Author(s): Koskela, Pekka; Zhang, Yi; Zhou, Yuan

Title: Morrey–Sobolev Extension Domains

Year: 2017

Version:

Please cite the original version:

Koskela, P., Zhang, Y., & Zhou, Y. (2017). Morrey–Sobolev Extension Domains. Journal of Geometric Analysis, 27(2), 1413-1434. https://doi.org/10.1007/s12220-016-9724-9

Morrey-Sobolev Extension Domains

Pekka Koskela, Yi Ru-Ya Zhang and Yuan Zhou

Dedicated to Professor Vladimir G. Maz'ya on his 75th birthday.

Abstract We show that every uniform domain of $\mathbb{R}^n$ with $n \geq 2$ is a Morrey-Sobolev $\mathscr{W}^{1,p}$-extension domain for all $p \in [1, n)$, and moreover, that this result is essentially best possible for each $p \in [1, n)$ in the sense that, given a simply connected planar domain or a domain of $\mathbb{R}^n$ with $n \geq 3$ that is quasiconformal equivalent to a uniform domain, if it is a $\mathscr{W}^{1,p}$-extension domain, then it must be uniform.

1 Introduction

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a domain, that is, an open connected subset. The Sobolev space $W^{1,p}(\Omega)$ with $p \in [1, \infty)$ is the collection of all functions $u \in L^1_{\text{loc}}(\Omega)$ with the seminorm

$$\|u\|_{W^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)} < \infty,$$

where ∇u is the distributional derivative of u. Modulo constant functions, $W^{1,p}(\Omega)$ forms a Banach space. We say that Ω is a $W^{1,p}$-extension domain if there exists a bounded (linear) operator $\Lambda : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ such that $\Lambda u|_\Omega = u$ for all $u \in W^{1,p}(\Omega)$. Regarding the issue of linearity in our definition we refer the reader to [4]. Notice that our definition of $W^{1,p}$ and the corresponding norm only deal with ∇u. The extension problem for the usual norm is equivalent to our definition when Ω is bounded [5].

Jones showed in his seminal paper [6] that every uniform domain of $\mathbb{R}^n$ is a $W^{1,p}$-extension domain for all $p \in [1, \infty)$. Let us recall the definition.

Definition 1.1. A domain Ω is called *uniform* if there exists a positive constant ϵ_0 such that for all $x, y \in \Omega$, there exists a rectifiable curve $\gamma \subset \Omega$ joining x, y and satisfying that

$$\ell(\gamma) \leq \frac{1}{\epsilon_0}|x-y| \text{ and } d(z, \Omega^\complement) \geq \epsilon_0 \frac{|x-z||z-y|}{|x-y|} \text{ for all } z \in \gamma. \tag{1.1}$$

Jones' idea was to construct an extension operator via a reflection technique for Whitney cubes, motivated by "quasiconformal reflections"; it was well-known that a simply connected planar domain is uniform if and only if it is the image of the unit disk under some quasiconformal mapping on $\mathbb{R}^2$, see [2].

2000 *Mathematics Subject Classification:* 42B35

Key words and phases: Morrey-Sobolev space, extension, LLC, uniform domain.

Pekka Koskela and Yi Zhang were supported by the Academy of Finland grant 120972; Yuan Zhou was supported by the New Teachers' Fund for Doctor Stations (# 20121102120031) and Program for New Century Excellent Talents in University (# NCET-11-0782) of Ministry of Education of China, and National Natural Science Foundation of China (# 11201015) .

In the case $p = n$, Jones' result is essentially best possible: a simply connected planar domain is a $W^{1,2}$-extension domain if and only if it is a uniform domain; see [13] and also [2] for a higher dimensional analog.

In the case $p > n$, the correct geometric condition is $\frac{p-n}{p-1}$-subhyperbolicity: each such domain is a $W^{1,p}$-extension domain and a simply connected planar domain is a $W^{1,p}$-extension domain if and only if it is a $\frac{p-2}{p-1}$-subhyperbolic domain; see [11] and also [8]. This class of domains is strictly larger than the class of uniform domains.

In the case $1 \le p < n$, it is also known that even a simply connected planar $W^{1,p}$-extension domain is not necessarily uniform (see [10] and related examples in [7, 9]), and it is still open to find a geometric characterization.

The geometric characterization for the planar case for $p = n$ can be obtained via the concept of linear local connectivity.

Definition 1.2. *A domain Ω is linearly locally connected (for short, LLC) if there exists a constant $b \in (0, 1]$ such that for all $z \in \mathbb{R}^n$ and $r > 0$,*

LLC(1) *points in $\Omega \cap B(z, r)$ can be joined by a rectifiable curve in $\Omega \cap B(z, r/b)$;*

LLC(2) *points in $\Omega \setminus B(z, r)$ can be joined by a rectifiable curve in $\Omega \setminus B(z, br)$.*

Linear local connectivity is intimately related to uniformity in the sense that each domain that is quasiconformally equivalent to a uniform domain is uniform if and only if LLC, see e.g. [2]. Hence a simply connected planar domain is LLC if and only if it is uniform. Regarding Sobolev extensions, one knows that each $W^{1,n}$-extension domain is LLC, see [2]. In fact, $W^{1,p}$-extension domains are known to be LLC(1) for $p \ge n$ (see [14]) and LLC(2) for $n - 1 < p \le n$ and also for $p = 1$ when $n = 2$; see [3, 7].

As a by-product of the proof of our main result Theorem 1.5, we are able to remove the restriction that $p > n - 1$ for concluding LLC(2).

Proposition 1.3. *For $p \in [1, n]$ every $W^{1,p}$-extension domain satisfies LLC(2).*

Let us move to the case of a Morrey-Sobolev space $\mathscr{W}^{1,p}$, $1 \le p \le n$, defined as follows.

Definition 1.4. Let $p \in [1, n]$ and let Ω be a domain of $\mathbb{R}^n$. Define the Morrey-Sobolev space $\mathscr{W}^{1,p}(\Omega)$ as the collection of all $u \in L^1_{\mathrm{loc}}(\Omega)$ with

$$\|u\|_{\mathscr{W}^{1,p}(\Omega)} = \sup_B \left(r_B^{p-n} \int_{B\cap\Omega} |\nabla u|^p \, dx \right)^{1/p} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$, r_B is the radius of B and ∇u is the distributional gradient of u. The above formula only gives a seminorm, but modulo constant functions, $\mathscr{W}^{1,p}(\Omega)$ is a Banach space.

Observe that $\mathscr{W}^{1,n}(\Omega) = W^{1,n}(\Omega)$, and that (via the Hölder inequality), for every $p \in [1, n)$, we always have $W^{1,n}(\Omega) \subset \mathscr{W}^{1,p}(\Omega)$. Moreover, for every $p \in [1, n)$, $\mathscr{W}^{1,p}(\Omega) \subset W^{1,p}(\Omega)$ when Ω is bounded, and $\mathscr{W}^{1,p}(\Omega) \subset W^{1,p}(\Omega \cap B)$ for all balls B when Ω is unbounded.

The main aim of this paper is to establish the following result for $\mathscr{W}^{1,p}$-extension domains. Below Ω is a (Morrey-Sobolev) $\mathscr{W}^{1,p}$-extension domain if there exists a bounded (linear) operator $\Lambda : \mathscr{W}^{1,p}(\Omega) \to \mathscr{W}^{1,p}(\mathbb{R}^n)$ such that $\Lambda u|_\Omega = u$ for all $u \in \mathscr{W}^{1,p}(\Omega)$.

