QUADRATIC BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT

In this thesis, we analyze backward stochastic differential equations. We begin by introducing stochastic processes, Brownian motion, stochastic integrals, and Itô's formula. After that, we move on to consider stochastic differential equations and finally backward stochastic differential equations. The backward stochastic differential equations are of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s \text{ for all } t \in I \text{ a.s.,}$$

where the function $f : \Omega \times I \times \mathbb{R}^2 \to \mathbb{R}$ is a random generator and the random variable ξ is a terminal value of the Y-process at time T. The main topic of this thesis are backward stochastic differential equations under quadratic assumptions. The assumptions we consider for the random generator f and the terminal value ξ of the quadratic backward stochastic differential equations are as follows:

(P1) There exists $\alpha, \beta \geq 0$, $\gamma > 0$ such that for all $(t, \omega) \in [0, T] \times \Omega$ the function $(y, z) \mapsto f(t, y, z)$ is continuous and $|f(\omega, t, y, z)| \leq \alpha + \beta|y| + \frac{\gamma}{2}|z|^2$ for all $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}^2$.

(P2) $\mathbb{E}\big[e^{\gamma e^{\beta T}|\xi|}\big] < \infty$.

(P3) There exists a $\lambda > \gamma e^{\beta T}$ such that $\mathbb{E}\big[e^{\lambda|\xi|}\big] < \infty$.

Obviously (P3) implies (P2). We prove that under these assumptions the backward stochastic differential equation has at least one solution (Y, Z) such that for some $C > 0$ and for all $t \in [0, T]$, a.s.,

$$-\frac{1}{\gamma} \log \mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t] \leq Y_t \leq \frac{1}{\gamma} \log \mathbb{E}[\phi_t(\xi)|\mathcal{F}_t],$$

where ϕ_t solves a special differential equation associated to this problem and

$$\mathbb{E}\Big[\int_t^T |Z_s|^2 ds \,\Big|\, \mathcal{F}_t\Big] \leq C\mathbb{E}\Big[e^{\lambda|\xi|} \,\Big|\, \mathcal{F}_t\Big].$$

TIIVISTELMÄ

Tässä tutkielmassa analysoimme takaperoisia stokastisia differentiaaliyhtälöitä. Aloitamme esittelemällä stokastiset prosessit, Brownin liikkeen, stokastiset integraalit ja Itôn kaavan. Tämän jälkeen siirrymme tarkastelemaan stokastisia differentiaaliyhtälöitä ja lopulta takaperoisia stokastisia differentiaaliyhtälöitä. Takaperoiset stokastiset differentiaaliyhtälöt ovat muotoa

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s \text{ kaikilla } t \in I \text{ m.v.},$$

missä funktio $f : \Omega \times I \times \mathbb{R}^2 \to \mathbb{R}$ on satunnaisgeneraattori ja satunnaismuuttuja ξ on Y-prosessin päätearvo hetkellä T. Tämän tutkielman pääaiheena on takaperoiset stokastiset differentiaaliyhtälöt kvadraattisilla oletuksilla. Käsittelemämme kvadraattisten stokastisten differentiaaliyhtälöiden oletukset koskien satunnaisgeneraattoria f ja päätearvoa ξ ovat seuraavat:

(P1) On olemassa $\alpha, \beta \geq 0$, $\gamma > 0$ siten, että kaikilla $(t, \omega) \in [0, T] \times \Omega$ funktio $(y, z) \mapsto f(t, y, z)$ on jatkuva ja $|f(\omega, t, y, z)| \leq \alpha + \beta|y| + \frac{\gamma}{2}|z|^2$ kaikilla $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}^2$.

(P2) $\mathbb{E}\big[e^{\gamma e^{\beta T}|\xi|}\big] < \infty$.

(P3) On olemassa $\lambda > \gamma e^{\beta T}$ siten, että $\mathbb{E}\big[e^{\lambda|\xi|}\big] < \infty$.

Selvästi oletuksesta (P3) seuraa oletus (P2). Todistamme, että näiden oletusten ollessa voimassa, takaperoisella stokastisella differentiaaliyhtälöllä on vähintään yksi ratkaisu (Y, Z) siten, että jollekin $C > 0$ ja kaikille $t \in [0, T]$, m.v.,

$$-\frac{1}{\gamma} \log \mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t] \leq Y_t \leq \frac{1}{\gamma} \log \mathbb{E}[\phi_t(\xi)|\mathcal{F}_t]$$

missä ϕ_t ratkaisee a tietyn ongelmaan liittyvän differentiaaliyhtälön, ja

$$\mathbb{E}\Big[\int_t^T |Z_s|^2 ds \,\Big|\, \mathcal{F}_t\Big] \leq C\mathbb{E}\Big[e^{\lambda|\xi|} \,\Big|\, \mathcal{F}_t\Big].$$

Contents

1. Introduction

In this thesis, we analyze Backward Stochastic Differential Equations (BSDEs), especially in the quadratic form. In Section 2 the most important definitions, such as of a stochastic process, a filtration, and a martingale, are introduced. The time set I for the stochastic processes is chosen to be the closed interval between 0 and some constant $T > 0$, i.e. $I := [0, T]$. This choice is because of the nature of most of the BSDEs that have a terminal condition on a finite time horizon $T > 0$.

After introducing the basic definitions, the Brownian motion will be defined. It is a fundamental stochastic process in the theory of stochastic differential equations. The main source regarding this topic is the monography of Karatzas and Shreve [9]. Also, the usual conditions and the augmentation of the natural filtration of a stochastic process are introduced. For technical reasons, the relation between the Brownian motion, the augmentation and the usual conditions is important.

Stochastic integration is a way to describe stochastic processes. The stochastic integral is a generalization of the Riemann-Stieltjes-integral, where the integrands and the integrators are stochastic processes. The space of suitable integrands is an $\mathcal{L}_2$-space, whereas the integrator is required to be a Brownian motion. Both of the requirements can be extended, but possible extensions are not covered by this thesis.

Itô's formula is a fundamental identity concerning the relation between stochastic processes and stochastic integrals. In stochastic analysis, it is a replacement of the mean value theorem from real analysis. It is an important tool for understanding and solving stochastic differential equations as well as backward stochastic differential equations. Itô's formula is applied repeatedly throughout this thesis.

As well as deterministic integrals can be generalized to stochastic counterparts, so in the case of differential equations. A Stochastic Differential Equation (shortened SDE and sometimes called a forward stochastic differential equation to be separated from a backward stochastic differential equation) determines a starting point for the process (the value of the stochastic process at time 0) and indicates certain random behaviour with respect to the time variable. The random behaviour includes two terms. One term consists of a Lebesgue-integral and the other term consists of a stochastic integral. By stochastic differential equations one can model stochastic processes that satisfy a desired random behavior. Solving a stochastic differential equation gives an explicit form of the modeled stochastic process. In Section

3 the stochastic integrals, Itô's formula, and the stochastic differential equations are introduced.

The main goal of this thesis and the topic of Section 4 is to study backward stochastic differential equations. As well as the forward stochastic differential equation, a backward stochastic differential equation models a certain random behaviour with respect to the time variable. However, instead of a starting point, the backward stochastic differential equation determines the terminal value of the stochastic process at time T. Because of the random nature of the process, in general, the terminal value of the process should be defined as a random variable instead of some constant.

BSDEs were first introduced by Bismut [1] and the theory was later extended by Pardoux and Peng in [12] and [13], El Karoui, Peng, and Quenez in [7], Pardoux in [11], and Briand, Delyon, Hu, Pardoux, and Stoica in [2]. Pardoux and Peng [12] were the first to prove an existence and uniqueness theorem for the backward stochastic differential equations under Lipschitz conditions. Kobylanski [10] obtained an existence theorem for the backward stochastic differential equations under quadratic conditions. Briand and Hu [3, 4] and Delbaen, Hu, and Richou [5, 6] studied the quadratic case further. In [3], Briand and Hu showed an existence theorem and certain regularity conditions for the solution of the backward stochastic differential equation. This existence result and the regularity conditions are proved at the end of this thesis.

2. Preliminaries

2.1. Basic definitions.

In this chapter, the most important definitions concerning general stochastic processes are introduced. A stochastic process is a fundamental object throughout this thesis. It can be considered to be a random function with a time variable. The stochastic processes can be used to model random behavior with respect to time.

Definition 2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $I := [0, T]$ for some $T \in (0, \infty)$. A* ***stochastic process*** *is a family of random variables $(X_t)_{t \in I}$ where $X_t : \Omega \to \mathbb{R}$.*

Remark 2.2.

(i) The value of the random variable X_t describes the state of the process at time t.
(ii) The time set I can be defined in other ways as well. For example, $I := [0, \infty)$ is a sufficient time set for a process that has no terminal time. For discrete processes $I := \{1, ..., n\}$ with $n \in \mathbb{N}$ or $I := \mathbb{N}$ are appropriate time sets. However, mainly the time set $I := [0, T]$ is considered in this thesis.

Definition 2.3. *Let $X = (X_t)_{t \in I}$ and $Y = (Y_t)_{t \in I}$ be stochastic processes. The processes X and Y are*

(i) ***indistinguishable*** *if $\mathbb{P}(X_t = Y_t \ \forall t \in I) = 1$, and*
(ii) ***modifications*** *of each other if $\mathbb{P}(X_t = Y_t) = 1$ for all $t \in I$.*

Definition 2.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A set of σ-algebras $(\mathcal{F}_t)_{t \in I}$ is called a* ***filtration*** *if $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $0 \leq s < t \leq T$.*

The filtration describes the idea of information at a certain time. The elements of a filtration $\mathcal{F}_t$ can be understood to be the known events of the probability space. As time moves forward the amount of information grows, since the filtration becomes finer, and thus, the amount of known events grows.

Definition 2.5. *A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t \in I}$ is called a* ***stochastic basis****.*

Definition 2.6. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in I})$ be a stochastic basis. A stochastic process $X = (X_t)_{t \in I}$ is called*

(i) ***adapted*** *with respect to $(\mathcal{F}_t)_{t \in I}$ if X_t is $\mathcal{F}_t$-measurable for all $t \in I$,*
(ii) ***progressively measurable*** *with respect to $(\mathcal{F}_t)_{t \in I}$ if $X : [0, s] \times \Omega \to \mathbb{R}$ with $X(t, \omega) := X_t(\omega)$ is $\mathcal{B}([0, s]) \otimes \mathcal{F}_s$-measurable for all $s \in I$,*

(iii) ***continuous*** *if the function* $t \mapsto X_t(\omega)$ *is continuous for all* $\omega \in \Omega$*,*
(iv) ***RCLL*** *if for all* $\omega \in \Omega$ *the function* $t \mapsto X_t(\omega)$ *is right-continuous and has left limits i.e.*

$$\lim_{s \to t+} X_s(\omega) = X_t(\omega)$$

for all $t \in [0, T)$ *and*

$$\lim_{s \to t-} X_s(\omega) \in \mathbb{R}$$

for all $t \in (0, T]$*,*
(v) ***predictable*** *if* $X : [0, T] \times \Omega \to \mathbb{R}$ *with* $X(t, \omega) := X_t(\omega)$ *is measurable with respect to the sigma-algebra generated by all continuous stochastic processes,*
(vi) ***integrable*** *if* $\mathbb{E}|X_t| < \infty$ *for all* $t \in I$*, and*
(vii) ***square integrable*** *if* $\mathbb{E}X_t^2 < \infty$ *for all* $t \in I$*.*

An adapted process can be interpreted to be a process, where at each time the past behaviour and the current situation of the process is known. On the other hand, a continuous process is a process which does not have jumps. An RCLL process is allowed to have a countable amount of jumps. Progressive measurability, integrability, and square integrability are technical concepts which are required for some of the results given later in this thesis.

Definition 2.7. *Let* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in I})$ *be a stochastic basis. A stochastic process* $M = (M_t)_{t \in I}$ *is called a* ***martingale*** *provided that*

(i) M *is adapted,*
(ii) M *is integrable, and*
(iii) $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ *a.s. for all* $0 \leq s \leq t \leq T$*.*

The space of martingales is denoted by $\mathcal{M}$ *and the space of continuous square integrable martingales* $M = (M_t)_{t \in I}$ *with* $M_0 = 0$ *a.s. is denoted by* $\mathcal{M}_2^{c,0}$*.*

Remark 2.8. The last property (iii) of a martingale is called the martingale property. The martingale property states that at each time t the conditional expectation given $\mathcal{F}_t$ of any future state of a martingale is the value of the martingale at time t. Thus, one can not predict the expected direction of a martingale by the historical behaviour of it. The martingales are, for example, used to model fair games where the expected value of wins and losses are balanced between the players of the game.

Definition 2.9. *A stochastic process* $M^* = (M_t^*)_{t \in I}$ *is called a* ***supermartingale*** *provided that*

(i) *M^* is adapted,*
(ii) *M^* is integrable, and*
(iii) *$\mathbb{E}[M_t^*|\mathcal{F}_s] \leq M_s^*$ a.s. for all $0 \leq s \leq t \leq T$.*

Definition 2.10. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$ be a stochastic basis. A random variable $\tau : \Omega \to I$ is called a **stopping time** if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in I$.*

2.2. **Brownian Motion.**
The Brownian motion is our driving stochastic process of the theory of stochastic differential equations. Aside of being a stochastic process itself, it is also used to define other processes. Later in this thesis, the Brownian motion is the basis to define stochastic integrals, stochastic differential equations, and finally backward stochastic differential equations.

Let us first give an idea of one possible construction of the Brownian motion. First we define a stochastic process $X = (X_n)_{n\in\mathbb{N}}$ in the following way. Let $N_1, N_2, N_3, ... \sim \mathcal{N}(0, 1)$ be independent normally distributed random variables with mean 0 and variance 1. Moreover, let $S_0 := 0$ and $S_n := \sum_{i=1}^{n} N_i$ for $n = 1, 2, 3, ...$ The process X is defined by

$$X_t := S_{\lfloor t\rfloor} + (t - \lfloor t\rfloor)N_{\lfloor t\rfloor+1}. \tag{1}$$

The process X defined above is a continuous stochastic process with linear transitions between time points 0,1,2,3,... The process X can be rescaled so that the set of base points is a finer set of time points, for example at time points $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, ...$ The modified process X^2 would be defined by

$$X_t^2 := \frac{X_{2t}}{\sqrt{2}}.$$

The latter process is distributed as the former one in the coarser set of time points $1, 2, 3, ...$

If we define stochastic processes X^n by

$$X_t^n := \frac{X_{nt}}{\sqrt{n}}$$

for $n = 1, 2, 3, ...,$ one gets piece-wise linear continuous stochastic processes based on the time grid $\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, ...\}$. It is actually possible to construct a continuous stochastic process that is distributed like any process X^n at the given time grid $\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, ...\}$. This continuous stochastic process is called a Brownian motion. We give the formal definition for the Brownian motion, where the time interval is set to be $I := [0, T]$.

Definition 2.11. *A stochastic process $B = (B_t)_{t\in I}$ is called a **Brownian motion** provided that*

(i) $B_0(\omega) = 0$ *for all* $\omega \in \Omega$,
(ii) B *is continuous,*
(iii) *the random variables* $B_{t_n} - B_{t_{n-1}}, ..., B_{t_1} - B_{t_0}$ *are independent for all* $n \in \mathbb{N}, 0 \leq t_0 \leq t_1 \leq ... \leq t_n \leq T$,
(iv) $B_t - B_s \sim \mathcal{N}(0, t-s)$ *for all* $0 \leq s \leq t \leq T$.

Remark 2.12. The Brownian motion is also often called a Wiener process. Robert Brown (1773-1858) described a physical phenomenon of particles moving randomly in water. Louis Bachelier (1870-1946) studied a similar phenomenon of stock market prices. Norbert Wiener (1894-1964) constructed mathematically the stochastic process modelling these phenomena.

Before giving an existence theorem for the Brownian motion, we define the weak convergence for the probability measures in a metric space.

Definition 2.13. *Let* (S, ρ) *be a metric space with a Borel* σ*-field* $\mathcal{B}(S)$. *Let* $(\mathbb{P}_n)_{n=1}^{\infty}$ *be a sequence of probability measures in* $(S, \mathcal{B}(S))$ *and let* $\mathbb{P}$ *be another probability measure in* $(S, \mathcal{B}(S))$. *Then the sequence* $(\mathbb{P}_n)_{n=1}^{\infty}$ *converges weakly to the probability measure* $\mathbb{P}$, *if*

$$\lim_{n\to\infty} \int_S f(s) d\mathbb{P}_n(s) = \int_S f(s) d\mathbb{P}(s)$$

for all bounded and continuous functions $f : S \to \mathbb{R}$.

The previous construction with $X_t^n := \frac{X_{nt}}{\sqrt{n}}$ has a limit process, which is a Brownian motion, as shown by the following invariance principle of Donsker. Before formulating the invariance principle of Donsker, we need to define a measurable space of continuous functions.

