\neq\emptyset}} \|\,\nabla^\beta u \,-\, \nabla^\beta P_i\|_{L^p(Q')} \\
&\lesssim \sum_{\beta<\alpha} 2^{m(k-|\beta|)} \sum_{Q\in\mathcal{Q}_i} \sum_{\substack{Q'\in\mathcal{A}(Q)\\ Q'\cap B_Q\neq\emptyset}} \|\,\nabla^\beta u \,-\, \nabla^\beta P_{Q'}\|_{L^p(Q')} \\
&\quad + \sum_{\beta<\alpha} 2^{m(k-|\beta|)} \sum_{Q\in\mathcal{Q}_i} \sum_{\substack{Q'\in\mathcal{A}(Q)\\ Q'\cap B_Q\neq\emptyset}} \|\,\nabla^\beta P_{Q'} \,-\, \nabla^\beta P_i\|_{L^p(Q')} \\
&\lesssim \sum_{Q\in\mathcal{Q}_i} \sum_{\substack{Q'\in\mathcal{A}(Q)\\ Q'\cap B_Q\neq\emptyset}} \sum_{Q''\in\mathcal{F}(Q_i,Q')} \|\nabla^k u\|_{L^p(Q'')}.
\end{aligned}
\tag{14}
$$

Secondly,

$$
\begin{aligned}
A_2' &\lesssim \sum_{\beta<\alpha} 2^{m(k-|\beta|)} \sum_{\substack{Q\in\mathcal{U}^{(m)}\\ B_Q\cap B_{Q'}\neq\emptyset}} \|\nabla^\beta P_Q \,-\, \nabla^\beta P_i\|_{L^p(B_Q\cap B_i)} \\
&\lesssim \sum_{\beta<\alpha} 2^{m(k-|\beta|)} \sum_{\substack{Q\in\mathcal{U}^{(m)}\\ B_Q\cap B_i\neq\emptyset}} \|\,\nabla^\beta P_Q \,-\, \nabla^\beta P_i\|_{L^p(Q_i)} \\
&\lesssim \sum_{\substack{Q\in\mathcal{U}^{(m)}\\ B_Q\cap B_i\neq\emptyset}} \sum_{Q'\in\mathcal{F}(Q,Q_i)} \|\nabla^k u\|_{L^p(Q')}.
\end{aligned}
\tag{15}
$$

Thirdly,

$$
\begin{aligned}
A_3' &\lesssim \sum_{\beta<\alpha} 2^{m(k-|\beta|)} \sum_{\substack{Q_j\in\mathcal{Q}^{(m)}\\ B_j\cap B_i\neq\emptyset}} \|\nabla^\beta P_j \,-\, \nabla^\beta P_i\|_{L^p(B_j\cap B_i)} \\
&\lesssim \sum_{\beta<\alpha} 2^{m(k-|\beta|)} \sum_{\substack{Q_j\in\mathcal{Q}^{(m)}\\ B_j\cap B_i\neq\emptyset}} \|\,\nabla^\beta P_j \,-\, \nabla^\beta P_i\|_{L^p(Q_i)} \\
&\lesssim \sum_{\substack{Q_j\in\mathcal{Q}^{(m)}\\ B_j\cap B_i\neq\emptyset}} \sum_{Q\in\mathcal{F}(Q_j,Q_i)} \|\nabla^k u\|_{L^p(Q)}
\end{aligned}
\tag{16}
$$

and finally,

$$
A_4' \leq \sum_{Q\in\mathcal{Q}_i} \sum_{\substack{Q'\in\mathcal{A}(Q)\\ Q'\cap B_Q\neq\emptyset}} \|\nabla^k u\|_{L^p(Q')}. \tag{17}
$$