Theorem 1.5. *(i) Every uniform domain of $\mathbb{R}^n$ is a $\mathscr{W}^{1,p}$-extension domain for all $p \in [1, n)$.*

(ii) Let $p \in [1, n)$. Then every $\mathscr{W}^{1,p}$-extension domain is linearly locally connected. In particular, a domain Ω in $\mathbb{R}^n$ that is quasiconformal equivalent to a uniform domain is a $\mathscr{W}^{1,p}$-extension domain if and only if it is uniform.

Theorem 1.5 shows that, from the point of view of extendability, the Morrey-Sobolev spaces $\mathscr{W}^{1,p}$, $p \in [1, n)$, are similar to $W^{1,n}$ instead of to $W^{1,p}$.

The proof of Theorem 1.5 (i) employs a slight modification to the extension operator constructed by Jones and several properties of uniform domains. The proof of Theorem 1.5 (ii) relies on a lower bound on the Morrey-Sobolev capacity established in Lemma 5.2, a measure density property obtained in Theorem 4.1, and some ideas from [2]. We stress here that (ii) is not an obvious consequence of the ideas in [2]: for $p \le n-1$ and $n \ge 3$, it may well happen that the Morrey-Sobolev capacity of a pair of continua is zero, see [15]. Up to now, this phenomenom for the usual variational capacity has been the obstacle for establishing Proposition 1.3.

This paper is organized as follows. In Section 2, we recall several properties of uniform domains. In Section 3, we prove Theorem 1.5(i). In Section 4, we obtain a strong measure density property of Morrey-Sobolev and Sobolev extension domains. In Section 5, we establish a simple lower bound on the Morrey-Sobolev capacity. In Section 6, we prove Proposition 1.3 and Theorem 1.5 (ii).

The notation used in what follows is standard. We denote by C a positive constant which is independent of the main parameters, but which may vary from line to line. Constants with subscripts, such as ϵ_0, do not change in different occurrences. The symbol $A \lesssim B$ or $B \gtrsim A$ means that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$. For any locally integrable function u and measurable set X, we denote by $⨍_X u$ the average of f on X, namely, $⨍_X f \equiv \frac{1}{|X|}\int_X f\,dx$. For a set Ω and $x \in \mathbb{R}^n$, we use $d(x, \Omega)$ to denote $\inf_{z\in\Omega}|x-z|$, the distance from x to Ω. Here λQ denotes the cube concentric with Q, with sides parallel to the axes, and with edge length $\ell(\lambda Q) = \lambda\ell(Q)$.

2 Uniform domains

We recall several facts about uniform domains and establish some lemmas which will be used to prove Theorem 1.5(i).

Let Ω be a uniform domain. It is well known that $|\partial\Omega| = 0$; see [6]. Since Ω is open it admits a Whitney decomposition, that is, there exists a collection $W_1 = \{S_j\}_{j\in\mathbb{N}}$ of countably many dyadic (closed) cubes such that

(i) $\Omega = \cup_{j\in\mathbb{N}} S_j$ and $(S_k)^\circ \cap (S_j)^\circ = \emptyset$ for all $j, k \in \mathbb{N}$ with $j \ne k$;

(ii) $\ell(S_k) \le \operatorname{dist}(S_k, \partial\Omega) \le 4\sqrt{n}\ell(S_k)$;

(iii) $\frac{1}{4}\ell(S_k) \le \ell(S_j) \le 4\ell(S_k)$ whenever $S_k \cap S_j \ne \emptyset$.

Whenever $(\Omega^\complement)^\circ \ne \emptyset$, we let $W_2 = \{Q_j\}_{j\in\mathbb{N}}$ be a Whitney decomposition of $(\Omega^\complement)^\circ$. If $\operatorname{diam}\Omega < \infty$, set

$$W_3 = \{Q_j \in W_2 : \ell(Q_j) \le \frac{\epsilon_0}{16n}\operatorname{diam}\Omega\};$$

otherwise, $W_3 = W_2$. For each $i = 1, 2, 3$ and $Q \in W_i$, set $W_i(Q) = \{Q_j \in W_i, Q_j \cap Q \neq \emptyset\}$, which is the collection of all neighbor cubes of Q in W_i. It is easy to see that there exists an integer N depending only on n such that

$$\sharp W_i(Q) \leq N$$

for all $Q \in W_i$ and $i = 1, 2, 3$.

Jones [6] derived from (1.1) that for each cube in W_3, we can pick a "reflected" cube in W_1 in the following sense: for every $Q_j \in W_3$, we can find at least one $S_k \in W_1$ satisfying that

$$\ell(Q_j) \leq \ell(S_k) \leq 4\ell(Q_j)$$

and

$$\mathrm{dist}(Q_j, S_k) \leq C_1 \ell(Q_j),$$

where C_1 is a constant depending on ϵ_0 and n but not on Q_j and S_k. Usually, for each $Q_j \in W_3$, there may exist more than one such S_k, but if $S_{k'} \in W_1$ is another cube as above, we have

$$\mathrm{dist}(S_k, S_{k'}) \leq \mathrm{dist}(S_k, Q_j) + \mathrm{diam}\, Q_j + \mathrm{dist}(Q_j, S_{k'}) \leq (2C_1 + \sqrt{n})\ell(Q_j). \tag{2.1}$$

Below we fix one of these cubes and denote it by Q_j^*. Then there is an integer N_1 depending only on C_1 and n such that for all $S_k \in W_1$, there are at most N_1 cubes $Q_j \in W_3$ such that $Q_j^* = S_k$, that is,

$$\sharp\{Q_j \in W_3 :\ Q_j{*} = S_k\} \leq N_1.$$

For every pair of Q_j, $Q_k \in W_3$ with $Q_j \cap Q_k \neq \emptyset$, by (2.1), we have

$$\begin{aligned} \mathrm{dist}(Q_j^*, Q_k^*) &\leq \mathrm{dist}(Q_j^*, Q_j) + \mathrm{diam}\, Q_j + \mathrm{diam}\, Q_k + \mathrm{dist}(Q_k, Q_k^*) \\ &\leq (5C_1 + 5\sqrt{n})\ell(Q_j). \end{aligned} \tag{2.2}$$

Moreover, based on (1.1), we have

Lemma 2.1. *For every pair of Q_j, $Q_k \in W_3$ with $Q_j \cap Q_k \neq \emptyset$, we can find a family $F_{j,k} = \{S_{i_1}, \cdots, S_{i_m}\} \subset W_1$ such that $S_{i_1} = Q_j^*$, $S_{i_m} = Q_k^*$, $S_{i_s} \cap S_{i_{s+1}}$ consists of an $(n-1)$-dimensional (closed) cube for all $s = 1, \cdots, m-1$, and $m \leq N_2$, where N_2 is an integer depending only on n and ϵ_0 but not on Q_j and Q_k.*

We call such a family $F_{j,k}$ a chain of length m joining Q_j^* and Q_k^*. Obviously, $\frac{1}{4}\ell(S_{i_{s+1}}) \leq \ell(S_{i_s}) \leq 4\ell(S_{i_{s+1}})$ for all $s = 1, \cdots, m-1$. We should point out that our chain $F_{j,k}$ is a slightly different from that in [6, Lemma 2.8], where it is only required that $S_{i_s} \cap S_{i_{s+1}} \neq \emptyset$. Lemma 2.1 can be obtained by a slight modification of [6, Lemma 2.8]; we omit the details.