Definition 2.14. *Let* $\big(C[0,T], d_u\big)$ *be the metric space of continuous functions on* $[0,T]$ *equipped with the uniform norm* d_u. *One can refer to the set of Borel-sets of the space* $\big(C[0,T], d\big)$ *by* $\mathcal{B}(C[0,T])$ *so that* $\big(C[0,T], \mathcal{B}(C[0,T])\big)$ *is a measurable space.*

Theorem 2.15 (The Invariance Principle of Donsker)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and let* $(\xi_i)_{i=1}^{\infty}$ *be a sequence of independent, identically distributed random variables with mean zero and finite variance* $\sigma^2 > 0$. *Let the random variables* S_n *be defined by* $S_n := \sum_{i=1}^{n} \xi_i$ *for* $n = 1, 2, 3, ...$ *and* $S_0 := 0$. *Let the processes* $X = (X_t)_{t\in I}$ *and* $X^n = (X_t)_{t\in I}^n$ *for* $n = 1, 2, 3, ...$ *be defined by* $X_t := S_{\lfloor t\rfloor} + (t - \lfloor t\rfloor) N_{\lfloor t\rfloor+1}$, *and* $X_t^n := \frac{X_{nt}}{\sqrt{n}}$. *Let* $\mathbb{P}_n$ *be the probability measure on the space of the continuous functions* $\big(C[0,T), \mathcal{B}(C[0,T])\big)$ *defined by* $\mathbb{P}_n(A) := \mathbb{P}(X^n \in$

A) for all $A \in \mathcal{B}(C[0,T])$. Then the sequence $(\mathbb{P}_n)_{n=1}^{\infty}$ of probability measures converges weakly to a measure $\mathbb{P}^*$. Moreover, the process $B = (B_t)_{t\in I}$, $B_t := \omega(t)$ is a Brownian motion on the probability space $\big(C[0,T], \mathcal{B}(C[0,T]), P^*\big)$.

The proof is given in [9, p. 70-71].

Remark 2.16. One can also give a corresponding theorem for the Brownian motion on the interval $[0,\infty)$. In that case the space of continuous functions $\big(C[0,\infty)), \mathcal{B}(C[0,\infty))\big)$ can be equipped with a metric defined by

$$d(\omega_1,\omega_2) := \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{t\in[n-1,n]} \big\{|\omega_1(t)-\omega_2(t)|, 1\big\}$$

for all $\omega_1, \omega_2 \in C[0,\infty)$.

Definition 2.17. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X = (X_t)_{t\in I}$ be a stochastic process. The filtration $(\mathcal{F}_t^X)_{t\in I}$ defined by $\mathcal{F}_t^X := \sigma(X_s : s \in [0,t])$ is called the **natural filtration** of X. Moreover let*

$$\mathcal{N} := \{A \subset \Omega : \text{ there exists } B \in \mathcal{F} \text{ such that } A \subset B \text{ and } \mathbb{P}(B) = 0\}.$$

*The filtration $(\mathcal{F}_t)_{t\in I}, \mathcal{F}_t := \sigma(\mathcal{F}_t^X \cup \mathcal{N})$ is called the **augmentation** of $(\mathcal{F}_t^X)_{t\in I}$.*

Remark 2.18. Note that the augmentation of $(\mathcal{F}_t^X)_{t\in I}$ is not necessarily a filtration in $(\Omega, \mathcal{F}, \mathbb{P})$. However, $(\Omega, \mathcal{F}, \mathbb{P})$ can be extended to include the sets from $\mathcal{N}$ by letting $\tilde{\mathcal{F}} := \sigma(\mathcal{F} \cup \mathcal{N})$ and $\tilde{\mathbb{P}}(A) := \mathbb{P}(B)$ when $B \triangle A \in \mathcal{N}$. Now $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is a probability space and the augmentation of $(\mathcal{F}_t^X)_{t\in I}$ is a filtration in $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. The probability space $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is called the **completion** of $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.19. *A stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$ satisfies the **usual conditions** provided that the following is satisfied:*

(i) *$\mathcal{N} \subset \mathcal{F}_0$,*
(ii) *the filtration $(\mathcal{F}_t)_{t\in I}$ is right-continuous i.e. $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0,T)$.*

Proposition 2.20. *Let B be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the stochastic basis $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, (\mathcal{F}_t)_{t\in I})$, where $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is the completion of $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t)_{t\in I}$ is the augmentation of $(\mathcal{F}_t^B)_{t\in I}$, satisfies the usual conditions.*

The first property (i) of the usual conditions is clear by the definitions of the augmentation and the completion. The right-continuity is proved in [9, p. 90-91]. The proof is given for strong Markov processes. In

[9, p. 86] it is shown that the Brownian motion is a strong Markov process.

Remark 2.21. Proposition 2.20 allows us to assume that the Brownian motion is defined on a stochastic basis that satisfies the usual conditions. From now on, the usual conditions are assumed whenever a stochastic basis is introduced. The assumption of the usual conditions is necessary for some results given later.

Proposition 2.22. *Let $B = (B_t)_{t\in I}$ be a Brownian motion and $(\mathcal{F}_t)_{t\in I}$ be the augmentation of $(\mathcal{F}_t^B)_{t\in I}$. Then*

(i) *B is $(\mathcal{F}_t)_{t\in I}$-adapted,*
(ii) *the random variable $B_t - B_s$ and the σ-algebra $\mathcal{F}_s$ are independent for all $0 \leq s \leq t \leq T$.*

The proof is given in [9, p. 116-117].

Definition 2.23. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$ be a stochastic basis. An $(\mathcal{F}_t)_{t\in I}$-adapted Brownian motion B is called $(\mathcal{F}_t)_{t\in I}$-**Brownian motion** provided that the random variable $B_t - B_s$ and the σ-algebra $\mathcal{F}_s$ are independent for all $0 \leq s \leq t \leq T$.*

Remark 2.24. Proposition 2.22 states, that a Brownian motion is an $(\mathcal{F}_t)_{t\in I}$-Brownian motion, where $(\mathcal{F}_t)_{t\in I}$ is the augmentation of the natural filtration of the Brownian motion.

3. Stochastic Differential Equations

3.1. **Stochastic Integration.**

Using stochastic integrals one can describe stochastic processes. The stochastic integral is a generalization of the Riemann-Stieltjes-integral, where the integrands and the integrators are stochastic processes. In this thesis the stochastic integrals are defined for $\mathcal{L}_2$-processes as integrands, which are defined in this chapter. Before considering the $\mathcal{L}_2$-processes, we consider simple processes. First, the stochastic integration is defined on the space of simple processes, and after that the definition is extended to the larger $\mathcal{L}_2$-space.

Definition 3.1. *Let $L = (L_t)_{t\in I}$ be a stochastic process on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$. The process L is called **simple** provided that there exist time points $0 =: t_0 < t_1 < ... < t_N := T$ and random variables $\xi_0, \xi_1, ...\xi_{N-1}$ such that*

(i) *ξ_i is $\mathcal{F}_{t_i}$-measurable for all $i \in \{0, ..., N-1\}$,*
(ii) *$\sup_{\omega\in\Omega} |\xi_i(\omega)| < \infty$ for all $i \in \{0, ..., N-1\}$,*
(iii) *$L_t = \sum_{i=1}^N \xi_{i-1} \mathbb{1}_{(t_{i-1},t_i]}(t)$ for all $t \in I$.*

The space of simple processes is denoted by $\mathcal{L}_0$.

Remark 3.2. A simple process L in $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$ is $(\mathcal{F}_t)_{t\in I}$-adapted.

Now, we are ready to give the first definiton for the stochastic integral.

Definition 3.3. *Let $L \in \mathcal{L}_0$ and B be a Brownian motion on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$. Then the **stochastic integral** of L is the stochastic process $(I_t(L))_{t\in I}$ where*

$$I_t(L) := \sum_{i=1}^{N} \xi_{i-1}(B_{t_i\wedge t} - B_{t_{i-1}\wedge t}) \text{ for all } t \in I.$$

Remark 3.4. The stochastic integral $(I_t(L))_{t\in I}$ for a simple process L is $(\mathcal{F}_t)_{t\in I}$-adapted which follows from the facts that the simple process L and the Brownian motion are adapted.

Now, let us define the $\mathcal{L}_2$-space and consider the stochastic integration on that.

Definition 3.5. *Let $\mathcal{L}_2$ be the space of the progressively measurable processes $L = (L_t)_{t\in I}$ with $\mathbb{E}\int_0^T L_t^2 dt < \infty$.*

Remark 3.6.

(i) Above, the integral $\int_0^T L_t^2 dt$ is a well-defined random variable which follows from the progressive measurability and Fubini's theorem. Therefore the expected value $\mathbb{E}\int_0^T L_t^2 dt$ can be defined.

(ii) The simple processes are in $\mathcal{L}_2$.

Proposition 3.7. *Let $L \in \mathcal{L}_2$.*

(i) *There exists a sequence $(L^n)_{n=0}^{\infty} \subset \mathcal{L}_0$ such that*
$$\lim_{n\to\infty} \mathbb{E} \int_0^T |L_t - L_t^n|^2 dt = 0.$$
There also exists an adapted continuous process $X = (X_t)_{t\in I}$ such that
$$\lim_{n\to\infty} \mathbb{E}|X_t - I_t(L^n)|^2 = 0.$$

(ii) *Let $(L^n)_{n=0}^{\infty}, (\hat{L}^n)_{n=0}^{\infty} \subset \mathcal{L}_0$ be such that*
$$\lim_{n\to\infty} \mathbb{E} \int_0^T |L_t - L_t^n|^2 dt = \lim_{n\to\infty} \mathbb{E} \int_0^T |L_t - \hat{L}_t^n|^2 dt = 0.$$
Then there exist continuous and adapted processes $X = (X_t)_{t\in I}$ and $\hat{X} = (\hat{X}_t)_{t\in I}$ such that
$$\lim_{n\to\infty} \mathbb{E}|X_t - I_t(L^n)|^2 = 0 \text{ and } \lim_{n\to\infty} \mathbb{E}|\hat{X}_t - I_t(\hat{L}^n)|^2 = 0.$$
Moreover, any such processes X and $\hat{X}$ are indistinguishable.

The proof of the part (i) is given in [9, p. 134-137] and the proof of the part (ii) is given in [9, p. 137-139].

Now, we are ready to define the stochastic integral for $\mathcal{L}_2$-processes.

Definition 3.8. *Let $L \in \mathcal{L}_2$ and $(L^n)_{n=0}^{\infty} \subset \mathcal{L}_0$ be such that*
$$\lim_{n\to\infty} \mathbb{E} \int_0^T |L_t - L_t^n|^2 dt = 0.$$
Then, the ***stochastic integral*** *of L is an adapted continuous process $(I_t(L))_{t\in I}$ for which*
$$\lim_{n\to\infty} \mathbb{E}|I_t(L) - I_t(L^n)|^2 = 0.$$

Remark 3.9.

(i) Proposition 3.7 verifies that the stochastic integral of an $\mathcal{L}_2$-process is well defined. The assertion (i) provides that the process $(I_t(L))_{t\in I}$ exists and the assertion (ii) provides that it is unique up to indistingushability.

(ii) The stochastic integral of an $\mathcal{L}_2$-process is a generalization of the stochastic integral of an $\mathcal{L}_0$-process. Therefore, the same notation $I_t(L)$ can be used for both integrals.

(iii) Often the stochastic integral $I_t(L)$ at time point t is denoted by $\int_0^t L_s dB_s$.

(iv) We also use the definition $\int_s^t L_u dB_u := I_t(L) - I_s(L)$ for $0 \leq s \leq t \leq T$.

According to the notation of the part (iii) of the last remark, the $\mathcal{L}_2$-space can be interpreted as the space of suitable integrands, where the integrator is required to be a Brownian motion. The definition of the stochastic integrals can be extended in a way that the space of suitable integrands is larger than the the $\mathcal{L}_2$-space. Also, the integrators can be generalized to be other stochastic processes than a Brownian motion. However, these extensions are not covered in this thesis.

Proposition 3.10. *Let $L, K \in \mathcal{L}_2$ and $\alpha, \beta \in \mathbb{R}$. Then*

(i) $(I_t(L))_{t\in I} \in \mathcal{M}_2^{c,0}$,
(ii) $I_t(\alpha L + \beta K) = \alpha I_t(L) + \beta I_t(K)$ *for all* $t \in I$ *a.s.*,
(iii) $\mathbb{E}|I_t(L)|^2 = \mathbb{E}\int_0^t L_u^2 du$ *for all* $t \in I$.

The proof is given in [9, p. 138-140].

Remark 3.11. The last property (iii) of Proposition 3.10 is called **Itô's isometry**.

Proposition 3.12. *Let $L, K \in \mathcal{L}_2$ such that $I_t(L) = I_t(K)$ a.s. for all $t \in I$. Then $L = K$ $d\mathbb{P} \otimes dt$-a.s.*

Proof. According to Itô's isometry

$$\begin{aligned} \mathbb{E}\int_0^T (L_u - K_u)^2 du &= \mathbb{E}|I_T(L-K)|^2 \\ &= 0, \end{aligned}$$

since $L - K \in \mathcal{L}_2$. □

3.2. Itô's Formula.

Itô's formula is a fundamental identity concerning the relation between stochastic processes and stochastic integrals. It is an important tool for understanding and solving stochastic differential equations as well as backward stochastic differential equations. Below, Itô's formula is stated for one process as well as for two processes.

Definition 3.13. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that all partial derivatives $\frac{\partial f}{\partial s}$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, where $(s,x) \in (0,T) \times \mathbb{R}$, are continuous and can be continuosly extended to $[0,T] \times \mathbb{R}$. Then it is said that f belongs to $C^{1,2}([0,T] \times \mathbb{R})$.*

Proposition 3.14 (Itô's Formula)**.** *Let $f \in C^{1,2}([0,T] \times \mathbb{R})$ and $X = (X_t)_{t\in I}$, $X_t := X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s$ be a stochastic process where*

(i) X_0 *is an* $\mathcal{F}_0$*-measurable random variable,*

(ii) *the process* $b = (b_s)_{s\in I}$ *is progressively measurable and satisfies* $\int_0^t |b_s|ds < \infty$ *a.s.,*
(iii) $\sigma = (\sigma_s)_{s\in I} \in \mathcal{L}_2$,
(iv) $\left(\frac{\partial f}{\partial x}(s, X_s)\sigma_s\right)_{s\in I} \in \mathcal{L}_2$.

Then we have that, a.s.,

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)b_s ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\sigma_s dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)\sigma_s^2 ds$$

for all $t \in I$.

This proposition is stated in [9, p. 153] and proved in [9, p. 230] with the help of the theorem stated and proved in [9, p. 149-153].

Remark 3.15. The stochastic integral can also be extended so that Itô's formula holds with a weaker assumption than $\left(\frac{\partial f}{\partial x}(s, X_s)\sigma_s\right)_{s\in I} \in \mathcal{L}_2$.

Definition 3.16. *Let* $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ *be a continuous function such that all partial derivatives* $\frac{\partial f}{\partial s}$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, *and* $\frac{\partial^2 f}{\partial x \partial y}$, *where* $(s, x, y) \in (0, T) \times \mathbb{R}^2$, *are continuous and can be continuosly extended to* $[0, T] \times \mathbb{R}^2$. *Then it is said that* f *belongs to* $C^{1,2}([0, T] \times \mathbb{R}^2)$.

Proposition 3.17 (Itô's Formula). *Let* $f \in C^{1,2}([0, T] \times \mathbb{R}^2)$ *and* $X = (X_t)_{t\in I}$, $X_t := X_0 + \int_0^t a_s ds + \int_0^t L_s dB_s$, $Y = (Y_t)_{t\in I}$, $Y_t := Y_0 + \int_0^t b_s ds + \int_0^t K_s dB_s$ *be a stochastic processes where*

(i) X_0 *is an* $\mathcal{F}_0$*-measurable random variable,*
(ii) *the process* $a = (a_s)_{s\in I}$ *is progressively measurable and satisfies* $\int_0^t |a_s|ds < \infty$ *a.s.,*
(iii) $L = (L_s)_{s\in I} \in \mathcal{L}_2$,
(iv) $\left(\frac{\partial f}{\partial x}(s, X_s, Y_s)\sigma_s\right)_{s\in I} \in \mathcal{L}_2$.
(v) Y_0 *is an* $\mathcal{F}_0$*-measurable random variable,*
(vi) *the process* $b = (b_s)_{s\in I}$ *is progressively measurable and satisfies* $\int_0^t |b_s|ds < \infty$ *a.s.,*
(vii) $K = (K_s)_{s\in I} \in \mathcal{L}_2$,
(viii) $\left(\frac{\partial f}{\partial y}(s, X_s, Y_s)\sigma_s\right)_{s\in I} \in \mathcal{L}_2$.

Then we have that, a.s.,

$$f(t, X_t, Y_t) = f(0, X_0, Y_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s, Y_s)ds$$

$$
\begin{aligned}
&+\int_0^t \frac{\partial f}{\partial x}(s,X_s,Y_s)a_s ds + \int_0^t \frac{\partial f}{\partial y}(s,X_s,Y_s)b_s ds \\
&+\int_0^t \frac{\partial f}{\partial x}(s,X_s,Y_s)L_s dB_s + \int_0^t \frac{\partial f}{\partial y}(s,X_s,Y_s)K_s dB_s \\
&+\frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,X_s,Y_s)L_s^2 ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial y^2}(s,X_s,Y_s)K_s^2 ds \\
&+\int_0^t \frac{\partial^2 f}{\partial x \partial y}(s,X_s,Y_s)L_s K_s ds
\end{aligned}
$$

for all $t \in I$*.*

This proposition as well as the previous one is proved in [9, p. 149-153] and [9, p. 230]. The statement in [9] is a general one and it contains both of the cases.

