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Author(s): Moonens, Laurent; Russ, Emmanuel; Tuominen, Heli

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REMOVABLE SINGULARITIES FOR $\operatorname{div} v = f$
IN WEIGHTED LEBESGUE SPACES

LAURENT MOONENS, EMMANUEL RUSS, AND HELI TUOMINEN

ABSTRACT. Let $w \in L^1_{loc}(\mathbb{R}^n)$ be a positive weight. Assuming a doubling condition and an L^1 Poincaré inequality on balls for the measure $w(x)dx$, as well as a growth condition on w, we prove that the compact subsets of $\mathbb{R}^n$ which are removable for the distributional divergence in $L^\infty_{1/w}$ are exactly those with vanishing weighted Hausdorff measure. We also give such a characterization for $L^p_{1/w}$, $1 < p < +\infty$, in terms of capacity. This generalizes results due to Phuc and Torres, Silhavy and the first author.

1. INTRODUCTION

In the past years, *removable singularities* of bounded vector fields satisfying $\operatorname{div} v = 0$ in the distributional sense have been studied, *e.g.* by the first author [26], Silhavy [33] and Phuc and Torres [30]. It has been shown, in particular, that a compact set $S \subset \mathbb{R}^n$ can contain a non void support of the distributional divergence of a bounded vector field on $\mathbb{R}^n$, if and only if its $(n-1)$-dimensional Hausdorff measure is positive. As a matter of fact, all those results have immediate counterparts for vector fields defined on an open subset Ω of $\mathbb{R}^n$, satisfying the equation $\operatorname{div} v = f$, where f is a locally integrable function on Ω, in case the latter equation admits at least one solution in $L^\infty(\Omega)$.

Given $n/(n-1) < p < \infty$, Phuc and Torres in [30] showed a corresponding result for L^p-vector fields. More precisely, given an open set $\Omega \subseteq \mathbb{R}^n$ and a locally integrable function f in Ω for which the equation $\operatorname{div} v = f$ is solvable in $L^p(\Omega)$, their results imply that a compact set $S \subseteq \Omega$ contains a non void support of the distributional divergence of an L^p-vector field in $\mathbb{R}^n$, if and only if $\operatorname{cap}_{p'}(S) > 0$, where $\operatorname{cap}_{p'}$ is the capacity associated to the Sobolev space $W^{1,p'}(\mathbb{R}^n)$ (see Definition 4.4 below) and p' is the conjugate exponent to p verifying $1/p + 1/p' = 1$.

On the other hand, given a (bounded) domain $\Omega \subset \mathbb{R}^n$, it may happen that it is *not* possible to find a constant $C > 0$ such that given any $f \in L^\infty(\Omega)$, the equation

$$\operatorname{div} v = f$$

admits a bounded solution $v \in L^\infty(\Omega, \mathbb{R}^n)$ satisfying $\|v\|_\infty \leqslant C\|f\|_\infty$. In fact, the existence in this context of an integrable weight $w > 0$ such that the divergence operator acting from the weighted Lebesgue space $L^\infty_{1/w}(\Omega, \mathbb{R}^n)$ to the usual space $L^\infty(\Omega)$, admits a continuous right inverse, has been shown by Duran, Muschietti, the second author and Tchamitchian in [13] to be *equivalent* to the integrability of the geodesic distance (in Ω) to a fixed point $x_0 \in \Omega$. Under the latter integrability

property, a similar invertibility result also holds when $L^\infty_{1/w}(\Omega, \mathbb{R}^n)$ and $L^\infty(\Omega)$ are replaced by $L^p_{1/w}(\Omega, \mathbb{R}^n)$ and $L^p(\Omega)$, respectively, with $1 < p < \infty$.

In order to get some understanding of how the introduction of a (locally) integrable weight w influences, in the associated weighted Lebesgue spaces, the set of singularities of a vector field having a prescribed divergence, we shall assume here that $\Omega = \mathbb{R}^n$ and $f = 0$ in the sequel (leaving the case where Ω is a bounded domain for a future work), and study first (see section 3) the possible sets of singularities of vector fields in $L^\infty_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$ solving $\operatorname{div} v = 0$ (or $\operatorname{div} v = f$ for some locally integrable function f on $\mathbb{R}^n$ yielding at least a solution in $L^\infty_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$)), and make a similar study in appropriate weighted L^p spaces.

More precisely, calling $L^p_{1/w}$-*removable* any compact subset of $\mathbb{R}^n$ that does *not* support any nonzero distributional divergence of a vector field $v \in L^p_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$, we show the following result (which combines our Theorems 3.24 and 4.19).

Theorem 1.1. *Assume S is a compact subset of $\mathbb{R}^n$.*

  (i) *If $p = \infty$ and if the weight w is 1-admissible and satisfies the growth condition (3.1) below, then S is $L^\infty_{1/w}$-removable if and only if $\mathscr{H}^h(S) = 0$, where $\mathscr{H}^h$ is the Hausdorff outer measure associated to w as in Section 3.1.*

  (ii) *If $1 < p < \infty$ and if one has $w^{p'-1} \in A_{p'}$, then S is $L^p_{1/w}$-removable if and only if one has $\operatorname{Cap}^{w^{p'-1}}_{p'}(S) = 0$, where $\operatorname{Cap}^{w^{p'-1}}_{p'}$ is the Sobolev capacity associated to $w^{p'-1}$ appearing in Definition 4.4.*

In the previous statement, we mean by saying that a weight is 1-*admissible*, that it is doubling and satisfies a $(1,1)$-Poincaré inequality (see Definition 2.2 below), while $A_{p'}$ stands for the Muckenhoupt class (see Definition 2.3). Note that any A_1 weight is 1-admissible (see Remark 2.4). Note that $\mathscr{H}^h$ is the classical (spherical) Hausdorff measure of dimension $n - 1$ is case $w = 1$.

A first remark about the previous theorem is that we recover, when $w = 1$, the result mentioned above stating that L^∞-removable (compact) sets are exactly those satisfying $\mathscr{H}^{n-1}(S) = 0$, and similarly in L^p.

An interesting case covered by our results is the one when the weight w equals $+\infty$ on a "large" set (*e.g.* on a set of positive Hausdorff dimension) — allowing the vector fields in $L^\infty_{1/w}$ to have singular pointwise behaviour on this "large" set. More precisely, as an interesting complement to [Theorem 1.1, (i)], we provide examples of A_1 weights w of the form $w(x) := \operatorname{dist}(x, C)^{-\alpha}$, with $\alpha > 0$, whose singular set C has positive Hausdorff dimension yet *is* $L^\infty_{1/w}$-removable for the equation $\operatorname{div} v = 0$ for some values of α related to the Hausdorff dimension of C and that of the ambient space. This is our Example 3.26.

The proof of both parts of Theorem 1.1 follow the same structure. Sufficient removability conditions are obtained by truncation arguments as in De Pauw [12] and [26] (case $p = \infty$) and [30] (case $1 < p < \infty$), but extra care is needed for we cannot, in this weighted context, rely on straightforward estimates relating the (weighted) perimeter of a ball B of radius r to the integral $\frac{1}{r} \int_B w$. Showing those conditions are also necessary for a set S to be removable is done by solving the

equation div $v = \mu$ in $L^p_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$ for some suitable measures μ supported in S, and showing it might admit non trivial solutions in case S does *not* satisfy the conditions in question. This is done, as in Bourgain and Brezis [8] and [30], by using a simple version of the closed range theorem. Note that the case where $p = +\infty$ cannot be dealt with using capacity arguments, see Remark 3.25 below.

The paper is organized as follows. In Section 2, we give definitions and basic properties of p-admissible weights and introduce the notations used in the paper. In Section 3, we study the removability question for the divergence equation for weighted L^∞-vector fields. In the proofs, we need theory of functions of bounded variation in the weighted case, the weighted Hausdorff measure of co-dimension one, the boxing inequality and a version of Frostman's lemma. Those, as well as some technical lemmas are presented before the main results of the section. Section 4 contains characterization of removable sets for the divergence equation for weighted L^p-vector fields. In this section, important tools are weighted Sobolev spaces, different capacities and some tools from the general theory of L^q-capacities, discussed before the main results.

2. Weights and notation

A locally integrable function $w\colon \mathbb{R}^n \to \mathbb{R}$ is *a weight* if $w(x) > 0$ for almost every $x \in \mathbb{R}^n$. We say that the weight w is *doubling* if there exists a constant $C_D \geqslant 1$ (called the *doubling constant of w*) such that for any $x \in \mathbb{R}^n$ and any $r > 0$ one has:

$$\int_{B(x,2r)} w \leqslant C_D \int_{B(x,r)} w,$$

where $B(x, r)$ denotes the Euclidean (open) ball with center x and radius r in $\mathbb{R}^n$ and where one integrates with respect to Lebesgue measure. An iteration of the doubling inequality then ensures that one has, for all $t > 0$:

$$(2.1) \qquad \int_{B(x,tr)} w \leqslant C_D t^{s_D} \int_{B(x,r)} w,$$

where $s_D := \log_2 C_D$ is the *doubling dimension of the weighted space* $(\mathbb{R}^n, w)$.

For $1 \leqslant p \leqslant \infty$, the weighted L^p-space, $L^p_w(\mathbb{R}^n, \mathbb{R}^k)$, consists of measurable functions $u\colon \mathbb{R}^n \to \mathbb{R}^k$ for which $|u|^p w \in L^1(\mathbb{R}^n, \mathbb{R}^k)$, and we let:

$$\|u\|_{p,w} := \left(\int_{\mathbb{R}^n} |u|^p w \right)^{1/p},$$

for $p < \infty$, and

$$\|u\|_{\infty,w} := \|uw\|_\infty.$$

We finally let $L^p(\mathbb{R}^n) := L^p(\mathbb{R}^n, \mathbb{R})$.

In the sequel we shall denote by $\mathrm{Lip}_c(\mathbb{R}^n)$ the set of all compactly supported (real valued) Lipschitz functions in $\mathbb{R}^n$. For a weight w, the weighted Euclidean space endowed with the Euclidean metric and the measure $d\mu = w\, dx$ is denoted by $(\mathbb{R}^n, w)$.

Definition 2.1. Let $1 \leqslant p < +\infty$ be a real number. We shall say that the weighted space $(\mathbb{R}^n, w)$ supports *a weighted $(1,p)$-Poincaré inequality* in case that there exist constants $C_P > 0$ and $\tau > 1$ such that for any $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$, any $x \in \mathbb{R}^n$ and any $r > 0$ we have:

$$\fint_{B(x,r)} |\varphi - \bar\varphi_{x,r}| w \leqslant C_P r \left(\fint_{B(x,\tau r)} |\nabla \varphi|^p w \right)^{1/p},$$

where we let $\bar\varphi_{x,r} := \fint_{B(x,r)} \varphi w$ and where $\fint_B \varphi w$ denotes the mean value $\frac{1}{\int_B w} \int_B \varphi w$ for any Borel set B satisfying $|B| > 0$.