For each pair of Q_j, $Q_k \in W_3$ with $Q_j \cap Q_k \neq \emptyset$, we fix one family $F_{j,k}$ as in Lemma 2.1 (notice that there may have more than one such family), and set

$$F(Q_j) = \bigcup_{Q_k \in W_3(Q_j)} F_{j,k},$$

where and in the sequel, by abuse of notation, we also denote the union $\cup_{k=1}^m S_{i_k}$ by $F_{j,k}$. One can easily derive from Lemma 2.1, (2.1) and (2.2) that

Lemma 2.2. *There exists a positive constant N_3 depending only on ϵ_0 and n such that for each $Q_j \in W_3$, we have*

$$\Big\| \sum_{Q_k \in W_3(Q_j)} \chi_{\cup F_{j,k}} \Big\|_{L^\infty} \le N_3,$$

$$\Big\| \sum_{Q_j \in W_3} \chi_{\cup F(Q_j)} \Big\|_{L^\infty} \le N_3,$$

and

$$\Big\| \sum_{Q_j \in W_2 \setminus W_3} \sum_{Q_k \in W_3(Q_j)} \chi_{Q_k^*} \Big\|_{L^\infty} \le N_3.$$

To prove Theorem 1.5 (i), we need further properties of uniform domains.

Lemma 2.3. *There exists C_3 depending only on n and ϵ_0 such that for every ball $B = B(x_0, r)$ and every $Q_j \in W_3$, if $Q_j \cap B \neq \emptyset$ and $\ell(Q_j) \le 32r$, then*

$$\bigcup_{Q_k \in W_3(Q_j)} F(Q_k) \subset B(x_0, C_3 r) \cap \Omega.$$

Proof. Given $Q_k \in W_3$, we claim that $\operatorname{diam} F(Q_k) \lesssim \ell(Q_k^*)$. Indeed, for each $Q_\ell \in W_3(Q_k)$, let $F_{j,k} = \{S_{i_1}, \cdots, S_{i_m}\} \subset W_1$ be a chain connecting Q_k^* and Q_ℓ^* as in Lemma 2.3. Then by $\frac{1}{4}\ell(S_{i_s}) \le \ell(S_{i_{s+1}}) \le 4\ell(S_{i_s})$, we have

$$4^{-s}\ell(Q_k^*) \le \ell(S_{i_s}) \le 4^s \ell(Q_k^*), \tag{2.3}$$

and hence

$$\operatorname{diam} F_{k,\ell} \le \sum_{s=1}^m \operatorname{diam} S_{i_s} \le \sum_{s=1}^m 4^s \sqrt{n}\,\ell(Q_k^*) \lesssim \ell(Q_k^*).$$

Since there at most N_1 cubes $Q_\ell \in W_3(Q_k)$, we have

$$\operatorname{diam} F(Q_k) \lesssim \sum_{Q_\ell \in W_3(Q_k)} \operatorname{diam} F_{k,\ell} \lesssim \ell(Q_k^*).$$

If $Q_j \in W_3$ with $B \cap Q_j \neq \emptyset$ and $\ell(Q_j) \le 32r$, we have $d(x_0, Q_j) \le r$ and hence, by $\ell(Q_j^*) \sim \ell(Q_j)$, for all $z \in Q_j^*$,

$$|z - x_0| \le \operatorname{diam}(Q_j^*) + d(Q_j^*, Q_j) + \operatorname{diam} Q_j + d(x_0, Q_j) \lesssim r.$$

For each $Q_k \in W_3(Q_j)$, we have $\ell(Q_j^*) \sim \ell(Q_j) \sim \ell(Q_k) \sim \ell(Q_k^*)$ which together with the above claim implies that $\operatorname{diam} F(Q_k) \lesssim \ell(Q_j^*)$; hence by (2.3), for each $z \in F(Q_k)$

$$|z - x_0| \le \operatorname{diam}(F_{j,k}) + \operatorname{dist}(Q_j^*, Q_j) + \operatorname{diam} Q_j^* + \operatorname{diam} Q_j + d(x_0, Q_j) \lesssim \ell(Q_j^*) \lesssim r$$

as desired. This finishes the proof of Lemma 2.3. $\square$

Lemma 2.4. *There exists a positive constant C_2 depending only on ϵ_0 and n such that for all $Q_j \in W_3$ and $u \in W^{1,p}(\Omega)$, we have*

$$\sum_{Q_k \in W_3(Q_j)} \left| u_{Q_j^*} - u_{Q_k^*} \right|^p \le C_2 \ell(Q_j)^{p-n} \int_{F(Q_j)} |\nabla u(z)|^p \, dz. \tag{2.4}$$

Proof. For $Q_j \in W_3$ and $Q_k \in W_3(Q_j)$, let $F_{j,k} = \{S_{i_1}, \cdots, S_{i_m}\} \subset W_1$ be a chain connecting Q_j^* and Q_k^* as in Lemma 2.3. Then

$$\left| u_{Q_j^*} - u_{Q_k^*} \right| \le \sum_{s=1}^{m-1} \left| u_{S_{i_s}} - u_{S_{i_{s+1}}} \right|.$$

It then suffices to show that

$$\left| u_{S_{i_s}} - u_{S_{i_{s+1}}} \right| \lesssim \ell(S_{i_s}) \left(\fint_{S_{i_s} \cup S_{i_{s+1}}} |\nabla u(z)|^p \, dz \right)^{1/p}. \tag{2.5}$$

Indeed, this, together with $\ell(S_{i_{s+1}})/4 \le \ell(S_{i_s}) \le 4\ell(S_{i_{s+1}})$, $\ell(Q_j) \sim \ell(Q_j^*)$ and $m \le N_2$, leads to that

$$\begin{aligned}
\left| u_{Q_j^*} - u_{Q_k^*} \right| &\lesssim \sum_{s=1}^{m-1} \ell(S_{i_s}) \left(\fint_{S_{i_s} \cup S_{i_{s+1}}} |\nabla u(z)|^p \, dz \right)^{1/p} \\
&\lesssim \sum_{s=1}^{m-1} \ell(S_{i_s})^{1-n/p} \left(\int_{F_{j,k}} |\nabla u(z)|^p \, dz \right)^{1/p} \\
&\lesssim \sum_{s=1}^{m-1} 4^{s(n/p-1)} \ell(Q_j^*)^{1-n/p} \left(\int_{F_{j,k}} |\nabla u(z)|^p \, dz \right)^{1/p} \\
&\lesssim \ell(Q_j)^{1-n/p} \left(\int_{F(Q_j)} |\nabla u(z)|^p \chi_{F_{j,k}} \, dz \right)^{1/p}.
\end{aligned}$$

Hence applying Lemma 2.2, we have

$$\begin{aligned}
\sum_{Q_k \in W_3(Q_j)} \left| u_{Q_j^*} - u_{Q_k^*} \right|^p &\lesssim C_2 \ell(Q_j)^{p-n} \int_{F(Q_j)} |\nabla u(z)|^p \sum_{Q_k \in W_3(Q_j)} \chi_{F_{j,k}} \, dz \\
&\lesssim C_2 \ell(Q_j)^{p-n} \int_{F(Q_j)} |\nabla u(z)|^p \, dz
\end{aligned}$$

as desired.

Towards (2.5), let R_s be the cube with $\ell(R_s) = \frac{1}{32}\ell(S_{i_s})$ and with the same center as the $(n-1)$-dimensional cube $S_{i_1} \cap S_{i_{s+1}}$. Also let $R_s^+ \subset S_{i_s}$ be the cube with same edge length as R_s and $|R_s^+ \cap R_s| = \frac{1}{2}|R_s|$. Then $R_s^+ \subset 2R_s \subset S_{i_s} \cup S_{i_{s+1}}$.

$$\left| u_{S_{i_s}} - u_{R_s} \right| \le \left| u_{S_{i_s}} - u_{R_s^+} \right| + \left| u_{R_s^+} - u_{2R_s} \right| + \left| u_{2R_s} - u_{R_s} \right|$$

$$
\begin{aligned}
&\lesssim \fint_{S_{i_s}} |u - u_{S_{i_s}}| + \fint_{2R_s} |u - u_{2R_s}| \\
&\lesssim \ell(S_{i_s}) \left(\fint_{S_{i_s}} |\nabla u|^p \right)^{1/p} + \ell(R_s) \left(\fint_{2R_s} |\nabla u|^p \right)^{1/p} \\
&\lesssim \ell(S_{i_s}) \left(\fint_{S_{i_s} \cup S_{i_{s+1}}} |\nabla u|^p \right)^{1/p}.
\end{aligned}
$$

A similar estimate holds for $\left| u_{S_{i_{s+1}}} - u_{R_s} \right|$. Combining these estimates gives the desired results, and hence completes the proof of Lemma 2.4. $\square$

Jones further established the following density result.