3.3. **Stochastic Differential Equations.**
As well as deterministic integrals can be generalized to stochastic counterparts, so is the case for the differential equations. A stochastic differential equation (sometimes called a forward stochastic differential equation to be separated from a backward stochastic differential equation) determines a starting point for the process (the value of the stochastic process at time 0) and indicates certain random behaviour with respect to the time variable. The random behaviour includes two terms. One term consists of a Lebesgue-integral and another term consists of a stochastic integral. By stochastic differential equations one can model stochastic processes that satisfy a certain random behavior. Solving a stochastic differential equation gives an explicit form of the modeled stochastic process. Below, the stochastic differential equation is defined formally.

Definition 3.18. *Let* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$ *be a stochastic basis that satisfies the usual conditions,* $x_0 \in \mathbb{R}$*,* $b, \sigma : I \times \mathbb{R} \to \mathbb{R}$ *be continuous functions and* $B = (B_t)_{t\in I}$ *be an* $(\mathcal{F}_t)_{t\in I}$*-Brownian motion. A stochastic process* $X = (X_t)_{t\in I}$ *is a solution to the* ***stochastic differential equation*** *(SDE)*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t,$$

$$X_0 = x_0$$

provided that

(i) X *is continuous and* $(\mathcal{F}_t)_{t\in I}$*-adapted,*
(ii) $X_0 = x_0$ *for all* $\omega \in \Omega$*,*
(iii) $X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \ \forall t \in I$ *a.s.,*

(iv) $\mathbb{E}\int_0^T |\sigma(s, X_s)|^2 ds < \infty$.

Remark 3.19. In the previous definition, the point x_0 describes the starting point of the process X. The function $b : I \times \mathbb{R} \to \mathbb{R}$ describes the drift behaviour of the process X, while the function $\sigma : I \times \mathbb{R} \to \mathbb{R}$ describes the volatility of the process X. If one omits the second term $\sigma(t, X_t)dB_t$, then the stochastic differential equation becomes a first order ordinary differential equation.

Next, we are going to formulate an existence and uniqueness theorem for the stochastic differential equations.

Theorem 3.20. *Let $x_0 \in \mathbb{R}$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$ be a stochastic basis, that satisfies the usual conditions, let $B = (B_t)_{t\in I}$ be an $(\mathcal{F}_t)_{t\in I}$-Brownian motion, and let $b, \sigma : I \times \mathbb{R} \to \mathbb{R}$ be continuous functions such that there exists $K > 0$ for which*

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \le K|x - y|$$

for all $t \in I, x, y \in \mathbb{R}$. Then there exists a stochastic process $X = (X_t)_{t\in I}$ which is a solution to the SDE

$$\begin{aligned} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dB_t, \\ X_0 &= x_0. \end{aligned}$$

Moreover, X is square integrable and unique up to indistinguishibility in $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$.

The proof is given in [9, p. 287-290].

We will need the following well known estimate for the solution of Theorem 3.20 (see for example [8, Proposition 4.3.1, p. 93-97]).

Proposition 3.21. *The solution $X = (X_t)_{t\in I}$ of Theorem 3.20 satisfies*

$$\mathbb{E}\sup_{t\in I} |X_t|^p < \infty$$

for all $p \in (0, \infty)$.

In the example below, Theorem 3.20 and Itô's formula are used to find a solution to linear stochastic differential equations.

Example 3.22. Let us find a solution to the following SDE:

$$\begin{aligned} dX_t &= (b_1 X_t + b_2)dt + (\sigma_1 X_t + \sigma_2)dB_t \\ X_0 &= x_0 \end{aligned} \tag{2}$$

where $b_1, b_2, \sigma_1, \sigma_2 \in \mathbb{R}$. Firstly, we note that

$$|(b_1 x + b_2) - (b_1 y + b_2)| + |(\sigma_1 x + \sigma_2) - (\sigma_1 y + \sigma_2)|$$

$$\begin{aligned} &= |b_1||x-y| + |\sigma_1||x-y| \\ &= (|b_1| + |\sigma_1|)|x-y| \\ &= K|x-y|, \end{aligned}$$

where

$$K := (|b_1| + |\sigma_1|).$$

According to Theorem 3.20, there exists a square integrable solution, which is unique up to indistinguishability. Let $X = (X_t)_{t\in I}$ be this solution. Now defining

$$f(s,x,y) := x \exp\Big(\big(\frac{\sigma_1^2}{2} - b_1\big)s - \sigma_1 y\Big)$$

gives by Itô's Formula (Proposition 3.17) that

$$\begin{aligned} f(t, X_t, B_t) &= x_0 + \int_0^t \left(\frac{\sigma_1^2}{2} - b_1\right) f(s, X_s, B_s) ds \\ &\quad + \int_0^t b_1 f(s, X_s, B_s) + b_2 \exp\Big(\big(\frac{\sigma_1^2}{2} - b_1\big)s - \sigma_1 B_s\Big) ds \\ &\quad + \int_0^t \sigma_1 f(s, X_s, B_s) + \sigma_2 \exp\Big(\big(\frac{\sigma_1^2}{2} - b_1\big)s - \sigma_1 B_s\Big) dB_s \\ &\quad - \int_0^t \sigma_1 f(s, X_s, B_s) dB_s + \frac{1}{2}\int_0^t \sigma_1^2 f(s, X_s, B_s) ds \\ &\quad - \int_0^t \sigma_1^2 f(s, X_s, B_s) + \sigma_1\sigma_2 \exp\Big(\big(\frac{\sigma_1^2}{2} - b_1\big)s - \sigma_1 B_s\Big) ds \\ &= x_0 + \int_0^t (b_2 + \sigma_1\sigma_2) \exp\Big(\big(\frac{\sigma_1^2}{2} - b_1\big)s - \sigma_1 B_s\Big) ds \\ &\quad + \int_0^t \sigma_2 \exp\Big(\big(\frac{\sigma_1^2}{2} - b_1\big)s - \sigma_1 B_s\Big) dB_s \end{aligned}$$

for all $t \in I$ a.s. Therefore, we can conclude that

$$\begin{aligned} X_t &= \exp\Big(\big(b_1 - \frac{\sigma_1^2}{2}\big)s + \sigma_1 B_s\Big) f(t, X_t, B_t) \\ &= e^{\left(b_1 - \frac{\sigma_1^2}{2}\right)s + \sigma_1 B_s}\Big[x_0 + \int_0^t (b_2 + \sigma_1\sigma_2) e^{\left(\frac{\sigma_1^2}{2} - b_1\right)s - \sigma_1 B_s} ds \\ &\quad + \int_0^t \sigma_2 e^{\left(\frac{\sigma_1^2}{2} - b_1\right)s - \sigma_1 B_s} dB_s\Big] \end{aligned}$$

for all $t \in I$ a.s.

4. Backward Stochastic Differential Equations

4.1. **Backward Stochastic Differential Equations.**

The stochastic differential equation (SDE) presented earlier deals with a process $X = (X_t)_{t\in I}$, where the starting point $X_0 = x_0$ is given and the process fullfills the equation

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \ \forall t \in I \text{ a.s.}$$

However, the setting can be also turned the other way round. Let us study the setting, where the terminal value of the process $X_T = \xi$ is given, and the process satisfies the backward stochastic differential equation (BSDE), where the integration intervall is $[t, T]$. The BSDEs were first introduced by Bismut [1] and the theory was later extended by Pardoux and Peng in [12] and [13].

Definition 4.1. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$ be a stochastic basis. Let $f : \Omega \times I \times \mathbb{R}^2 \to \mathbb{R}$ be a function, where the function $(y, z) \mapsto f(\omega, t, y, z)$ is continuous for all $\omega \in \Omega, t \in I$, the stochastic process $(\omega, t) \mapsto f(\omega, t, y, z)$ is predictable for all $y, z \in \mathbb{R}$. Then f is called a* ***random generator****.*

Definition 4.2. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$ be a stochastic basis, let $B = (B_t)_{t\in I}$ be a Brownian motion such that $(\mathcal{F}_t)_{t\in I}$ is the augmentation of the natural filtration of B, let $f : \Omega \times I \times \mathbb{R}^2 \to \mathbb{R}$ be a random generator, and let ξ be an $\mathcal{F}_T$-measurable, square integrable random variable. Then the pair $(Y, Z) = ((Y_t)_{t\in I}, (Z_t)_{t\in I})$ is said to be the solution to the* ***backward stochastic differential equation*** *$BSDE(f, \xi)$ provided that*

(i) *Y is a continuous and $(\mathcal{F}_t)_{t\in I}$-adapted stochastic process,*
(ii) *$Z \in \mathcal{L}_2$,*
(iii) *$\mathbb{E} \int_0^T |f(t, Y_t, Z_t)|dt < \infty$, and*
(iv) *$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s \ \forall t \in I$ a.s.*

Definition 4.3. *The random variable ξ of the previous definition is called* ***terminal value****.*

Remark 4.4.

(i) Often the first argument of a random generator $f(\omega, t, y, z)$ is omitted. When using the form $f(t, y, z)$ one has to remember that the function is not deterministic in general.

(ii) The random variable ξ corresponds the terminal value of the stochastic process Y at time T:

$$Y_T = \xi + \int_T^T f(s, Y_s, Z_s)ds - \int_T^T Z_s dB_s = \xi \text{ a.s.}$$

(iii) Notice that in the forward stochastic differential equation the terminal value X_T is a square integrable random variable. Similarly, here it is reasonable to demand ξ to be a square integrable random variable.

(iv) The $BSDE(f, \xi)$ can be formally be written as

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s.$$

Example 4.5. Let $X = (X_t)_{t \in I}$ be a solution to the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dX_t,$$

$$X_0 = x_0.$$

Then

$$\begin{aligned} X_t - X_T &= x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \\ &\quad -x_0 - \int_0^T b(s, X_s)ds - \int_0^T \sigma(s, X_s)dB_s \\ &= -\int_t^T b(s, X_s)ds - \int_t^T \sigma(s, X_s)dB_s \ \forall t \in I \text{ a.s.}, \end{aligned}$$

which implies that

$$X_t = X_T - \int_t^T b(s, X_s)ds - \int_t^T \sigma(s, X_s)dB_s \ \forall t \in I \text{ a.s.},$$

Therefore $(X, \sigma(s, X))$ is a solution to the $BSDE(f, \xi)$, where $f(t, y, z) := -b(t, y)$ and $\xi := X_T$.

Remark 4.6. As Example 4.5 illustrates, the setting of BSDEs is more general than the setting of SDEs. The given form of a BSDE with two processes Y and Z has become established in the literature.

Theorem 4.7 (Martingale representation theorem)**.**
Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in I})$ be a stochastic basis that satisfies the usual conditions, let X be a square integrable, $\mathcal{F}_T$-measurable random variable, and let $B = (B_t)_{t \in I}$ be a Brownian motion such that $(\mathcal{F}_t)_{t \in I}$ is the augmentation of the natural filtration of B. Then there exists a stochastic

process $\Psi = (\Psi_t)_{t\in I} \in \mathcal{L}_2$ *such that*

$$\mathbb{E}[X|\mathcal{F}_t] = \mathbb{E}[X] + \int_0^t \Psi_s dB_s$$

a.s. for all $t \in I$.

The proof is given in [9, p. 185].

Example 4.8. Let us analyze the $BSDE(0,\xi)$, i.e.

$$Y_t = \xi - \int_t^T Z_s dB_s,$$

where ξ is an $\mathcal{F}_T$-measurable, square integrable, random variable. Let $X = (X_t)_{t\in I}$ be a stochastic process defined by $X_t := \mathbb{E}[\xi|\mathcal{F}_t]$. Then, X is $(\mathcal{F}_t)_{t\in I}$-adapted. According to the martingale representation theorem (Theorem 4.7) there exists a stochastic process $Z = (Z_t)_{t\in I} \in \mathcal{L}_2$ such that

$$X_t = \mathbb{E}[\xi|\mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t Z_s dB_s$$

a.s. for all $t \in I$. Now, we can define $Y = (Y_t)_{t\in I}$ by

$$Y_t := \mathbb{E}[\xi] + \int_0^t Z_s dB_s.$$

Therefore, Y is a continuous modification of X that is $(\mathcal{F}_t)_{t\in I}$-adapted. Moreover, we have that

$$\begin{aligned} Y_t - Y_T &= \mathbb{E}[\xi] + \int_0^t Z_s dB_s - \mathbb{E}[\xi] - \int_0^T Z_s dB_s \\ &= -\int_t^T Z_s dB_s \end{aligned}$$

for all $t \in I$ a.s., which implies that

$$\begin{aligned} Y_t &= Y_T - \int_t^T Z_s dB_s \\ &= \mathbb{E}[\xi|\mathcal{F}_T] - \int_t^T Z_s dB_s \\ &= \xi - \int_t^T Z_s dB_s, \end{aligned}$$

for all $t \in I$ a.s., and therefore, (Y, Z) is a solution to the $BSDE(0,\xi)$.

Definition 4.9. *Let* $\mathcal{S}_2$ *be the space of continuous and adapted stochastic processes* $X = (X_t)_{t\in I}$ *with* $\mathbb{E}\big[\sup_{t\in[0,T]} X_t^2\big] < \infty$. *Let* $\mathcal{S}_\infty$ *be the space of bounded continuous, and adapted stochastic processes.*

In [2] Briand, Delyon, Hu, Pardoux, and Stoica stated the following existence and uniqueness theorem.

Theorem 4.10 (Theorem 4.1, [2]). *Let the following assumptions hold:*

(i) *Let ξ be an $\mathcal{F}_T$-measurable, square integrable random variable.*
(ii) *Let $f : \Omega \times I \times \mathbb{R}^2 \to \mathbb{R}$ be a random generator.*
(iii) $\mathbb{E}\big(\int_0^T f(t,0,0)dt\big)^2 < \infty$
(iv) *There exists a $C > 0$ such that for all $y, z, \overline{y}, \overline{z} \in \mathbb{R}$, $\omega \in \Omega$, and $t \in I$ one has that*

$$|f(\omega,t,y,z) - f(\omega,t,\overline{y},\overline{z})| \leq C(|y-\overline{y}| + |z-\overline{z}|)$$

Then there exists a pair $(Y,Z) = ((Y_t)_{t\in I},(Z_t)_{t\in I})$, where $Y \in \mathcal{S}_2$ and $Z \in \mathcal{L}_2$, such that (Y,Z) solves the backward stochastic differential equation $BSDE(f,\xi)$. Moreover, the processes Y and Z are unique in $\mathcal{S}_2$ and $\mathcal{L}_2$, which means that for such solutions (Y,Z) and (Y^*,Z^*) one has

$$\mathbb{E}\big[\sup_{t\in[0,T]}(Y_t - Y_t^*)^2\big] = 0$$

and

$$\mathbb{E}\int_0^T (Z_t - Z_t^*)^2 dt = 0.$$

Theorem 4.10 follows from the work of Pardoux and Peng [12, 13], El Karoui, Peng, and Quenez [7], Pardoux [11], and Briand, Delyon, Hu, Pardoux, and Stoica [2].

Example 4.11. Let us analyze the BSDE

$$Y_t = \xi + \int_t^T (a_s + b_sY_s + c_sZ_s)ds - \int_t^T Z_s dB_s,$$

where $(b_s)_{s\in I}$ and $(c_s)_{s\in I}$ are bounded and progressively measurable stochastic processes, $(a_s)_{s\in I} \in \mathcal{L}_2$, and ξ is an $\mathcal{F}_T$-measurable, square integrable random variable.

Let $\Gamma = (\Gamma_t)_{t\in I}$ be the $\mathcal{S}_2$-solution to the SDE

$$dX_t = b_tX_tdt + c_tX_tdB_t,$$
$$X_0 = 1.$$

i.e.

$$\Gamma_t = 1 + \int_0^t b_s\Gamma_s ds + \int_0^t c_s\Gamma_s dB_s$$

for all $t \in I$ a.s. We claim that

$$Y_t = \mathbb{E}\left[\frac{\Gamma_T}{\Gamma_t}\xi + \int_t^T a_s\frac{\Gamma_s}{\Gamma_t}ds\middle|\mathcal{F}_t\right]$$

for all $t \in I$ a.s.