We use the class of p-admissible weights as in [16, Section 1.1] and [6, Definition A.6]. Such weights are important in the nonlinear potential theory developed in [16], see also [6, Appendix A].

Definition 2.2. Let $1 \leqslant p < \infty$ be a real number. A weight w is said to be *p-admissible* in case it is doubling and the weighted space $(\mathbb{R}^n, w)$ supports a $(1,p)$-Poincaré inequality.

It follows from Heinonen, Kilpeläinen and Martio [16, Corollary 20.9] (in the second edition of their book) that p-admissible weights for $p > 1$ satisfy several other interesting properties, among which the following Poincaré inequality for compactly supported functions (see [16, Section 1.4] and [34, Corollary 2.1.5]): there exists $\kappa > 0$ such that for any ball $B = B(x,r) \subset \mathbb{R}^n$ and any $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$ supported in $B(x,r)$, we get:

$$(2.2) \qquad \int_{B(x,r)} |\varphi|^p w \leqslant \kappa \, r^p \int_{B(x,r)} |\nabla \varphi|^p w.$$

An important class of p-admissible weights are the A_p-weights, which were defined by Muckenhoupt in [28], where he showed that when $1 < p < \infty$, the Hardy–Littlewood maximal operator is L_w^p-bounded if and only if $w \in A_p$.

Definition 2.3. A weight w is *an A_p-weight*, $1 < p < \infty$, if

$$\sup_B \left(\fint_B w \right) \left(\fint_B w^{1/(1-p)} \right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. Note that the A_p-condition implies that $w \in L^{1/(1-p)}(\mathbb{R}^n)$ *locally*.

A weight w is *an A_1-weight*, if

$$\sup_B \left(\fint_B w \right) \operatorname{ess\,sup}_B \frac{1}{w} < \infty.$$

Remark 2.4. The fact that A_p-weights are p-admissible has been proved in [16, Theorem 15.21] in case $p > 1$ and in [7, Theorem 4] in case $p = 1$. The doubling property follows easily from the A_p-condition but the validity of a weighted $(1,p)$-Poincaré inequality requires more work.

For further properties of A_p-weights, see for example [11], [18], [32, Chapter V] and [34, Remark 1.2.4] and for examples of p-admissible weights that are not A_p-weights, see for example [10] and [15].

Example 2.5. It is classical that for $-n < \eta \leqslant 0$, $w_\eta(x) := |x|^\eta$ is an A_1-weight. Moreover any weight of the form $w_\eta(x) = |x|^\eta$ for $\eta > -n$ is doubling (while it may *not* be A_1). See *e.g.* [34, Example 1.2.5].

For a set $A \subset \mathbb{R}^n$, $\mathcal{M}_+(A)$ is the set of locally finite (nonnegative) Radon measures supported in A.

3. The case of weighted L^∞ vector fields

In this section, we study the removability question for the divergence equation for weighted L^∞-vector fields. We start by defining some tools and proving results needed in the proofs - those include weighted Hausdorff content and measure of codimension 1 and functions of bounded variation in the weighted setting. Our main results in this section hold for doubling weights that satisfy a $(1,1)$-Poincaré inequality. In Theorem 3.17, we show that any compact set $S \subset \mathbb{R}^n$ with $\mathscr{H}^h(S) = 0$, is $L^\infty_{1/w}$-removable for $\operatorname{div} v = 0$. Indeed, vanishing Hausdorff measure almost characterizes removable sets - if the weight w satisfies an additional mild growth condition, then a compact set $S \subset \mathbb{R}^n$ is $L^\infty_{1/w}$-removable for $\operatorname{div} v = 0$ if and only if $\mathscr{H}^h(S) = 0$, see Theorem 3.24.

3.1. **Hausdorff contents.** Let w be a weight. Associated to w, define a (spherical) measure function h on (closed) balls $B(x, r)$ by:

$$h(B(x,r)) := \frac{1}{r} \int_{B(x,r)} w.$$

According to the usual Carathéodory construction (see [23, Section 4.1]), we also define a weighted co-dimension 1 (spherical) Hausdorff outer measure (as in Turesson [34, Section 2.3] and Nieminen [29]) by letting first, for $0 < \delta \leqslant \infty$ and $E \subset \mathbb{R}^n$:

$$\mathscr{H}^h_\delta(E) := \inf \sum_{j \in J} h(B(x_j, r_j)),$$

where the infimum is taken on all countable coverings of E by balls $(B(x_j, r_j))_{j \in J}$ satisfying $r_j \leqslant \delta$ for all $j \in J$. We define then

$$\mathscr{H}^h(E) := \lim_{\delta \to 0} \mathscr{H}^h_\delta(E).$$

It follows from [34, Proposition 2.3.3] that $\mathscr{H}^h_\delta$ is an outer measure for any $0 < \delta \leqslant \infty$, and that $\mathscr{H}^h$ is a Borel regular outer measure.

The following straightforward lemma will be useful in the sequel.

Lemma 3.1. *Let h be associated to the weight w as above and assume that the compact set $S \subset \mathbb{R}^n$ satisfies $\mathscr{H}^h(S) = 0$. Then S is Lebesgue-negligible.*

Proof: if $\eta > 0$ is an arbitrary positive number, let $(B(x_j, r_j))_{j \in J}$ be a finite family of balls covering S, verifying $r_j \leqslant 1$ for each $j \in J$ as well as:

$$\sum_{j \in J} h(B(x_j, r_j)) \leqslant \eta.$$

We then compute:

$$\int_S w \leqslant \int_{\bigcup_{j\in J} B(x_j,r_j)} w \leqslant \sum_{j\in J} \int_{B(x_j,r_j)} w = \sum_{j\in J} r_j h(B(x_j,r_j)) \leqslant \sum_{j\in J} h(B(x_j,r_j)) \leqslant \eta,$$

so that one has $\int_S w = 0$, since $\eta > 0$ is arbitrary. It follows that the set $\{x \in S : w(x) > 0\}$ is Lebesgue-negligible, and hence that S itself is Lebesgue-negligible for we assumed than one has $w > 0$ a.e. on $\mathbb{R}^n$.

When $\mathbb{R}^n$ is endowed with a doubling weight which grows fast enough, we have the following version of Frostman's lemma, which is a particular case of [34, Theorem 3.4.27].

Lemma 3.2. *Assume that w is a doubling weight on $\mathbb{R}^n$. If moreover, for any $x \in \mathbb{R}^n$:*

(3.1) $$\lim_{r\to\infty} h(B(x,r)) = \infty,$$

and if $B \subset \mathbb{R}^n$ is a Borel set verifying $\mathscr{H}^h_\infty(B) > 0$, then there exists a nontrivial measure $\mu \in \mathcal{M}_+(B)$ satisfying the following inequality:

$$\mu(B(x,r)) \leqslant Ch(B(x,r)),$$

for any $x \in \mathbb{R}^n$ and $r > 0$.

Example 3.3. An easy computation shows that the weight w_η defined in Example 2.5 also satisfies condition (3.1) in case $1 - n < \eta \leqslant 0$.

In fact, assuming the growth condition (3.1) only for $x = 0$, it is equivalent for a bounded set B to satisfy $\mathscr{H}^h_\infty(B) = 0$ or $\mathscr{H}^h(B) = 0$.

Lemma 3.4. *Assume that w is doubling and that one has:*

(3.2) $$\lim_{r\to\infty} h(B(0,r)) = \infty.$$

For any bounded set $B \subset \mathbb{R}^n$, the equalities $\mathscr{H}^h(B) = 0$ and $\mathscr{H}^h_\infty(B) = 0$ are equivalent.

Proof: fix a bounded set $B \subset \mathbb{R}^n$. Since one has $\mathscr{H}^h_\infty(B) \leqslant \mathscr{H}^h(B)$, it is clear that $\mathscr{H}^h_\infty(B) = 0$ provided that $\mathscr{H}^h(B) = 0$.

Conversely, assume that $\mathscr{H}^h_\infty(B) = 0$. Choose $R_0 > 0$ such that one has $B \subset B(0,R_0)$. Let $f(r) := h(B(0,r))$ for all $r > 0$. Since (3.2) yields

$$\lim_{r\to+\infty} f(r) = +\infty,$$

we are allowed to choose $R > R_0$ such that $f(r) > 3^{s_D} C_D$ for all $r > R$, where $C_D > 0$ and $s_D := \log_2 C_D$ are the doubling constants of w (see (2.1) above).

Fix now $\delta > 0$ and let:

$$c_\delta := \max\left(3, 2 + \frac{R_0}{\delta}\right).$$

Choose then $\varepsilon > 0$ with:

$$\varepsilon < \min\left(1, \frac{1}{c_\delta^{s_D} C_D R} \int_{B(0,\delta)} w\right).$$

Since one has $\mathscr{H}_\infty^h(B) = 0$, there exists an (at most countable) family of balls $(B(x_j, r_j))_{j \in J}$ covering B and satisfying:

$$\sum_{j \in J} h(B(x_j, r_j)) = \sum_{j \in J} \frac{1}{r_j} \int_{B(x_j, r_j)} w \leqslant \varepsilon.$$

One may assume that, for all $j \in J$, one has $B(x_j, r_j) \cap B \neq \emptyset$, so that one computes $|x_j| \leqslant r_j + R_0$. By the doubling property (2.1), we get for any $j \in J$ such that $r_j > R_0$:

$$\int_{B(0,r_j)} w \leqslant \int_{B(x_j, r_j + |x_j|)} w \leqslant C_D \left(1 + \frac{|x_j|}{r_j} \right)^{s_D} \int_{B(x_j, r_j)} w$$

$$\leqslant C_D \left(2 + \frac{R_0}{r_j} \right)^{s_D} \int_{B(x_j, r_j)} w \leqslant 3^{s_D} C_D \int_{B(x_j, r_j)} w,$$

so that:

$$\sum_{j \in J} f(r_j) \leqslant 3^{s_D} C_D \varepsilon < 3^{s_D} C_D.$$

We hence get $r_j \leqslant R$ for all $j \in J$, for the latter inequality is obvious in case $j \in J$ is such that one has $r_j \leqslant R_0 < R$.

