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## Research Article

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# Jacobian of weak limits of Sobolev homeomorphisms 

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#### Abstract

Let $\Omega$ be a domain in $\mathbb{R}^{n}$, where $n=2$, 3. Suppose that a sequence of Sobolev homeomorphisms $f_{k}: \Omega \rightarrow \mathbb{R}^{n}$ with positive Jacobian determinants, $J\left(x, f_{k}\right)>0$, converges weakly in $W^{1, p}\left(\Omega, \mathbb{R}^{n}\right)$, for some $p \geqslant 1$, to a mapping $f$. We show that $J(x, f) \geqslant 0$ a.e. in $\Omega$. Generalizations to higher dimensions are also given.


Keywords: Sobolev homeomorphism, weak limits, Jacobian
MSC 2010: Primary 26B10; secondary 46E35

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## 1 Introduction

The main goal of this note is to establish when the sign of the Jacobian is preserved under $W^{1, p^{-}}$-weak convergence. Such a question pops out naturally in the variational approach to Geometric Function Theory (GFT) [2, 14, 22] and Nonlinear Elasticity (NE) [1, 4, 6, 19, 24, 25]. Both theories GFT and NE deal with minimizing sequences of Sobolev homeomorphisms. In the context of NE, one typically deals with twodimensional or three-dimensional models and require that the deformation gradients belong to $M_{+}^{n \times n}$, where $M^{m \times n}=\{$ real $m \times n$ matrices $\}$, and $M_{+}^{n \times n}=\left\{A \in M^{n \times n}: \operatorname{det} A>0\right\}$. The infimum of the energy is not attained, in general, at a homeomorphism; interpenetration of matter may occur. Even in a special case of Dirichlet energy injectivity is often lost when passing to the weak limit of the minimizing sequence, $[3,13,15,16]$. Further examinations are needed to know the properties of such singular minimizers.

Throughout this text $\Omega$ will be a domain in $\mathbb{R}^{n}$. The class of Sobolev mappings $f: \Omega \rightarrow \mathbb{R}^{n}$ with nonnegative Jacobian determinant, $J(x, f)=\operatorname{det} D f(x) \geqslant 0$ almost everywhere, is closed under the weak convergence in $W^{1, p}\left(\Omega, \mathbb{R}^{n}\right)$ provided $p \geqslant n$ (see [14, Theorem 8.4.2]). However, if $p<n$, passing to the weak $W^{1, p}$-limit of a sequence with nonnegative Jacobians one may lose the sign of the Jacobian. Indeed, there exists a sequence of Sobolev mappings $f_{k}: \Omega \rightarrow \mathbb{R}^{n}$ with $J\left(x, f_{k}\right)>0$ almost everywhere such that the sequence converges weakly in $W^{1, p}\left(\Omega, \mathbb{R}^{n}\right), p<n$, to the mapping $f(x)=\left(-x_{1}, x_{2}, \ldots, x_{n}\right)$, see [14, p. 181]. Moreover, following the construction in [18] such mappings $f_{k}$ can be made continuous. However, it is not obvious at all as to whether one can make a similar example with $f_{k}$ being homeomorphisms. This is the subject of our result here. Here $\left[\frac{n}{2}\right]$ denotes the integer part, i.e. $\left[\frac{2}{2}\right]=1,\left[\frac{3}{2}\right]=1$ and so on.

Theorem 1. Let $\Omega \subset \mathbb{R}^{n}$ be a domain and let $p \geqslant 1$ for $n \in\{2,3\}$ and $p>\left[\frac{n}{2}\right]$ for $n \geqslant 4$. Suppose that a sequence of Sobolev homeomorphisms $f_{k}: \Omega \rightarrow \mathbb{R}^{n}$ with $J\left(x, f_{k}\right) \geqslant 0$ converges weakly in $W^{1, p}\left(\Omega, \mathbb{R}^{n}\right)$ to a mapping $f$ and further assume that $J\left(x, f_{k}\right)$ is not a.e. zero. Then $J(x, f) \geqslant 0$ a.e. in $\Omega$.

[^0]It is worth noting that in Theorem 1 the Jacobian $J(x, f)$ can have very different behavior than the Jacobians in the sequence without knowing that $J\left(x, f_{k}\right)>0$ on a set of positive measure. Indeed, there exists a sequence of Sobolev homeomorphisms $f_{k}$ with $J\left(x, f_{k}\right)=0$ a.e., converging weakly in $W^{1, p}\left(\Omega, \mathbb{R}^{n}\right), 1 \leqslant p<n$, to the mapping $f(x)=x$. Let us briefly sketch this using the construction from [10]: we cover $\Omega$ by cubes of diameter less than $\frac{1}{k}$ and on each cube we follow the construction from [10] to obtain a homeomorphism with zero Jacobian a.e. It is possible to make the $W^{1, p}$-norm of the sequence uniformly bounded and hence find a weakly convergent subsequence. Furthermore, it follows from the construction that the sequence $f_{k}$ converges uniformly to the identity. This also shows that there is a sequence with $J\left(x, f_{k}\right)=0$ a.e. converging weakly in $W^{1, p}\left(\Omega, \mathbb{R}^{n}\right)$, $1 \leqslant p<n$, to $f(x)=\left(-x_{1}, x_{2}, \ldots, x_{n}\right)$.