Combining (9), (10), (11), (12), (14), (15), (16) and (17) we get

$$\begin{aligned}\|u_m\|_{L^{k,p}(\bigcup_{Q\in\mathcal{U}^{(m)}} B_Q \cup \bigcup_{\mathcal{Q}_i\in\mathcal{Q}^{(m)}} B_i)} \lesssim & \sum_{Q\in\mathcal{P}_m} \sum_{\substack{Q'\in\mathcal{A}(Q)\\ Q'\cap B_Q\neq\emptyset}} \sum_{Q''\in\mathcal{F}(Q,Q')} \|\nabla^k u\|_{L^p(Q'')} \\ &+ \sum_{Q\in\mathcal{U}^{(m)}} \sum_{\substack{Q'\in\mathcal{U}^{(m)}\\ B_Q\cap B_{Q'}\neq\emptyset}} \sum_{Q''\in\mathcal{F}(Q,Q')} \|\nabla^k u\|_{L^p(Q'')} \\ &+ \sum_{\mathcal{Q}_i\in\mathcal{Q}^{(m)}} \sum_{Q\in\mathcal{Q}_i} \sum_{\substack{Q'\in\mathcal{A}(Q)\\ Q'\cap B_Q\neq\emptyset}} \sum_{Q''\in\mathcal{F}(Q_i,Q')} \|\nabla^k u\|_{L^p(Q'')} \\ &+ \sum_{\mathcal{Q}_i\in\mathcal{Q}^{(m)}} \sum_{\substack{Q\in\mathcal{U}^{(m)}\\ B_Q\cap B_i\neq\emptyset}} \sum_{Q'\in\mathcal{F}(Q,Q_i)} \|\nabla^k u\|_{L^p(Q')} \\ &+ \sum_{\mathcal{Q}_i\in\mathcal{Q}^{(m)}} \sum_{\substack{Q_j\in\mathcal{Q}^{(m)}\\ B_j\cap B_i\neq\emptyset}} \sum_{Q\in\mathcal{F}(Q_j,Q_i)} \|\nabla^k u\|_{L^p(Q)}.\end{aligned} \tag{18}$$

Therefore, for $|\alpha| = k$

$$\|u_m\|_{L^{k,p}(\bigcup_{Q\in\mathcal{U}^{(m)}} B_Q \cup \bigcup_{\mathcal{Q}_i\in\mathcal{Q}^{(m)}} B_i)} \lesssim \|\nabla^k u\|_{L^p(\Omega\setminus\Omega^{(1)}_{\alpha m})}, \tag{19}$$

where $0 < \alpha = \alpha(c_0, c, n) < 1$; the inequality in (19) follows from interchanging the order of summation in (18) using Lemma 4.11. Hence by choosing m large enough, we get the required error bound. □

5. Gromov Hyperbolic and related domains

In this section we show that the hypotheses of Theorem 4.1 are satisfied by Gromov hyperbolic domains in $\mathbb{R}^n$. For this we use the uniformization of Gromov hyperbolic domains by a conformal deformation of the quasihyperbolic metric. This idea was developed in [4] (see also [16]), where it was proved among other things that the domain equipped with the deformed metric is a uniform space. It was also shown that the resulting metric space is Loewner (see Definition 5.4 below) when equipped with a suitable measure, compatible with the metric deformation. We use this information to prove a diameter version of the length Gehring-Hayman Theorem (Theorem 5.6) which, in turn implies the hypothesis of Theorem 4.1. In the following we discuss the main concepts needed for our purpose.

Given a domain $\Omega \subset \mathbb{R}^n$, consider the metric space (Ω, k) and conformal deformations of Ω by densities, denoted ρ_ϵ, for $\epsilon > 0$ and defined as (see page 28 in [4])

$$\rho_\epsilon(x) := \exp\{-\epsilon k(x, x_0)\},$$

for all $x \in \Omega$, where $x_0 \in \Omega$ is a fixed basepoint. The metric d_ϵ induced by ρ_ϵ is defined by

$$\mathrm{d}_\epsilon(x, y) := \inf \int_\gamma \rho_\epsilon ds_k,$$

where the infimum is taken over all curves joining x and y in the domain Ω. Here the measure ds_k is induced by the quasihyperbolic metric k, that is we have

$$\int_\gamma \rho_\epsilon ds_k = \int_\gamma \frac{\rho_\epsilon(\gamma(t))}{d_\Omega(\gamma(t))} dt$$

where γ is parametrized by arc-length in the domain Ω in the usual sense.

Theorem 5.1 (Proposition 4.5, [4])**.** *The conformal deformations (Ω, d_ϵ) of a δ-hyperbolic domain $\Omega \subset \mathbb{R}^n$ are bounded $A(\delta)$-uniform spaces for $0 < \epsilon < \epsilon_0(\delta)$.*

For our purpose we fix $\epsilon = \epsilon_0(\delta)/2$ and take x_0 as defined in Section 4 to be the fixed base point for the conformal deformation metric ρ_ϵ and denote ρ_ϵ by ρ below. We also denote the metric d_ϵ as d_ρ for our chice of ϵ.