Lemma 2.5. *Suppose that Ω is a bounded uniform domain. Then $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\Omega)$ for all $p \in [1, \infty)$.*

Finally, we need a Sobolev-Poincaré inequality for suitable subdomains of uniform domains. This result follows by combining the main result in [1] with a localization result, see 1.9 in [12].

Lemma 2.6. *Suppose that Ω is a uniform domain and $1 \le p < n$. Set $p* = pn/(n-p)$. Then there exist positive constants C_4, C_5 depending only on n and Ω so that the following holds. Given any $x \in \Omega$ and $r > 0$, there is a subdomain $\Omega_r \subset \Omega$ with*

$$
B(x, r/C_4) \cap \Omega \subset \Omega_r \subset B(x, C_4 r) \cap \Omega
$$

and such that for all $u \in W^{1,p}(\Omega_r)$, we have $u \in L^1(\Omega_r)$ and

$$
\left(\fint_{\Omega_r} |u(z) - u_\Omega|^{p*} \, dz \right)^{1/p*} \le C_5 \operatorname{diam}(\Omega_r) \left(\fint_{\Omega_r} |\nabla u(z)|^p \, dz \right)^{1/p}.
$$

3 Proof of Theorem 1.5 (i)

Let Ω be a uniform domain. We divide the proof into the following 4 steps.

Step 1. We employ Jones's construction of extension operator on uniform domains (with a slight modification when $\operatorname{diam}\Omega < \infty$). Let W_1, W_2 and W_3 be as in Section 2. For each $Q_j \in W_3$, let $\phi_j \in C^\infty(\mathbb{R}^n)$ such that ϕ_j is supported on $\frac{9}{8}Q_j$, $0 \le \phi_j \le 1$, $\sum_{Q_j \in W_3} \phi_j = 1$ on $\cup_{Q_j \in W_3} Q_j$ and

$$
|\nabla \phi_j| \le C \ell(Q_j)^{-1}, \tag{3.1}
$$

where C is a constant independent of Q_j.

For $u \in \mathscr{W}^{1,p}(\Omega)$, we define an extension Λu as:

$$
\Lambda u(x) = \begin{cases} u_\Omega + \sum_{Q_j \in W_3} (u_{Q_j^*} - u_\Omega)\phi_j, & x \in \mathbb{R}^n \setminus \overline{\Omega} \\ u(x), & x \in \Omega, \end{cases}
$$

where $Q_j^* \in W_1$ corresponds to Q_j as in Section 2, and $u_\Omega = \frac{1}{|\Omega|}\int_\Omega u(z)\,dz$ if $\operatorname{diam}\Omega < \infty$ and $u_\Omega = 0$ if $\operatorname{diam}\Omega = \infty$. Notice that Λ is linear and independent of p, and Λu is defined on $\mathbb{R}^n \setminus \partial\Omega$. Now we have to show that Λu induces another function $\widetilde{\Lambda} u$ defined on entire $\mathbb{R}^n$ such that $\widetilde{\Lambda} u = \Lambda u$ on $\mathbb{R}^n \setminus \partial\Omega$, $\widetilde{\Lambda} u \in \mathscr{W}^{1,\,p}(\mathbb{R}^n)$ and $\|\widetilde{\Lambda} u\|_{\mathscr{W}^{1,\,p}(\mathbb{R}^n)} \lesssim \|u\|_{\mathscr{W}^{1,\,p}(\Omega)}$.

Step 2. We first show that $\Lambda u \in \mathscr{W}^{1,\,p}(\mathbb{R}^n \setminus \overline{\Omega})$ and $\|\Lambda u\|_{\mathscr{W}^{1,\,p}(\mathbb{R}^n\setminus\overline{\Omega})} \lesssim \|u\|_{\mathscr{W}^{1,\,p}(\Omega)}$. It suffices to show that $\Lambda u \in W^{1,\,p}_{\mathrm{loc}}(\overline{\Omega}^\complement)$ and for every ball $B = B(x_0,\, r)$, we have

$$\mathbf{H} = r^{n-p}\int_{B\cap\overline{\Omega}^\complement} |\nabla\Lambda u(z)|^p\,dz \lesssim \|u\|_{\mathscr{W}^{1,\,p}(\Omega)}. \tag{3.2}$$

Without loss of generality, we may assume that $(\overline{\Omega})^\complement \neq \emptyset$.

We first show that $\Lambda u \in W^{1,\,p}_{\mathrm{loc}}(\overline{\Omega}^\complement)$. Indeed, for every ball B with $2B \subset \overline{\Omega}^\complement$, there are finitely many $Q_i \in W_2$ such that $Q_i \cap B \neq \emptyset$. For each $Q_i \in W_2$, notice that if $\phi_j \neq 0$ on Q_i, then $Q_i \in W_3(Q_j)$, that is, $Q_j \in W_3(Q_i)$. So on B

$$\Lambda u = u_\Omega + \sum_{\substack{Q_i\cap B\neq\emptyset\\ Q_j\in W_3(Q_i)}} (u^*_{Q_j} - u_\Omega)\phi_j$$

as a finite sum. Hence Λu is differentiable on B and

$$|\nabla\Lambda u| \lesssim \sum_{\substack{Q_i\cap B\neq\emptyset\\ Q_j\in W_3(Q_i)}} |u^*_{Q_j} - u^*_{Q_i}|\ell(Q_j)^{-1} < \infty$$

on B, that is, $\Lambda u \in W^{1,\,p}_{\mathrm{loc}}(B)$. So we have $\Lambda u \in W^{1,\,p}_{\mathrm{loc}}(\overline{\Omega}^\complement)$.

To see (3.2) for arbitrary $B \subset \mathbb{R}^n$, write

$$\begin{aligned}\mathbf{H} =& \sum_{Q_i\in W_2} r^{p-n}\int_{B\cap Q_i} |\nabla\Lambda u(z)|^p dz\\ =& \sum_{Q_i\in W_3} r^{p-n}\int_{B\cap Q_i} |\nabla\Lambda u(z)|^p dz + \sum_{Q_i\in W_2\setminus W_3} r^{p-n}\int_{B\cap Q_i} |\nabla\Lambda u(z)|^p dz\\ =& \mathbf{H}_1 + \mathbf{H}_2.\end{aligned}$$

Observe that if $\operatorname{diam}\Omega = \infty$, then $W_3 = W_2$ and hence $\mathbf{H}_2 = 0$. Recall that on each $Q_i \in W_3$, we have $\sum_{Q_j\in W_3}\phi_j = 1$. Thus, by (3.1), we have

$$\mathbf{H}_1 = \sum_{Q_i\in W_3} r^{p-n}\int_{B\cap Q_i}\left|\sum_{Q_j\in W_3}\left[u_{Q_j^*} - u_{Q_i^*}\right]\nabla\phi_j(z)\right|^p dz$$