Firstly, we notice that the assumptions (i)-(vi) of Theorem 4.10 hold in the setting given above. Therefore, there exists a unique solution $(Y, Z) = ((Y_t)_{t\in I}, (Z_t)_{t\in I}) \in (\mathcal{S}_2, \mathcal{L}_2)$. Now, we have that

$$\begin{aligned} Y_t - Y_0 &= \xi + \int_t^T (a_s + b_s Y_s + c_s Z_s)ds - \int_t^T Z_s dB_s \\ &\quad -\xi - \int_0^T (a_s + b_s Y_s + c_s Z_s)ds + \int_0^T Z_s dB_s \\ &= -\int_0^t (a_s + b_s Y_s + c_s Z_s)ds + \int_0^t Z_s dB_s \end{aligned}$$

for all $t \in I$ a.s., which implies that

$$Y_t = Y_0 - \int_0^t (a_s + b_s Y_s + c_s Z_s)ds + \int_0^t Z_s dB_s$$

for all $t \in I$ a.s. Now defining

$$f(s, x, y) := xy$$

gives by Itô's Formula (Proposition 3.17)

$$\begin{aligned} f(t, \Gamma_t, Y_t) &= Y_0 + \int_0^t Y_s b_s \Gamma_s ds - \int_0^t \Gamma_s (a_s + b_s Y_s + c_s Z_s)ds \\ &\quad + \int_0^t Y_s c_s \Gamma_s dB_s + \int_0^t \Gamma_s Z_s dB_s + \int_0^t c_s \Gamma_s Z_s ds \\ &= Y_0 - \int_0^t a_s \Gamma_s ds + \int_0^t (c_s Y_s + Z_s)\Gamma_s dB_s \end{aligned}$$

for all $t \in I$ a.s. Therefore,

$$\begin{aligned} \Gamma_t Y_t - \Gamma_T Y_T &= f(t, \Gamma_t, Y_t) - f(T, \Gamma_T, Y_T) \\ &= Y_0 - \int_0^t a_s \Gamma_s ds + \int_0^t (c_s Y_s + Z_s)\Gamma_s dB_s \\ &\quad -Y_0 + \int_0^T a_s \Gamma_s ds - \int_0^T (c_s Y_s + Z_s)\Gamma_s dB_s \\ &= \int_t^T a_s \Gamma_s ds - \int_t^T (c_s Y_s + Z_s)\Gamma_s dB_s \end{aligned}$$

for all $t \in I$ a.s., which implies that

$$\Gamma_t Y_t = \Gamma_T Y_T + \int_t^T a_s \Gamma_s ds - \int_t^T (c_s Y_s + Z_s)\Gamma_s dB_s$$

for all $t \in I$ a.s. Taking conditional expectation with respect to $\mathcal{F}_t$ from both sides gives

$$\Gamma_t Y_t = \mathbb{E}\Big[\Gamma_T Y_T + \int_t^T a_s \Gamma_s ds \Big| \mathcal{F}_t\Big]$$

for all $t \in I$ a.s., since $\mathbb{E}[\Gamma_t Y_t | \mathcal{F}_t] = \Gamma_t Y_t$ and $\mathbb{E}\Big[\int_t^T (c_s Y_s + Z_s)\Gamma_s dB_s \Big| \mathcal{F}_t\Big] = 0$ for all $t \in I$. The latter is true, since $\Big(\int_0^t (c_s Y_s + Z_s)\Gamma_s dB_s\Big)_{t\in I}$ is a martingale (see Lemma 4.12 below).

Finally, dividing both sides by Γ_t and replacing Y_T with ξ gives

$$Y_t = \mathbb{E}\Big[\frac{\Gamma_T}{\Gamma_t}\xi + \int_t^T a_s \frac{\Gamma_s}{\Gamma_t} ds \Big| \mathcal{F}_t\Big]$$

for all $t \in I$ a.s.

Lemma 4.12. *The stochastic process* $\Big(\int_0^t (c_s Y_s + Z_s)\Gamma_s dB_s\Big)_{t\in I}$ *of the Example 4.11 is a martingale.*

Proof. Firstly, one realizes that

$$\begin{aligned}\int_0^T ((c_s Y_s + Z_s)\Gamma_s)^2 ds &\leq 2 \sup_{s\in I} \Gamma_s^2 \big(T \sup_{s\in I} c_s^2 \sup_{s\in I} Y_s^2 + \int_0^T Z_s^2 ds\big) \\ &< \infty\end{aligned}$$

a.s., since $(c_s)_{s\in I}$ is bounded, $(\Gamma_s)_{s\in I}$ and $(Y_s)_{s\in I}$ are continuous stochastic processes and $(Z_s)_{s\in I} \in \mathcal{L}_2$. Therefore, by Itô's isometry (Proposition 3.10) $\Big(\int_0^t (c_s Y_s + Z_s)\Gamma_s dB_s\Big)_{t\in I}$ is a local martingale (see Definition A.1 of Appendix A).

Next, we conclude that

$$\begin{aligned}\mathbb{E}\Big(\int_0^T ((c_s Y_s + Z_s)\Gamma_s)^2 ds\Big)^{\frac{1}{2}} &\leq \mathbb{E} \sup_{s\in I} |\Gamma_s| \Big(\int_0^T (c_s Y_s + Z_s)^2 ds\Big)^{\frac{1}{2}} \\ &\leq \Big(\mathbb{E} \sup_{s\in I} |\Gamma_s|^2\Big)^{\frac{1}{2}} \Big(\mathbb{E}\int_0^T (c_s Y_s + Z_s)^2 ds\Big)^{\frac{1}{2}} \\ &< \infty,\end{aligned}$$

since $(c_s)_{s\in I}$ is bounded, $(Y_s)_{s\in I} \in \mathcal{S}_2$, $(Z_s)_{s\in I} \in \mathcal{L}_2$, and $\mathbb{E}\sup_{s\in I} |\Gamma_s|^2 < \infty$ by Proposition 3.21.

According to Theorem A.4 of Appendix A, there exists $C > 0$ such that

$$\mathbb{E}\sup_{t\in I}\Big|\int_0^t (c_sY_s + Z_s)\Gamma_s dB_s\Big| \le C\mathbb{E}\Big(\int_0^T ((c_sY_s + Z_s)\Gamma_s)^2 ds\Big)^{\frac{1}{2}} < \infty,$$

which implies that the family $\Big\{\int_0^\sigma (c_sY_s + Z_s)\Gamma_s dB_s : \sigma \in \Sigma\Big\}$ is uniformly integrable, when Σ is the class of all stopping times $\sigma : \Omega \to I$. Therefore, according to Proposition A.5 of Appendix A $\Big(\int_0^t (c_sY_s + Z_s)\Gamma_s dB_s\Big)_{t\in I}$ is a martingale.

□

In Example 4.11 we were able to find an explicit solution for the Y-process of the BSDE. In Example 4.13 below we have a special case of the BSDE of Example 4.11, where we can find an explicit solution for the Z-process as well.

Example 4.13. Let us find a solution to the following BSDE:

$$Y_t = e^{B_T+T} + \int_t^T Y_s ds - \int_t^T Z_s dB_s.$$

We claim that

$$Y_t = e^{\frac{5}{2}T-\frac{3}{2}t+B_t}$$

for all $t \in I$ a.s. and

$$Z_t = e^{\frac{5}{2}T-\frac{3}{2}t+B_t}$$

$d\mathbb{P} \otimes dt$-a.s.

In the Example 4.11 we were able to find an explicite representation of the process Y. However, the solution can be simplified in this special case. We have

$$Y_t = \mathbb{E}\Big[\frac{\Gamma_T}{\Gamma_t} e^{B_T+T}\Big|\mathcal{F}_t\Big],$$

for all $t \in I$ a.s., where

$$\Gamma_t = 1 + \int_0^t \Gamma_s ds.$$

Above, we have an ordinary differential equation with a solution $\Gamma_t = e^t$ for all $t \in I$. Therefore, we can deduce that, a.s. for all $t \in I$,

$$\begin{aligned} Y_t &= e^{T-t}\mathbb{E}\big[e^{B_T+T}\big|\mathcal{F}_t\big] \\ &= e^{T-t}\mathbb{E}\big[e^{B_T-B_t}e^{B_t+T}\big|\mathcal{F}_t\big] \end{aligned}$$

$$\begin{aligned}
&= e^{T-t}e^{B_t+T}\mathbb{E}\big[e^{B_T-B_t}\big|\mathcal{F}_t\big]\\
&= e^{2T-t+B_t}\mathbb{E}\big[e^{B_T-B_t}\big]\\
&= e^{2T-t+B_t}\mathbb{E}\big[e^{B_{T-t}}\big]\\
&= e^{2T-t+B_t}e^{(T-t)/2}\\
&= e^{\frac{5}{2}T-\frac{3}{2}t+B_t}.
\end{aligned}$$

If $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is defined by $f(s,x):=e^{x-\frac{3}{2}s}$, we have by Itô's formula (Proposition 3.14) that, a.s.,

$$\begin{aligned}
e^{B_T-\frac{3}{2}T} &= f(T,B_T)\\
&= 1-\int_0^T \frac{3}{2}e^{B_s-\frac{3}{2}s}ds+\int_0^T e^{B_s-\frac{3}{2}s}dB_s+\frac{1}{2}\int_0^T e^{B_s-\frac{3}{2}s}ds\\
&= 1+\int_0^T e^{B_s-\frac{3}{2}s}dB_s-\int_0^T e^{B_s-\frac{3}{2}s}ds.
\end{aligned}$$

Multiplying both sides by $e^{\frac{5}{2}T}$ gives, a.s.,

$$e^{B_T+T}=e^{\frac{5}{2}T}+\int_0^T e^{\frac{5}{2}T-\frac{3}{2}s+B_s}dB_s-\int_0^T e^{\frac{5}{2}T-\frac{3}{2}s+B_s}ds.$$

On the other hand, inserting the solution of the process Y into the original BSDE at $t=0$ gives, a.s.,

$$e^{\frac{5}{2}T}=e^{B_T+T}+\int_0^T e^{\frac{5}{2}T-\frac{3}{2}s+B_s}ds-\int_0^T Z_s dB_s.$$

Combining the equations above gives, a.s.,

$$\int_0^T Z_s dB_s=\int_0^T e^{\frac{5}{2}T-\frac{3}{2}s+B_s}dB_s.$$

Now, Proposition 3.12 gives

$$Z_t=e^{\frac{5}{2}T-\frac{3}{2}t+B_t}$$

$d\mathbb{P}\otimes dt$-a.s.

4.2. **Quadratic Backward Stochastic Differential Equations.** We start with formulating assumptions on the random generator f and the terminal value ξ of a BSDE:

(P1) There exists $\alpha,\beta\geq 0$, $\gamma>0$ such that for all $(t,\omega)\in[0,T]\times\Omega$ the function $(y,z)\mapsto f(t,y,z)$ is continuous and $|f(\omega,t,y,z)|\leq\alpha+\beta|y|+\frac{\gamma}{2}|z|^2$ for all $(\omega,t,y,z)\in\Omega\times[0,T]\times\mathbb{R}^2$.

(P2) $\mathbb{E}\big[e^{\gamma e^{\beta T}|\xi|}\big]<\infty$.

(P3) There exists $\lambda > \gamma e^{\beta T}$ such that $\mathbb{E}\left[e^{\lambda|\xi|}\right] < \infty$.

Remark 4.14. Notice that (P3) implies (P2). In some places we will only assume (P1) or (P1) and (P2). In the following we need to assume that $\alpha \geq \frac{\beta}{\gamma}$. However, by setting α large enough, this assumption can be satisfied without losing any of the properties (P1)-(P3).

Theorem 4.15 (Optional sampling theorem)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$ be a stochastic basis, let $X = (X_t)_{t\in I}$ be a continuous martingale, and let $\tau : \Omega \to I$ be a stopping time. Then we have that*

$$\mathbb{E}X_\tau = \mathbb{E}X_0.$$

The proof is given in [9, p. 19-20].

Example 4.16. Let us analyze the BSDE

$$Y_t = \xi + \frac{1}{2}\int_t^T Z_s^2 ds - \int_t^T Z_s dB_s,$$

where ξ and e^ξ are square integrable random variables, and where we assume that $\mathbb{E}\int_0^T Z_s ds < \infty$. We have

$$\begin{aligned} Y_t - Y_0 &= \xi + \frac{1}{2}\int_t^T Z_s^2 ds - \int_t^T Z_s dB_s \\ &\quad -\xi - \frac{1}{2}\int_0^T Z_s^2 ds + \int_0^T Z_s dB_s \\ &= -\frac{1}{2}\int_0^t Z_s^2 ds + \int_0^t Z_s dB_s, \end{aligned}$$

which implies that

$$Y_t = Y_0 - \frac{1}{2}\int_0^t Z_s^2 ds + \int_0^t Z_s dB_s.$$

Let $f(s,x) := e^x$. Then by Itô's formula (Proposition 3.14)

$$\begin{aligned} e^{Y_t} &= e^{Y_0} - \frac{1}{2}\int_0^t e^{Y_s} Z_s^2 ds + \int_0^t e^{Y_s} Z_s dB_s + \frac{1}{2}\int_0^t e^{Y_s} Z_s^2 ds \\ &= e^{Y_0} + \int_0^t e^{Y_s} Z_s dB_s \end{aligned}$$

as long as $\left(e^{Y_s} Z_s\right)_{s\in I} \in \mathcal{L}_2$. Therefore,

$$e^{Y_t} - e^{Y_T} = e^{Y_0} + \int_0^t e^{Y_s} Z_s dB_s - e^{Y_0} - \int_0^T e^{Y_s} Z_s dB_s$$

$$= \quad -\int_t^T e^{Y_s} Z_s dB_s,$$

which implies that

$$e^{Y_t} = e^{\xi} - \int_t^T e^{Y_s} Z_s dB_s.$$

Let us define $\xi^* := e^{\xi}$, $Y^* = (Y_t^*)_{t\in I}$ by $Y_t^* := e^{Y_t}$, and $Z^* = (Z_t^*)_{t\in I}$ by $Z_t^* := e^{Y_t} Z_t$. Then the above equation can be written in the following way:

$$Y_t^* = \xi^* - \int_t^T Z_s^* dB_s.$$

According to the Example 4.8, there exists a solution (Y^*, Z^*) to the $BSDE(0, \xi^*)$ above. Let us show that $Y_t^* > 0$ for all $t \in I$ a.s.

Let us define

$$\tau_0(\omega) := \inf\{t \in I : Y_t^*(\omega) = 0\} \wedge T.$$

We have that $Y_T^* > 0$ a.s. by assumption. Now, according to Theorem 4.15 (Optional sampling theorem) we have that

$$\mathbb{E} Y_\tau^* = \mathbb{E} Y_0^*.$$

On the other hand, we notice that

$$\mathbb{E} Y_\tau = \mathbb{E} \mathbb{1}_{\{\tau_0 < T\}} 0 + \mathbb{E} \mathbb{1}_{\{\tau_0 = T\}} \xi = \mathbb{E} \mathbb{1}_{\{\tau_0 = T\}} \xi^*$$

and

$$\mathbb{E} Y_0^* = \mathbb{E} Y_T^* = \mathbb{E} \xi^*.$$

Therefore, we have that

$$\mathbb{E} \mathbb{1}_{\{\tau_0 = T\}} \xi^* = \mathbb{E} \xi^*,$$

which implies that $\tau_0 = T$ a.s. and therefore $Y_t^* > 0$ for all $t \in I$ a.s.

Now, setting $Y_t := \log Y_t^*$ and $Z_t := \frac{Z_t^*}{Y_t^*}$ gives a candidate for solution to the original BSDE. We can define that $Y_t := 0$ and $Z_t := 0$ for all $t \in I$ on the set $\{\omega \in \Omega : Y_t^*(\omega) \leq 0 \text{ for some } t \in I\}$ if needed.

4.3. **Existence of Quadratic BSDEs.**

We are going to formulate and prove some existence results and regularity conditions for the backward stochastic differential equations under the conditions (P1)-(P3).

For the following, we need to introduce the functions ϕ_t and H.

Definition 4.17. *Let $\alpha, \beta \geq 0, \gamma > 0$ be the parameters defined in (P1)-(P3). The function $H : \mathbb{R} \to \mathbb{R}$ is defined by*

$$H(\phi) := \gamma\alpha \mathbb{1}_{(-\infty,1)}(\phi) + \phi(\gamma\alpha + \beta \log \phi) \mathbb{1}_{[1,\infty)}(\phi).$$

For $t \in I$, the function $\phi_t : \mathbb{R} \to \mathbb{R}$ is defined by

$$\phi_t(z) := \exp\left(\gamma\alpha \frac{e^{\beta(T-t)} - 1}{\beta}\right) \exp\left(z\gamma e^{\beta(T-t)}\right),$$

if $\beta > 0$ and $z \geq 0$,

$$\phi_t(z) := e^{\gamma\alpha(T-t)} e^{\gamma z}$$

if $\beta = 0$ and $z \geq 0$,

$$\phi_t(z) := e^{\gamma z} + \gamma\alpha(T-t),$$

if $e^{\gamma z} + \gamma\alpha T \leq 1$ and $z < 0$,

$$\phi_t(z) := [e^{\gamma z} + \gamma\alpha(T-t)]\mathbb{1}_{(S,T]}(t) + \exp\left(\gamma\alpha \frac{e^{\beta(S-t)} - 1}{\beta}\right)\mathbb{1}_{[0,S]}(t),$$

where $S \in I$ is a constant such that $e^{\gamma z} + \gamma\alpha(T-S) = 1$, if $\beta > 0, e^{\gamma z} + \gamma\alpha T > 1$ and $z < 0$, and

$$\phi_t(z) := [e^{\gamma z} + \gamma\alpha(T-t)]\mathbb{1}_{(S,T]}(t) + e^{\gamma\alpha(S-t)}\mathbb{1}_{[0,S]}(t),$$

where $S \in I$ is a constant such that $e^{\gamma z} + \gamma\alpha(T-S) = 1$, if $\beta = 0, e^{\gamma z} + \gamma\alpha T > 1$ and $z < 0$.