Theorem 5.1 says that for any domain Ω which is Gromov hyperbolic with the quasihyperbolic metric (induced by the density $1/\mathrm{d}_\Omega(\cdot)$), one can find a metric d_ϵ induced by multiplying the density $1/\mathrm{d}_\Omega(\cdot)$ with a suitable weight, with which the domain Ω is a bounded and uniform metric space. Hence Ω equipped with the deformed metric d_ρ is also quasihyperbolic with the metric k_ρ, induced by the density $1/\mathrm{d}_\rho(\cdot)$, where

$$\mathrm{d}_\rho(x) = \inf \mathrm{d}_\rho(x, y),$$

and the infimum is taken over all points in the topological boundary, denoted $\partial_\rho\Omega$, of the metric space $(\Omega, \mathrm{d}_\rho)$; see for example Theorem 3.6 in [4]. The boundary $\partial_\rho\Omega$ with the metric d_ρ extended to the boundary is shown to be quasisymmetrically equivalent to the Gromov boundary (equipped with its quasisymmetric gauge) of (Ω, k) in Proposition 4.13 of [4]. The boundary is thus stretched out by the deformation to make the interior uniform. We state next as a lemma, a fact which follows from Proposition 4.37 and Lemma 7.8 in [4].

Lemma 5.2. *The metric spaces (Ω, k_ρ) and (Ω, k) are $C(\delta)$-bilipschitz equivalent.*

Definition 5.3 (Conformal modulus)**.** Let $Q > 1$. Let X be a rectifiably connected metric space. Let μ be a Borel measure in X. The Q-modulus of a family $\mathcal{G}$ of curves in X is

$$\mathrm{mod}_Q(\mathcal{G}) = \inf \int_X f^Q d\mu,$$

where the infimum is taken over all Borel functions $f : X \to [0, \infty]$ such that

$$\int_\gamma f ds \geq 1$$

for all $\gamma \in \mathcal{G}$.

Definition 5.4 (Loewner Spaces)**.** Let $Q > 1$. Let X be a rectifiably connected metric space. Let μ be a Borel measure in X. Then X is Loewner space if the function

$$\varphi(t) = \inf\{\mathrm{mod}_Q(E, F; X) : \Delta(E, F) \leq t\}$$

is positive for each $t > 0$, where E and F are any non-degenerate disjoint continua in X with

$$\Delta(E, F) = \frac{\mathrm{dist}(E, F)}{\mathrm{diam}\, E \wedge \mathrm{diam}\, F}$$

and $(E, F; U)$ is the family of all curves in U joining the sets $E, F \subset U$.

The crucial fact from [4] for us is that the resulting uniform space $(\Omega, \mathrm{d}_\rho, \mu_\rho)$ obtained through the deformation is n-Loewner. This is Proposition 7.14 in [4]. Here $d\mu_\rho = \rho^n dx$. We note that conformal modulus is preserved in the deformed space.

Lemma 5.5. *Let $\Omega \subset \mathbb{R}^n$ be a δ-hyperbolic domain, $\delta > 0$. Then there exists $M = M(n, \delta)$ such that for any pair of points $x, y \in \Omega$ we have that*

$$diam_k(\Gamma_i) \leq \log 2$$

where Γ_i are the connected components of $\Gamma \backslash B(x, M\delta_\Omega(x, y))$ and Γ is a quasihyperbolic geodesic from x to y in Ω.

Proof. Let $(\Omega, \mathrm{d}_\rho, \mu_\rho)$ be the A-uniform metric measure space obtained by conformal deformation (recall $\epsilon = \epsilon(\delta)/2$ was fixed), where $A = A(\delta)$ as in Theorem 5.1. Let $C' > 1$ and consider the annulus

$$A_{C'} := B(x, 2C'\delta_\Omega(x, y)) \backslash B(x, 2\delta_\Omega(x, y)).$$

Let C be the supremum of all numbers C' such that there exists a connected subcurve $\Gamma_0 \in [x, y] \backslash B(x, 2C'\delta_\Omega(x, y))$ such that

$$\mathrm{diam}_k(\Gamma_0) > \log 2.$$

If $-\infty \leq C < 2$, then we again have the claim with $M = 6$, so let us assume that $C \geq 3$ and fix Γ_0 be as above.