Recently it was shown in [12] and [5] that a Jacobian of a Sobolev homeomorphism can change sign in dimension $n \geq 4$ for $1 \leq p<\left[\frac{n}{2}\right]$.

## 2 Preliminaries

### 2.1 Degree and Jacobian

There are two basic approaches to the notion of local degree for a mapping, the algebraic (see e.g. Dold [7]) and the analytic (see e.g. Lloyd [17]). Both of these notions try to capture the idea of counting the preimages of a target point. For a continuous mapping $f: \Omega \rightarrow \mathbb{R}^{n}$ and $y_{\circ} \in \mathbb{R}^{n} \backslash f(\partial \Omega)$ the degree of $f$ at $y$ 。 with respect to $\Omega$ is denoted by $\operatorname{deg}\left(f, \Omega, y_{0}\right)$. If $f: \Omega \rightarrow \mathbb{R}^{n}$ is a homeomorphism, then $\operatorname{deg}\left(f, \Omega, y_{0}\right)$ is either 1 or -1 for all $y_{\circ} \in f(\Omega)$, see e.g. [17, Section IV.5] or [21, Section II.2.4, Theorem 3]. We say that a homeomorphism $f$ is sense-preserving if $\operatorname{deg}\left(f, \Omega, y_{0}\right) \equiv 1$. For a linear $\operatorname{map} A: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ with $\operatorname{det} A \neq 0$, it is easy to check from the definition that

$$
\begin{equation*}
\operatorname{deg}\left(A, \Omega, y_{\mathrm{o}}\right)=\operatorname{sgn} \operatorname{det} A . \tag{1}
\end{equation*}
$$

We recall the following corollary [2, Corollary 2.8.2]. Given a homeomorphism $f: \Omega \rightarrow \mathbb{R}^{n}$ suppose that $f$ is differentiable at $x_{\circ}$ with $J\left(x_{\circ}, f\right) \neq 0$. Then we have

$$
\begin{equation*}
\operatorname{deg}\left(f, \Omega, f\left(x_{\circ}\right)\right)=\operatorname{sgn} J\left(x_{\circ}, f\right) . \tag{2}
\end{equation*}
$$

We will use the fact that the topological degree is stable under homotopy. That is for every continuous mapping $H: \bar{\Omega} \times[0,1] \rightarrow \mathbb{R}^{n}$ and $y \circ \in \mathbb{R}^{n}$ such that $y \circ \notin H(\partial \Omega, t)$ for all $t \in[0,1]$ we have

$$
\begin{equation*}
\operatorname{deg}\left(H(\cdot, 0), \Omega, y_{\circ}\right)=\operatorname{deg}\left(H(\cdot, 1), \Omega, y_{\circ}\right) . \tag{3}
\end{equation*}
$$

### 2.2 Differentiability of Sobolev mappings

A Sobolev homeomorphism $f \in W^{1, p}\left(\Omega, \mathbb{R}^{n}\right)$ is differentiable almost everywhere if $p>n-1, n \geqslant 3$, and $p \geqslant 1$ for $n=2$, see $[9,20,26]$. We will also need a generalization of the concept of differentiability, which is obtained by replacing the ordinary limit by an approximate limit, see e.g. [8, Section 6.1.3]. It is known that a Sobolev mapping $f \in W_{\text {loc }}^{1,1}\left(\Omega, \mathbb{R}^{n}\right)$ is approximatively differentiable almost everywhere, see e.g. [8, Section 6.1.2, Theorem 2]. Moreover, such a mapping is $L^{1}$-differentiable almost everywhere [27]; that is, for almost every $x_{\circ} \in \Omega$ we have

$$
\begin{equation*}
\lim _{r \rightarrow 0} f_{B\left(x_{0}, r\right)}\left|\frac{f(x)-f\left(x_{\circ}\right)-D f\left(x_{\circ}\right)\left(x-x_{\circ}\right)}{r}\right| \mathrm{d} x=0 . \tag{4}
\end{equation*}
$$

Hereafter, the notation $f_{B\left(x_{0}, r\right)}$ means the integral average over the $n$-dimensional ball

$$
B\left(x_{\circ}, r\right)=\left\{x \in \mathbb{R}^{n}:\left|x-x_{\circ}\right|<r\right\} .
$$

In order to illustrate our ideas and for reader's comprehension, we first prove Theorem 1 in the simpler cases $p \geq 1, n=2$; and $p>n-1, n \geq 3$, where we can avoid some technicalities.

## 3 Proof of Theorem 1 for $p>n-1, n \geqslant 3$, and $p \geqslant 1, n=2$

Each homeomorphism $f_{j}$ is either sense-preserving or sense-reversing. Under our assumptions there exists a point $x_{j}$ such that $f_{j}$ is differentiable at $x_{j}$, see Section 2.2 , and $J\left(x_{j}, f_{j}\right)>0$. By (2) we know that the degree of $f_{j}$ is one and hence each $f_{j}$ is sense-preserving.