Proposition 4.18. *Let $\alpha, \beta \geq 0, \gamma > 0$. Then the function ϕ_t of the Definition 4.17 satisfies the following differential equation:*

$$\phi_t(z) = e^{\gamma z} + \int_t^T H\big(\phi_s(z)\big)ds$$

for all $t \in I, z \in \mathbb{R}$, where H is as in the Definition 4.17. Moreover, the following properties hold for the functions H and ϕ_t:

- (i) *The function $\phi \mapsto H(\phi)$ is increasing, continuous, and convex on $\mathbb{R}$.*
- (ii) *The function $t \mapsto \phi_t(z)$ is decreasing and continuous for $z \in \mathbb{R}$.*
- (iii) *The function $z \mapsto \phi_t(z)$ is increasing and continuous for $t \in I$.*

Proof. The properties (i)-(iii) are straightforward to check. To show that the differential equation is satisfied, it is sufficient to show, that

- (i) the function $t \mapsto \phi_t(z)$ is the antiderivative of the function $t \mapsto -H\big(\phi_t(z)\big)$ on I, and
- (ii) $\phi_T(z) = e^{\gamma z}$,

since then the fundamental theorem of calculus gives

$$\phi_t(z) - \phi_T(z) = \int_t^T H\big(\phi_s(z)\big)ds,$$

which implies that

$$\phi_t(z) = \phi_T(z) + \int_t^T H\big(\phi_s(z)\big)ds = e^{\gamma z} + \int_t^T H\big(\phi_s(z)\big)ds.$$

Case $\beta > 0$ and $z \geq 0$: For all $t \in I$ one has

$$\begin{aligned}
\frac{\partial \phi_t(z)}{\partial t} &= \frac{\partial}{\partial t}\left[\exp\left(\gamma\alpha\frac{e^{\beta(T-t)}-1}{\beta}\right)\exp\left(z\gamma e^{\beta(T-t)}\right)\right] \\
&= -\gamma e^{\beta(T-t)}(\beta z + \alpha)\left[\exp\left(\gamma\alpha\frac{e^{\beta(T-t)}-1}{\beta}\right)\exp\left(z\gamma e^{\beta(T-t)}\right)\right] \\
&= -\left[\gamma\alpha + \beta\left(\gamma\alpha\frac{e^{\beta(T-t)}-1}{\beta} + z\gamma e^{\beta(T-t)}\right)\right]\phi_t(z) \\
&= -\Big[\gamma\alpha + \beta\log\phi_t(z)\Big]\phi_t(z) \\
&= -H\big(\phi_t(z)\big)
\end{aligned}$$

and

$$\begin{aligned}
\phi_T(z) &= \exp\left(\gamma\alpha\frac{e^{\beta(T-T)}-1}{\beta}\right)\exp\left(z\gamma e^{\beta(T-T)}\right) \\
&= e^{\gamma z}.
\end{aligned}$$

Case $\beta = 0$ and $z \geq 0$: For all $t \in I$

$$\begin{aligned}
\frac{\partial \phi_t(z)}{\partial t} &= \frac{\partial}{\partial t}\Big[e^{\gamma\alpha(T-t)}e^{\gamma z}\Big] \\
&= -\gamma\alpha\Big[e^{\gamma\alpha(T-t)}e^{\gamma z}\Big] \\
&= -\gamma\alpha\phi_t(z) \\
&= -H\big(\phi_t(z)\big)
\end{aligned}$$

and

$$\phi_T(z) = e^{\gamma\alpha(T-T)}e^{\gamma z} = e^{\gamma z}.$$

Case $z < 0$ and $e^{\gamma z} + \gamma\alpha T \leq 1$: For all $t \in I$

$$\frac{\partial \phi_t(z)}{\partial t} = \frac{\partial}{\partial t}\Big[e^{\gamma z} + \gamma\alpha(T-t)\Big] = -\gamma\alpha = -H\big(\phi_t(z)\big)$$

and

$$\phi_T(z) \;=\; e^{\gamma z} + \gamma\alpha(T - T) = e^{\gamma z}.$$

Case $\beta > 0, z < 0$, and $e^{\gamma z} + \gamma\alpha T > 1$: Let $S \in I$ be such that $e^{\gamma z} + \gamma\alpha(T - S) = 1$. Then, for all $t \in [0, S)$,

$$\begin{aligned}
\frac{\partial\phi_t(z)}{\partial t} &= \frac{\partial}{\partial t}\left[\exp\left(\gamma\alpha\frac{e^{\beta(S-t)} - 1}{\beta}\right)\right] \\
&= -\gamma\alpha e^{\beta(S-t)}\left[\exp\left(\gamma\alpha\frac{e^{\beta(S-t)} - 1}{\beta}\right)\right] \\
&= -\left[\gamma\alpha + \beta\left(\gamma\alpha\frac{e^{\beta(S-t)} - 1}{\beta}\right)\right]\phi_t(z) \\
&= -\Big[\gamma\alpha + \beta\log\phi_t(z)\Big]\phi_t(z) \\
&= -H\big(\phi_t(z)\big)
\end{aligned}$$

and for all $t \in (S, T]$ one has

$$\frac{\partial\phi_t(z)}{\partial t} \;=\; \frac{\partial}{\partial t}\Big[e^{\gamma z} + \gamma\alpha(T - t)\Big] = -\gamma\alpha = -H\big(\phi_t(z)\big).$$

Moreover, since $\phi \mapsto H(\phi)$ and $t \mapsto \phi_t(z)$ are continuous functions, we have that

$$\begin{aligned}
\lim_{t\to S-}\frac{\partial\phi_t(z)}{\partial t} &= \lim_{t\to S-}\big[-H\big(\phi_t(z)\big)\big] \\
&= \lim_{t\to S+}\big[-H\big(\phi_t(z)\big)\big] \\
&= \lim_{t\to S+}\frac{\partial\phi_t(z)}{\partial t}
\end{aligned}$$

and therefore, for all $t \in I$

$$\frac{\partial\phi_t(z)}{\partial t} = -H\big(\phi_t(z)\big).$$

Finally

$$\phi_T(z) \;=\; e^{\gamma z} + \gamma\alpha(T - T) = e^{\gamma z}.$$

Case $\beta = 0, z < 0$, and $e^{\gamma z} + \gamma\alpha T > 1$: Let $S \in I$ be such that $e^{\gamma z} + \gamma\alpha(T - S) = 1$. Then, for all $t \in [0, S)$

$$\begin{aligned}\frac{\partial \phi_t(z)}{\partial t} &= \frac{\partial}{\partial t}\Big[e^{\gamma\alpha(S-t)}\Big] \\ &= -\gamma\alpha\Big[e^{\gamma\alpha(S-t)}\Big] \\ &= -\gamma\alpha\phi_t(z) \\ &= -H\big(\phi_t(z)\big).\end{aligned}$$

In the case $t \in (S,T]$ we get, as for $\beta > 0$,

$$\frac{\partial \phi_t(z)}{\partial t} = -H\big(\phi_t(z)\big).$$

Moreover, as in the previous case we have that

$$\lim_{t\to S-}\frac{\partial \phi_t(z)}{\partial t} = \lim_{t\to S+}\frac{\partial \phi_t(z)}{\partial t}$$

and therefore, for all $t \in I$

$$\frac{\partial \phi_t(z)}{\partial t} = -H\big(\phi_t(z)\big).$$

Finally

$$\phi_T(z) \;=\; e^{\gamma z} + \gamma\alpha(T-T) = e^{\gamma z}.$$

□

Theorem 4.19 (Comparison theorem)**.** *Let the following assumptions hold:*

(i) *Let $f, \hat{f} : \Omega \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be random generators.*
(ii) *Let the function $(\omega,t) \mapsto f(\omega,t,0,0)$ be in $\mathcal{L}_2$.*
(iii) *Let the function $(\omega,t) \mapsto \hat{f}(\omega,t,0,0)$ be in $\mathcal{L}_2$.*
(iv) *For all $y, z, \overline{y}, \overline{z} \in \mathbb{R}$ and $t \in I$, there exists $C > 0$ such that,*

$$|f(\omega,t,y,z) - f(\omega,t,\overline{y},\overline{z})| \le C(|y-\overline{y}| + |z-\overline{z}|) \;\; a.s.$$

(v) *For all $y, z, \overline{y}, \overline{z} \in \mathbb{R}$ and $t \in I$, there exists $\hat{C} > 0$ such that,*

$$|\hat{f}(\omega,t,y,z) - \hat{f}(\omega,t,\overline{y},\overline{z})| \le \hat{C}(|y-\overline{y}| + |z-\overline{z}|) \;\; a.s.$$

(vi) *Let (Y,Z) and $(\hat{Y},\hat{Z})$ be solutions of $BSDE(f,\xi)$ and $BSDE(\hat{f},\hat{\xi})$ in the sence of Theorem 4.10.*
(vii) *Let $\xi \ge \hat{\xi}$ a.s. and $f(t,\hat{Y}_t,\hat{Z}_t) \ge \hat{f}(t,\hat{Y}_t,\hat{Z}_t)$ $d\mathbb{P}\otimes dt$-a.s.*

Then $Y_t \ge \hat{Y}_t$ a.s. for all $t \in I$.

The proof is given in [7, Theorem 2.2].

Theorem 4.20. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$ be a stochastic basis that satisfies the usual conditions, and let $M = (M_t)_{t\in I}$ be a super-martingale. Then the super-martingale M has an RCLL modification which is also a super-maringale if and only if the function $t \mapsto \mathbb{E}M_t$ is right-continuous.*

The proof is given in [9, p. 16-17].

Lemma 4.21. *Assume that the property (P1) holds for a random generator f, let ξ be a bounded $\mathcal{F}_T$-measurable random variable, and let (Y, Z) be a solution to the $BSDE(f, \xi)$ such that Y is a bounded, continuous, and $(\mathcal{F}_t)_{t\in I}$-adapted stochastic process. Then we have that, a.s.,*

$$-\frac{1}{\gamma}\log \mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t] \le Y_t \le \frac{1}{\gamma}\log \mathbb{E}[\phi_t(\xi)|\mathcal{F}_t]. \tag{3}$$

Proof. Define $\Phi_t := \mathbb{E}[\phi_t(\xi)|\mathcal{F}_t]$ for all $t \in I$. Because for $0 \le s \le t \le T$ we have

$$\mathbb{E}[\Phi_t|\mathcal{F}_s] = \mathbb{E}\big[\mathbb{E}[\phi_t(\xi)|\mathcal{F}_t]|\mathcal{F}_s\big] \le \mathbb{E}[\phi_s(\xi)|\mathcal{F}_s] = \Phi_s$$

a.s., the stochastic process $(\Phi_t)_{t\in I}$ is a super-martingale. Moreover, by dominated convergence, the map $t \to \mathbb{E}\Phi_t = \mathbb{E}\phi_t(\xi)$ is continuous. Applying Theorem 4.20 there exists an RCLL modification of $(\Phi_t)_{t\in I}$ which is taken in the sequel.

The martingale representation theorem (Theorem 4.7) gives a stochastic process $\Psi = (\Psi_t)_{t\in I} \in \mathcal{L}_2$ such that

$$\begin{aligned} &\mathbb{E}\Big[e^{\gamma\xi} + \int_0^T \mathbb{E}\big[H(\phi_s(\xi))\big|\mathcal{F}_s\big]ds\Big|\mathcal{F}_t\Big] \\ &= \mathbb{E}\Big[e^{\gamma\xi} + \int_0^T \mathbb{E}\big[H(\phi_s(\xi))\big|\mathcal{F}_s\big]ds\Big] + \int_0^t \Psi_s dB_s \end{aligned} \tag{4}$$

for all $t \in I$ a.s. Now it follows from Proposition 4.18 that, a.s.,

$$\begin{aligned} \Phi_t &= \mathbb{E}[\phi_t(\xi)|\mathcal{F}_t] \\ &= \mathbb{E}\Big[e^{\gamma\xi} + \int_t^T H(\phi_s(\xi))ds\Big|\mathcal{F}_t\Big] \\ &= \mathbb{E}\Big[e^{\gamma\xi} + \int_t^T \mathbb{E}\big[H(\phi_s(\xi))\big|\mathcal{F}_s\big]ds\Big|\mathcal{F}_t\Big] \\ &= \mathbb{E}\Big[e^{\gamma\xi} + \int_0^T \mathbb{E}\big[H(\phi_s(\xi))\big|\mathcal{F}_s\big]ds\Big|\mathcal{F}_t\Big] - \int_0^t \mathbb{E}\big[H(\phi_s(\xi))\big|\mathcal{F}_s\big]ds \\ &= \mathbb{E}\Big[e^{\gamma\xi} + \int_0^T \mathbb{E}\big[H(\phi_s(\xi))\big|\mathcal{F}_s\big]ds\Big] + \int_0^t \Psi_s dB_s \end{aligned}$$

$$
\begin{aligned}
& -\int_0^t \mathbb{E}\big[H(\phi_s(\xi))\big|\mathcal{F}_s\big]ds \\
= \;& e^{\gamma\xi} + \int_t^T \mathbb{E}\big[H(\phi_s(\xi))\big|\mathcal{F}_s\big]ds - \int_t^T \Psi_s dB_s.
\end{aligned}
$$

On the other hand, a.s.,

$$
\begin{aligned}
Y_t - Y_0 &= \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s \\
&\quad -\xi - \int_0^T f(s, Y_s, Z_s)ds + \int_0^T Z_s dB_s \\
&= -\int_0^t f(s, Y_s, Z_s)ds + \int_0^t Z_s dB_s,
\end{aligned}
$$

which implies that, a.s.,

$$
Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s)ds + \int_0^t Z_s dB_s.
$$

Let $P_t := e^{\gamma Y_t}$ and $Q_t := \gamma e^{\gamma Y_t} Z_t$. Then by Itô's formula (Proposition 3.14), a.s.,

$$
\begin{aligned}
P_t &= e^{\gamma Y_t} \\
&= e^{\gamma Y_0} + \int_0^t \gamma e^{\gamma Y_s}(-f(s, Y_s, Z_s))ds \\
&\quad + \int_0^t \gamma e^{\gamma Y_s} Z_s dB_s + \frac{1}{2}\int_0^t \gamma^2 e^{\gamma Y_s} Z_s^2 ds \\
&= P_0 - \int_0^t F(s, P_s, Q_s)ds + \int_0^t Q_s dB_s,
\end{aligned}
$$

where $F(s,p,q) := \mathbb{1}_{p>0}\Big[\gamma p f\Big(s, \frac{\log p}{\gamma}, \frac{q}{\gamma p}\Big) - \frac{q^2}{2p}\Big]$. Therefore, a.s.,

$$
\begin{aligned}
P_t - P_T &= P_0 - \int_0^t F(s, P_s, Q_s)ds + \int_0^t Q_s dB_s \\
&\quad -P_0 + \int_0^T F(s, P_s, Q_s)ds - \int_0^T Q_s dB_s \\
&= \int_t^T F(s, P_s, Q_s)ds - \int_t^T Q_s dB_s,
\end{aligned}
$$

which implies that, a.s.,

$$\begin{aligned} P_t &= P_T + \int_t^T F(s, P_s, Q_s)ds - \int_t^T Q_s dB_s \\ &= e^{\gamma\xi} + \int_t^T F(s, P_s, Q_s)ds - \int_t^T Q_s dB_s. \end{aligned}$$

Let

$$A(s, y, z) := H(y) + \mathbb{E}\big[H(\phi_s(\xi))\big|\mathcal{F}_s\big] - H(\Phi_s)$$

and

$$B(s, y, z) := H(y) + F(s, P_s, Q_s) - H(P_s).$$

From the convexity of H we get that, a.s.,

$$\begin{aligned} &\mathbb{E}\big[H(\phi_s(\xi))\big|\mathcal{F}_s\big] - H(\Phi_s) \\ &= \mathbb{E}\big[H(\phi_s(\xi))\big|\mathcal{F}_s\big] - H(\mathbb{E}[\phi_s(\xi)|\mathcal{F}_s]) \geq 0. \end{aligned} \tag{5}$$