Let λ be a curve connecting x and y which lies inside the ball $B(x, 2\delta_\Omega(x, y))$. We have the following modulus estimate (see for example Väisälä [22]).

$$\mathrm{mod}(\lambda, \Gamma_0, \Omega) \leq \mathrm{mod}(B(x, 2C\delta_\Omega(x, y)) \backslash B(x, 2\delta_\Omega(x, y)), \mathbb{R}^n) \leq \frac{1}{(\log C)^{n-1}}. \tag{20}$$

Next we find a lower bound for $\mathrm{mod}_\rho(\lambda, \Gamma_0, \Omega)$, where mod_ρ denotes the conformal modulus of paths computed in $(\Omega, \mathrm{d}_\rho, \mu_\rho)$. We have by Lemma 5.2 that

$$\mathrm{diam}_{k_\rho}(\Gamma_0) \geq \frac{\log 2}{C(\delta)} \tag{21}$$

where $C(\delta)$ is the constant from Lemma 5.2. Let x_0 and y_0 be points in Γ_0 such that $k_\rho(x_0, y_0) = k_\rho(\Gamma_0)$. Since Γ is A-uniform we have

$$\mathrm{diam}_{\mathrm{d}_\rho}(\Gamma_0) \geq \mathrm{d}_\rho(x_0, y_0) \geq (2^{\frac{1}{4A^2}} - 1)\mathrm{d}_\rho(x_0) \tag{22}$$

and

$$\mathrm{dist}_\rho(\Gamma_0, \lambda) \leq A\, \mathrm{d}_\rho(x_0). \tag{23}$$

We again have by uniformity that

$$A\, \mathrm{diam}_{\mathrm{d}_\rho}(\lambda) \geq l_\rho(\Gamma) \geq \mathrm{d}_\rho(x_0, y_0). \tag{24}$$

Thus we obtain from the Loewner property a lower bound as required which depends only on δ. This gives an upper bound for C in terms of δ. The theorem follows with $M = 2C + 1$. □

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^n$ be a δ-Gromov hyperbolic domain. Then there exists a constant $M = M(\delta, n)$ such that for any pair of points $x, y \in \Omega$ we have*

$$diam(\Gamma) \leq M\delta_\Omega(x, y) \tag{25}$$

for any quasihyperbolic geodesic Γ joining x and y in Ω.

Proof. Choose M as in Lemma 5.5. Choose a component Γ_0 of Γ that lies outside the ball $B(x, M\delta_\Omega(x,y))$. If no such component exists then the claim in the theorem follows. Let $w_0 \in \Gamma_0$ be the midpoint of Γ_0 in the metric k. For Γ_0 we have that

$$\mathrm{diam}_k(\Gamma_0) \le \log 2 \tag{26}$$

from which it follows that

$$\Gamma_0 \subset B(w_0, \mathrm{d}_\Omega(w_0)/2).$$

If $x_0 \in \Gamma_0$ is the point where Γ_0 leaves $B(x, M\delta_\Omega(x,y))$, then

$$\mathrm{d}_\Omega(w_0) \le 2\mathrm{d}_\Omega(x_0) \le 2(M\delta_\Omega(x,y) + \mathrm{d}_\Omega(x)) \le (2M+4)\delta_\Omega(x,y). \tag{27}$$

Next, let λ be a curve in $B(x, 2\delta_\Omega(x,y)) \cap \Omega$ joining x and y. We have by the ball separation property that

$$\mathrm{dist}(w_0, \lambda) \le c_0\mathrm{d}_\Omega(w_0) \le c_0(2M+4)\delta_\Omega(x,y). \tag{28}$$

The same holds for any other component that falls under this case.
Thus $\Gamma \subset B(x, 3Mc_0\,\delta_\Omega(x,y))$.

□

Proof of Theorem 1.1. In this case Ω is (c_0, c, ∞)-radially hyperbolic for some $c_0 = c_0(\delta)$ and $c = c(\delta)$ by Theorem 3.6 and Theorem 5.6. The claim follows by Theorem 4.1. □

Proof of Corollary 1.2. Using the C^0-boundary assumption we may find a sequence of Lipschitz domains $\{\Omega_k\}_{k\in\mathbb{N}}$ such that $\Omega \subset \Omega_{k+1} \Subset \Omega_k$, for each $k \in \mathbb{N}$ (see for example Corollary 1.2 of [14] or Corollary 1.3 of [19]). The proof then proceeds as the proof of Corollary 1.3 of [19].

□

It remains now to prove Lemma 4.7.

Proof of Lemma 4.7. Fix $x_1 \in Q_1$ and $x_2 \in Q_2$ such that $(\Gamma_{x_1} \cap \Gamma_{x_2}) \setminus \{x_0\} \neq \emptyset$, and fix $x \in \Gamma_{x_1} \cap \Gamma_{x_2}$. We have by Lemma 4.6 and definitions that

$$\delta_\Omega(x_1, x_2) \lesssim_{c_0,n} \mathrm{d}_\Omega(x_1) \simeq \mathrm{d}_\Omega(x_2),$$