As $f_{j} \rightharpoonup f$ in $L^{p}, p>1$, we know that $\int_{\Omega}|D f|^{p}$ is uniformly bounded and hence we can find a Radon measure $\mu$ and a subsequence (which we will denote again as $f_{j}$ ) such that

$$
\left|D f_{j}\right|^{p} \stackrel{w^{*}}{\rightharpoonup} \mu \text { in measures. }
$$

Moreover, for $p=1$ we can use De La Vale Pousin characterization of weak convergence in $L^{1}$ and we can find an continuous convex function $\Phi:[0, \infty) \rightarrow[0, \infty)$ such that

$$
\begin{equation*}
\frac{\Phi(t)}{t} \text { is increasing, } \quad \lim _{t \rightarrow \infty} \frac{\Phi(t)}{t}=\infty \quad \text { and } \quad \int_{\Omega} \Phi\left(\left|D f_{j}\right|\right) \leq 1 \tag{5}
\end{equation*}
$$

It follows that we can find a Radon measure $\mu$ and a subsequence (which we will denote again as $f_{j}$ ) such that

$$
\Phi\left(\left|D f_{j}\right|\right) \stackrel{w^{*}}{\rightharpoonup} \mu \text { in measures. }
$$

It is well known that for almost every $x_{\circ} \in \Omega$ we have

$$
\begin{equation*}
M \mu\left(x_{\circ}\right):=\sup _{r>0} \frac{\mu\left(B\left(x_{\circ}, r\right) \cap \Omega\right)}{\left|B\left(x_{\circ}, r\right)\right|}<\infty \tag{6}
\end{equation*}
$$

Let $\delta>0$. For the contrary we suppose that there is $x_{\circ} \in \Omega$ such that (4) and (6) hold at $x_{\circ}$ and

$$
J\left(x_{\circ}, f\right)<0
$$

Without loss of generality we may and do assume that

$$
D f\left(x_{\circ}\right)=\left(\begin{array}{cccc}
1 & 0 & \ldots & 0  \tag{7}\\
0 & 1 & \ldots & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \ldots & -1
\end{array}\right)
$$

Using (4) we can find $0<r_{1}$ small enough such that for all $0<r<r_{1}$ we have

$$
\oint_{B\left(x_{\mathrm{o}}, r\right)}\left|\frac{f(x)-f\left(x_{\mathrm{o}}\right)-D f\left(x_{\mathrm{o}}\right)\left(x-x_{\mathrm{o}}\right)}{r}\right| \mathrm{d} x<\frac{\delta^{n}}{2}
$$

Since the sequence of mappings $f_{j}$ converges to $f$ weakly in $W^{1, p}\left(\Omega, \mathbb{R}^{n}\right)$, we have that the sequence of mappings $f_{j}$ converges to $f$ strongly in $L_{\text {loc }}^{1}\left(\Omega, \mathbb{R}^{n}\right)$. Now, we may pick up an index $j_{0}$ large enough such that for all $j \geq j_{\text {o }}$,

$$
\int_{\Omega}\left|f(x)-f_{j}(x)\right| \mathrm{d} x<|B(0,1)| r^{n+1} \frac{\delta^{n}}{2}
$$

The last two inequalities imply that for all $0<r<r_{1}$ we have

$$
\begin{equation*}
\int_{B\left(x_{\circ}, r\right)}\left|\frac{f_{j}(x)-f\left(x_{\circ}\right)-D f\left(x_{\circ}\right)\left(x-x_{\circ}\right)}{r}\right| \mathrm{d} x<\delta^{n} \tag{8}
\end{equation*}
$$

Our next goal is to prove the following:
(i) if $p>n-1$, then there exists a constant $C$ (depending only on $p$ and $n$ ) such that for all $0<r<r_{1}$ and $j \geq j_{0}$,

$$
\delta^{n-1-p} r^{n} \leqslant C \int_{B\left(x_{o}, r\right)}\left|D f_{j}\right|^{p}
$$

(ii) if $n=2$ and $p=1$, there exist a constant $C$ and such that for all $0<r<r_{1}$ and $j \geq j_{0}$ there is a set $A \subset B\left(x_{\circ}, r\right)$ such that

$$
|A|<C \delta\left|B\left(x_{\circ}, r\right)\right| \quad \text { and } \quad r^{2} \leqslant C \int_{A}\left|D f_{j}\right| .
$$

These would lead to a desired contradiction. Indeed, choose $0<r<r_{1}$ such that $\mu\left(\partial B\left(x_{\mathrm{o}}, r\right)\right)=0$ and in case (i) we obtain after passing to a limit in $j$ that

$$
\delta^{n-1-p} \leq C \lim _{j \rightarrow \infty} \int_{B\left(x_{\circ}, r\right)}\left|D f_{j}\right|^{p}=C \frac{\mu\left(B\left(x_{\circ}, r\right) \cap \Omega\right)}{\left|B\left(x_{\circ}, r\right)\right|} \leq C M \mu\left(x_{\circ}\right) .
$$

After passing $\delta \rightarrow 0+$ we obtain a contradiction with (6). In case (ii) we can use Jensen's inequality and (5) to obtain

$$
f_{B\left(x_{0}, r\right)} \Phi\left(\left|D f_{j}\right|\right) \geq \frac{|A|}{r^{2}} f_{A} \Phi\left(\left|D f_{j}\right|\right) \geq \frac{|A|}{r^{2}} \Phi\left(f_{A}\left|D f_{j}\right|\right) \geq \frac{|A|}{r^{2}} \Phi\left(\frac{C r^{2}}{|A|}\right) \geq C \delta \Phi\left(\frac{C}{\delta}\right)
$$