Assume now that $j \in J$ is such that $r_j > \delta$. If moreover one has $r_j > R_0$, the computations above show that:

$$\int_{B(0,r_j)} w \leqslant 3^{s_D} C_D r_j \cdot \frac{1}{r_j} \int_{B(x_j, r_j)} w \leqslant 3^{s_D} C_D R \varepsilon \leqslant c_\delta^{s_D} C_D R \varepsilon.$$

In case one has $\delta < r_j \leqslant R_0$, we compute using again (2.1):

$$\int_{B(0,r_j)} w \leqslant C_D \left(2 + \frac{R_0}{r_j} \right)^{s_D} \int_{B(x_j, r_j)} w \leqslant \left(2 + \frac{R_0}{\delta} \right)^{s_D} C_D r_j \varepsilon \leqslant c_\delta^{s_D} C_D R \varepsilon.$$

Hence in both cases we have $\int_{B(0,r_j)} w \leqslant c_\delta^{s_D} C_D R \varepsilon$. We hence get:

$$\int_{B(0,\delta)} w \leqslant \int_{B(0,r_j)} w \leqslant c_\delta^{s_D} C_D R \varepsilon < \int_{B(0,\delta)} w,$$

which is impossible.

Therefore $r_j \leqslant \delta$ for each $j \in J$, so that we get:

$$\mathscr{H}_\delta^h(B) \leqslant \sum_{j \in J} h(B(x_j, r_j)) \leqslant \varepsilon.$$

Since ε is arbitrary small, this yields $\mathscr{H}_\delta^h(B) = 0$. Finally, we get $\mathscr{H}^h(B) = 0$ for the previous estimates yield $\mathscr{H}_\delta^h(B) = 0$ for any $\delta > 0$.

Remark 3.5. In case w is 1-admissible, the previous lemma can be obtained by combining results [20, Lemma 7.6, Remark 7.4] by Kinnunen, Korte, Shanmugalingam and the third author.

3.2. **Miranda's BV-functions.** In the more general context of metric measure spaces, M. Miranda introduced in [25] the notion of *function with bounded variation*. We shall in the sequel particularize some results obtained by Miranda to the context where $\mathbb{R}^n$ is endowed by an appropriate weight; to this purpose, we need to introduce some terminology.

Following Miranda [25], read in this weighted context by Camfield [9], define the *metric (weighted) variation of $u \in L_w^1(\mathbb{R}^n)$* by:

$$\|Du\|_w := \inf \left\{ \varliminf_{k\to\infty} \int_{\mathbb{R}^n} |\nabla\varphi_k| w : (\varphi_k) \subseteq \mathrm{Lip}(\mathbb{R}^n), \varphi_k \to u \text{ in } L_w^1 \right\}.$$

The following Theorem, stated here for reader's convenience but unnecessary for our purposes, is a direct consequence of two deep results by Camfield [9, Theorems 3.2.6 and 3.4.5]. It shows that, under some regularity conditions on the weight w, the metric variation of a Lipschitz function is identical to the L_w^1 norm of its gradient.

Theorem 3.6 (Camfield). *Assume that w is locally integrable and lower semicontinuous. If moreover there exists an $\mathscr{H}^h$-negligible set E outside which w is continuous and (strictly) positive, then for any $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$ we have:*

$$\text{(3.3)} \qquad \|D\varphi\|_w = \|\nabla\varphi\|_{1,w} = \int_{\mathbb{R}^n} |\nabla\varphi| w.$$

For 1-admissible weights, the equality (3.3) can be replaced by a comparison between the two quantities involved, at least for Lipschitz functions (see [25, p. 992, (19) and below]).

Proposition 3.7. *Assume that the weight w is 1-admissible. Then for any $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$ we have:*

$$c\|\nabla\varphi\|_{1,w} \leqslant \|D\varphi\|_w \leqslant \|\nabla\varphi\|_{1,w},$$

where $c > 0$ is independent of φ.

The following proposition is a particular case of M. Miranda's Coarea formula [25, Proposition 4.2]. The *(weighted) perimeter* of a Borel set B, denoted by $P_w(B)$, is defined by:

$$P_w(B) := \|D\chi_B\|_w.$$

Proposition 3.8. *Assume that w is a 1-admissible weight. Then, for any $u \in L_w^1(\mathbb{R}^n)$ verifying $\|Du\|_w < +\infty$, we have:*

$$\|Du\|_w = \int_{\mathbb{R}} P_w(\{u > t\}) \, dt.$$

We shall also make use of the following boxing inequality, due (in the more general framework of measure metric spaces) to Kinnunen, Korte, Shanmugalingam and the third author [20, Theorem 3.1].

Theorem 3.9 (Boxing inequality). *Assume that w is a 1-admissible weight. There exists a constant $C_B = C_B(C_D, C_P) > 0$ such that for any open set $U \subseteq \mathbb{R}^n$ verifying $\int_U w < \infty$, we can find sequences $(x_i) \subseteq U$ and $(r_i) \subseteq (0, +\infty)$ satisfying the following conditions:*

(i) $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$ for $i \neq j$;

(ii) $U \subseteq \bigcup_{i \in \mathbb{N}} B(x_i, 5r_i)$;

(iii) $\sum_{i=0}^{\infty} h(B(x_i, 5r_i)) = \sum_{i=0}^{\infty} \frac{1}{5r_i} \int_{B(x_i, 5r_i)} w \leqslant C_B P_w(U)$.

We are now ready to study removable singularities of divergence-free vector fields

3.3. **Removable singularities.** Assume in this whole section that w is a weight. Given a vector field $v \in L^{\infty}_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$ and $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$, it is clear that we have, a.e. on $\mathbb{R}^n$:

$$|v \cdot \nabla \varphi| \leqslant \left|\frac{v}{w}\right| |\nabla \varphi| w \chi_{\mathrm{supp}\,\varphi} \leqslant \|v\|_{\infty, 1/w} \|\nabla \varphi\|_{\infty} w \chi_{\mathrm{supp}\,\varphi}.$$

Since w is locally integrable, one can define the (extended) distributional divergence of v by:

$$\langle \mathrm{div}\, v, \varphi \rangle := -\int_{\mathbb{R}^n} v \cdot \nabla \varphi,$$

for any $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$.

Definition 3.10. Let $S \subset \mathbb{R}^n$ be compact and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The set S is said to be $L^{\infty}_{1/w}$-*removable with respect to the equation* $\mathrm{div}\, v = f$ in case for any $v \in L^{\infty}_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$, the equality

(3.4) $$\langle \mathrm{div}\, v, \varphi \rangle = \int_{\mathbb{R}^n} f \varphi$$

for any $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$ with $\mathrm{supp}\,\varphi \cap S = \emptyset$ (which we shall abbreviate "$\mathrm{div}\, v = f$ outside S") implies that (3.4) also holds for any $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$ (which we shall abbreviate "$\mathrm{div}\, v = f$ in $\mathbb{R}^n$").

Remark 3.11. Assume that w is an A_1 weight. According to the fact that given $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$, one has $\varphi \in L^1_w(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\nabla \varphi \in L^1_w(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$, [34, Corollary 2.1.5] ensures that the above definition remains unchanged in case $\mathrm{Lip}_c(\mathbb{R}^n)$ is replaced by $C^{\infty}_c(\mathbb{R}^n)$; yet it is not clear that the two definitions are equivalent without this extra assumption on w.

Remark 3.12. Assume that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ yields a solution $v_0 \in L^{\infty}_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$ to the equation $\mathrm{div}\, v = f$, in the sense that $\mathrm{div}\, v_0 = f$ in $\mathbb{R}^n$. Assume also that S is $L^{\infty}_{1/w}$-removable for the equation $\mathrm{div}\, v = 0$. Fix now $v \in L^{\infty}_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$ and assume that $\mathrm{div}\, v = f$ outside S. Since it is clear that one has $\mathrm{div}(v - v_0) = 0$ outside S, the removability assumption made on S ensures that one has $\mathrm{div}(v - v_0) = 0$ in $\mathbb{R}^n$; we hence get $\mathrm{div}\, v = f$ in $\mathbb{R}^n$, and S is $L^{\infty}_{1/w}$-removable for the equation $\mathrm{div}\, v = f$. Conversely, one shows in a similar fashion that any $L^{\infty}_{1/w}$-removable set for $\mathrm{div}\, v = f$ is also removable for $\mathrm{div}\, v = 0$. Hence we shall assume in the sequel that $f = 0$.

3.4. **A sufficient condition for a set to be $L^{\infty}_{1/w}$-removable.** Let us observe first that it suffices, in order to show that a set is removable, that one is able to construct, in any of its neighborhoods, an appropriate Lipschitz approximation of its characteristic function.

Lemma 3.13. *Assume that S is compact and that for any $\varepsilon > 0$, there exists a neighborhood U of S satisfying $\int_U w \leqslant \varepsilon$ together with a Lipschitz function $\chi \in \operatorname{Lip}_c(\mathbb{R}^n)$ satisfying $\operatorname{supp}\chi \subseteq U$, $0 \leqslant \chi \leqslant 1$ in U, $\chi = 1$ in a neighborhood of S as well as $\|\nabla\chi\|_{1,w} \leqslant \varepsilon$. Then S is $L^\infty_{1/w}$-removable for the equation $\operatorname{div} v = 0$.*

Proof: assume that one has $\langle \operatorname{div} v, \varphi \rangle = 0$ for any $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n)$ satisfying $\operatorname{supp}\varphi \cap S = \emptyset$. Fix then $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n)$, let $\varepsilon > 0$ and let χ be associated to U and ε as in the above assumption. Observe that one has, by hypothesis:

$$\begin{aligned}
\langle \operatorname{div} v, \varphi \rangle &= \langle \operatorname{div} v, \varphi\chi \rangle \\
&\leqslant \|v\|_{\infty,1/w}\Big(\|\nabla\varphi\|_\infty \int_U w + \|\varphi\|_\infty \|\nabla\chi\|_{1,w}\Big) \\
&\leqslant \varepsilon\|v\|_{\infty,1/w}(\|\nabla\varphi\|_\infty + \|\varphi\|_\infty).
\end{aligned}$$

It then follows that $\langle \operatorname{div} v, \varphi \rangle = 0$ for $\varepsilon > 0$ is arbitrary. This establishes that S is $L^\infty_{1/w}$-removable for the equation $\operatorname{div} v = 0$.

The following proposition will be useful while showing that compact sets with $\mathscr{H}^h(S) = 0$ are removable for the divergence equation.