Observe that $\phi_j \neq 0$ on Q_i only happens when $Q_j \in W_3(Q_i)$. Indeed, we have

$$\operatorname{supp}\phi_j \subset \frac{17}{16}Q_j \subset \bigcup_{Q_k\in W_3(Q_j)} Q_k.$$

So if $\phi_j \neq 0$ on Q_i, then $Q_i \in W_3(Q_j)$, that is, $Q_j \in W_3(Q_i)$. By $\sharp W_3(Q_i) \lesssim 1$, $\ell(Q_i) \sim \ell(Q_j)$ for all $Q_j \in W_3(Q_i)$ and Lemma 2.4, we obtain

$$\begin{aligned}
\mathbf{H}_1 &\lesssim \sum_{Q_i \in W_3} r^{p-n} \int_{B\cap Q_i} \sum_{Q_j \in W_3(Q_i)} \left| u_{Q_j^*} - u_{Q_i^*} \right|^p |\nabla \phi_j(z)|^p dz \\
&\lesssim \sum_{Q_i \in W_3} r^{p-n} \ell(Q_i)^{-p} |B \cap Q_i| \sum_{Q_j \in W_3(Q_i)} \left| u_{Q_j^*} - u_{Q_i^*} \right|^p \\
&\lesssim \sum_{Q_i \in W_3} r^{p-n} \ell(Q_i)^{-n} |B \cap Q_i| \int_{F(Q_i)} |\nabla u(z)|^p \, dz \\
&\lesssim \sum_{\substack{Q_i \in W_3,\, Q_i \cap B \neq \emptyset \\ \ell(Q_i) \leq 32r}} r^{p-n} \int_{F(Q_i)} |\nabla u(z)|^p \, dz \\
&\quad + \sum_{\substack{Q_i \in W_3,\, Q_i \cap B \neq \emptyset \\ \ell(Q_i) > 32r}} r^p \ell(Q_i)^{-n} \int_{F(Q_i)} |\nabla u(z)|^p \, dz \\
&= \mathbf{H}_{1,1} + \mathbf{H}_{1,2}.
\end{aligned}$$

By Lemma 2.2 and Lemma 2.3, we obtain

$$\mathbf{H}_{1,1} \lesssim r^{p-n} \int_{B(x_0, Cr)\cap\Omega} |\nabla u(z)|^p \, dz \lesssim \|u\|_{\mathscr{W}^{1,p}(\Omega)}.$$

To estimate $\mathbf{H}_{1,2}$, we may assume that $B \cap Q_{i_0} \neq \emptyset$ for some $Q_{i_0} \in W_3$ with $32r < \ell(Q_{i_0})$. Observe that $32r < \ell(Q_{i_0})$ implies that $B \subset \frac{9}{8} Q_{i_0}$, and hence, applying Lemma 2.3 to the ball $B = B(x_{Q_{i_0}}, \ell(Q_{i_0}))$, we have

$$\begin{aligned}
\mathbf{H}_{1,2} &\lesssim \ell(Q_{i_0})^{p-n} \sum_{Q_i \in W_3(Q_{i_0})} \int_{F(Q_i)} |\nabla u(z)|^p \, dz \\
&\lesssim \ell(Q_{i_0})^{p-n} \int_{B(x_{Q_{i_0}}, C\ell(Q_{i_0}))} |\nabla u(z)|^p \, dz \\
&\lesssim \|u\|_{\mathscr{W}^{1,p}(\Omega)}.
\end{aligned}$$

If $\operatorname{diam}\Omega = \infty$, by $\mathbf{H}_2 = 0$, the proof of (3.2) is finished. Assume that $\operatorname{diam}\Omega < \infty$. Suppose that $u \neq 0$ on Q_i for some $Q_i \in W_2 \setminus W_3$. Observing that $\phi_j \neq 0$ on Q_i only happens when $Q_j \in W_3(Q_i)$, we have

$$u = u_\Omega + \sum_{Q_j \in W_3(Q_i)} (u_{Q_j^*} - u_\Omega)\phi_j$$

on Q_i. Notice that for each $Q_j \in W_3(Q_i)$,

$$\ell(Q_j) \leq \frac{1}{16nC_0} \operatorname{diam}(\Omega) < \ell(Q_i) \leq 16\ell(Q_j),$$

and hence, by Lemma 2.6,

$$\begin{aligned}|u_{Q_j^*} - u_\Omega| &\lesssim \fint_{Q_j^*} |u(z) - u_\Omega|\, dz \\ &\lesssim \fint_{\Omega} |u(z) - u_\Omega|\, dz \\ &\lesssim \operatorname{diam}(\Omega) \left(\fint_\Omega |\nabla u(z)|^p\, dz \right)^{1/p} \\ &\lesssim \operatorname{diam}(\Omega)^{1-n/p} \left(\int_\Omega |\nabla u(z)|^p\, dz \right)^{1/p}.\end{aligned}$$

So, by $\sharp W_3(Q_i) \lesssim 1$,

$$\begin{aligned}\mathbf{H}_2 &\le \sum_{\substack{Q_i \in W_2 \setminus W_3 \\ B\cap Q_i \neq \emptyset}} r^{p-n} \int_{B\cap Q_i} \sum_{Q_j \in W_3(Q_i)} |u_{Q_j^*} - u_\Omega|^p |\nabla \phi_j|^p dz \\ &\le \sum_{\substack{Q_i \in W_2 \setminus W_3 \\ B\cap Q_i \neq \emptyset}} r^{p-n} |B \cap Q_i| (\operatorname{diam}\Omega)^{-n} \int_\Omega |\nabla u(z)|^p\, dz.\end{aligned}$$

Notice that $\sharp\{Q_i \in W_2 \setminus W_3 : \ B \cap Q_i \neq \emptyset\} \lesssim 1$ due to $\ell(Q_i) \sim \operatorname{diam}\Omega$ and $\operatorname{dist}(Q_i, \Omega) \gtrsim \ell(Q_i)$. If $r < \frac{1}{2nC_0} \operatorname{diam}\Omega$, then

$$\mathbf{H}_2 \lesssim r^p (\operatorname{diam}\Omega)^{-n} \int_\Omega |\nabla u(z)|^p\, dz \lesssim (\operatorname{diam}\Omega)^{p-n} \int_\Omega |\nabla u(z)|^p\, dz \lesssim \|u\|_{\mathscr{W}^{1,p}(\Omega)}.$$

If $r \ge \frac{1}{2nC_0} \operatorname{diam}\Omega$, then

$$\mathbf{H}_2 \lesssim r^{p-n} \int_\Omega |\nabla u(z)|^p\, dz \lesssim (\operatorname{diam}\Omega)^{p-n} \int_\Omega |\nabla u(z)|^p\, dz \lesssim \|u\|_{\mathscr{W}^{1,p}(\Omega)}.$$

Combining the estimates of $\mathbf{H}_1$ and $\mathbf{H}_2$, we obtain the desired result.

Step 3. For $u \in \mathscr{W}^{1,p}(\Omega) \cap C_0^1(\mathbb{R}^n)$, we define $\widetilde{\Lambda} u = \Lambda u$ on $\mathbb{R}^n \setminus \Omega$ and $\widetilde{\Lambda} u = u$ on $\partial\Omega$. We claim that $\widetilde{\Lambda} u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$.

To see that $\widetilde{\Lambda} u \in W^{1,p}(B)$ for every ball $B \subset \mathbb{R}^n$, it suffices to show that $\widetilde{\Lambda} u$ is locally Lipschitz on $\mathbb{R}^n$, that is,

$$|\widetilde{\Lambda} u(x) - \widetilde{\Lambda} u(y)| \lesssim |x - y| \tag{3.3}$$

whenever $x, y \in A \subset\subset \mathbb{R}^n$.