Similarly as above we obtain in the limit that

$$
C \delta \Phi\left(\frac{C}{\delta}\right) \leq C M \mu\left(x_{\circ}\right)
$$

and now passing to a limit $\delta \rightarrow 0+$ we obtain a contradiction using (5).
Proof of (i). We simplify the notation and write

$$
\varphi_{j}(x)=\left|f_{j}(x)-f\left(x_{\circ}\right)-D f\left(x_{\circ}\right)\left(x-x_{\circ}\right)\right| \quad \text { and } \quad B_{s}=B\left(x_{\circ}, s\right) .
$$

In the following we use the notation $\mathcal{H}^{k}(A)$ for the $k$-dimensional Hausdorff measure of the set $A$. We claim that the set of radii

$$
I_{G}=\left\{s \in[0, r]: \mathcal{H}^{n-1}\left(\left\{x \in \partial B_{s}: \varphi_{j}(x) \geqslant \delta r\right\}\right)<5^{n} \delta^{n-1} \mathcal{H}^{n-1}\left(\partial B_{s}\right)\right\}
$$

has measure at least $\frac{3 r}{4}$, i.e. $\left|I_{G}\right| \geqslant \frac{3 r}{4}$, otherwise

$$
f_{B_{r}}\left|\frac{\varphi_{j}(x)}{r}\right| \mathrm{d} x \geq \frac{1}{\left|B_{r}\right|} \int_{0}^{\frac{r}{4}} 5^{n} \delta^{n-1} \mathcal{H}^{n-1}\left(\partial B_{s}\right)\left|\frac{\delta r}{r}\right| \mathrm{d} s=5^{n} \delta^{n} \frac{\left|B_{\frac{r}{4}}\right|}{\left|B_{r}\right|}
$$

which contradicts (8).
On the other hand, the key point in our argument is that for $x_{\circ} \in \Omega_{\circ}$ and for every $s \in(0, r)$ we can find $\beta=\beta(s) \in \partial B_{s}$ such that

$$
\begin{equation*}
\varphi_{j}(\beta) \geqslant \frac{4}{5} s \quad \text { for every } j=1,2, \ldots \tag{9}
\end{equation*}
$$

Finding such a point $\beta$ is the only place where we use the homeomorphism assumption of $f_{j}$. Suppose on the contrary that (9) fails for every $\beta \in \partial B_{s}$ and for some $j \in\{1,2, \ldots\}$. For $x \in \partial B_{s}$ and $t \in[0,1]$ we consider the following homotopy:

$$
H(x, t):=(1-t)\left(f_{j}(x)-f\left(x_{\circ}\right)\right)+t D f\left(x_{\circ}\right)\left(x-x_{\circ}\right) .
$$

By (7) we know that $D f\left(x_{\circ}\right)$ is an isometry and thus $\left|D f\left(x_{\circ}\right) z\right|=|z|$. Furthermore, if (9) does not hold, then for all $x \in \partial B_{s}$ we have

$$
|H(x, t)| \geqslant\left|D f\left(x_{0}\right)\left(x-x_{0}\right)\right|-(1-t)\left|f_{j}(x)-f\left(x_{0}\right)-D f\left(x_{0}\right)\left(x-x_{0}\right)\right| \geq s-(1-t) \frac{4}{5} s>0 .
$$

It follows that $H(x, t) \neq 0$ for every $x \in \partial B_{s}$ and all $t \in[0,1]$. Thus, by (3) and (1),

$$
\operatorname{deg}\left(f_{j}, B_{s}, f\left(x_{o}\right)\right)=\operatorname{sgn} \operatorname{det}\left(D f\left(x_{\circ}\right)\right)=-1
$$

This contradicts the fact that $f_{j}$ is sense-preserving.

We apply the Sobolev embedding theorem [8, Theorem 3 (i), p. 143] on the ( $n-1$ )-dimensional spheres. This way for almost every $s \in\left(0, r_{0}\right)$ and for all $z_{1}, z_{2} \in \partial B\left(x_{0}, s\right)$ we have

$$
\begin{equation*}
\left|f_{j}\left(z_{1}\right)-f_{j}\left(z_{2}\right)\right| \leqslant C(n, p)\left|z_{1}-z_{2}\right|^{1-\frac{n-1}{p}}\left(\int_{\partial B_{s}}\left|D f_{j}\right|^{p}\right)^{\frac{1}{p}} \tag{10}
\end{equation*}
$$

Now let us fix $s \in I_{G}$ so that (10) is satisfied on the sphere $\partial B_{s}$. Since $s \in I_{G}$, we find $\alpha=\alpha(s) \in \partial B_{s}$ satisfying

$$
\varphi_{j}(\alpha)<\delta r \quad \text { and } \quad|\alpha-\beta| \leqslant C_{0} \delta s
$$

where $C_{0}$ is some fixed constant (which depends only on $n$ ). Combining this with (9) we have found $\alpha, \beta \in \partial B_{s}$ such that

$$
\frac{4}{5} s-\delta r-2 C_{0} \delta s \leqslant\left|\varphi_{j}(\beta)\right|-\left|\varphi_{j}(\alpha)\right|-2|\alpha-\beta| \leqslant\left|f_{j}(\alpha)-f_{j}(\beta)\right|
$$