Proposition 3.14. *Let w be a 1-admissible weight and assume that $V' \subset\subset V \subset\subset U \subset\subset \mathbb{R}^n$ are open sets, and assume that $|\partial V| = |\partial V'| = 0$. For any $\varepsilon > 0$, there exists a Lipschitz function $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n)$ satisfying $\operatorname{supp}\varphi \subseteq U$, $\varphi = 1$ on V' and*

$$\|\nabla\varphi\|_{1,w} \leqslant \varepsilon + 2P_w(V).$$

Proof: start by choosing $\psi \in \operatorname{Lip}_c(\mathbb{R}^n)$ satisfying $\chi_{V'} \leqslant \psi \leqslant \chi_V$. Choose, according to the definition of $P_w(V)$, a sequence $(\varphi_k) \subseteq \operatorname{Lip}(\mathbb{R}^n)$ converging in $L^1_w(\mathbb{R}^n)$ to χ_V and such that for any $k \in \mathbb{N}$, one has:

$$\|\nabla\varphi_k\|_{1,w} \leqslant \frac{\varepsilon}{4} + P_w(V).$$

Replacing if necessary φ_k by $\min[1, \max(\varphi_k, 0)]$ (which does not increase the norm of the gradient of φ_k on the complement of a negligible set), one can assume that one has $0 \leqslant \varphi_k \leqslant 1$ for each k. Replacing if necessary φ_k by $\theta\varphi_k$ for $k \in \mathbb{N}$, where $\theta \in \operatorname{Lip}_c(\mathbb{R}^n)$ satisfies $0 \leqslant \theta \leqslant 1$, as well as $\theta = 1$ on V and $\theta = 0$ outside U, and observing that one has, for any $k \in \mathbb{N}$:

$$|\nabla(\theta\varphi_k)|w \leqslant |\nabla\theta||\varphi_k - \chi_V|w + |\nabla\varphi_k|w \leqslant \|\nabla\theta\|_\infty|\varphi_k - \chi_V|w + |\nabla\varphi_k|w,$$

we may also assume that $\operatorname{supp}\varphi_k \subseteq U$ for any $k \in \mathbb{N}$.

Define now, for $k \in \mathbb{N}$:

$$\tilde{\varphi}_k := \psi + (1 - \psi)\varphi_k.$$

It is clear that $\tilde{\varphi}_k$ is Lipschitz and has compact support for any $k \in \mathbb{N}$; moreover one computes, for a.e. $x \in \mathbb{R}^n$ (recall that one has $|\partial V'| = |\partial V| = 0$):

$$|\nabla\tilde{\varphi}_k(x)| \leqslant \begin{cases} 0 & \text{if } x \in V', \\ 2|\nabla\varphi_k(x)| + \|\nabla\psi\|_\infty|1 - \varphi_k(x)| & \text{if } x \in V \setminus V', \\ |\nabla\varphi_k(x)| & \text{if } x \in \complement V. \end{cases}$$

We hence get, a.e. on $\mathbb{R}^n$, for any $k \in \mathbb{N}$:

$$|\nabla \tilde{\varphi}_k| w \leqslant 2|\nabla \varphi_k| w + \|\nabla \psi\|_\infty |\chi_V - \varphi_k| w.$$

Since we have $\varphi_k \to \chi_V$ in $L^1_w(\mathbb{R}^n)$, there exists $k \in \mathbb{N}$ such that:

$$\|\nabla \tilde{\varphi}_k\|_{1,w} \leqslant \varepsilon + 2P_w(V).$$

We can hence take $\varphi := \tilde{\varphi}_k$, and the proof is complete.

The next lemma, taken from [21, Lemma 6.2], will be used in the proof of Proposition 3.16. We include its proof for the sake of clarity.

Lemma 3.15. *Assume that w is a 1-admissible weight. Given $x \in \mathbb{R}^n$ and $r > 0$, there exists $\rho \in [r, 2r]$ such that one has:*

$$P_w(B(x, \rho)) \leqslant Ch(B(x, \rho)),$$

where $C > 0$ is independent of x and ρ.

Proof: fix $x \in \mathbb{R}^n$ and $r > 0$. Define $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n)$ by the formula:

$$\varphi(y) := \max\left[0, \min\left(2 - \frac{1}{r}|y - x|, 1\right)\right],$$

for $y \in \mathbb{R}^n$. It is clear that one has $|\nabla \varphi| \leqslant \frac{1}{r}\chi_{B(x,2r) \setminus B(x,r)}$ on $\mathbb{R}^n$. Hence we get, using M. Miranda's coarea formula (Proposition 3.8) and Proposition 3.7:

$$\int_0^1 P_w(B(x, (2-t)r))\, dt = \int_{\mathbb{R}} P_w(\{\varphi > t\}) = \|D\varphi\|_w$$
$$\leqslant \int_{\mathbb{R}^n} |\nabla \varphi| w \leqslant \frac{1}{r}\int_{B(x,2r) \setminus B(x,r)} w \leqslant C_D h(B(x, r)).$$

If we now choose $\rho \in [r, 2r]$ such that one has

$$P_w(B(x, \rho)) \leqslant \int_0^1 P_w(B(x, (2-t)r))\, dt,$$

we get:

$$P_w(B(x, \rho)) \leqslant C_D h(B(x, r)) \leqslant 2C_D h(B(x, \rho)),$$

and the lemma is proved.

The next proposition is a first step towards showing $\mathscr{H}^h$-negligible sets are removable.

Proposition 3.16. *Assume that w is a 1-admissible weight. If S is compact and satisfies $\mathscr{H}^h(S) = 0$, then for any $\varepsilon > 0$ and any neighborhood U of S, one can find an open subset $V \subset\subset U$, satisfying $S \subseteq V$ as well as:*

$$P_w(V) \leqslant \varepsilon.$$

Moreover V can be chosen to be a finite union of balls.

Proof: fix $\eta > 0$. Let $0 < \delta < \frac{1}{4}\operatorname{dist}(S, \complement U)$ and observe that $\mathscr{H}_\delta^h(S) = 0$. Hence there are balls $B(x_j, r_j)$ for some $x_j \in \mathbb{R}^n$, $r_j > 0$, $j \in J$ covering S (since S is compact we can assume J to be finite), satisfying $r_j \leqslant \delta$ for each $j \in J$ and:

$$\sum_{j \in J} h(B(x_j, r_j)) \leqslant \eta.$$

According to Lemma 3.15, choose for each $j \in J$ a radius $\rho_j \in [r_j, 2r_j]$ for which one has:

$$P_w(B(x_j, \rho_j)) \leqslant \frac{C}{\rho_j} \int_{B(x_j, \rho_j)} w;$$

observe that the doubling property of w yields, for any $j \in J$:

$$P_w(B(x_j, \rho_j)) \leqslant \frac{CC_D}{r_j} \int_{B(x_j, r_j)} w = CC_D h(B(x_j, r_j)).$$

Letting $V := \bigcup_{j \in J} B(x_j, \rho_j) \subset\subset U$, we hence get, using the subadditivity of the weighted perimeter ([25, Proposition 4.7]):

$$P_w(V) \leqslant \sum_{j \in J} P_w(B(x_j, \rho_j)) \leqslant CC_D \sum_{j \in J} h(B(x_j, r_j)) \leqslant CC_D \eta,$$

from which the desired inequality readily follows.

The previous proposition together with Lemma 3.13 and Proposition 3.14, yield a sufficient removability condition.

Theorem 3.17. *Assume that w is a 1-admissible weight. Then any compact set $S \subset \mathbb{R}^n$ verifying $\mathscr{H}^h(S) = 0$, is $L_{1/w}^\infty$-removable for $\operatorname{div} v = 0$.*

Proof: according to Lemma 3.1, the set S is Lebesgue-negligible. So we may fix $\varepsilon > 0$, and let U be a neighborhood of S for which one has $\int_U w \leqslant \varepsilon$ (we may moreover assume that U consists of a finite union of balls). Start, according to Proposition 3.16, by choosing an open set $V \subset\subset U$ satisfying $P_w(V) \leqslant \frac{\varepsilon}{4}$ as well as $S \subseteq V$ (and consisting of a finite union of balls). Choose, according to Proposition 3.14, a Lipschitz function $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n)$ equal to 1 in a neighborhood $V' \subset\subset V$ of S and verifying $\|\nabla \chi\|_{1,w} \leqslant \frac{\varepsilon}{2} + 2P_w(V)$. One now computes:

$$\|\nabla \chi\|_{1,w} \leqslant \frac{\varepsilon}{2} + 2P_w(V) \leqslant \varepsilon,$$

and it follows from Lemma 3.13 that S is $L_{1/w}^\infty$-removable for $\operatorname{div} v = 0$.

3.5. **A necessary condition for a set to be $L_{1/w}^\infty$-removable.** We first state the following estimate for measures satisfying a weighted-Frostman condition. It is similar to [34, Theorem 2.6.3], although we do not require here w to be A_1.

Proposition 3.18. *Assume that w is a 1-admissible weight. Let μ be a (nonnegative) Radon measure on $\mathbb{R}^n$ and assume that for any $x \in \mathbb{R}^n$ and any $r > 0$, we have :*

$$(3.5) \qquad\qquad \mu(B(x, r)) \leqslant Ch(B(x, r)).$$

Then for any $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n)$ we have:

$$\int_{\mathbb{R}^n} |\varphi| \, d\mu \leqslant M \|\nabla \varphi\|_{1,w},$$

where $M > 0$ is independent of φ.

Proof: fix $t > 0$ and let U_t be the bounded open set defined by $U_t := \{x \in \mathbb{R}^n : |\varphi(x)| > t\}$. According to the weighted boxing inequality (Theorem 3.9), there exists $(x_i) \subseteq U_t$ and $(r_i) \subseteq (0, \infty)$ satisfying conditions (i) to (iii) in Theorem 3.9, with U_t instead of U. Write then

$$\mu(U_t) \leqslant \sum_{i=0}^{\infty} \mu(B(x_i, 5r_i)) \leqslant C \sum_{i=0}^{\infty} \frac{1}{5r_i} \int_{B(x_i, 5r_i)} w \leqslant C C_B P_w(U_t).$$

Now we have, using Cavalieri's principle and Miranda's coarea formula (Proposition 3.8):

$$\int_{\mathbb{R}^n} |\varphi| \, d\mu = \int_0^{\infty} \mu(U_t) \, dt \leqslant C C_B \int_0^{\infty} P_w(U_t) \, dt = C C_B \|D|\varphi|\|_w.$$

According to Proposition 3.7, we have

$$\|D|\varphi|\|_w \leqslant \|\nabla|\varphi|\|_{1,w} = \|\nabla \varphi\|_{1,w},$$

since one has $|\nabla|\varphi|| = |\nabla \varphi|$ almost everywhere. This finishes the proof since one can take $M := C C_B > 0$.

The following lemma, of functional analytic nature, avoids unnecessary technicalities in the sequel, and has been suggested by J. Boël.

Lemma 3.19. *Let X and Y be two normed spaces, and assume that $T \colon X \to Y$ is linear and* isometric, *i.e. that one has $\|T(x)\|_Y = \|x\|_X$ for every $x \in X$. Then its adjoint map $T^* \colon Y^* \to X^*$ is surjective.*

Remark 3.20. Observe that one does not need, in the above statement, any kind of completeness to be satisfied, neither by X nor by Y.