Case $x, y \in \overline{\Omega}$. Obviously, $u \in C_0^1(\mathbb{R}^n)$ implies (3.3).

Case $x \in \overline{\Omega}$ and $y \in \mathbb{R}^n \setminus \overline{\Omega}$. Then $y \in Q_i$ for some $Q_i \in W_2$. Notice that if $\phi_j \neq 0$ on Q_i, then $Q_i \in W_3(Q_j)$, that is, $Q_j \in W_3(Q_i)$. So

$$\Lambda u(x) - \Lambda u(y) = (u(x) - u_\Omega) + \sum_{Q_j \in W_3(Q_i)} (u^*_{Q_j} - u_\Omega)\phi_j.$$

If $Q_i \in W_3$, since $\sum_{Q_j \in W_3(Q_i)} \phi_j = \sum_{Q_j \in W_3} \phi_j = 1$ on Q_i, we have

$$
\begin{aligned}
|\widetilde{\Lambda}u(x) - \widetilde{\Lambda}u(y)| &= |\sum_{Q_j \in W_3(Q_i)} (u(x) - u^*_{Q_j})\phi_j| \\
&\le \|\nabla u\|_{L^\infty(\mathbb{R}^n)} (\sharp W_3(Q_i)) \sup_{Q_j \in W_3(Q_i)} \sup_{z \in Q^*_j} |x - z|.
\end{aligned}
$$

For each $z \in Q^*_j$ with $Q_j \in W_3(Q_i)$, since $|x - y| \ge d(y, \partial\Omega) \ge \ell(Q_i) \gtrsim \ell(Q_j)$, we have

$$
\begin{aligned}
|x - z| &\le |x - y| + |y - z| \le |x - y| + \sqrt{n}\ell(Q_j) + \operatorname{dist}(Q_j, Q^*_j) + \sqrt{n}\ell(Q^*_j) \\
&\lesssim |x - y| + \ell(Q_j) \lesssim |x - y|.
\end{aligned}
$$

We conclude that $|\widetilde{\Lambda}u(x) - \widetilde{\Lambda}u(y)| \lesssim |x - y|$.

If $Q_i \in W_2 \setminus W_3$, we have $|x-y| \ge d(y, \partial\Omega) \ge \ell(Q_i) \gtrsim \ell(Q_j)$ for all $Q_j \in W_3(Q_i)$, which together with $\sum_{Q_j \in W_2(Q_i)} |\phi_j| \le 1$ and $|u(x) - u_\Omega| + |u^*_{Q_j} - u_\Omega| \le 2\|\nabla u\|_{L^\infty(\mathbb{R}^n)} \operatorname{diam}\Omega$ gives (3.3).

Case $x, y \in \mathbb{R}^n \setminus \overline{\Omega}$. Notice that $\widetilde{\Lambda}u \in C^1(\mathbb{R}^n \setminus \overline{\Omega})$ and $|\nabla \widetilde{\Lambda}u| \lesssim \|\nabla u\|_{L^\infty(\mathbb{R}^n)}$ on $\mathbb{R}^n \setminus \overline{\Omega}$. Indeed, for $z \in Q_i$ with $Q_i \in W_3$, by $\sum_{Q_j \in W_3(Q_i)} \phi_j = 1$ on Q_i, we have

$$
\begin{aligned}
|\nabla \Lambda u(z)| &= |\sum_{Q_j \in W_3(Q_i)} (u_{Q^*_j} - u_{Q^*_i})\nabla\phi_j| \\
&\lesssim \sum_{Q_j \in W_3(Q_i)} \|\nabla u\|_{L^\infty(\mathbb{R}^n)} \sqrt{n}[\ell(Q^*_i) + \ell(Q^*_j)]\ell(Q_j)^{-1} \\
&\lesssim 1;
\end{aligned}
$$

for $z \in Q_i$ with $Q_i \in W_2 \setminus W_3$, by $\ell(Q_j) \sim \ell(Q_i) \sim \operatorname{diam}\Omega$ we have

$$
|\nabla \widetilde{\Lambda}u(z)| = |\sum_{Q_j \in W_3(Q_i)} (u_{Q^*_j} - u_\Omega)\nabla\phi_j| \lesssim \sum_{Q_j \in W_3(Q_i)} \|u\|_{C^1(\mathbb{R}^n)} (\operatorname{diam}\Omega)\ell(Q_j)^{-1} \lesssim 1.
$$

Thus (3.3) follows.

Step 4. Finally, for arbitrary $u \in \mathscr{W}^{1,p}(\Omega)$, we define $\widetilde{\Lambda}$ with $\widetilde{\Lambda}u = \Lambda u$ on $\mathbb{R}^n \setminus \partial\Omega$ as above.

Since $|\partial\Omega| = 0$, for every ball $B \subset \mathbb{R}^n$, we have

$$
\int_B |\nabla \widetilde{\Lambda}u(z)|^p \, dz = \int_{B \cap \Omega} |\nabla \Lambda u(z)|^p \, dz + \int_{B \cap \overline{\Omega}^{\complement}} |\nabla \Lambda u(z)|^p \, dz.
$$

By this and the estimate from Step 2, our claim would follow if we knew that $\widetilde{\Lambda}u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$.

We show that $\widetilde{\Lambda}u$ has this degree of regularity relying on Step 3. We separate our argument into two cases.

Case of $\operatorname{diam}\Omega < \infty$. Then $u \in \mathscr{W}^{1,p}(\Omega)$ implies that $u \in W^{1,p}(\Omega)$ and from Lemma 2.6 we further deduce that $u \in L^p(\Omega)$. From Lemma 2.5 we conclude that $C^1_0(\mathbb{R}^n)$ is dense in $W^{1,p}(\Omega)$ and since $u \in L^p(\Omega)$ it easily follows via the Poincaré inequality that there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C_0^1(\mathbb{R}^n)$ such that $\|u-u_k\|_{W^{1,p}(\Omega)} + \|u-u_k\|_{L^p(\Omega)} \lesssim 2^{-k}$. By Step 3, $\widetilde{\Lambda}u_k$ is well-defined, and $\widetilde{\Lambda}u_k - \widetilde{\Lambda}u_\ell = \widetilde{\Lambda}(u_k-u_\ell)$ for all $k,\ell \in \mathbb{N}$. Moreover, by Jones [6],

$$\|\widetilde{\Lambda}u_k - \widetilde{\Lambda}u_\ell\|_{W^{1,p}(\mathbb{R}^n)} = \|\widetilde{\Lambda}(u_k-u_\ell)\|_{W^{1,p}(\mathbb{R}^n)} \lesssim \|\widetilde{\Lambda}u_k - \widetilde{\Lambda}u_\ell\|_{W^{1,p}(\Omega)} \lesssim 2^{-k}.$$

By the Poincaré inequality, we know that there exists a function still denoted by $\widetilde{\Lambda}u$ such that $\widetilde{\Lambda}u_k$ converges to $\widetilde{\Lambda}u$ in $L^p_{\rm loc}(\mathbb{R}^n)$ and in $W^{1,p}(\mathbb{R}^n)$. Obviously on Ω, we have $\Lambda u = u$. Hence for each $S_j \in W_1$, $u_{S_j} = \lim_{k\to\infty}(u_k)_{S_j}$. Thus, $\widetilde{\Lambda}u_k(z) = \Lambda u_k(z)$ converges to $\Lambda u(z)$ for all $z \in \overline{\Omega}^{\complement}$, that is, $\widetilde{\Lambda}u(z) = \Lambda u(z)$ for all $z \in \overline{\Omega}^{\complement}$.