This together with (10) implies that for $s \in I_{G} \cap\left[\frac{r}{2}, r\right]$ and $\delta$ small enough

$$
\begin{equation*}
C s^{p} \leqslant\left(\frac{4}{5} s-\delta r-2 C_{0} \delta s\right)^{p} \leqslant C(n, p)(\delta s)^{p-n+1} \int_{\partial B_{s}}\left|D f_{j}\right|^{p} \tag{11}
\end{equation*}
$$

Integrating inequality (11) over the set $I_{G} \cap\left[\frac{r}{2}, r\right]$ we obtain (i), finishing the proof of Theorem 1 in the case $p>n-1$.

Proof of (ii). We proceed as above. For $s \in I_{G}$ we can find $\beta=\beta(s) \in \partial B_{s}$ so that (9) holds. In fact we consider the measurable set

$$
A:=\left\{x \in B_{r}: \varphi_{j}(x)>\delta r\right\} .
$$

By Chebyshev's inequality and (8) we obtain

$$
|A| \leq \frac{1}{\delta r} \int_{B_{r}}\left|\varphi_{j}(x)\right| \mathrm{d} x \leq \frac{1}{\delta r} \delta^{2} r^{2} r=C \delta\left|B_{r}\right|
$$

Let $s \in I_{G} \cap\left[\frac{r}{2}, r\right]$. The point $\beta \in \partial B_{s}$ with (9) clearly belongs to $A \cap \partial B_{s}$ and the closest point $\alpha$ on the relative boundary of $\partial B_{s} \cap A$ satisfies

$$
\left|\varphi_{j}(\alpha)\right|=\delta r
$$

by the definition of $A$. It follows that for every $s \in I_{G} \cap\left[\frac{r}{2}, r\right]$ we have

$$
s \leqslant C \int_{\partial B_{s} \cap A}\left|D f_{j}\right| .
$$

Integrating this over $I_{G} \cap\left[\frac{r}{2}, r\right]$ we obtain

$$
r^{2} \leqslant C \int_{A}\left|D f_{j}\right|
$$

finishing the proof of (ii).
The above proof was based on the Sobolev embedding theorem on spheres and therefore does not work for $p<n-1$. To overcome these difficulties we follow Hencl and Malý [11] and use the theory of linking numbers and its topological invariance. For the convenience of the reader we recall the needed properties of linking numbers here.

## 4 Linking number

We use the notation $\mathbb{B}_{d}$ for the unit ball in $\mathbb{R}^{d}$ and $\mathbb{S}_{d-1}$ for the unit sphere. By $\overline{\mathbb{B}}_{d}(c, r)$ we denote the closed ball with center $c$ and radius $r>0$.

Let $n, t, q$ be positive integers with $t+q=n-1$. Let us consider the mapping $\Phi(\xi, \eta): \overline{\mathbb{B}}_{t+1} \times \overline{\mathbb{B}}_{q+1} \rightarrow \mathbb{R}^{n}$ defined coordinatewise as $\Phi(\xi, \eta)=x$, where

$$
\begin{aligned}
x_{1} & =\left(2+\eta_{1}\right) \xi_{1}, \\
& \vdots \\
x_{t+1} & =\left(2+\eta_{1}\right) \xi_{t+1}, \\
x_{t+2} & =\eta_{2} \\
& \vdots \\
x_{t+q+1} & =\eta_{q+1} .
\end{aligned}
$$

Denote by $\mathbb{A}$ the anuloid

$$
\Phi\left(\mathbb{S}_{t} \times \mathbb{B}_{q+1}\right)=\left\{x \in \mathbb{R}^{n}:\left(\sqrt{x_{1}^{2}+\cdots+x_{t+1}^{2}}-2\right)^{2}+x_{t+2}^{2}+\cdots+x_{n}^{2}<1\right\}
$$

Of course, given $x \in \overline{\mathbb{A}}$ we can find a unique $\xi \in \mathbb{S}_{t}$ and $\eta \in \overline{\mathbb{B}}_{q+1}$ such that $\Phi(\xi, \eta)=x$. We will denote these as $\xi(x)$ and $\eta(x)$.

A link is a pair $(\varphi, \psi)$ of parametrized surfaces $\varphi: \mathbb{S}_{t} \rightarrow \mathbb{R}^{n}, \psi: \mathbb{S}_{q} \rightarrow \mathbb{R}^{n}$. The linking number of the link $(\varphi, \psi)$ is defined as the topological degree

$$
\mathcal{L}(\varphi, \psi)=\operatorname{deg}(L, \mathbb{A}, 0)
$$

where the mapping $L=L_{\varphi, \psi}: \overline{\mathbb{A}} \rightarrow \mathbb{R}^{n}$ is defined as

$$
L(x)=\varphi(\xi(x))-\bar{\psi}(-\eta(x)),
$$

or equivalently

$$
L(\Phi(\xi, \eta))=\varphi(\xi)-\bar{\psi}(-\eta), \quad \xi \in \mathbb{S}_{t}, \eta \in \mathbb{B}_{q+1}
$$

where $\bar{\psi}$ is an arbitrary continuous extension of $\psi$ to $\overline{\mathbb{B}}_{q+1}$ (of course, the degree does not depend on the way how we extend $\psi$, it depends only on the values on the boundary $\partial \mathbb{A}=\Phi\left(\mathbb{S}_{t} \times \mathbb{S}_{q}\right)$ ). Geometrically speaking, for $t=q=1$, the linking number is the number of loops of a curve $\varphi$ around a curve $\psi$ counting orientation into account as +1 or -1 . For the introductions to the linking number in $\mathbb{R}^{3}$ and its application to the theory of knots see [23].