Proof: fix $f \in X^*$. It is clear by assumption that T is injective; hence the formula:

$$\langle g_0, T(x) \rangle := \langle f, x \rangle,$$

defines a linear map $g_0 \colon T(X) \to \mathbb{R}$ satisfying

$$|\langle g_0, T(x) \rangle| \leqslant \|f\| \|x\|_X = \|f\| \|T(x)\|_Y$$

for any $x \in X$. The Hahn-Banach Theorem hence ensures the existence of $g \in Y^*$ verifying $g \restriction T(X) = g_0$, meaning that one has:

$$\langle T^* g, x \rangle := \langle g, T(x) \rangle = \langle g_0, T(x) \rangle = \langle f, x \rangle,$$

for any $x \in X$. This shows that $T^* g = f$ and establishes the surjectivity of T^*.

We are now able to show that any Radon measure satisfying a weighted Frostman condition is the divergence of a vector field in $L^{\infty}_{1/w}$.

Theorem 3.21. *Let w be a 1-admissible weight. If μ is a (nonnegative) Radon measure verifying condition (3.5) above, then there exists $v \in L^\infty_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$ such that, for any $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$, one has:*

$$\langle \mathrm{div}\, v, \varphi \rangle = \int_{\mathbb{R}^n} \varphi \, d\mu;$$

in particular, $\mathrm{div}\, v = \mu$ holds on $\mathbb{R}^n$, in the distributional sense.

Proof: it is inspired by [30, Theorem 3.3]. Denote by X the space $\mathrm{Lip}_c(\mathbb{R}^n)$ endowed with the norm $\|\varphi\|_X := \|\nabla \varphi\|_{1,w}$, and define an operator

$$T \colon X \to L^1_w(\mathbb{R}^n, \mathbb{R}^n), u \mapsto -\nabla u.$$

Since T is clearly isometric, its adjoint operator

$$\mathrm{div} \colon [L^1_w(\mathbb{R}^n, \mathbb{R}^n)]^* = L^\infty_{1/w}(\mathbb{R}^n, \mathbb{R}^n) \to X^*$$

is surjective. Yet Proposition 3.18 ensures that we have $\mu \in X^*$. The proof is complete.

Remark 3.22. To obtain the isometric isomorphism between $[L^1_w(\mathbb{R}^n, \mathbb{R}^n)]^*$ and $L^\infty_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$, it suffices to notice that given $f \in [L^1_w(\mathbb{R}^n, \mathbb{R}^n)]^*$, the formula $v \mapsto \langle f, \frac{v}{w} \rangle$ defines a bounded linear map on $L^1(\mathbb{R}^n, \mathbb{R}^n)$ with the same norm as f. Hence there exists $g \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\|g\|_\infty = \|f\|$ such that for any $u \in L^1_w(\mathbb{R}^n, \mathbb{R}^n)$ one has

$$f(u) = \int_{\mathbb{R}^n} u \cdot gw.$$

Yet the function $h := gw$ belongs to $L^\infty_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$ and $\|h\|_{L^\infty_{1/w}} = \|f\|$.

We now have a necessary condition on a compact set S for it to be $L^\infty_{1/w}$-removable.

Theorem 3.23. *Assume that w is a 1-admissible weight that satisfies condition (3.1). If the compact set $S \subset \mathbb{R}^n$ is $L^\infty_{1/w}$-removable for $\mathrm{div}\, v = 0$, then $\mathscr{H}^h_\infty(S) = 0$.*

Proof: to show this, assume that one has $\mathscr{H}^h_\infty(S) > 0$. According to the above weighted version of Frostman's lemma (Lemma 3.2), there exists a nontrivial Radon measure μ on $\mathbb{R}^n$ supported in S and satisfying:

$$\mu(B(x,r)) \leqslant Ch(B(x,r)),$$

for any $x \in \mathbb{R}^n$ and $r > 0$. Since Theorem 3.21 ensures the existence of a vector field $v \in L^\infty_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$ such that $\mathrm{div}\, v = \mu$, we get $\mathrm{div}\, v \neq 0$ while $\langle \mathrm{div}\, v, \varphi \rangle = 0$ holds in case $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$ satisfies $\mathrm{supp}\, \varphi \cap S = \emptyset$. Hence S cannot be $L^\infty_{1/w}$-removable for $\mathrm{div}\, v = 0$.

According to Lemma 3.4, Theorems 3.17 and 3.23 give a complete characterization of $L^\infty_{1/w}$-removable (compact) subsets of $\mathbb{R}^n$ for the equation $\mathrm{div}\, v = 0$, in case w satisfies (3.1).

Theorem 3.24. *Assume that w is a 1-admissible weight and that w satisfies condition (3.1). Let $S \subset \mathbb{R}^n$ be compact. Under those assumptions, S is $L^\infty_{1/w}$-removable for $\mathrm{div}\, v = 0$ if and only if one has $\mathscr{H}^h(S) = 0$.*

Remark 3.25. The reader may wonder why we did not use the capacity theory to establish Theorem 3.24. Indeed, recall that in the classical unweighted case, $\mathscr{H}^{n-1}$, Hausdoff measure of dimension $n-1$ and 1-capacity cap_1 (defined by (4.2) below) have same zero sets, see [14, Theorem 3, p. 193]. A corresponding result holds in the setting of metric spaces: by [20, Theorem 3.5] Hausdorff content of codimension 1 and 1-capacity are comparable for compact sets. The reason why we cannot use this result here is that, in [20], admissible functions for 1-capacity belong to the Sobolev space *defined using weak upper gradients*, and it is not known if $|\nabla u|$ is a 1-minimal weak upper gradient of each locally Lipschitz function u, see [6, Appendix A2].

Before discussion on L^p-analogues of the previous results, let us mention an interesting example.

Example 3.26. Given a cube Q, we denote by $\sigma(Q)$ the length of its edges. For $0 < \lambda < 1/2$, we let $\mathscr{E}(Q, \lambda)$ stand for the collection of 2^n cubes contained in Q whose edges have length $\lambda\sigma(Q)$, arranged in such a way that each cube of $\mathscr{E}(Q, \lambda)$ has a common vertex with Q.

Fix a sequence $\lambda = (\lambda_k)_{k \geqslant 1} \subseteq (0, 1/2)$. Write $E_0 := [0,1]^n$ and $\mathscr{E}_0 = \{E_0\}$. For each $k \geqslant 1$, define inductively

$$\mathscr{E}_k(\lambda_1, \ldots, \lambda_k) := \cup\{\mathscr{E}(Q, \lambda_k) : Q \in \mathscr{E}_{k-1}(\lambda_1, \ldots, \lambda_{k-1})\}$$

and $E_k(\lambda_1, \ldots, \lambda_k) := \cup\mathscr{E}_k(\lambda_1, \ldots, \lambda_k)$ (or briefly $\mathscr{E}_k$ and E_k when the underlying sequence is clear). In particular, each cube in $\mathscr{E}_k$ has edges of length $\sigma_k := \lambda_1 \ldots \lambda_k$. We finally let

$$E(\lambda) = \bigcap_{k=1}^{\infty} E_k.$$

Associated to λ, we choose a nondecreasing function $h \colon [0, \infty) \to [0, \infty)$ satisfying $h(0) = 0$ and $h(\sigma_k) = 2^{-kn}$ for each $k \geqslant 1$. Recall that the Hausdorff h-measure of $E(\lambda)$ is positive and finite (see [23, Section 4.9]).

If for example we fix a real number $0 < s < n$ and take $\lambda_k = 2^{-\frac{n}{s}}$ for each k, this yields $\sigma_k = 2^{-\frac{kn}{s}}$ for each k, and we may take $h(t) = t^s$. Hence in this case the set $E := E(\lambda)$ has Hausdorff dimension s and is s-Ahlfors regular according to [22, Theorem 8.3.2] (see also [27] and [17]). It hence follows from [5, Theorem 7] that for any $0 \leqslant \gamma < 1$, the map w_γ defined for a.e. $x \in \mathbb{R}^n$ by

$$w_\gamma(x) := \operatorname{dist}(x, E)^{\gamma(s-n)},$$

is an A_1-weight.

Observing now that E_k consists of 2^{kn} cubes of side length σ_k, compute, for any one of those cubes Q and $B \supseteq Q$, the smallest ball containing Q having the same center:

$$\frac{1}{|B|} \int_B w_\gamma \leqslant A_\gamma \operatorname*{ess\,inf}_B w_\gamma,$$

where $A_\gamma > 0$ is a constant associated with the A_1-property of w_γ. Since the set:

$$\left\{ x \in B : \operatorname{dist}(x, E) \geqslant \frac{1}{4}\left(\frac{\sigma_k}{2} - \sigma_{k+1}\right) \right\}$$

has positive Lebesgue measure, this yields:

$$\frac{1}{|B|} \int_B w_\gamma \leqslant A_\gamma 4^{\gamma(n-s)} \left(\frac{\sigma_k}{2} - \sigma_{k+1} \right)^{\gamma(s-n)}.$$

Since 2^{kn} of those balls suffice to cover E_k, we compute:

$$\mathscr{H}^h(E_k) \leqslant 2^{kn} h(B) = 2^{kn} \cdot \frac{\sqrt{2}}{\sqrt{n}\sigma_k} \int_B w_\gamma$$

$$\leqslant C(n,\gamma,s) 2^{kn} \sigma_k^{n-1} \left(\frac{\sigma_k}{2} - \sigma_{k+1} \right)^{\gamma(s-n)}$$

$$= C(n,\gamma,s) 2^{kn} \sigma_k^{n-1+\gamma(s-n)} \left(\frac{1}{2} - 2^{-\frac{n}{s}} \right)^{\gamma(n-s)}$$

$$= C'(n,\gamma,s) 2^{-k\left[(1-\gamma)\frac{n^2}{s} - \frac{n}{s} + (\gamma-1)n \right]}.$$

The last expression tends to zero for $k \to \infty$, provided that one has:

$$(3.6) \qquad\qquad s < n - \frac{1}{1-\gamma}.$$

This yields $\mathscr{H}^h(E) = 0$ if inequality (3.6) is satisfied. In particular, the set E is then L^∞_{1/w_γ}-removable for the equation $\operatorname{div} v = 0$ while we have $w_\gamma = +\infty$ on E, which is a "large" set in the sense that it has positive Hausdorff dimension (see the introduction).

4. The case of weighted L^p vector fields

In this section, we study the removability question for the equation $\operatorname{div} v = 0$ for $L^p_{1/w}$-vector fields, that is measurable functions $v : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\|v\|_{L^p_{1/w}} := \left(\int_{\mathbb{R}^n} |v(x)|^p \frac{1}{w(x)} dx \right)^{1/p} < +\infty,$$

$1 < p < +\infty$. In Theorems 4.17 and 4.18, we give a characterization for removability by showing that, under suitable assumptions on the weight w, a compact set S is $L^p_{1/w}$-removable for $\operatorname{div} v = 0$ if and only if $\operatorname{Cap}_{p'}^{w^{p'-1}}(S) = 0$. Before that, we recall the definition of weighted Sobolev spaces, different capacities and prove some lemmas. In the whole section, p and p' are conjugate exponents satisfying $1/p + 1/p' = 1$.