Case of $\operatorname{diam}\Omega = \infty$. Observe that $u \in W^{1,p}(\Omega \cap B_\lambda)$ for all balls $B_\lambda = B(x_0, \lambda)$ with $x_0 \in \partial\Omega$ and $\lambda \in \mathbb{N}$. Fix λ and choose $\eta_\lambda \in C_0^1(B_\lambda)$ so that $\eta_\lambda = 1$ on $B_{\lambda/2}$. Applying Lemma 2.6 for the ball $B(x_0, C_8\lambda)$ we see that $u\eta_\lambda \in W^{1,p}(\Omega) \cap L^p(\Omega)$.

Since $u\eta_\lambda$ has bounded support, we conclude as in our previous case that there exists a sequence $\{u_{\lambda,k}\}_{k\in\mathbb{N}} \subset C_0^1(\mathbb{R}^n)$ such that $\|u\eta_\lambda - u_{\lambda,k}\|_{W^{1,p}(\Omega)} + \|u\eta_\lambda - u_{\lambda,k}\|_{L^p(\Omega)} \lesssim 2^{-k}$. As above, we obtain an extension $\widetilde{\Lambda}(u\eta_\lambda) \in W^{1,p}(\mathbb{R}^n)$ such that $\widetilde{\Lambda}(u\eta_\lambda) = \Lambda(u\eta_\lambda)$ on $\mathbb{R}^n \setminus \partial\Omega$.

Fix a ball $B \subset \mathbb{R}^n$. Observe that whenever $\lambda, \tau \in \mathbb{N}$ with $\lambda < \tau$, we have $\widetilde{\Lambda}(u\eta_\lambda) = \widetilde{\Lambda}(u\eta_\tau)$ on $B_{\lambda/2} \cap \Omega$. Choose now λ so that $B \subset B_{\lambda/2}$. If $\overline{\Omega}^{\complement} \neq \emptyset$, then $B \cap \overline{\Omega}^{\complement} \neq \emptyset$ can be covered by cubes in $Q_i \in W_2$ with $\operatorname{diam} Q_i \le 4\sqrt{n}\lambda$. There exists a positive integer N such that for all $\tau \ge N\lambda$, $F(Q_i) \subset B_{\tau/2}$, and hence $u_{Q_j^*} = (u\eta_\tau)_{Q_j^*} = u_{Q_j^*}$, which further implies that $\widetilde{\Lambda}(u\eta_\tau) = \Lambda u$ on Q_i and hence on $B \cap \overline{\Omega}^{\complement}$. So $\widetilde{\Lambda}(u\eta_\tau) = \widetilde{\Lambda}(u\eta_{N\lambda}) = \Lambda u$ on $B \setminus \partial\Omega \subset B_\tau \setminus \partial\Omega$. Since $|\partial\Omega| = 0$, we conclude that $\widetilde{\Lambda}u \in W^{1,p}(B)$.

The claim follows by noticing from above that we may define $\widetilde{\Lambda}u$ via the extensions $\widetilde{\Lambda}(u\eta_{N\lambda})$

4 Measure density of $\mathscr{W}^{1,p}$- and $W^{1,p}$-extension domains

We first show the following strong measure density property for $\mathscr{W}^{1,p}$-extension domains, which will be employed to prove Theorem 1.5(ii).

Theorem 4.1. *Let $p \in [1, n)$ and Ω be a $\mathscr{W}^{1,p}$-extension domain of $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that for every component Ω_0 of $\Omega \setminus K$, where K is an arbitrary closed set of $\mathbb{R}^n$, and for all $x \in \Omega_0$ and $0 < r \le \min\{\operatorname{dist}(x, \partial\Omega_0 \setminus \partial\Omega), \operatorname{diam}\Omega\}$ we have*

$$|\Omega_0 \cap B(x, r)| \ge Cr^n.$$

Theorem 4.1 means that the components of $\Omega \setminus K$ for an arbitrary closed set K of $\mathbb{R}^n$ are Ahlfors n-regular to some extent. To prove this, we use ideas from [4].

Lemma 4.2. *Let $p \in [1, n)$ and Ω be a $\mathscr{W}^{1,p}$-extension domain of $\mathbb{R}^n$. There exist positive constants C_6 and C_7 such that for each $B \subset \mathbb{R}^n$ and for all $u \in \mathscr{W}^{1,p}(\Omega)$,*

$$\inf_{c\in\mathbb{R}} \int_{B\cap\Omega} \exp\left(\frac{C_6|u-c|}{\|u\|_{\mathscr{W}^{1,p}(\Omega)}}\right) dx \le C_7|B|.$$

Proof. Since Ω is a $\mathscr{W}^{1,p}$-extension domain of $\mathbb{R}^n$, denoting the extension operator by Λ, for every $u \in \mathscr{W}^{1,p}(\Omega)$, we have that $\Lambda u \in \mathscr{W}^{1,p}(\mathbb{R}^n)$ with

$$\|\Lambda u\|_{\mathscr{W}^{1,p}(\mathbb{R}^n)} \le C_0 \|u\|_{\mathscr{W}^{1,p}(\Omega)},$$

where C_0 is a fixed constant. Applying Poincaré's inequality, we have $\Lambda u \in BMO(\mathbb{R}^n)$ with

$$\|\Lambda u\|_{BMO(\mathbb{R}^n)} = \sup_B \fint_B |\Lambda u - (\Lambda u)_B| \le \widetilde{C}_0 \sup_B r_B \left(\fint_B |\nabla \Lambda u|^p \right)^{1/p} \le \widetilde{C}_0 \|\Lambda u\|_{\mathscr{W}^{1,p}(\mathbb{R}^n)},$$

where $\widetilde{C}_0$ is a fixed constant. Moreover, by the John-Nirenberg theorem for BMO-functions, for each ball B,

$$\fint_B \exp\left(\frac{\widetilde{\widetilde{C}}_0 |\Lambda u - (\Lambda u)_B|}{\|\Lambda u\|_{BMO(\mathbb{R}^n)}} \right) dx \lesssim 1,$$

where $\widetilde{\widetilde{C}}_0$ is a fixed constant. Thus, choosing $C_6 = \widetilde{\widetilde{C}}_0 / C_0 \widetilde{C}_0$, we have

$$\inf_{c \in \mathbb{R}} \frac{1}{|B|} \int_{B \cap \Omega} \exp\left(\frac{C_6 |u - c|}{\|u\|_{\mathscr{W}^{1,p}(\Omega)}} \right) dx \le \fint_B \exp\left(\frac{C_0 |\Lambda u - (\Lambda u)_B|}{\|\Lambda u\|_{BMO(\mathbb{R}^n)}} \right) dx \lesssim 1$$

as desired. $\square$

Proof of Theorem 4.1. Without loss of generality, we may assume that $r < \infty$. Let $b_0 = 1$ and $b_j \in (0, 1)$ for $j \in \mathbb{N}$ be such that

$$|B(x, b_j r) \cap \Omega_0| = 2^{-1} |B(x, b_{j-1} r) \cap \Omega_0| = 2^{-j} |B(x, r) \cap \Omega_0|. \tag{4.1}$$

Define a function u on Ω by setting

$$u(y) \equiv \begin{cases} 1, & y \in B(x, b_2 r) \cap \Omega_0, \\ \dfrac{b_1 r - |y - x|}{b_1 r - b_2 r}, & y \in (B(x, b_1 r) \setminus B(x, b_2 r)) \cap \Omega_0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $B(x, r) \cap \Omega_0 \cap K = \emptyset$ and $B(x, r)$ has empty intersection with other components of $\Omega \setminus K$, we know that u is well defined. Then

$$\|u\|_{\mathscr{W}^{1,p}(\Omega)} \le \|u\|_{W^{1,n}(\Omega)} \le \frac{|(B(x, b_1 r) \setminus B(x, b_2 r)) \cap \Omega_0|^{1/n}}{b_1 r - b_2 r} \lesssim \frac{|B(x, r) \cap \Omega_0|^{1/n}}{b_1 r - b_2 r}.$$