The canonical link is the pair $(\mu, v)$, where

$$
\begin{array}{lll}
\mu(\xi)=\Phi(\xi, 0), & \xi \in \mathbb{S}_{t}, \\
v(\eta)=\Phi\left(\mathbf{e}_{1}, \eta\right), & \eta \in \mathbb{S}_{q} .
\end{array}
$$

For example in dimension $n=3$ we get that

$$
\begin{aligned}
\mu\left(\mathbb{S}_{1}\right) & =\left\{x \in \mathbb{R}^{3}: x_{3}=0, x_{1}^{2}+x_{2}^{2}=4\right\} \\
v\left(\mathbb{S}_{1}\right) & =\left\{x_{2}=0,\left(x_{1}-2\right)^{2}+x_{3}^{2}=1\right\}
\end{aligned}
$$

It is well known that the linking number is a topological invariant. The simple proof of the following proposition can be found in [11].

Proposition 2. Let $n, t, q$ be positive integers with $t+q=n-1$. Let $f: \mathbb{B}_{n}(4) \rightarrow \mathbb{R}^{n}$ be a homeomorphism. Then $\mathcal{L}(f \circ \mu, f \circ v)$ is 1 if $f$ is sense preserving and -1 if $f$ is sense reversing.

Analogously, we can pick $a \in \overline{\mathbb{B}}_{q+1}\left(0, \frac{1}{10}\right)$ and $b \in \overline{\mathbb{B}}_{t+1}\left(\mathbf{e}_{1}, \frac{1}{10}\right) \cap \overline{\mathbb{B}}_{t+1}$ and consider the pair

$$
\begin{array}{ll}
\mu_{a}(\xi)=\Phi(\xi, a), & \xi \in \mathbb{S}_{t} \\
v_{b}(\eta)=\Phi(b, \eta), & \eta \in \mathbb{S}_{q}
\end{array}
$$

Similarly to the previous proposition we have:
Proposition 3. Let $n, t$, $q$ be positive integers with $t+q=n-1, a \in \overline{\mathbb{B}}_{q+1}\left(0, \frac{1}{10}\right)$ and $b \in \overline{\mathbb{B}}_{t+1}\left(\mathbf{e}_{1}, \frac{1}{10}\right) \cap \overline{\mathbb{B}}_{t+1}$. Let $f: \mathbb{B}_{n}(4) \rightarrow \mathbb{R}^{n}$ be a homeomorphism. Then $\mathcal{L}\left(f \circ \mu_{a}, f \circ v_{b}\right)$ is 1 iff is sense preserving and -1 iff is sense reversing.

## 5 Proof of Theorem 1 for $p>\left[\frac{n}{2}\right], n \geqslant 3$, and $p \geq 1, n=3$

Our argument is similar to the proof given in Section 3 and therefore some details are only sketched. By $\mu$ we again denote the $w^{*}$ limit of (some subsequence) $\int\left|D f_{j}\right|^{p}$ for $p>\left[\frac{n}{2}\right]$ and of $\int \Phi\left(\left|D f_{j}\right|\right)$ for $p=1$ and $n=3$.

By $C_{1}$ and $C_{2}$ we denote a fixed constants whose exact value will be determined later. We fix $\delta>0$ and we choose a point $x_{0}$ such that (4) and (6) hold and without loss of generality we assume that the derivative of $f$ at $x_{\circ}$ is given by (7).

We fix $r_{1}>0$ such that for all $0<r<r_{1}$ we have

$$
\int_{B\left(x_{\mathrm{o}}, 4 r\right)}\left|\frac{f(x)-f\left(x_{\mathrm{\circ}}\right)-D f\left(x_{\mathrm{\circ}}\right)\left(x-x_{\mathrm{\circ}}\right)}{r}\right| \mathrm{d} x<C_{1} \frac{\delta^{n}}{2}
$$

and again for all $j \geq j$ 。 we obtain

$$
\begin{equation*}
\int_{B\left(x_{0}, 4 r\right)}\left|\frac{f_{j}(x)-f\left(x_{\mathrm{o}}\right)-D f\left(x_{\mathrm{o}}\right)\left(x-x_{\mathrm{o}}\right)}{r}\right| \mathrm{d} x<C_{1} \delta^{n} \tag{12}
\end{equation*}
$$