and therefore $R = R(c_0, n)$ may be chosen based on the previous inequality, such that if Ω is (c, R)-radially hyperbolic with only diameter Gehring-Hayman in requirement (ii), then by Lemma 3.10, the conclusion of Lemma 4.7 follows. We still need to show the existence of R such that also allowing the (c, R)-length Gehring-Hayman condition provides the claim (cf. Remark 3.13). Towards this end we need to show that

$$\lambda_\Omega(x_1, x_2) \lesssim_{c_0,c,n} \mathrm{d}_\Omega(x_1) \simeq \mathrm{d}_\Omega(x_2),$$

when $x_1 \in Q_1$ and $x_2 \in Q_2$ such that $(\Gamma_{x_1} \cap \Gamma_{x_2}) \setminus \{x_0\} \neq \emptyset$. This will be achieved through the lemmas below. For convenience of notation we write Γ_1 for Γ_{x_1} and Γ_2 for Γ_{x_2}.

Lemma 5.7. *If $z \in \Gamma_i(x_0, x_i)$ for $i = 1$ or $i = 2$, then*

$$5(c_0+1)d_\Omega(z) \ge d_\Omega(x_i). \tag{29}$$

Proof. This is a consequence of the separation property. We consider competing curves, denoted λ_i for $i = 1, 2$ which join x_i to x_0 in $\Omega_m^{(1)}$. These curves exist because $\Omega_m^{(1)}$ was defined to be connected. Suppose $z \in \Gamma_1(x_0, x_1)$. Then, by the ball separation property of Γ_1, there exists a point x' in λ_1 such that

$$\lambda_\Omega(z, x') \le c_0\mathrm{d}_\Omega(z).$$

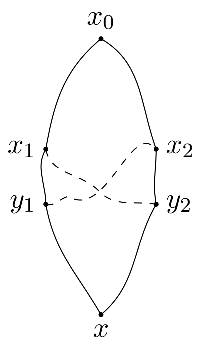

FIGURE 2. Illustrating the curves Γ_1, Γ_2, $\Gamma_{x_1y_2}$ and $\Gamma_{x_2y_1}$.

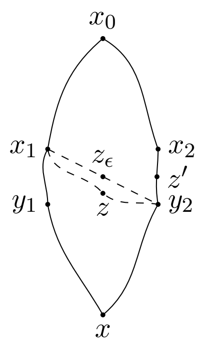

FIGURE 3. Illustrating the competing curves γ_1^ϵ and $\Gamma_{x_1y_2}$.

On the other hand, since $x' \in \Omega_m^{(1)}$, we also have $d_\Omega(x') \geq \frac{1}{5}\mathrm{d}_\Omega(x_2)$ and the claim follows. □

Next, fix $y_1 \in \Gamma_1$ and $y_2 \in \Gamma_2$ such that

$$\lambda_\Omega(x_1, y_2) = \inf_{y\in\Gamma_2} \lambda_\Omega(x_1, y) \leq c_0\mathrm{d}_\Omega(x_1) \tag{30}$$

and

$$\lambda_\Omega(x_2, y_1) = \inf_{y\in\Gamma_1} \lambda_\Omega(y, x_2) \leq c_0\mathrm{d}_\Omega(x_2), \tag{31}$$

where the upper bounds are due to the ball separation property, since Γ_1 and Γ_2 are competing quasihyperbolic geodesics (see Figure 2).

We note that by the triangle inequality

$$\mathrm{d}_\Omega(y_1) \leq (c_0 + 1)\mathrm{d}_\Omega(x_2) \simeq \mathrm{d}_\Omega(x_1) \tag{32}$$

and

$$\mathrm{d}_\Omega(y_2) \leq (c_0 + 1)\mathrm{d}_\Omega(x_1) \simeq \mathrm{d}_\Omega(x_2). \tag{33}$$

Lemma 5.8. *If $y_i \in \Gamma_i(x_0, x_i)$ for $i = 1$ or $i = 2$, then the claim of Lemma 4.7 is true.*

Proof. To this end, suppose $y_2 \in \Gamma_2(x_0, x_2)$. Since we have that $\delta_\Omega(x_1, x_2) \lesssim \mathrm{d}_\Omega(x_1)$, it follows by the triangle inequality and (30) that

$$\delta_\Omega(x_2, y_2) \lesssim \mathrm{d}_\Omega(x_1) \lesssim \mathrm{d}_\Omega(x_2). \tag{34}$$