On the other hand, for any $p > 0, q \in \mathbb{R}$, (P1) implies that

$$\begin{aligned} F(s, p, q) &= \gamma p f\Big(s, \frac{\log p}{\gamma}, \frac{q}{\gamma p}\Big) - \frac{q^2}{2p} \\ &\leq p(\gamma\alpha + \beta|\log p|) + \frac{q^2}{2p} - \frac{q^2}{2p} \\ &= p(\gamma\alpha + \beta|\log p|) \\ &\leq \gamma\alpha \mathbb{1}_{(-\infty,1)}(p) + p(\gamma\alpha + \beta\log p)\mathbb{1}_{[1,\infty)}(p) \\ &= H(p), \end{aligned}$$

since $\alpha \geq \frac{\beta}{\gamma}$ as we have assumed in the Remark 4.14. The fact

$$F(s, P_s, Q_s) - H(P_s) \leq 0$$

together with the fact

$$\mathbb{E}\big[H(\phi_s(\xi))\big|\mathcal{F}_s\big] - H(\Phi_s) \geq 0$$

implies that

$$A(s, P_s, Q_s) \geq B(s, P_s, Q_s)$$

for all $s \in I, \omega \in \Omega$. We note that, a.s.,

$$\Phi_t = e^{\gamma\xi} + \int_t^T A(s, \Phi_s, \Psi_s)ds - \int_t^T \Psi_s dB_s$$

and

$$P_t = e^{\gamma\xi} + \int_t^T B(s, P_s, Q_s)ds - \int_t^T Q_s dB_s.$$

Since Φ and P are bounded and therefore square integrable, it is possible to use the comparison theorem (Theorem 4.19) to get $\Phi_t \geq P_t$

a.s., which implies that $Y_t \leq \frac{1}{\gamma}\log(\mathbb{E}[\phi_t(\xi)|\mathcal{F}_t])$ a.s. The other inequality $Y_t \geq -\frac{1}{\gamma}\log(\mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t])$ follows from the fact, that the property (P1) is satisfied also with the random generator $(\omega, t, y, z) \mapsto -f(\omega, t, -y, -z)$, and therefore, a.s.,

$$-Y_t \leq \frac{1}{\gamma}\log(\mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t]).$$

$\square$

Proposition 4.22 (Theorem 2.3, [10]). *Assume (P1) and assume that the terminal value ξ is bounded. Then the $BSDE(f, \xi)$ has at least one solution (Y, Z) in $\mathcal{S}_\infty \times \mathcal{L}_2$. Moreover, there exists a minimal solution (Y^*, Z^*) to the $BSDE(f, \xi)$ such that given any $\xi' \geq \xi$ a.s., where ξ' is bounded as well, and (Y', Z') is a solution as above to the $BSDE(f, \xi')$, one has $Y^*_t \leq Y'_t$ for all $t \in I$ a.s.*

Proposition 4.23 (Proposition 2.4, [10]). *Let $f, f^1, f^2, f^3, \ldots : \Omega \times I \times \mathbb{R}^2 \to \mathbb{R}$ be a set of random generators and $\xi, \xi^1, \xi^2, \xi^3, \ldots$ be a set of bounded $\mathcal{F}_T$-measurable random variables such that for some $C, K \geq 0, M > 0$*

(i) *the sequence $(f^n)_{n\in\mathbb{N}}$ converges a.s. locally uniformly to f,*
(ii) *the sequence $(\xi^n)_{n\in\mathbb{N}}$ converges a.s. to ξ,*
(iii) *for all $(\omega, t) \in \Omega \times I$ and all $y, z \in \mathbb{R}$ one has*

$$|f^n(\omega, t, y, z)| \leq K + C|z|^2,$$

(iv) *for all $n \in \mathbb{N}$ the $BSDE(f^n, \xi^n)$ has a solution (Y^n, Z^n) in $\mathcal{S}_\infty \times \mathcal{L}_2$ such that for all $t \in I$ one has $Y^n_t \leq Y^{n+1}_t \leq M$ a.s.*

Then, there exists $(Y, Z) \in S_\infty \times \mathcal{L}_2$ which is a solution to the $BSDE(f, \xi)$, such that $\lim_{n\to\infty} Y^n = Y$ uniformly on $[0, T]$ and $\lim_{n\to\infty} Z^n = Z$ in $\mathcal{L}_2$ i.e., a.s.,

$$\lim_{n\to\infty} \mathbb{E}\int_0^T (Z^n_t - Z_t)^2 dt = 0.$$

Corollary 4.24. *Let $f, f^1, f^2, f^3, \ldots : \Omega \times I \times \mathbb{R}^2 \to \mathbb{R}$ be random generators and $\xi, \xi^1, \xi^2, \xi^3, \ldots$ be bounded $\mathcal{F}_T$-measurable random variables such that for some $M > 0$*

(i) *the sequence $(f^n)_{n\in\mathbb{N}}$ converges a.s. locally uniformly to f,*
(ii) *the sequence $(\xi^n)_{n\in\mathbb{N}}$ converges a.s. to ξ,*
(iii) $\sup_{n\in\mathbb{N},\omega\in\Omega} |\xi^n(\omega)| < \infty$,
(iv) $\sup_{n\in\mathbb{N}} |f^n(\omega, t, y, z)| \leq \alpha + \beta|y| + \frac{\gamma}{2}|z|^2$ *for all $\omega \in \Omega, t \in I, y, z \in \mathbb{R}$,*
(v) *for all $n \in \mathbb{N}$ the $BSDE(f^n, \xi^n)$ has a solution (Y^n, Z^n) in $S_\infty \times \mathcal{L}_2$ such that for all $t \in I$ one has $Y^n_t \leq Y^{n+1}_t \leq M$ a.s.*

Then, there exists $(Y, Z) \in S_\infty \times \mathcal{L}^2$*, which is a solution to the* $BSDE(f, \xi)$*, such that* $\lim_{n\to\infty} Y_n = Y$ *a.s. in* $C([0,T])$ *and* $\lim_{n\to\infty} Z_n = Z$ *in* $\mathcal{L}^2$*.*

Proof. Let $r := \frac{1}{\gamma} \log \phi_0(M) < \infty$. Then according to the Lemma 4.21, a.s.,

$$
\begin{aligned}
|Y_t^n| &\le \frac{1}{\gamma} \log \mathbb{E}[\phi_t(|\xi^n|)|\mathcal{F}_t] \\
&\le \frac{1}{\gamma} \log \mathbb{E}[\phi_0(\sup_{n\in\mathbb{N}} |\xi^n|)|\mathcal{F}_t] \\
&\le \frac{1}{\gamma} \log \phi_0(M) \\
&= r
\end{aligned}
\tag{6}
$$

for all $t \in I, n \in \mathbb{N}$. Let $p : \mathbb{R} \to \mathbb{R}$ be defined by

$$
p(x) := \frac{xr}{\max\{r, |x|\}}.
$$

Then, $p(x) = x$ for $|x| \le r$, which implies, that (Y^n, Z^n) solves the $BSDE(g^n, \xi^n)$, where $g^n : \Omega \times I \times \mathbb{R}^2 \to \mathbb{R}$ is defined by $g^n(t, y, z) := f^n(t, p(y), z)$. Moreover,

$$
\begin{aligned}
|g^n(t, y, z)| &= |f^n(t, p(y), z)| \\
&\le \alpha + \beta|p(y)| + \frac{\gamma}{2}|z|^2 \\
&\le \alpha + \beta r + \frac{\gamma}{2}|z|^2.
\end{aligned}
\tag{7}
$$

Also, the sequence $(g^n)_{n\in\mathbb{N}}$ converges a.s. locally uniformly to $g : \Omega \times I \times \mathbb{R}^2 \to \mathbb{R}$ defined by $g(t, y, z) := f(t, p(y), z)$. Now, Proposition 4.23 can be applied to get $(Y, Z) \in S_\infty \times \mathcal{L}_2$ such that, a.s.,

$$
Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \tag{8}
$$

and $\lim_{n\to\infty} Y_n = Y$ a.s. in $C([0,T])$ Therefore, $|Y_t| \le r$ a.s., which implies that $p(Y_t) = Y_t$ and $f(t, Y_t, Z_t) = g(t, Y_t, Z_t)$ a.s. for all $t \in I$. Therefore, a.s.,

$$
Y_t = \xi + \int_t^T f(s, Y, Z) ds - \int_t^T Z_s dB_s. \tag{9}
$$

□

Proposition 4.25. *Let* $M = (M_t)_{t\in I}$ *be an RCLL martingale. Then* $\lim_{t\to T} M_t = M_T$ *a.s.*

Proposition 4.25 is stated in [9, p. 18-19] and proved in [9, p. 42].

Corollary 4.26. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$ be a stochastic basis that satisfies the usual conditions, and let X be an integrable random variable. Then the stochastic process $\big(\mathbb{E}[X|\mathcal{F}_t]\big)_{t\in I}$ has an RCLL modification.*

Proof. The statement follows from Theorem 4.20 since the function $t \to \mathbb{E}\big[\mathbb{E}[X|\mathcal{F}_t]\big] = \mathbb{E}X$ is continuous. □

The following theorem is a modified version of the first part of [3, Theorem 2]. We provide a proof with slight differences.

Theorem 4.27. *Assume (P1) and (P2) and define for $k \in \mathbb{N}$ the stopping times*

$$\tau_k := \inf\Big\{t \in I : \frac{1}{\gamma}\log \mathbb{E}[\phi_0(|\xi|)|\mathcal{F}_t] \geq k\Big\} \wedge T.$$

Then one has the following:

(1) $\lim_{k\to\infty} \mathbb{P}\big(\{\omega \in \Omega : \tau_k(\omega) < T\}\big) = 0$.

(2) *There exists an adapted and continuous process $Y = (Y_t)_{t\in I}$ such that $Y_T = \xi$ a.s. and such that for all $t \in I$ and $k \in \mathbb{N}$ a.s.*

$$-\frac{1}{\gamma}\log \mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t] \leq Y_t \leq \frac{1}{\gamma}\log \mathbb{E}[\phi_t(\xi)|\mathcal{F}_t]$$

and a progressively measurable $Z = (Z_t)_{t\in I}$ such that

(a) $\int_0^T |Z_t|^2 dt < \infty$ *a.s.*,

(b) $\mathbb{E}\int_0^{\tau_k} |Z_t|^2 dt < \infty$ *for all $k \in \mathbb{N}$*,

(c) $Y_{t\wedge\tau_k} = Y_{\tau_k} + \int_{t\wedge\tau_k}^{\tau_k} f(s, Y_s, Z_s)ds - \int_{t\wedge\tau_k}^{\tau_k} Z_s dB_s$ *for all $t \in I, k \in \mathbb{N}$ a.s.*

Proof. First we check that

$$\lim_{k\to\infty} \mathbb{P}\big(\{\omega \in \Omega : \tau_k(\omega) < T\}\big) = 0.$$

For this we observe that

$$\begin{aligned}
&\lim_{k\to\infty} \mathbb{P}\big(\{\omega \in \Omega : \tau_k(\omega) < T\}\big) \\
= &\lim_{k\to\infty} \mathbb{P}\Big(\Big\{\omega \in \Omega : \exists t \in [0,T) \text{ such that } \frac{1}{\gamma}\log \mathbb{E}[\phi_0(|\xi|)|\mathcal{F}_t] \geq k\Big\}\Big) \\
= &\ \mathbb{P}\Big(\Big\{\omega \in \Omega : \sup_{t\in[0,T)} \frac{1}{\gamma}\log \mathbb{E}[\phi_0(|\xi|)|\mathcal{F}_t] = \infty\Big\}\Big) \\
= &\ 0
\end{aligned}$$

because $\mathbb{E}[\phi_0(|\xi|)] < \infty$ by (P2) and because the stochastic process $\big(\mathbb{E}[\phi_0(|\xi|)|\mathcal{F}_t]\big)_{t\in I}$ can be assumed to be continuous (see [14, p. 200, Theorem 3.4]) by choosing a continuous modification.

Let us show the rest of the theorem. (Case 1): Let $\xi \geq 0$ a.s. For $n \in \mathbb{N}$, let $\xi^n := \xi \wedge n$, and let (Y^n, Z^n) be the minimal solution of the $BSDE(f, \xi^n)$ given in Theorem 4.22. Then

$$Y_t^n = \xi^n + \int_t^T f(s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n dB_s$$

for $t \in I$ a.s. According to the Lemma 4.21 we have that, a.s.,

$$-\frac{1}{\gamma} \log \mathbb{E}[\phi_t(-\xi^n)|\mathcal{F}_t] \leq Y_t^n \leq \frac{1}{\gamma} \log \mathbb{E}[\phi_t(\xi^n)|\mathcal{F}_t]. \tag{10}$$

Theorem 4.22 implies that $Y_t^n \leq Y_t^{n+1}$ a.s. for all $t \in I, n \in \mathbb{N}$. Let $Y = (Y_t)_{t \in I}$ be defined by

$$Y_t := \sup_{n \in \mathbb{N}} Y_t^n.$$

Now, $0 \leq \phi_t(-\xi^n) \leq \phi_0(|\xi|)$ and $0 \leq \phi_t(\xi^n) \leq \phi_0(|\xi|)$. Because $\mathbb{E}[\phi_0(|\xi|)] < \infty$, the dominated convergence theorem implies that, a.s.,

$$-\frac{1}{\gamma} \log \mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t] = -\frac{1}{\gamma} \log \left(\lim_{n \to \infty} \mathbb{E}[\phi_t(-\xi^n)|\mathcal{F}_t] \right) \leq \lim_{n \to \infty} Y_t^n = Y_t,$$

and similarly, a.s.,

$$\frac{1}{\gamma} \log \mathbb{E}[\phi_t(\xi)|\mathcal{F}_t] = \frac{1}{\gamma} \log \left(\lim_{n \to \infty} \mathbb{E}[\phi_t(\xi^n)|\mathcal{F}_t] \right) \geq \lim_{n \to \infty} Y_t^n = Y_t.$$

Therefore we have that, a.s.,

$$-\frac{1}{\gamma} \log \mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t] \leq Y_t \leq \frac{1}{\gamma} \log \mathbb{E}[\phi_t(\xi)|\mathcal{F}_t].$$

The latter inequality implies, that for each $S \in (0, T)$ one has, a.s.,

$$\begin{aligned}
\limsup_{t \to T} Y_t &\leq \limsup_{t \to T, t \geq S} \frac{1}{\gamma} \log \mathbb{E}[\phi_t(\xi)|\mathcal{F}_t] \\
&\leq \lim_{t \to T, t \geq S} \frac{1}{\gamma} \log \mathbb{E}[\phi_S(\xi)|\mathcal{F}_t] \\
&= \frac{1}{\gamma} \log \mathbb{E}[\phi_S(\xi)|\mathcal{F}_T] \\
&= \frac{1}{\gamma} \log \phi_S(\xi),
\end{aligned} \tag{11}$$

where the second last equality follows from Proposition 4.25 and Corollary 4.26. Therefore, by $S \nearrow T$, we have that, a.s.,

$$\limsup_{t\to T} Y_t \le \lim_{S\to T} \frac{1}{\gamma} \log \phi_S(\xi) = \xi. \tag{12}$$

Similar arguments can be applied to $\liminf_{t\to T} Y_t$. For each $S \in (0,T)$ one has, a.s.,

$$\begin{aligned} \liminf_{t\to T} Y_t &\ge \liminf_{t\to T, t\ge S} \Big(-\frac{1}{\gamma} \log \mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t] \Big) \\ &\ge \lim_{t\to T, t\ge S} \Big(-\frac{1}{\gamma} \log \mathbb{E}[\phi_S(-\xi)|\mathcal{F}_t] \Big) \\ &= -\frac{1}{\gamma} \log \mathbb{E}[\phi_S(-\xi)|\mathcal{F}_T] \\ &= -\frac{1}{\gamma} \log \phi_S(-\xi). \end{aligned} \tag{13}$$

and therefore, a.s.,

$$\liminf_{t\to T} Y_t \ge -\lim_{S\to T} \frac{1}{\gamma} \log \phi_S(-\xi) = \xi. \tag{14}$$