We fix $t, q \leqslant\left[\frac{n}{2}\right]$ such that $t+q=n-1$ (e.g. $t=q=\frac{n-1}{2}$ for $n$ odd and $t=\frac{n-2}{2}, q=\frac{n}{2}$ for $n$ even). Our goal is to prove the following:
(i) if $p>\left[\frac{n}{2}\right]$ and $n \geqslant 3$, then there exists a constant $C$ (depending only on $p$ and $n$ ) such that for all $0<r<r_{1}$ and $j \geq j_{0}$,

$$
\delta^{\min \{t, q\}-p} r^{n} \leqslant C \int_{B\left(x_{\circ}, 4 r\right)}\left|D f_{j}\right|^{p}
$$

(ii) if $p=1$ and $n=3$, we have $A \subset B\left(x_{\circ}, 4 r\right)$ such that

$$
|A|<C_{2} \delta\left|B\left(x_{\circ}, 4 r\right)\right| \quad \text { and } \quad r^{3} \leqslant C \int_{A}\left|D f_{j}\right|
$$

Analogously to reasoning in Section 3 we obtain a contradiction using $\min \{t, q\}-p<0$ for $p>\left[\frac{n}{2}\right]$ and (5) for $p=1$ and $n=3$.
Proof of (i). Without loss of generality we will assume that $x_{\circ}=0$. We write

$$
\varphi_{j}(x)=\left|f_{j}(r x)-f(0)-D f(0) r x\right|
$$

Let us fix $y \in \mu_{a}\left(\mathbb{S}_{t}\right)$ and denote

$$
B_{\mu_{a}\left(\mathbb{S}_{t}\right)}(y, \delta)=\left\{x \in \mu_{a}\left(\mathbb{S}_{t}\right):|x-y|<\delta\right\}
$$

the ball of radius $\delta$ on the link $\mu_{a}\left(\mathbb{S}_{t}\right)$. We can clearly choose a constant $C_{1}$ small enough at the beginning of the proof so that (12) implies that the set of good links

$$
\begin{aligned}
& I_{a}=\left\{a \in \overline{\mathbb{B}}_{q+1}\left(0, \frac{1}{10}\right): \mathcal{H}^{t}\left(x \in \mu_{a}\left(\mathbb{S}_{t}\right): \varphi_{j_{o}}(x) \geqslant \delta r\right)<\mathcal{H}^{t}\left(B_{\mu_{a}\left(\mathbb{S}_{t}\right)}(y, \delta)\right)\right\} \\
& I_{b}=\left\{b \in \overline{\mathbb{B}}_{t+1}\left(\mathbf{e}_{1}, \frac{1}{10}\right) \cap \overline{\mathbb{B}}_{t+1}: \mathcal{H}^{q}\left(x \in v_{b}\left(\mathbb{S}_{q}\right): \varphi_{j_{o}}(x) \geqslant \delta r\right)<\mathcal{H}^{q}\left(B_{v_{b}\left(\mathbb{S}_{q}\right)}(y, \delta)\right)\right\}
\end{aligned}
$$

has measure at least

$$
\mathcal{H}^{q+1}\left(I_{a}\right)>\frac{1}{2}\left|\mathbb{B}_{q+1}\left(0, \frac{1}{10}\right)\right| \quad \text { and } \quad \mathcal{H}^{t+1}\left(I_{b}\right)>\frac{1}{2}\left|\mathbb{B}_{t+1}\left(\mathbf{e}_{1}, \frac{1}{10}\right) \cap \mathbb{B}_{t+1}\right|
$$

The key point of our argument is that for every $a \in \overline{\mathbb{B}}_{q+1}\left(0, \frac{1}{10}\right)$ and every $b \in \overline{\mathbb{B}}_{t+1}\left(\mathbf{e}_{1}, \frac{1}{10}\right) \cap \overline{\mathbb{B}}_{t+1}$ we can find $\xi \in \mathbb{S}_{t}$ and $\eta \in \mathbb{S}_{q}$ such that

$$
\begin{align*}
& \varphi_{j}\left(\mu_{a}(\xi)\right)=\left|f_{j}\left(r \mu_{a}(\xi)\right)-f(0)-D f(0) r \mu_{a}(\xi)\right|>\frac{r}{10} \quad \text { or }  \tag{13}\\
& \varphi_{j}\left(v_{b}(\eta)\right)=\left|f_{j}\left(r v_{b}(\eta)\right)-f(0)-D f(0) r v_{b}(\eta)\right|>\frac{r}{10}
\end{align*}
$$

We prove the observation by contradiction and we suppose that (13) does not hold. We define

$$
f_{s}(x)=(1-s)(f(0)+D f(0) r x)+s f_{j}(r x)
$$

and we consider the homotopy $H(\overline{\mathbb{A}} \times[0,1]) \rightarrow \mathbb{R}^{n}$ defined as

$$
H(\Phi(\xi, \eta), s)=\left(f_{s} \circ \mu_{a}\right)(\xi)-\overline{\left(f_{s} \circ v_{b}\right)}(-\eta),
$$

where $\overline{\left(f_{s} \circ v_{b}\right)}$ denotes a continuous extension of $f_{s} \circ v_{b}$ to $\overline{\mathbb{B}}_{q+1}$ as in the definition of the linking number, which in addition depends continuously on $s$. From [11] we know that the mapping $f_{j} \in W^{1, p}, p>\left[\frac{n}{2}\right]$, with nonnegative and nonzero Jacobian is sense preserving. By Proposition 3 we get that