From the previous claim we also have

$$\mathrm{d}_\Omega(x_2) \leq 5(c_0 + 1)\mathrm{d}_\Omega(y_2). \tag{35}$$

Thus, by (33), (35) and since $x_2, y_2 \in \Gamma_2$, there exists $R_1 = R_1(c_0)$ such that if Ω is (c, R)-radially hyperbolic for $R \geq R_1$, then by Lemma 3.10, we get

$$k_\Omega(y_2, x_2) \leq c'(c_0, c, n),$$

which implies $\lambda_\Omega(y_2, x_2) \lesssim \mathrm{d}_\Omega(x_2)$, by (2). This, together with (30) gives

$$\lambda_\Omega(x_1, x_2) \lesssim_{c_0,c,n} \mathrm{d}_\Omega(x_1) \simeq \mathrm{d}_\Omega(x_2).$$

Thus, since $\Gamma_1 \cap \Gamma_2 \neq \emptyset$, there exists $R_2 = R_2(c_0, c, n)$ such that if Ω is (c, R)-radially hyperbolic for $R \geq \max_{1\leq i\leq 2} R_i$, then Lemma 4.7 follows by arguments in Lemma 3.10. Indeed, we only need to observe that there is a curve competing with $\Gamma_{x_1x_2}$ and joining x_1 and x_2 in $\Omega_m^{(1)}$.

□

Therefore we may assume $y_i \in \Gamma_i(x_i, x)$ for $i = 1, 2$. Fix $\Gamma_{x_1y_2}$, a quasihyperbolic geodesic joining x_1 to y_2 in Ω and and $\Gamma_{x_2y_1}$, a quasihyperbolic geodesic joining x_2 to y_1 in Ω.

Lemma 5.9. *If* $z \in \Gamma_{x_1y_2}$ *then*

$$100(c_0+1)^3 d_\Omega(z) \geq d_\Omega(y_2). \tag{36}$$

Proof. Fix $z \in \Gamma_{x_1y_2}$. Let μ_1 denote the competing curve $\tilde{\Gamma}_1(x_1, x_0) * \Gamma_2(x_0, y_2)$ joining x_1 to y_2 (where $\tilde{\Gamma}$ denotes the reversed curve of Γ). Choose a point $z' \in \mu_1$ such that

$$\lambda_\Omega(z, z') \leq c_0 \mathrm{d}_\Omega(z), \tag{37}$$

which we get from ball separation. If $z' \notin \Gamma_2(x_2, y_2)$, then by (29) and (33), we have that

$$100(c_0+1)^2 \mathrm{d}_\Omega(z') \geq \mathrm{d}_\Omega(x_1)(c_0+1) \geq \mathrm{d}_\Omega(y_2) \tag{38}$$

and by the triangle inequality with equation (37) that

$$\mathrm{d}_\Omega(z)(c_0+1) \geq \mathrm{d}_\Omega(z'). \tag{39}$$

The claim follows by combining (38) and (39).

So we assume next that $z' \in \Gamma_2(x_2, y_2)$ (see Figure 3). We use now the fact that y_2 is taken to be at minimal intrinsic distance from x_1; see equation (33). Let γ_1^ϵ be a curve joining x_1 and y_2 such that

$$\lambda_\Omega(x_1, y_2) \geq l(\gamma_1^\epsilon) - \epsilon \tag{40}$$

Choose a point $z_\epsilon \in \gamma_1^\epsilon$ such that

$$\lambda_\Omega(z, z_\epsilon) \leq c_0 \mathrm{d}_\Omega(z), \tag{41}$$

obtained by applying ball separation to the geodesic $\Gamma_{x_1y_2}$ with respect to the competing curve γ_1^ϵ. From equations (37) and (41) we have

$$\lambda_\Omega(z_\epsilon, z') \leq 2c_0 \mathrm{d}_\Omega(z) \tag{42}$$

From equations (40) and the choice of y_2 we have

$$\begin{aligned} \lambda_\Omega(x_1, z_\epsilon) + \lambda_\Omega(z_\epsilon, y_2) &\leq l(\gamma_1^\epsilon) \\ &\leq \lambda_\Omega(x_1, y_2) + \epsilon \\ &\leq \lambda_\Omega(x_1, z') + \epsilon \\ &\leq \lambda_\Omega(x_1, z_\epsilon) + \lambda_\Omega(z_\epsilon, z') + \epsilon \end{aligned} \tag{43}$$

and therefore by equations (41),(42) and (43) and passing to the limit we get

$$\begin{aligned} \mathrm{d}_\Omega(y_2) &\leq \mathrm{d}(z, y_2) + \mathrm{d}_\Omega(z) \\ &\leq \lim_{\epsilon \to 0} \left(\mathrm{d}(z, z_\epsilon) + \mathrm{d}(z_\epsilon, y_2) + \mathrm{d}_\Omega(z) \right) \\ &\leq (3c_0+1)\mathrm{d}_\Omega(z) \end{aligned} \tag{44}$$

which was the claim. □

Similarly for points $z \in \Gamma_{x_2y_1}$, we get

$$\mathrm{d}_\Omega(y_1) \leq 100(c_0+1)^3 \mathrm{d}_\Omega(z). \tag{45}$$

Lemma 5.10. *For $i = 1, 2$, we have*

$$d_\Omega(y_i) \le 100(c_0 + 1)^4 d_\Omega(z) \tag{46}$$

for any $z \in \Gamma_i(x_i, y_i)$.