Because

$$\limsup_{t\to T} Y_t \le \xi \le \liminf_{t\to T} Y_t,$$

a.s., we can conclude that, a.s.,

$$\lim_{t\to T} Y_t = \xi.$$

For $k \ge 1$, let τ_k be the stopping time defined above. Moreover, let $(Y_k^n, Z_k^n) = \big((Y_k^n(t), Z_k^n(t))\big)_{t\in I}$ be defined by

$$(Y_k^n(t), Z_k^n(t)) := (Y_{t\wedge\tau_k}^n, Z_t^n \mathbb{1}_{\{t\le\tau_k\}}(t))$$

for all $t \in I, k, n \in \mathbb{N}$ and let $\xi_k^n := Y_{\tau_k}^n$ for all $k, n \in \mathbb{N}$. Then we have that, a.s. for all $t \in I$,

$$Y_k^n(t) = \xi_k^n + \int_t^T \mathbb{1}_{\{s\le\tau_k\}}(s) f(s, Y_k^n(s), Z_k^n(s)) ds - \int_t^T Z_k^n(s) dB_s, \tag{15}$$

where $k, n \in \mathbb{N}$. Now, Corollary 4.24 implies that, a.s. for all $t \in I$,

$$Y_k(t) = \xi_k + \int_t^T \mathbb{1}_{\{s\le\tau_k\}}(s) f(s, Y_k(s), Z_k(s)) ds - \int_t^T Z_k(s) dB_s,$$

where $k, n \in \mathbb{N}$, $(Y_k) = (Y_k(t))_{t\in I}$ is defined by $Y_k(t) := \sup_{n\in\mathbb{N}} Y_k^n(t)$ and $(Z_k) = (Z_k(t))_{t\in I}$ is defined by $Z_k(t) := \lim_{n\to\infty} Z_k^n(t)$ in $\mathcal{L}^2$. By definition, we have that

$$\begin{aligned} &Y_{t\wedge\tau_k} = \sup_{n\in\mathbb{N}} Y^n_{t\wedge\tau_k}, \\ &Y_{k+1}(t\wedge\tau_k) = \sup_{n\in\mathbb{N}} Y^n_{k+1}(t\wedge\tau_k) = \sup_{n\in\mathbb{N}} Y^n_{t\wedge\tau_k\wedge\tau_{k+1}} = \sup_{n\in\mathbb{N}} Y^n_{t\wedge\tau_k}, \text{ and} \\ &Y_k(t) = \sup_{n\in\mathbb{N}} Y^n_k(t) = \sup_{n\in\mathbb{N}} Y^n_{t\wedge\tau_k} \end{aligned} \tag{16}$$

for all $\omega \in \Omega, t \in I, k \in \mathbb{N}$, which implies that

$$Y_{t\wedge\tau_k} = Y_{k+1}(t\wedge\tau_k) = Y_k(t).$$

for all $t \in I, k \in \mathbb{N}$. Therefore for $0 \le t < T$, one can set a random variable k^* large enough to satisfy $t < \tau_{k^*} \le T$ to realize that

$$Y_t = Y_k(t)$$

for all $t \in I, k \ge k^*$. Thus, Y is a continuous process on $[0, T)$. On the other hand,

$$\begin{aligned} &Z_{k+1}(t)\mathbb{1}_{\{t\le\tau_k\}}(t) = \lim_{n\to\infty} Z^n_{k+1}(t)\mathbb{1}_{\{t\le\tau_k\}}(t) \text{ in } \mathcal{L}_2, \text{ and} \\ &Z_k(t) = \lim_{n\to\infty} Z^n_k(t) \text{ in } \mathcal{L}_2. \end{aligned} \tag{17}$$

for all $k \in \mathbb{N}$. Now, one can realize that

$$Z^n_{k+1}(t)\mathbb{1}_{\{t\le\tau_k\}}(t) = Z^n_t\mathbb{1}_{\{t\le\tau_{k+1}\}}(t)\mathbb{1}_{\{t\le\tau_k\}}(t) = Z^n_t\mathbb{1}_{\{t\le\tau_k\}}(t) = Z^n_k(t),$$

for all $\omega \in \Omega, t \in I, k, n \in \mathbb{N}$, which implies that $Z_{k+1}(t)\mathbb{1}_{\{t\le\tau_k\}}(t) = Z_k(t)\ d\mathbb{P}\otimes dt$-a.s. Finally, one can define the process $Z = (Z_t)_{t\in I}$. For $t \in [0, T)$, let Z_t be defined by $Z_t := Z_k(t)$, where $t \in [\tau_{k-1}, \tau_k)$ and let $Z_T := 0$. Then, it follows from the equation (15) that (Y, Z) gives a solution to the BSDE

$$Y_{t\wedge\tau_k} = Y_{\tau_k} + \int_{t\wedge\tau_k}^{\tau_k} f(s, Y_s, Z_s)ds - \int_{t\wedge\tau_k}^{\tau_k} Z_s dB_s. \tag{18}$$

(Case 2): Let ξ be allowed to have negative values as well, and let us indicate the proof in this case.

For $n, p \in \mathbb{N}$, let $\xi^{n,p} := \xi^+ \wedge n - \xi^- \wedge p$, and let $(Y^{n,p}, Z^{n,p})$ be the minimal solution of the $BSDE(f, \xi^{n,p})$ given in Theorem 4.22. Then, for $n, p \in \mathbb{N}$,

$$Y^{n,p}_t = \xi^{n,p} + \int_t^T f(s, Y^{n,p}_s, Z^{n,p}_s)ds - \int_t^T Z^{n,p}_s dB_s$$

for all $t \in I$ a.s. According to Lemma 4.21, a.s.,

$$-\frac{1}{\gamma}\log \mathbb{E}[\phi_t(-\xi^{n,p})|\mathcal{F}_t] \leq Y_t^{n,p} \leq \frac{1}{\gamma}\log \mathbb{E}[\phi_t(\xi^{n,p})|\mathcal{F}_t]. \tag{19}$$

Theorem 4.22 implies that

$$Y_t^{n,p+1} \leq Y_t^{n,p} \leq Y_t^{n+1,p}$$

a.s. for all $t \in I, n, p \in \mathbb{N}$. Let $Y = (Y_t)_{t\in I}$ be defined by

$$Y_t := \inf_{p\in\mathbb{N}} \big(\sup_{n\in\mathbb{N}} Y_t^{n,p}\big).$$

Now, $0 \leq \phi_t(-\xi^{n,p}) \leq \phi_0(|\xi|)$ and $0 \leq \phi_t(\xi^{n,p}) \leq \phi_0(|\xi|)$. Because $\mathbb{E}[\phi_0(|\xi|)] < \infty$, the dominated convergence theorem gives that, a.s.,

$$\begin{aligned} -\frac{1}{\gamma}\log \mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t] &= -\frac{1}{\gamma}\log\Big(\lim_{p\to\infty}\lim_{n\to\infty}\mathbb{E}[\phi_t(-\xi^{n,p})|\mathcal{F}_t]\Big) \\ &\leq \lim_{p\to\infty}\lim_{n\to\infty} Y_t^{n,p} \\ &= Y_t, \end{aligned}$$

and similarly, a.s.,

$$\begin{aligned} \frac{1}{\gamma}\log \mathbb{E}[\phi_t(\xi)|\mathcal{F}_t] &= \frac{1}{\gamma}\log\Big(\lim_{p\to\infty}\lim_{n\to\infty}\mathbb{E}[\phi_t(\xi^{n,p})|\mathcal{F}_t]\Big) \\ &\geq \lim_{p\to\infty}\lim_{n\to\infty} Y_t^{n,p} \\ &= Y_t. \end{aligned}$$

We conclude that, a.s.,

$$-\frac{1}{\gamma}\log \mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t] \leq Y_t \leq \frac{1}{\gamma}\log \mathbb{E}[\phi_t(\xi)|\mathcal{F}_t].$$

As in (Case 1), we have that a.s.,

$$\lim_{t\to T} Y_t = \xi.$$

Now, a.s.,

$$Y_{t\wedge\tau_k}^{n,p} = Y_{\tau_k}^{n,p} + \int_t^T \mathbb{1}_{\{s\leq\tau_k\}}(s) f(s, Y_s^{n,p}, Z_s^{n,p}) ds - \int_t^T \mathbb{1}_{\{s\leq\tau_k\}}(s) Z_s^{n,p} dB_s, \tag{20}$$

and Corollary 4.24 gives that

$$Y_t^k = \xi^k + \int_t^T \mathbb{1}_{\{s \le \tau_k\}}(s) f(s, Y_s^k, Z_s^k) ds - \int_t^T Z_s^k dB_s \tag{21}$$

a.s., where $(Y^k) = (Y_t^k)_{t \in I}$ is defined by $Y_t^k := \inf_{p \in \mathbb{N}} \big(\sup_{n \in \mathbb{N}} Y_{t \wedge \tau_k}^{n,p} \big)$ and $(Z^k) = (Z_t^k)_{t \in I}$ is defined by $Z_t^k := \lim_{p \to \infty} \lim_{n \to \infty} \mathbb{1}_{\{t \le \tau_k\}}(t) Z_t^{n,p}$ in $\mathcal{L}_2$. Now, by definition

$$\begin{aligned} Y_{t \wedge \tau_k} &= \inf_{p \in \mathbb{N}} \big(\sup_{n \in \mathbb{N}} Y_{t \wedge \tau_k}^{n,p} \big), \\ Y_{t \wedge \tau_k}^{k+1} &= \inf_{p \in \mathbb{N}} \big(\sup_{n \in \mathbb{N}} Y_{t \wedge \tau_{k+1} \wedge \tau_k}^{n,p} \big) = \inf_{p \in \mathbb{N}} \big(\sup_{n \in \mathbb{N}} Y_{t \wedge \tau_k}^{n,p} \big), \text{ and} \\ Y_t^k &= \inf_{p \in \mathbb{N}} \big(\sup_{n \in \mathbb{N}} Y_{t \wedge \tau_k}^{n,p} \big), \end{aligned} \tag{22}$$

which implies that

$$Y_{t \wedge \tau_k} = Y_{t \wedge \tau_k}^{k+1} = Y_t^k.$$

Therefore for $0 \le t < T$, one can set a random variable k^* large enough to satisfy $t < \tau_{k^*} \le T$ to realize that

$$Y_t = Y_t^k$$

for all $k \ge k^*$. Thus, Y is a continuous process on $[0, T)$. On the other hand,

$$\begin{aligned} Z_t^{k+1} \mathbb{1}_{\{t \le \tau_k\}}(t) &= \lim_{p \to \infty} \lim_{n \to \infty} \mathbb{1}_{\{t \le \tau_{k+1}\}}(t) Z_t^{n,p} \mathbb{1}_{\{t \le \tau_k\}}(t) \text{ in } \mathcal{L}_2, \text{, and} \\ Z_t^k &= \lim_{p \to \infty} \lim_{n \to \infty} \mathbb{1}_{\{t \le \tau_k\}}(t) Z_t^{n,p} \text{ in } \mathcal{L}_2. \end{aligned} \tag{23}$$

for all $k \in \mathbb{N}$. Now, one can realize that

$$\mathbb{1}_{\{t \le \tau_{k+1}\}}(t) Z_t^{n,p} \mathbb{1}_{\{t \le \tau_k\}}(t) = \mathbb{1}_{\{t \le \tau_k\}}(t) Z_t^{n,p} = Z_t^n \mathbb{1}_{\{t \le \tau_k\}}(t)$$

for all $\omega \in \Omega, t \in I, k, n, p \in \mathbb{N}$, which implies that $Z_t^{k+1} \mathbb{1}_{\{t \le \tau_k\}}(t) = Z_t^k$ $d\mathbb{P} \otimes dt$-a.s. Finally, one can define the process $Z = (Z_t)_{t \in I}$. For $t \in [0, T)$, let Z_t be defined by $Z_t := Z_t^k$, where $t \in [\tau_{k-1}, \tau_k)$ and let $Z_T := 0$. Then (Y, Z) gives a solution to the BSDE

$$Y_{t \wedge \tau_k} = Y_{\tau_k} + \int_{t \wedge \tau_k}^{\tau_k} f(s, Y_s, Z_s) ds - \int_{t \wedge \tau_k}^{\tau_k} Z_s dB_s, \tag{24}$$

□

Proposition 4.28 (Doob's maximal inequality)**.** *Let $(M_t)_{t\in I}$ be an RCLL martingale and $1 < p < \infty$. Then*

$$\mathbb{E}\Big[\sup_{t\in I}|M_t|^p\Big] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_T|^p.$$

Proposition 4.28 is stated in [9, p. 13-14].

Proposition 4.29. *Let $(M_t)_{t\in I}$ be an RCLL martingale and $1 < p < \infty$. Then, a.s.,*

$$\mathbb{E}\Big[\sup_{s\in[t,T]}|M_s|^p\Big|\mathcal{F}_t\Big] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\big[|M_T|^p\big|\mathcal{F}_t\big].$$

Proof. Let $A_t \in \mathcal{F}_t$. Then

$$\begin{aligned}\int_{A_t}\sup_{t\in[t,T]}|M_s|^p d\mathbb{P} &= \mathbb{E}\sup_{s\in[t,T]}|\mathbb{1}_{A_t}M_s|^p \\ &\leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|\mathbb{1}_{A_t}M_T|^p \\ &= \int_{A_t}\left(\frac{p}{p-1}\right)^p |M_T|^p d\mathbb{P}\end{aligned} \tag{25}$$

because $(\mathbb{1}_{A_t}M_s)_{s\in[t,T]}$ is a martingale. □

The following proposition is a generalization of a result given in the proof of [3, Theorem 2].

Proposition 4.30. *Assuming (P1) and (P2), the processes Y and Z of Theorem 4.27 satisfy the following inequality a.s. for all $0 \leq r \leq t \leq T$:*

$$\frac{1}{2}\mathbb{E}\Big[\int_r^t |Z_s|^2 ds\Big|\mathcal{F}_r\Big] \leq \mathbb{E}\Big[\frac{1}{\gamma^2}\sup_{s\in[r,t]}e^{\gamma|Y_s|} + \frac{1}{\gamma}\int_r^t e^{\gamma|Y_s|}(\alpha+\beta|Y_s|)ds\Big|\mathcal{F}_r\Big].$$

Proof. The inequality

$$-\frac{1}{\gamma}\log\mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t] \leq Y_t \leq \frac{1}{\gamma}\log\mathbb{E}[\phi_t(\xi)|\mathcal{F}_t]$$

of Theorem 4.27 implies that, a.s.,

$$e^{\gamma|Y_t|} \leq M_t$$

where $M_t := \mathbb{E}[\phi_0(|\xi|)|\mathcal{F}_t)]$ can be assumed to be an RCLL martingale according to Corollary 4.26. Now Doob's maximal inequality (Proposition 4.28) gives for $1 < p < \infty$ that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in I} e^{p\gamma|Y_t|}\right] &\leq \mathbb{E}\left[\sup_{t\in I}|M_t|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_T|^p \\ &= \left(\frac{p}{p-1}\right)^p \mathbb{E}[\phi_0(|\xi|)]^p < \infty,\end{aligned} \tag{26}$$

where the last inequality follows from property (P2).

Let τ_n be defined by

$$\tau_n := \inf\left\{t \geq 0 : \int_0^t e^{2\gamma|Y_s|}|Z_s|^2 ds \geq n\right\} \wedge T$$

for all $n \in \mathbb{N}$. Then τ_n is a stopping time for all $n \in \mathbb{N}$. Moreover, the function $s \mapsto Y_s(\omega)$ is continuous for all $\omega \in \Omega$ and we may assume that $\int_0^T |Z_s(\omega)|^2 ds < \infty$ for all $\omega \in \Omega$. Therefore

$$\int_0^T e^{2\gamma|Y_s|}|Z_s(\omega)|^2 ds < \infty$$

for all $\omega \in \Omega$ and there exists $n(\omega) \in \mathbb{N}$ such that

$$\int_0^T e^{2\gamma|Y_s|}|Z_s(\omega)|^2 ds < n(\omega)$$

and $\lim_{n\to\infty} \tau_n(\omega) = T$.

Moreover, let $u : [0, \infty] \mapsto \mathbb{R}$ be defined by

$$u(x) := \frac{1}{\gamma^2}(e^{\gamma x} - 1 - \gamma x).$$

Then $g : [0, T] \times \mathbb{R} \to \mathbb{R}$, $g(s, x) := u(|x|)$ belongs to $C^{1,2}([0, T] \times \mathbb{R})$. Now Itô's formula (Proposition 3.14) gives for $0 \leq r \leq t \leq T$ that, a.s.,

$$\begin{aligned}& u(|Y_{t\wedge\tau_n}|) - u(|Y_{r\wedge\tau_n}|) \\ = \;& g(t\wedge\tau_n, Y_{t\wedge\tau_n}) - g(r\wedge\tau_n, Y_{r\wedge\tau_n}) \\ = \;& g(0, Y_0) + \int_0^{t\wedge\tau_n} \frac{\partial g}{\partial s}(s, Y_s) ds - \int_0^{t\wedge\tau_n} \frac{\partial g}{\partial x}(s, Y_s) f(s, Y_s, Z_s) ds \\ & + \int_0^{t\wedge\tau_n} \frac{\partial g}{\partial x}(s, Y_s) Z_s dB_s + \frac{1}{2}\int_0^{t\wedge\tau_n} \frac{\partial^2 g}{\partial x^2}(s, Y_s) Z_s^2 ds\end{aligned}$$

$$
\begin{aligned}
&-g(0,Y_0)-\int_0^{r\wedge\tau_n}\frac{\partial g}{\partial s}(s,Y_s)ds+\int_0^{r\wedge\tau_n}\frac{\partial g}{\partial x}(s,Y_s)f(s,Y_s,Z_s)ds\\
&-\int_0^{r\wedge\tau_n}\frac{\partial g}{\partial x}(s,Y_s)Z_sdB_s-\frac{1}{2}\int_0^{r\wedge\tau_n}\frac{\partial^2 g}{\partial x^2}(s,Y_s)Z_s^2ds.\\
= \quad &\int_{r\wedge\tau_n}^{t\wedge\tau_n}\Big[-u'(|Y_s|)\mathrm{sgn}(Y_s)f(s,Y_s,Z_s)+\frac{1}{2}u''(|Y_s|)Z_s^2\Big]ds\\
&+\int_{r\wedge\tau_n}^{t\wedge\tau_n}u'(|Y_s|)\mathrm{sgn}(Y_s)Z_sdB_s.
\end{aligned}
$$