$$
\operatorname{deg}(H(x, 1), \mathbb{A}, 0)=1
$$

On the other hand

$$
\operatorname{deg}(H(x, 0), \mathbb{A}, 0)=-1
$$

since the affine mapping $f(0)+D f(0) r x$ is sense reversing. To obtain a contradiction (with the preservation of the degree under homotopy) it is now enough to show that for every $\xi \in \mathbb{S}_{t}$, for every $\eta \in \mathbb{S}_{q}$ and for every $s \in[0,1]$ we have $H(\Phi(\xi, \eta), s) \neq 0$. It is easy to see that

$$
\operatorname{dist}\left(\left(f_{0} \circ \mu_{a}\right)\left(\mathbb{S}_{t}\right),\left(f_{0} \circ v_{b}\right)\left(\mathbb{S}_{q}\right)\right) \geqslant \operatorname{dist}\left(\left(f_{0} \circ \mu\right)\left(\mathbb{S}_{t}\right),\left(f_{0} \circ v\right)\left(\mathbb{S}_{q}\right)\right)-\frac{6 r}{10} \geq \frac{3 r}{10}
$$

Since (13) does not hold, we obtain from the definition of $f_{s}$ that

$$
\operatorname{dist}\left(\left(f_{s} \circ \mu_{a}\right)\left(\mathbb{S}_{t}\right),\left(f_{s} \circ v_{b}\right)\left(\mathbb{S}_{q}\right)\right) \geqslant \frac{3 r}{10}-\frac{r}{10}-\frac{r}{10}
$$

which implies $H(\Phi(\xi, \eta), s) \neq 0$.
By (13) and the symmetry we may assume without loss of generality that

$$
\tilde{I}_{a}=\left\{a \in I_{a}: \text { there exists } \xi \in \mathbb{S}_{t} \text { such that } \varphi_{j}\left(\mu_{a}(\xi)\right)>\frac{r}{10}\right\}
$$

satisfies $\mathcal{H}^{q+1}\left(\tilde{I}_{a}\right)>\frac{1}{4}\left|\mathbb{B}_{q+1}\left(0, \frac{1}{10}\right)\right|$. Since $p>\left[\frac{n}{2}\right] \geq t$, we can use the Sobolev embedding theorem on the $t$-dimensional space $r \mu_{a}\left(\mathbb{S}_{t}\right)$ and we have for almost every $a \in \tilde{I}_{a}$ and for all $z_{1}, z_{2} \in r \mu_{a}\left(\mathbb{S}_{t}\right)$,

$$
\begin{equation*}
\left|f_{j}\left(z_{1}\right)-f_{j}\left(z_{2}\right)\right| \leqslant C\left|z_{1}-z_{2}\right|^{1-\frac{t}{p}}\left(\int_{r \mu_{a}\left(\mathbb{S}_{t}\right)}\left|D f_{j}\right|^{p}\right)^{\frac{1}{p}} \tag{14}
\end{equation*}
$$

Now let us fix $a \in \tilde{I}_{a}$ so that (14) is satisfied and find $\xi \in \mathbb{S}_{t}$ so that for $\beta=\mu_{a}(\xi)$ we have $\varphi_{j}(\beta)>\frac{r}{10}$ as in the definition of $\tilde{I}_{a}$. Using $a \in I_{a}$ we find $\alpha \in \mu_{a}\left(\mathbb{S}_{t}\right)$ satisfying

$$
\varphi_{j}(\alpha)<\delta r \quad \text { and } \quad|\alpha-\beta| \leqslant \delta
$$

Thus we have found $\alpha, \beta \in \mu_{a}\left(\mathbb{S}_{t}\right)$ such that

$$
\frac{r}{10}-3 \delta r \leqslant\left|\varphi_{j}(\beta)\right|-\left|\varphi_{j}(\alpha)\right|-2 r|\alpha-\beta| \leqslant\left|f_{j}(r \alpha)-f_{j}(r \beta)\right| .
$$

This together with (14) implies that for almost every $a \in \tilde{I}_{a}$ and $\delta$ small enough we have

$$
\begin{equation*}
r^{t} \leqslant C \delta^{p-t} \int_{r \mu_{a}\left(\mathbb{S}_{t}\right)}\left|D f_{j}\right|^{p} \tag{15}
\end{equation*}
$$

Integrating inequality (15) over the set $\tilde{I}_{A}$ we obtain (i).
Proof of (ii). If $p=1$ and $n=3$, then for each $a \in \tilde{I}_{a}$ we can find $\xi \in \mathbb{S}(t)$ so that (13) holds for $\beta=\mu_{a}(\xi)$. The measurable set

$$
A:=\left\{x \in B\left(x_{\circ}, 4 r\right): \varphi_{j}(x)>\delta r\right\}
$$

satisfies $|A| \leq C \delta\left|B\left(x_{\circ}, 4 r\right)\right|$ by inequality (12) and Chebyshev’s inequality. Now clearly $\xi$ with (13) satisfies $r \beta=r \mu_{a}(\xi) \in A$. In (14) and (15) instead of integrating over the entire $r \mu_{a}\left(\mathbb{S}_{t}\right)$ we integrate only over the set $r \mu_{a}\left(\mathbb{S}_{t}\right) \cap A$. Integrating over $\tilde{I}_{a}$ we obtain the desired conclusion.

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