Proof. This follows by comparison with the competing curves $\tilde{\Gamma}_2(x_0, x_2) * \Gamma_1(x_0, x_1) * \Gamma_{x_1 y_2}$ and $\tilde{\Gamma}_1(x_0, x_1) * \Gamma_1(x_0, x_2) * \Gamma_{x_2 y_1}$ respectively for the geodesics $\Gamma_2(x_2, y_2)$ and $\Gamma_1(x_1, y_1)$ and using the ball separation property. □

Next, we assume without loss of generality that

$$k_\Omega(\Gamma_1(y_1, x)) \le k_\Omega(\Gamma_2(y_2, x)), \tag{47}$$

and complete the proof of Lemma 4.7 by considering the following two possible cases.
Case 1: Assume first that $\mathrm{d}_\Omega(y_i) \ge \frac{\mathrm{d}_\Omega(x_i)}{10(c_0+10)^2}$ for either $i = 1$ or $i = 2$; say

$$\mathrm{d}_\Omega(y_1) \ge \frac{\mathrm{d}_\Omega(x_1)}{10(c_0 + 10)^2} \tag{48}$$

In this case we have

$$\delta_\Omega(x_1, y_1) \le \delta_\Omega(x_1, x_2) + \delta_\Omega(y_1, x_2) \lesssim \mathrm{d}_\Omega(y_1) \simeq \mathrm{d}_\Omega(x_1),$$

where the second equivalence comes from our assumption and (31). Thus there exists $R_3 = R_3(c_0, c)$ such that if Ω is (c, R)-radially hyperbolic for $R \ge \max_{1 \le i \le 3} R_i$, then by Lemma 3.10 (which we may apply by Lemma 5.10), we have $k_\Omega(x_i, y_i) \le c'(c_0, c, n)$ and

$$l(\Gamma_1(x_1, y_1)) \lesssim \mathrm{d}_\Omega(x_1). \tag{49}$$

Now by (49) we get

$$\lambda_\Omega(x_1, x_2) \le l(\Gamma_1(x_1, y_1)) + \lambda_\Omega(y_1, x_2) \lesssim \mathrm{d}_\Omega(x_1).$$

Thus there exists $R_4 = R_4(c_0, c, n)$ such that if Ω is (c, R)-radially hyperbolic for $R \ge \max_{1 \le i \le 4} R_i$, we have $l(\Gamma_{x_1 x_2}) \lesssim \mathrm{d}_\Omega(x_1)$, where $\Gamma_{x_1 x_2}$ is any quasihyperbolic geodesic joining x_1 and x_2. It then follows by arguments similar to those in Lemma 3.10, that $k_\Omega(x_1, x_2) \le c'(c_0, c, n)$ which is the claim.

Case 2: Next we consider the case

$$\mathrm{d}_\Omega(y_i) < \frac{\mathrm{d}_\Omega(x_i)}{10(c_0 + 10)^2} \tag{50}$$

for $i = 1, 2$. By the ball separation property of Γ_1, there exists a point $z_2 \in \Gamma_2$ (see Figure 4) such that

$$\lambda_\Omega(y_1, z_2) \le c_0 \mathrm{d}_\Omega(y_1). \tag{51}$$

Due to our assumption that $\mathrm{d}_\Omega(y_1)$ is small enough compared to $\mathrm{d}_\Omega(x_1)$, Lemma 5.7 tells that $z_2 \notin \Gamma_2(x_0, x_2)$. Indeed, note that

$$\mathrm{d}_\Omega(x_2) \le 5(c_0 + 1)\mathrm{d}_\Omega(z_2) \le 5(c_0 + 1)^2 \mathrm{d}_\Omega(y_2)$$

contradicts (50).