Remark 4.1. It follows from the Hölder inequality that if $w \in L^{p'-1}_{\mathrm{loc}}(\mathbb{R}^n)$, then the distributional divergence of $v \in L^p_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$ is well defined by the formula:

$$\langle \operatorname{div} v, \varphi \rangle := - \int_{\mathbb{R}^n} v \cdot \nabla \varphi,$$

for $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n)$.

Removability in the L^p-case is defined similarly as for L^∞-vector fields.

Definition 4.2. Let $S \subset \mathbb{R}^n$ be compact and $1 < p < \infty$. Let w be a weight. The set S is *L_w^p-removable with for the equation* div $v = 0$ (in the distributional sense) if for any $v \in L_w^p(\mathbb{R}^n, \mathbb{R}^n)$, the validity of the equality

$$\text{(4.1)} \qquad\qquad\qquad\qquad \langle \text{div}\, v, \varphi \rangle = 0$$

for any $\varphi \in \text{Lip}_c(\mathbb{R}^n)$ with supp $\varphi \cap S = \emptyset$ implies that (4.1) also holds for any $\varphi \in \text{Lip}_c(\mathbb{R}^n)$.

Remark 4.3. Given $1 < p < \infty$ and an A_p-weight w, we can again (as in Remark 3.11) rely on [34, Corollary 2.1.5] (see [24, Lemma 2.4] for a proof in this case) to claim that the above definition remains unchanged in case $\text{Lip}_c(\mathbb{R}^n)$ is replaced by $C_c^\infty(\mathbb{R}^n)$.

4.1. **Weighted Sobolev spaces.** There are two ways to define weighted Sobolev spaces: using distributional derivatives or the closure of smooth compactly supported functions in the weighted Sobolev norm. The space $W_w^{1,p}(\mathbb{R}^n)$ consists of functions $u \in L_w^p(\mathbb{R}^n)$ whose distributional derivatives of order one belong to $L_w^p(\mathbb{R}^n)$. The space is equipped with norm

$$\|u\|_{W_w^{1,p}(\mathbb{R}^n)} := \|u\|_{p,w} + \|\nabla u\|_{p,w},$$

where ∇u is the distributional gradient of u. The space $H_w^{1,p}(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ under the norm $\|\cdot\|_{W_w^{1,p}(\mathbb{R}^n)}$. For A_p-weights, $1 < p < \infty$, H and W definitions of weighted Sobolev spaces give the same space by [19, Theorem 2.5].

4.2. **Potentials and capacities.** We need three different capacities: Sobolev capacity and the weighted versions of Riesz and Bessel capacity. The capacity theory in weighted Sobolev spaces has been developed to study nonlinear potential theory, see [16], [2] and [34]. For the properties of Riesz and Bessel potentials, we refer, in the classical case to [3], [31, Chapter V] and [35] and in the weighted case to [2], [4], [34]. As in the unweighted Euclidean space, Riesz and Bessel potentials and the corresponding capacities are closely related to Sobolev spaces.

Definition 4.4. Let w be a weight and let $1 \le p < \infty$. The *weighted Sobolev p-capacity* of a set E is

$$\text{Cap}_p^w(E) := \inf \left\{ \int_{\mathbb{R}^n} |\varphi|^p w + \int_{\mathbb{R}^n} |\nabla \varphi|^p w \right\},$$

where the infimum is taken over all functions $\varphi \in \mathcal{A}(E)$, where we let:

$$\mathcal{A}(E) := \left\{ \varphi \in C_c^\infty(\mathbb{R}^n) : \varphi \ge 1 \text{ in a neighbourhood of } E \right\},$$

where as usual $C_c^\infty(\mathbb{R}^n)$ denotes the space of all compactly supported smooth functions in $\mathbb{R}^n$.

Remark 4.5. Assume that E is compact and that $w \in A_p$. We claim that

$$\text{Cap}_p^w(E) = \inf \left\{ \int_{\mathbb{R}^n} |\varphi|^p w + \int_{\mathbb{R}^n} |\nabla \varphi|^p w \right\},$$

where the infimum is taken over all functions $\varphi \in \mathcal{B}(E)$, defined by

$$\mathcal{B}(E) := \left\{ \varphi \in C_c^\infty(\mathbb{R}^n) : 0 \leqslant \varphi \leqslant 1 \text{ in } \mathbb{R}^n \text{ and } \varphi = 1 \text{ in a neighbourhood of } E \right\}.$$

Indeed, let $\varphi \in \mathcal{A}(E)$. Define $\psi := \max(0, \inf(\varphi, 1))$. The function ψ belongs to $\mathrm{Lip}_c(\mathbb{R}^n)$ with $|\nabla \psi| \leqslant |\nabla \varphi|$ almost everywhere, satisfies $0 \leqslant \psi \leqslant 1$ in $\mathbb{R}^n$ and $\psi = 1$ in a neighbourhood of E and

$$\int_{\mathbb{R}^n} |\psi|^p w + \int_{\mathbb{R}^n} |\nabla \psi|^p w \leqslant \int_{\mathbb{R}^n} |\varphi|^p w + \int_{\mathbb{R}^n} |\nabla \varphi|^p w.$$

According to [34, Corollary 2.1.5] (see also [24, Lemma 2.4] for the proof in our case where $p > 1$) and since ψ is compactly supported, for all $\varepsilon > 0$, there exists a function ψ_ε belonging to $C_c^\infty(\mathbb{R}^n)$ (obtained by mollification of ψ with a suitable convolution kernel), satisfying $0 \leqslant \psi_\varepsilon \leqslant 1$ in $\mathbb{R}^n$ and $\psi_\varepsilon = 1$ in a neighbourhood of E, such that

$$\|\psi_\varepsilon\|_{p,w}^p \leqslant \varepsilon + \|\psi\|_{p,w}^p \quad \text{and} \quad \|\nabla \psi_\varepsilon\|_{p,w}^p \leqslant \varepsilon + \|\nabla \psi\|_{p,w}^p,$$

which proves the claim, for we hence get:

$$\int_{\mathbb{R}^n} |\psi_\varepsilon|^p w + \int_{\mathbb{R}^n} |\nabla \psi_\varepsilon|^p w \leqslant 2\varepsilon + \int_{\mathbb{R}^n} |\varphi|^p w + \int_{\mathbb{R}^n} |\nabla \varphi|^p w.$$

Remark 4.6. Assume that $1 < p < n$ is a real number and that w is a p-admissible weight. According to [16, Corollary 2.39], the weighted Sobolev capacity Cap_p^w has the same zero sets as the capacity cap_p^w defined by

$$(4.2) \qquad \mathrm{cap}_p^w(E) := \inf \left\{ \int_{\mathbb{R}^n} |\nabla \varphi|^p w \right\},$$

where the infimum is taken over all functions $\varphi \in \mathcal{A}(E)$.

Riesz potential and capacity. Let $0 < R \leqslant \infty$. The $(R$-truncated$)$ Riesz potential (of order 1) of a nonnegative, locally integrable function u is the function $I_{1,R}u \colon \mathbb{R}^n \to [0, \infty]$ given by

$$I_{1,R}u(x) := \int_{B(x,R)} \frac{u(y)}{|x - y|^{n-1}} \, dy,$$

and the $(R$-truncated$)$ Riesz potential of a measure $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ is $I_{1,R}\mu \colon \mathbb{R}^n \to [0, \infty]$, given by

$$I_{1,R}\mu(x) := \int_{B(x,R)} \frac{1}{|x - y|^{n-1}} \, d\mu(y).$$

Taking $R = \infty$ gives the usual Riesz potential as defined in [3] and [34] (note that usually the integrals above are multiplied by a constant $\gamma(n)$; since we are not interested in the exact values of potentials and capacities, we omit the constant).

Definition 4.7. Let w be a weight and let $1 < p < \infty$. For $0 < R \leqslant \infty$, we define the *$(R$-truncated$)$ weighted Riesz capacity* of a set $E \subset \mathbb{R}^n$ by

$$R_{1,p;R}^w(E) := \inf \left\{ \|f\|_{p,w}^p : f \geq 0, \ I_{1,R}f \geq 1 \text{ on } E \right\},$$

and we let $R_{1,p}^w(E) := R_{1,p;\infty}^w(E)$.

If $I_{1,R}\mu \in L_w^p(\mathbb{R}^n)$, then we say that measure μ has *a finite weighted $(p, w; R)$-energy* (or finite weighted (p, w)-energy in case $R = \infty$).

Bessel potential and capacity. The Bessel kernel G_1 is the tempered distribution whose Fourier transform is $\hat{G}_1(x) = (1 + |x|^2)^{-1/2}$. It is actually a function with the same singularity at 0 as the Riesz kernel $I_1(x) = |x|^{1-n}$, but has more rapid decay at infinity. Similarly as for Riesz potentials, the Bessel potential of a measurable function u is convolution $\mathcal{G}_1 u = G_1 * u$ and of measure $\mu \in \mathcal{M}_+(\mathbb{R}^n)$, $\mathcal{G}_1 \mu = G_1 * \mu$.

Riesz potentials are easier to handle but Bessel potentials have better mapping properties. The most important properties for us are the inequality

$$(4.3) \qquad\qquad 0 < G_1(x) < C I_1(x)$$

for all $x \in \mathbb{R}$, the fact that

$$G_1(x) = I_1(x) + o(I_1(x))$$

as $x \to 0$ (for these two facts, see [34, Section 3.1.2]), and the following Theorem from [24, Theorem 3.3], [29] which says that the Bessel potential space equals the weighted Sobolev space for A_p-weights.

Theorem 4.8. *Let $1 < p < \infty$. Let w be an A_p-weight. A function u belongs to $H_w^{1,p}(\mathbb{R}^n)$ if and only if there is a function $f \in L_w^p(\mathbb{R}^n)$ such that $u = \mathcal{G}_1 f$. Moreover, there is a constant $C = C(n, p, w) > 0$ such that*

$$\frac{1}{C}\|f\|_{p,w} \le \|u\|_{H_w^{1,p}(\mathbb{R}^n)} \le C\|f\|_{p,w}.$$

Definition 4.9. *Let $1 < p < \infty$. Let w be a weight. The weighted Bessel capacity of a set $E \subset \mathbb{R}^n$ is*

$$B_{1,p}^w(E) = \inf\left\{\|f\|_{p,w}^p : f \ge 0,\ \mathcal{G}_1 f \ge 1 \text{ on } E\right\}.$$

4.3. **Connection between different capacities.** In the proof of Theorem 4.18, we need the property that weighted Sobolev, Bessel and Riesz capacities have same zero sets. Connection between Sobolev and Bessel capacities is much stronger; Theorem 4.8 implies that the weighted Bessel and Sobolev capacities for an A_p-weight are comparable. For a proof, see [34, Theorem 3.5.2].