Rearranging terms, property (P1), and the facts that $u'(x)\geq 0$, $u''(x)-\gamma u'(x)=1$ for all $x\geq 0$, imply that, a.s.,

$$
\begin{aligned}
0 \quad = \quad & u(|Y_{t\wedge\tau_n}|)-u(|Y_{r\wedge\tau_n}|)\\
&+\int_{r\wedge\tau_n}^{t\wedge\tau_n}\Big[u'(|Y_s|)\mathrm{sgn}(Y_s)f(s,Y_s,Z_s)-\frac{1}{2}u''(|Y_s|)Z_s^2\Big]ds\\
&-\int_{r\wedge\tau_n}^{t\wedge\tau_n}u'(|Y_s|)\mathrm{sgn}(Y_s)Z_sdB_s\\
\leq \quad & u(|Y_{t\wedge\tau_n}|)-u(|Y_{r\wedge\tau_n}|)\\
&+\int_{r\wedge\tau_n}^{t\wedge\tau_n}\Big[u'(|Y_s|)(\alpha+\beta|Y_s|)-\frac{1}{2}\big(u''(|Y_s|)-\gamma u'(|Y_s|)\big)Z_s^2\Big]ds\\
&-\int_{r\wedge\tau_n}^{t\wedge\tau_n}u'(|Y_s|)\mathrm{sgn}(Y_s)Z_sdB_s\\
= \quad & u(|Y_{t\wedge\tau_n}|)-u(|Y_{r\wedge\tau_n}|)+\int_{r\wedge\tau_n}^{t\wedge\tau_n}\Big[u'(|Y_s|)(\alpha+\beta|Y_s|)-\frac{1}{2}Z_s^2\Big]ds\\
&-\int_{r\wedge\tau_n}^{t\wedge\tau_n}u'(|Y_s|)\mathrm{sgn}(Y_s)Z_sdB_s.
\end{aligned}
$$

Now it follows that, a.s.,

(27)
$$\begin{aligned}\frac{1}{2}\int_{r\wedge\tau_n}^{t\wedge\tau_n} Z_s^2 ds &\le u(|Y_{t\wedge\tau_n}|) - u(|Y_{r\wedge\tau_n}|) + \int_{r\wedge\tau_n}^{t\wedge\tau_n} u'(|Y_s|)(\alpha+\beta|Y_s|)ds \\ &\quad - \int_{r\wedge\tau_n}^{t\wedge\tau_n} u'(|Y_s|)\mathrm{sgn}(Y_s)Z_s dB_s \\ &\le \frac{1}{\gamma^2}\sup_{s\in[r,t]} e^{\gamma|Y_s|} + \frac{1}{\gamma}\int_{r\wedge\tau_n}^{t\wedge\tau_n} e^{\gamma|Y_s|}(\alpha+\beta|Y_s|)ds \\ &\quad - \int_{r\wedge\tau_n}^{t\wedge\tau_n} u'(|Y_s|)\mathrm{sgn}(Y_s)Z_s dB_s.\end{aligned}$$

Taking conditional expectation with respect to $\mathcal{F}_r$ from both sides gives, a.s.,

$$\frac{1}{2}\mathbb{E}\Big[\int_{r\wedge\tau_n}^{t\wedge\tau_n} |Z_s|^2 ds\Big|\mathcal{F}_r\Big] \le \mathbb{E}\Big[\frac{1}{\gamma^2}\sup_{s\in[r,t]} e^{\gamma|Y_s|}+\frac{1}{\gamma}\int_{r\wedge\tau_n}^{t\wedge\tau_n} e^{\gamma|Y_s|}(\alpha+\beta|Y_s|)ds\Big|\mathcal{F}_r\Big],$$

since $\big(u'(|Y_s|)\mathrm{sgn}(Y_s)Z_s\big)_{s\in I} \in \mathcal{L}_2$ and therefore, a.s.,

$$\mathbb{E}\Big[\int_{r\wedge\tau_n}^{t\wedge\tau_n} u'(|Y_s|)\mathrm{sgn}(Y_s)Z_s dB_s \,\Big|\, \mathcal{F}_r\Big] = 0.$$

Since the inequality above holds for all $n \in \mathbb{N}$ and $\tau_n \to T$ as $n \to \infty$ it follows that, a.s.,

$$\frac{1}{2}\mathbb{E}\Big[\int_r^t |Z_s|^2 ds\Big|\mathcal{F}_r\Big] \le \mathbb{E}\Big[\frac{1}{\gamma^2}\sup_{s\in[r,t]} e^{\gamma|Y_s|}+\frac{1}{\gamma}\int_r^t e^{\gamma|Y_s|}(\alpha+\beta|Y_s|)ds\Big|\mathcal{F}_r\Big].$$

□

Lemma 4.31. *For all $\alpha,\beta \ge 0$ and $\gamma,\varepsilon > 0$, there exists $\kappa > 0$ such that*

$$e^{\gamma|y|}(\alpha+\beta|y|) \le \kappa e^{(\gamma+\varepsilon)|y|}$$

for all $y \in \mathbb{R}$.

Proof. Let $h : [0,\infty] \mapsto \mathbb{R}$ be defined by $h(x) := (\alpha+\beta x)e^{-\varepsilon x}$. Since h is a continuous function with $\lim_{x\to\infty} h(x) = 0$, there exists κ such that $\kappa \ge h(x)$ for all $x \ge 0$. Now we get that

(28)
$$\begin{aligned}&h(|y|) \le \kappa \text{ for all } y \in \mathbb{R} \\ \iff &(\alpha+\beta|y|)e^{-\varepsilon|y|} \le \kappa \text{ for all } y \in \mathbb{R} \\ \iff &e^{\gamma|y|}(\alpha+\beta|y|) \le \kappa e^{(\gamma+\varepsilon)|y|} \text{ for all } y \in \mathbb{R}.\end{aligned}$$

□

Now, we are ready to prove the main theorem of this thesis. The following theorem states that there exists a solution for the quadratic BSDE. It is a modified version of the of [3, Theorem 2]. We provide some details for the proof of the last part of [3, Theorem 2] here with the help of Theorem 4.27 and Proposition 4.30.

Theorem 4.32. *Assuming (P1) and (P3), the processes Y and Z of Proposition 4.30 give a solution to the $BSDE(f,\xi)$. Moreover there exists $C>0$ such that for all $t \in I$ we have, a.s.,*

$$-\frac{1}{\gamma}\log \mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t] \le Y_t \le \frac{1}{\gamma}\log \mathbb{E}[\phi_t(\xi)|\mathcal{F}_t] \tag{29}$$

and

$$\mathbb{E}\Big[\int_t^T |Z_s|^2 ds \,\Big|\, \mathcal{F}_t\Big] \le C\mathbb{E}\Big[e^{\lambda|\xi|}\,\Big|\,\mathcal{F}_t\Big]. \tag{30}$$

Proof. We have proved the inequality (29) in Theorem 4.27.

Let us prove the inequality (30). Theorem 4.27, Proposition 4.29, Proposition 4.30, and Lemma 4.31 imply that for $\epsilon > 0$ such that $(\gamma+\epsilon)e^{\beta T} \le \lambda$ one has

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\Big[\int_t^T |Z_s|^2 ds \,\Big|\, \mathcal{F}_t\Big] \\
\le\ & \mathbb{E}\left[\frac{1}{\gamma^2}\sup_{s\in[t,T]} e^{\gamma|Y_s|} + \frac{1}{\gamma}\int_t^T e^{\gamma|Y_s|}(\alpha+\beta|Y_s|)ds \,\middle|\, \mathcal{F}_t\right] \\
\le\ & \mathbb{E}\left[\frac{1}{\gamma^2}\sup_{s\in[t,T]} e^{\gamma|Y_s|} + \frac{\kappa}{\gamma}\int_t^T e^{(\gamma+\varepsilon)|Y_s|}ds \,\middle|\, \mathcal{F}_t\right] \\
\le\ & \left(\frac{1}{\gamma^2}+\frac{\kappa(T-t)}{\gamma}\right)\mathbb{E}\left[\sup_{s\in[t,T]} e^{(\gamma+\varepsilon)|Y_s|}\,\middle|\,\mathcal{F}_t\right] \\
=\ & \left(\frac{1}{\gamma^2}+\frac{\kappa(T-t)}{\gamma}\right)\mathbb{E}\left[\left(\sup_{s\in[t,T]} e^{\gamma|Y_s|}\right)^{\frac{\gamma+\varepsilon}{\gamma}}\,\middle|\,\mathcal{F}_t\right] \\
\le\ & \left(\frac{1}{\gamma^2}+\frac{\kappa(T-t)}{\gamma}\right)\mathbb{E}\left[\left(\sup_{s\in[t,T]} \mathbb{E}[\phi_0(|\xi|)|\mathcal{F}_s)]\right)^{p}\,\middle|\,\mathcal{F}_t\right] \\
\le\ & \left(\frac{1}{\gamma^2}+\frac{\kappa(T-t)}{\gamma}\right)\left(\frac{p}{p-1}\right)^p\mathbb{E}\left[\Big(\phi_0(|\xi|)\Big)^{p}\,\middle|\,\mathcal{F}_t\right] \\
=\ & C\mathbb{E}\left[\left(e^{\gamma e^{\beta T}|\xi|}\right)^{\frac{\gamma+\varepsilon}{\gamma}}\,\middle|\,\mathcal{F}_t\right] \\
=\ & C\mathbb{E}\left[\left(e^{(\gamma+\varepsilon)e^{\beta T}|\xi|}\right)\,\middle|\,\mathcal{F}_t\right]
\end{aligned}$$

$$\leq \quad C\mathbb{E}\Big[\Big(e^{\lambda|\xi|}\Big)\Big|\mathcal{F}_t\Big]$$

where $p := \frac{\gamma+\varepsilon}{\gamma} \in (1,\infty)$ and

$$C := \left(\frac{1}{\gamma^2} + \frac{\kappa(T-t)}{\gamma}\right)\left(\frac{p}{p-1}\right)^p \exp\left(p\gamma\alpha\frac{e^{\beta T}-1}{\beta}\right),$$

if $\beta > 0$ and

$$C := \left(\frac{1}{\gamma^2} + \frac{\kappa(T-t)}{\gamma}\right)\left(\frac{p}{p-1}\right)^p e^{p\gamma\alpha T},$$

if $\beta = 0$.

Finally, we can show that (Y,Z) is a solution to the $BSDE(f,\xi)$. One can see that $Z \in \mathcal{L}_2$ since according to (P3)

$$\mathbb{E}\int_t^T |Z_s|^2 ds \leq 2C\mathbb{E}\big[e^{\lambda|\xi|}\big] < \infty.$$

According to Theorem 4.27 we have that for all $t \in I$ and $k \in \mathbb{N}$, a.s.,

$$Y_{t\wedge\tau_k} = Y_{\tau_k} + \int_{t\wedge\tau_k}^{\tau_k} f(s,Y_s,Z_s)ds - \int_{t\wedge\tau_k}^{\tau_k} Z_s dB_s,$$

where

$$\tau_k := \inf\left\{t \in I : \frac{1}{\gamma}\log\mathbb{E}[\phi_0(|\xi|)|\mathcal{F}_t] \geq k\right\} \wedge T.$$

By sending $k \to \infty$ we get that $\tau_k \to T$ and for all $t \in I$, a.s.,

$$Y_t = Y_T + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dB_s.$$

□

4.4. **Uniqueness of Quadratic BSDEs.**

The previous subsection considered the existence of the quadratic BSDEs. The results were mainly based on the article [3] of Briand and Hu and the article [10] of Kobylanski. After those papers, Briand and Hu considered the uniqueness of quadratic BSDEs with convex generators in [4]. In [5] and [6], Delbaen, Hu, and Richou continued the work on the uniqueness. Next, we state the main uniqueness results of [4], [5], and [6] without proofs.

Theorem 4.33. *Let the property (P1) hold. Moreover, let us assume that*

(i) *the function $z \mapsto f(\omega,t,y,z)$ is convex for all $\omega \in \Omega, t \in I, y \in \mathbb{R}$,*
(ii) *$f(t,y,z) - f(t,\hat{y},z) \leq \beta|y-\hat{y}|$ for all $\omega \in \Omega, t \in I, y, \hat{y}, z \in \mathbb{R}$, and*

(iii) $e^{r|\xi|} < \infty$ *for all* $r > 0$.

Then there exists a pair $(Y, Z) = ((Y_t)_{t\in I}, (Z_t)_{t\in I})$, *where* $Y \in \mathcal{S}_2$ *and* $Z \in \mathcal{L}_2$ *such, that* (Y, Z) *solves the backward stochastic differential equation* $BSDE(f, \xi)$. *Moreover, the processes* Y *and* Z *are unique in* $\mathcal{S}_2$ *and* $\mathcal{L}_2$, *which means that for solutions* (Y, Z) *and* (Y^*, Z^*) *one has*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} (Y_t - Y_t^*)^2\Big] = 0$$

and

$$\mathbb{E}\int_0^T (Z_t - Z_t^*)^2 dt = 0.$$

The proof is given in [4, Corollary 6].

Theorem 4.34. *Let the property (P1) hold. Moreover, let us assume that*

(i) *the function* $z \mapsto f(\omega, t, y, z)$ *is convex for all* $\omega \in \Omega, t \in I, y \in \mathbb{R}$, *and*
(ii) *there exists* $K > 0$ *such that* $f(\omega, t, y, z) - f(\omega, t, \hat{y}, z) \le K|y - \hat{y}|$ *for all* $\omega \in \Omega, t \in I, y, \hat{y}, z \in \mathbb{R}$.

Suppose that there exists a solution $(Y, Z) = ((Y_t)_{t\in I}, (Z_t)_{t\in I})$, *where* $Y \in \mathcal{S}_2$ *and* $Z \in \mathcal{L}_2$ *such, that* (Y, Z) *solves the backward stochastic differential equation* $BSDE(f, \xi)$ *and that there exists* $p > \gamma, \epsilon > 0$ *such that*

$$\mathbb{E}\Big[e^{p\sup_{t\in I} Y_t^-} + e^{\epsilon sup_{t\in I} Y_t^+}\Big] < \infty.$$

Then the processes Y *and* Z *are unique in* $\mathcal{S}_2$ *and* $\mathcal{L}_2$ *among the processes satisfying the condition above.*

The proof is given in [5, Theorem 3.3].

Theorem 4.35. *Let the following assumptions hold:*

(i) ξ *is a square integrable random variable such that* $\mathbb{E}e^{-\gamma\xi} < \infty$.
(ii) *The function* $g : \mathbb{R} \to \mathbb{R}$ *is* C^2, *convex, and satisfies* $g(0) = 0$.
(iii) *There exists* $C > 0$ *such that* $g(z) \le C + \frac{\gamma}{2}z^2$ *for all* $z \in \mathbb{R}$.
(iv) *There exists an* $R \ge 0$ *and* $\epsilon > 0$ *such that for all* $z \in \mathbb{R}$ *with* $|z| > R$ *one has* $g''(z) \ge \epsilon$.

Then the $BSDE(f, \xi)$, *where* $f(t, y, z) := g(z)$ *has a unique solution* (Y, Z) *such that* $\{Y_\tau : \tau : \Omega \to I$ *stopping time* $\}$ *and* $\{e^{-Y_\tau} : \tau : \Omega \to I$ *stopping time* $\}$ *are uniformly integrable and that* $\int_0^T |Z_s|^2 ds < \infty$ *a.s.*

The proof is given in [6, Theorem 4.1].

Appendix A.

Definition A.1. *Let $M = (M_t)_{t\in I}$ be a continuous and adapted stochastic process. M is called a **local martingale** provided that there exists an increasing sequence $(\sigma_n)_{n=1}^{\infty}$ of stopping times such that $0 \leq \sigma_1 \leq \sigma_2 \leq ... \leq T$, $\mathbb{P}(\exists n \in \mathbb{N} : \sigma_n = T) = 1$, and $(M_{t\wedge\sigma_n})_{t\in I}$ is a martingale for all $n \in \mathbb{N}$.*

Proposition A.2. *Let $X = (X_t)_{t\in I}$ be a progressively measurable stochastic process with $\int_0^T X_s^2 ds < \infty$ a.s. Then $(\int_0^t X_s dB_s)_{t\in I}$ is a local martingale.*

The proof is given in [9, p. 144-147].

Remark A.3. We have not defined the stochastic integral for the progressively measurable processes with $\int_0^T X_s^2 ds < \infty$ a.s. However, this extension is possible such that $(\int_0^t X_s dB_s)_{t\in I}$ is a local martingale. The details are left out of this thesis.

Proposition A.4 (Burkholder-Davis-Gundy inequalities)**.** *Let $M = (M_t)_{t\in I}$ be the local martingale of the form $(\int_0^t X_s dB_s)_{t\in I}$ from Remark A.3. Then*

$$c\mathbb{E}\Big[\int_0^T X_t^2 ds\Big]^{\frac{1}{2}} \leq \mathbb{E}\sup_{t\in I}|M_t| \leq C\mathbb{E}\Big[\int_0^T X_t^2 ds\Big]^{\frac{1}{2}},$$

where $c, C > 0$ are constants.

The proof is given in [14, p. 160-166].

Proposition A.5. *Let Σ be the class of all stopping times $\sigma : \Omega \to I$. Let $M = (M_t)_{t\in I}$ be a local martingale such that the family $\{M_\sigma : \sigma \in \Sigma\}$ is uniformly integrable, i.e. for all $\epsilon > 0$ there exists $c > 0$ such that $\sup_{\sigma\in\Sigma}\mathbb{E}\big[\mathbb{1}_{\{|M_\sigma|>c\}}|M_\sigma|\big] < \epsilon$. Then M is a martingale.*

Proposition A.5 is stated in [9, p. 36].

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