We next assume that that $z_2 \in \Gamma_2(y_2, x)$. Then by (31),(50) and (51) we have that

$$\lambda_\Omega(x_2, y_2) \le \lambda_\Omega(x_2, z_2) \le \lambda_\Omega(x_2, y_1) + \lambda_\Omega(y_1, z_2) \lesssim \mathrm{d}_\Omega(x_2). \tag{52}$$

We have by (31) and (52)

$$\lambda_\Omega(x_1, x_2) \le \lambda_\Omega(x_1, y_2) + \lambda_\Omega(y_2, x_2) \lesssim \mathrm{d}_\Omega(x_2). \tag{53}$$

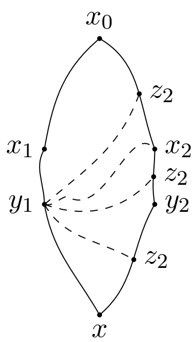

FIGURE 4. Illustrating the three possibilities for the position of z_2.

Thus there exists $R_5 = R_5(c_0, c, n)$ such that if Ω is (c, R)-radially hyperbolic for $R \geq \max_{1\leq i\leq 5} R_i$ then $k_\Omega(x_1, x_2) \leq c'(c_0, c, n)$ and the claim of the lemma is true in this case.

Next, note that if

$$\mathrm{d}_\Omega(y_1) \leq \frac{\mathrm{d}_\Omega(y_2)}{200(c_0+1)^5}$$

and $z_2 \in \Gamma_2(x_0, y_2)$, then we get a contradiction between the consequences of ball separation for Γ_1; (51), that

$$\mathrm{d}_\Omega(y_1)(c_0+1) \geq \mathrm{d}_\Omega(y_1) + \lambda_\Omega(y_1, z_2) \geq \mathrm{d}_\Omega(z_2), \tag{54}$$

and that of Lemma 5.10. This forces $z_2 \in \Gamma_2(y_2, x)$, which has been considered previously.

We interchange the roles of the pairs (y_1, Γ_1) and (y_2, Γ_2) in above argument to observe that the only remaining case is when $\mathrm{d}_\Omega(y_1) \simeq_{c_0} \mathrm{d}_\Omega(y_2)$.

Thus we only need to check the claim in the case when $z_2 \in \Gamma_2(x_2, y_2)$ and $\mathrm{d}_\Omega(y_1) \simeq_{c_0} \mathrm{d}_\Omega(y_2)$. Let $\Gamma_{y_1z_2}$ be a fixed geodesic joining y_1 and z_2. Then, by (54),

$$100(c_0+1)^4\mathrm{d}_\Omega(z_2) \geq \mathrm{d}_\Omega(y_2) \simeq \mathrm{d}_\Omega(y_1),$$

(coming from Lemma 5.10) and since y_1 and z_2 lie on intersecting radial geodesics, there exists $R_6 = R_6(c_0)$ such that ifΩ is (c, R)-radially hyperbolic for $R \geq \max_{1\leq i\leq 6} R_i$ we get

$$l(\Gamma_{y_1z_2}) \lesssim \mathrm{d}_\Omega(y_1), \tag{55}$$

and arguments similar to the ones in Lemma 5.9 provide

$$\mathrm{d}_\Omega(y_1) \lesssim \mathrm{d}_\Omega(w) \tag{56}$$

for all $w \in \Gamma_{y_1z_2}$. Then by a covering argument considering equations (55) and (56) we get $k_\Omega(y_1, z_2) \leq c'(c_0, c, n)$. Comparing now the quasihyperbolic lengths of the curves $\tilde{\Gamma}_{y_1z_2} * \Gamma_1(y_1, x)$ and $\Gamma_2(z_2, x)$ and recalling assumption (47), we get $k_\Omega(z_2, y_2) \leq c'(c_0, c, n)$ and thus

$$\lambda_\Omega(z_2, y_2) \lesssim \mathrm{d}_\Omega(y_2) \tag{57}$$

by (2). Therefore by (30), (31), (51) and (57) we get

$$\begin{aligned}\lambda_\Omega(x_1, x_2) &\leq \lambda_\Omega(x_1, y_2) + \lambda_\Omega(y_2, x_2)\\ &\leq \lambda_\Omega(x_1, y_2) + \lambda_\Omega(y_2, z_2) + \lambda_\Omega(z_2, y_1) + \lambda_\Omega(y_1, x_2)\\ &\lesssim \mathrm{d}_\Omega(x_1)\end{aligned} \tag{58}$$

Hence there exists $R_7 = R_7(c_0, c, n)$ such that if Ω is (c, R)-radially hyperbolic for $R \geq \max_{1\leq i\leq 7} R_i$, then by arguments in Lemma 3.10, we get the desired claim, namely $k_\Omega(x_1, x_2) \leq c'(c_0, c, n)$. This completes the proof of Lemma 4.7. □

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