Theorem 4.10. *Let $1 < p < \infty$. Let w be an A_p-weight. There is a constant $C > 0$ such that*

$$\frac{1}{C}\operatorname{Cap}_p^w(K) \le B_{1,p}^w(K) \le C\operatorname{Cap}_p^w(K)$$

for all compact sets $K \subset \mathbb{R}^n$.

Concerning Riesz capacities, we have, by (4.3), that

$$R_{1,p}^w(E) \le C B_{1,p}^w(E)$$

for all sets $E \subset \mathbb{R}^n$. In the unweighted case, Bessel and Riesz capacities have same sets of finite capacity by [1, Theorem 1]. This Theorem says that if $p < n$, then

$$B_{1,p}(E) \le C\left(R_{1,p}(E) + R_{1,p}(E)^{n/(n-p)}\right).$$

In the weighted case, Bessel and Riesz capacities are locally comparable if the A_p-weight satisfies an additional integrability condition, see [4, Lemma 20]. The following lemma (resulting from [34, Theorem 3.3.7 and Lemma 3.3.8]) together with Theorem 4.10 implies that for A_p-weights, weighted Sobolev, Bessel and (localized) Riesz capacities are comparable.

Lemma 4.11. *Let $1 < p < \infty$ and $R > 0$ be real numbers. Let w be an A_p-weight. There is a constant $C_R > 0$ such that*

$$\frac{1}{C_R} R^w_{1,p;R}(K) \leq B^w_{1,p}(K) \leq C_R\, R^w_{1,p;R}(K),$$

for all compact sets $K \subset \mathbb{R}^n$.

Remark 4.12. We could use the general theory for L^q-capacities and dual definitions of capacity from [3, Chapter 2] for weighted Riesz capacities, see also [2] and [34, Chapter 3]. We do not repeat the details of the theory because the only property we need is that for each (compact) set $S \subseteq B(0,R) \subseteq \mathbb{R}^n$ of positive weighted $(3R)$-localized Riesz capacity, there is a nonnegative non-zero measure supported in S with finite energy.

Indeed, by [3, Theorem 2.5.3], for each compact set $K \subset \mathbb{R}^n$, there is a measure $\mu_K \in \mathcal{M}_+(K)$ such that

$$(4.4) \qquad \mathrm{Cap}_{g,q}(K) = \mu_K(K) = \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} g(x,y)\, d\mu_K(x) \Big)^{q'} d\nu(y)$$

where $1 < q < \infty$, $1/q + 1/q' = 1$, capacity $\mathrm{Cap}_{g,q}$ is defined using a kernel function $g(x,y)$ and ν is a positive measure in $\mathbb{R}^n$.

Let $S \subseteq B(0,R)$ be a compact set with $\mathrm{Cap}_{p'}^{w^{p'-1}}(S) > 0$. A non-zero measure supported in S with finite $(p, 1/w; 3R)$ energy is obtained using (4.4) and selecting $q' = p$, because for

$$d\nu = w^{p'-1}\, dy \quad \text{and} \quad g(x,y) = \frac{\chi_{B(0,3R)}(x-y)}{|x-y|^{n-1}} w(y)^{1-p'},$$

we have

$$\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} g(x,y)\, d\mu(x) \Big)^{p} d\nu(y)$$
$$= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \frac{\chi_{B(0,3R)}(x-y)}{|x-y|^{n-1}} w(y)^{1-p'}\, d\mu(x) \Big)^{p} w(y)^{p'-1}\, dy$$
$$= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \frac{\chi_{B(0,3R)}(x-y)}{|x-y|^{n-1}}\, d\mu(x) \Big)^{p} w(y)^{(1-p')p-1+p'}\, dy$$
$$= \int_{\mathbb{R}^n} \big(I_{1,3R}\mu \big)^{p} w(y)^{-1}\, dy.$$

4.4. **Removable singularities.** The following result is a counterpart of Proposition 3.18 in the L^p-case. In the unweighted case, compare it with [30, Theorem 3.2].

Proposition 4.13. *Let w be a weight and let $1 < p < \infty$ and $R > 0$ be real numbers. Assume that $w^{p'-1}$ is p'-admissible. If $\mu \in \mathcal{M}_+(B(0,R))$ has a finite $(p, 1/w; 3R)$-energy, then*

$$\int_{\mathbb{R}^n} |\varphi| \, d\mu \leq C(n) \|I_{1,3R}\mu\|_{p,w^{-1}} \|\nabla\varphi\|_{p',w^{p'-1}}$$

for each $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n)$.

Proof: assume first that $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n)$ satisfies $\operatorname{supp}\varphi \subseteq B(0,2R)$. Fix $x \in B(0,R)$. Since for any $y \in B(0,R)$ one has $x - y \in B(0,3R)$, and since, according to [31, (18) p. 125], there is a constant $C = C(n) > 0$ such that, for all $x \in \mathbb{R}^n$,

$$\varphi(x) = C(n) \int_{\mathbb{R}^n} \frac{\nabla\varphi(y) \cdot (x-y)}{|x-y|^n} \, dy,$$

we get, using the fact that φ vanishes outside $B(0,2R)$ that, for all $x \in \mathbb{R}^n$,

$$\varphi(x) = C(n) \int_{\mathbb{R}^n} \frac{\nabla\varphi(y) \cdot (x-y)}{|x-y|^n} \chi_{B(0,3R)}(x-y) \, dy.$$

Hence we have, using the Fubini Theorem and the Hölder inequality,

$$\int_{\mathbb{R}^n} |\varphi| \, d\mu \leq C(n) \int_{\mathbb{R}^n} I_{1,3R}|\nabla\varphi| \, d\mu = C(n) \int_{\mathbb{R}^n} |\nabla\varphi| I_{1,3R}\mu$$

$$\leq C(n) \left(\int_{\mathbb{R}^n} |\nabla\varphi|^{p'} w^{p'-1} \right)^{1/p'} \left(\int_{\mathbb{R}^n} (I_{1,3R}\mu)^p w^{-1} \right)^{1/p},$$

and the claim follows in case one has $\operatorname{supp}\varphi \subseteq B(0,2R)$.

Given a general $\varphi \in \operatorname{Lip}_c(\mathbb{R}^n)$, choose $\chi \in \operatorname{Lip}_c(\mathbb{R}^n)$ satisfying $\chi_{B(0,R)} \leq \chi \leq \chi_{B(0,2R)}$ as well as $\|\nabla\varphi\|_\infty \leq 2/R$. According to the computations before and to the fact that $\operatorname{supp}\mu \subseteq B(0,R)$, we know that one has:

$$\int_{\mathbb{R}^n} |\varphi| \, d\mu = \int_{\mathbb{R}^n} |\varphi\chi| \, d\mu \leq C(n) \|I_{1,3R}\mu\|_{p,w^{-1}} \|\nabla(\varphi\chi)\|_{p',w^{p'-1}}.$$

Since we have $\nabla(\varphi\chi) = \chi\nabla\varphi + \varphi\nabla\chi$ a.e. on $\mathbb{R}^n$ and since $\chi = 0$ outside $B(0,2R)$, we get:

$$\|\nabla(\varphi\chi)\|_{p',w^{p'-1}} \leq \|\nabla\varphi\|_{p',w^{p'-1}} + \frac{2}{R} \left(\int_{B(0,2R)} |\varphi|^{p'} w^{p'-1} \right)^{1/p'}.$$

Yet $w^{p'-1}$ being p'-admissible, we get:

$$\int_{B(0,2R)} |\varphi|^{p'} w^{p'-1} \leq \kappa(2R)^{p'} \int_{\mathbb{R}^n} |\nabla\varphi|^{p'} w^{p'-1},$$

where κ is a constant associated to the p'-admissibility of the weight $w^{p'-1}$ according to Poincaré inequality (2.2). This finally gives rise to the following inequality:

$$\|\nabla(\varphi\chi)\|_{p',w^{p'-1}} \leq (1 + 4\kappa^{1/p'}) \|\nabla\varphi\|_{p',w^{p'-1}}$$

and the proof is complete.

Remark 4.14. We have assumed that $w^{p'-1} \in A_{p'}$. By the definition of conjugate exponents $1/p + 1/p' = 1$ and A_q-weights, this means that

$$\sup_B \left(\fint_B w^{p'-1} \right) \left(\fint_B w^{-1} \right)^{1/(p'-1)} < \infty$$

and hence that $1/w$ is locally integrable on $\mathbb{R}^n$.

A counterpart of Theorem 3.21 in the L^p-case says that any nonnegative Radon measure with finite $(p, 1/w)$-energy is the divergence of a vector field in $L^p_{1/w}$.

Theorem 4.15. *Let $w \in L^{p'-1}_{\text{loc}}(\mathbb{R}^n)$ be a weight and let $R > 0$ and $1 < p < \infty$ be real numbers. Assume that $w^{p'-1}$ is p'-admissible. If $\mu \in \mathcal{M}_+(B(0,R))$ is a measure with finite $(p, 1/w; 3R)$-energy, then there exists $v \in L^p_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{div} v = \mu$ in $\mathbb{R}^n$ in the distributional sense.*

Proof: let $X := \operatorname{Lip}_c(\mathbb{R}^n)$ be endowed with the norm

$$\|\varphi\|_X = \|\nabla \varphi\|_{p', w^{p'-1}}$$

(notice that it is clear that this norm is well defined), and let $T \colon X \to L^{p'}_{w^{p'-1}}(\mathbb{R}^n, \mathbb{R}^n)$, $Tu = -\nabla u$. Since T is a linear isometry, Lemma 3.19 implies that the adjoint operator

$$T^* \colon \left(L^{p'}_{w^{p'-1}}(\mathbb{R}^n, \mathbb{R}^n) \right)^* \to X^*$$

is surjective. A standard argument of functional analysis recalled below shows that $\left(L^{p'}_{w^{p'-1}}(\mathbb{R}^n, \mathbb{R}^n) \right)^* = L^p_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$. Since

$$\langle T^* v, \varphi \rangle = \langle v, T\varphi \rangle = -\int_{\mathbb{R}^n} v \cdot \nabla \varphi$$

for all $\varphi \in X$, we have that $T^* = \operatorname{div}$ in the distributional sense. The claim follows because $\mu \in X^*$ by Proposition 4.13.

Remark 4.16. To obtain the isometric isomorphism $[L^{p'}_{w^{p'-1}}(\mathbb{R}^n, \mathbb{R}^n)]^* = L^p_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$, it suffices to notice that given $f \in [L^{p'}_{w^{p'-1}}(\mathbb{R}^n, \mathbb{R}^n)]^*$, the formula $v \mapsto \langle f, \frac{v}{w^{1/p}} \rangle$ defines a bounded linear map on $L^{p'}(\mathbb{R}^n, \mathbb{R}^n)$ with the same norm as f. Hence there exists $g \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ with $\|g\|_{L^p(\mathbb{R}^n)} = \|f\|$ such that for any $u \in L^{p'}_{w^{p'-1}}(\mathbb{R}^n, \mathbb{R}^n)$ one has

$$\langle f, u \rangle = \int_{\mathbb{R}^n} u \cdot g w^{1/p}.$$

Yet the function $h := g w^{1/p}$ belongs to $L^p_{1/w}(\mathbb{R}^n, \mathbb{R}^n)$ and $\|h\|_{L^p_{1/w}} = \|f\|$.

Theorem 4.17. *Let $1 < p < \infty$ be a real number. Let $w \in L^{p'-1}_{\text{loc}}(\mathbb{R}^n)$ be a weight such that $w^{p'-1} \in A_{p'}$. Let $S \subset \mathbb{R}^n$ be a compact set. If $\operatorname{Cap}^{w^{p'-1}}_{p'}(S) = 0$, then S is $L^p_{1/w}$-removable for*

$$(4.5) \qquad\qquad\qquad \operatorname{div} v = 0.$$

Proof: since $\mathrm{Cap}_{p'}^{w^{p'-1}}(S) = 0$, by Remark 4.5, there are functions $u_k \in C_0^\infty(\mathbb{R}^n)$, $k \in \mathbb{N}$, such that $0 \le u_k \le 1$ and $u_k = 1$ in a neighborhood of S for all k, $\|\nabla u_k\|_{p', w^{p'-1}} \to 0$ and $u_k \to 0$ as $k \to \infty$ almost everywhere. Let $v \in L_{1/w}^p(\mathbb{R}^n, \mathbb{R}^n)$ be a solution of (4.5) in $\mathbb{R}^n \setminus S$. Let $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$. We have to show that

$$\int_{\mathbb{R}^n} v \cdot \nabla \varphi = 0.$$

Since div $v = 0$ outside S and $(1 - u_k)\varphi \in \mathrm{Lip}_c(\mathbb{R}^n \setminus S)$, we have

$$\int_{\mathbb{R}^n} v \cdot \nabla \varphi = \int_{\mathbb{R}^n} v \cdot \nabla((1 - u_k)\varphi) + \int_{\mathbb{R}^n} v \cdot \nabla(u_k \varphi)$$

$$= \int_{\mathbb{R}^n} v \cdot \nabla(u_k \varphi).$$

Hence, using the Hölder inequality, we obtain

$$\left| \int_{\mathbb{R}^n} v \cdot \nabla \varphi \right| \le \int_{\mathbb{R}^n} |v \cdot \nabla(u_k \varphi)|$$

$$\le \|v\|_{p, w^{-1}} \Big(\|u_k \nabla \varphi\|_{p', w^{p'-1}} + \|\varphi \nabla u_k\|_{p', w^{p'-1}} \Big) \to 0$$

as $k \to \infty$ by the properties of functions u_k and φ. This shows that S is removable for div $v = 0$.

Theorem 4.18. *Let $1 < p < \infty$ be a real number. Let $w \in L_{\mathrm{loc}}^{p'-1}(\mathbb{R}^n)$ be a weight such that $w^{p'-1} \in A_{p'}$. Let $S \subset \mathbb{R}^n$ be a compact set. If S is $L_{1/w}^p$-removable for div $v = 0$, then $\mathrm{Cap}_{p'}^{w^{p'-1}}(S) = 0$.*

Proof: since S is compact, there exists $R > 0$ such that $S \subseteq B(0, R)$.

If $\mathrm{Cap}_{p'}^{w^{p'-1}}(S) > 0$, then, by Theorem 4.10 and Lemma 4.11, $R_{1, p'; 3R}^{w^{p'-1}}(S) > 0$. Hence, by Remark 4.12, there is a nonnegative non-zero measure μ_S supported in S such that:

$$\int_{\mathbb{R}^n} \big(I_{1, 3R} \mu_S\big)^p w(y)^{-1} \, dy < +\infty.$$

Theorem 4.15 implies that there exists $v \in L_{1/w}^p(\mathbb{R}^n, \mathbb{R}^n)$ such that div $v = \mu_S$ in $\mathbb{R}^n$ in the distributional sense. This shows that S is not removable for div $v = 0$. Hence we have that $\mathrm{Cap}_{p'}^{w^{p'-1}}(S) = 0$.

In summary, we proved the following result.

Theorem 4.19. *Let $1 < p < \infty$ be a real number. Let $w \in L_{\mathrm{loc}}^{p'-1}(\mathbb{R}^n)$ be a weight such that $w^{p'-1} \in A_{p'}$. Let $S \subset \mathbb{R}^n$ be a compact set. Under those assumptions, the set S is $L_{1/w}^p$-removable for div $v = 0$ if and only if one has $\mathrm{Cap}_{p'}^{w^{p'-1}}(S) = 0$.*

References

[1] David R. Adams. Quasi-additivity and sets of finite L^p-capacity. *Pacific J. Math.* 79 (2): 283–291, 1978.

[2] David. R. Adams. Weighted nonlinear potential theory. *Trans. Amer. Math. Soc.* 297 (1): 73–94, 1986.

[3] David R. Adams and Lars Inge Hedberg. Function spaces and potential theory. Grundlehren der Mathematischen Wissenschaften, 314. Springer-Verlag, Berlin, 1996.

[4] Hiroaki Aikawa. On weighted Beppo Levi functions–integral representations and behavior at infinity. *Analysis* 9 (4): 323–346, 1989.

[5] Hugo Aimar, Marilina Carena, Ricardo Durán and Marisa Toschi. Powers of distances to lower dimensional sets as Muckenhoupt weights. *Acta Math. Hungar.* 143 (1): 119–137, 2014.

[6] Anders Björn and Jana Björn. Nonlinear potential theory on metric spaces. EMS Tracts in Mathematics, 17. European Mathematical Society (EMS), Zürich, 2011.

[7] Jana Björn. Poincaré inequalities for powers and products of admissible weights. *Ann. Acad. Sci. Fenn. Math.* 26 (1): 175–188, 2001.

[8] Jean Bourgain and Haïm Brezis. On the equation $\operatorname{div} Y = f$ and application to control of phases. *J. Amer. Math. Soc.* 16(2): 393–426, 2003.

[9] Christopher Scott Camfield. *Comparison of BV norms in weighted Euclidean spaces and metric measure spaces*. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–University of Cincinnati.

[10] Sagun Chanillo and Richard Wheeden. Poincaré inequalities for a class of non-A_p weights. *Indiana Univ. Math. J.* 41 (3): 605–623, 1992.

[11] Ronald. R. Coifman and Charles L. Fefferman. Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.* 51: 241–250, 1974.

[12] Thierry De Pauw. On the exceptional sets of the flux of a bounded vectorfield. *J. Math. Pures Appl.* 82 (9): 1191–1217, 2003.

[13] Ricardo Duran, Maria-Amelia Muschietti, Emmanuel Russ, and Philippe Tchamitchian. Divergence operator and Poincaré inequalities on arbitrary bounded domains. *Complex Var. Elliptic Equ.*, 55 (8-10): 795–816, 2010.

[14] Lawrence C. Evans and Ronald F. Gariepy. Measure theory and fine properties of functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[15] Bruno Franchi, Cristian E. Gutiérrez and Richard Wheeden. Weighted Sobolev-Poincaré inequalities for Grushin type operators. *Comm. Partial Differential Equations* 19 (3-4): 523–604, 1994).

[16] Juha Heinonen, Tero Kilpeläinen and Olli Martio. Nonlinear potential theory of degenerate elliptic equations. Oxford Mathematical Monographs, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[17] John E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.* 30 (5): 713–747, 1981.

[18] Peter W. Jones. Factorization of A_p weights. *Ann. of Math. (2)* 111 (3): 511–530, 1980.

[19] Tero Kilpeläinen. Weighted Sobolev spaces and capacity. *Ann. Acad. Sci. Fenn.* 19 (1): 95–113, 1994.

[20] Juha Kinnunen, Riikka Korte, Nageswari Shanmugalingam, and Heli Tuominen. Lebesgue points and capacities via the boxing inequality in metric spaces. *Indiana Univ. Math. J.*, 57 (1): 401–430, 2008.

[21] Juha Kinnunen, Riikka Korte, Nageswari Shanmugalingam, and Heli Tuominen. A characterization of Newtonian functions with zero boundary values. *Calc. Var. Partial Differential Equations*, 43: 507–528, 2012.

[22] John M. Mackay and Jeremy T. Tyson. *Conformal dimension. Theory and application.* University Lecture Series, 54. American Mathematical Society, Providence, RI, 2010.

[23] Pertti Mattila. Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability. Cambridge Studies in Advanced Mathematics 44, Cambridge University Press, Cambridge, 1995.

[24] Nicholas. Miller. Weighted Sobolev spaces and pseudodifferential operators with smooth symbols, *Trans. Amer. Math. Soc.* 269 (1): 91–109, 1982.

[25] Michele Miranda, Jr. Functions of bounded variation on "good" metric spaces. *J. Math. Pures Appl. (9)*, 82 (8): 975–1004, 2003.

[26] Laurent Moonens. Removable singularities for the equation div $v = 0$. *Real Anal. Exchange*, (30th Summer Symposium Conference): 125–132, 2006.

[27] P. A. P. Moran. Additive functions of intervals and Hausdorff measure. *Proc. Cambridge Philos. Soc.* 42: 15–23, 1946.

[28] Benjamin Muckenhoupt. Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* 165: 207–226, 1972.

[29] Esko Nieminen. Hausdorff measures, capacities, and Sobolev spaces with weights. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes* No. 81, 1991.

[30] Nguyen Cong Phuc and Monica Torres: Characterizations of the existence and removable singularities of divergence-measure vector fields. *Indiana Univ. Math. J.* 57 (4): 1573–1597, 2008.

[31] Elias M. Stein. Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970.

[32] Elias M. Stein. Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[33] Miroslav Šilhavý. Divergence measure fields and Cauchy's stress theorem. *Rend. Sem. Mat. Univ. Padova*, 113: 15–45, 2005.

[34] Bengt Ove Turesson. Nonlinear potential theory and weighted Sobolev spaces. Lecture Notes in Mathematics, 1736. Springer-Verlag, Berlin, 2000.

[35] William P. Ziemer. Weakly differentiable functions. Sobolev spaces and functions of bounded variation, Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.

L.M., Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, CNRS UMR8628, Université Paris-Saclay, Bâtiment 425, F-91405 Orsay Cedex, France
Laurent.Moonens@math.u-psud.fr

E.R., Université Joseph Fourier, Institut Fourier, 100 rue des Maths, B.P. 74, F-38402 Saint-Martin-d'Hères, France
Emmanuel.Russ@ujf-grenoble.fr

H.T., Department of Mathematics and Statistics, P.O. Box 35, FI-40014 University of Jyväskylä, Finland
heli.m.tuominen@jyu.fi