By Lemma 4.2 and (4.1), we have

$$\inf_{c \in \mathbb{R}} \int_{B(x, r) \cap \Omega_0} \exp\left(C \frac{|u(y) - c|(b_1 r - b_2 r)}{|B(x, b_1 r) \cap \Omega_0|^{1/n}} \right) dy \le C_2 r^n.$$

Observe that, for each $c \in \mathbb{R}$, $|u - c| \geq 1/2$ either on $(B(x, r) \setminus B(x, b_1 r)) \cap \Omega_0$ or on $B(x, b_2 r) \cap \Omega_0$. By (4.1), we conclude that

$$|B(x, b_1 r) \cap \Omega_0| \exp\left(\frac{C(b_1 r - b_2 r)}{|B(x, b_1 r) \cap \Omega_0|^{1/n}}\right) \lesssim r^n,$$

which implies that

$$b_1 r - b_2 r \leq |B(x, b_1 r) \cap \Omega_0|^{1/n} \log\left(\frac{Cr^n}{|B(x, b_1 r) \cap \Omega_0|}\right).$$

Similar inequalities also hold for $b_j r - b_{j+1} r$ with $j \geq 2$. This leads to

$$\begin{aligned} b_1 r &= \sum_{j \in \mathbb{N}} (b_j r - b_{j+1} r) \lesssim \sum_{j \in \mathbb{N}} |B(x, b_j r) \cap \Omega_0|^{1/n} \log\left(\frac{C(b_{j-1} r)^n}{|B(z, b_j r) \cap \Omega_0|}\right) \\ &\lesssim \sum_{j \in \mathbb{N}} 2^{-j/n} |B(x, r) \cap \Omega_0|^{1/n} \log\left(2^j \frac{Cr^n}{|B(x, r) \cap \Omega_0|}\right) \\ &\lesssim |B(x, r) \cap \Omega_0|^{1/n} \log\left(\frac{Cr^n}{|B(x, r) \cap \Omega_0|}\right). \end{aligned}$$

If $b_1 \geq 1/10$, observing that $t \log \frac{1}{t} \geq 1$ implies that $t \gtrsim 1$, we have

$$|B(x, r) \cap \Omega_0|^{1/n} \gtrsim r$$

as desired. If $b_1 \leq 1/10$, we choose $R = 2r/5$ and a point $y \in B(x, r) \cap \Omega_0$ such that $|y - x| = b_1 r + R/2$. Then

$$B(x, b_1 r) \subset B(y, R) \subset B(x, r)$$

but $B(y, R/2) \cap B(x, b_1 r) = \emptyset$. Therefore, if

$$|B(y, \widetilde{b}_1 R) \cap \Omega_0| = \frac{1}{2} |B(y, R) \cap \Omega_0|,$$

then by

$$|B(x, b_1 r) \cap \Omega_0| \geq \frac{1}{2} |B(y, R) \cap \Omega_0|,$$

we have $\widetilde{b}_1 \geq 1/2$. Applying the result when $b_1 \geq 1/10$, we conclude that

$$|B(y, R) \cap \Omega_0| \gtrsim R^n,$$

which implies $|B(x, r) \cap \Omega_0| \gtrsim r^n$ as desired. $\square$

Theorem 4.1 implies the following result.

Corollary 4.3. *Let $p \in [1, n)$ and Ω be a $\mathscr{W}^{1,p}$-extension domain of $\mathbb{R}^n$. Then Ω has the measure density property, that is, there exists a constant $C > 0$ such that for all $x \in \Omega$ and $0 < r < \operatorname{diam} \Omega$, we have*

$$|\Omega \cap B(x, r)| \geq Cr^n.$$

For $W^{1,p}$-extension domains with $p \in [1, n)$, a result similar to Corollary 4.3 was established in [4]. But to show the LLC(2)-property of Sobolev $W^{1,p}$-extension domains when $p \in [1, n-1]$, we need the following stronger version, which is similar to Theorem 4.1; see Section 6 below.

Theorem 4.4. *Let $p \in [1, n)$ and Ω be a $W^{1,p}$-extension domain of $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that for every component Ω_0 of $\Omega \setminus K$, where K is an arbitrary closed set of $\mathbb{R}^n$, and for all $x \in \Omega_0$ and $0 < r \leq \min\{\operatorname{dist}(x, \partial\Omega_0 \setminus \partial\Omega), \operatorname{diam}\Omega\}$ we have*

$$|\Omega_0 \cap B(x, r)| \geq Cr^n.$$

Proof of Theorem 4.4. Without loss of generality, we may assume that $r < \infty$. The proof is similar to that of Theorem 4.1; we sketch it. Let b_j for $j \in \mathbb{N} \cup \{0\}$ and the function u be exactly as in Theorem 4.1. Then

$$\|u\|_{W^{1,p}(\Omega)} \leq \frac{|(B(x, b_1 r) \setminus B(x, b_2 r)) \cap \Omega_0|^{1/p}}{b_1 r - b_2 r} \lesssim \frac{|B(x, r) \cap \Omega_0|^{1/p}}{b_1 r - b_2 r}.$$

Since Ω is a $W^{1,p}$-extension domain, we may extend u to $\widetilde{u} \in W^{1,p}(\mathbb{R}^n)$ with a norm bound. Applying the Sobolev-Poincaré inequality, we have

$$\begin{aligned}\left(\int_{B(x,r)} |\widetilde{u} - \widetilde{u}_{B(x,r)}|^{np/(n-p)}\right)^{(n-p)/np} &\lesssim \left(\int_{B(x,r)} |\nabla \widetilde{u}|^p\right)^{1/p} \\ &\lesssim \|\widetilde{u}\|_{W^{1,p}(\mathbb{R}^n)} \lesssim \|\widetilde{u}\|_{W^{1,p}(\Omega)}.\end{aligned}$$

Observe that, for each $c \in \mathbb{R}$, $|u - c| \geq 1/2$ either on $(B(x, r) \setminus B(x, b_1 r)) \cap \Omega_0$ or on $B(x, b_2 r) \cap \Omega_0$. By (4.1), we have

$$|B(x, b_1 r) \cap \Omega_0|^{(n-p)/np} \lesssim \frac{|B(x, r) \cap \Omega_0|^{1/p}}{b_1 r - b_2 r},$$

which implies that

$$b_1 r - b_2 r \lesssim |B(x, b_1 r) \cap \Omega_0|^{1/n}.$$

Similar inequalities also hold for $b_j r - b_{j+1} r$ with $j \geq 2$. This leads to

$$\begin{aligned} b_1 r &= \sum_{j=1}^{\infty} (b_j r - b_{j+1} r) \lesssim \sum_{j \in \mathbb{N}} |B(x, b_{j+1} r) \cap \Omega_0|^{1/n} \\ &\leq \sum_{j \in \mathbb{N}} 2^{-j/n} |B(x, r) \cap \Omega_0|^{1/n} \\ &\lesssim |B(x, r) \cap \Omega_0|^{1/n}. \end{aligned}$$

Applying the same argument on b_1 as Theorem 4.1, we have $|B(x, r) \cap \Omega_0|^{1/n} \gtrsim r$ as desired. $\square$

5 Morrey-Sobolev Capacity

We obtain a lower bound for the Morrey-Sobolev capacity, which will be used in the proof of Theorem 1.5 (ii). Since we will only deal with estimates in terms of the usual Lebesgue measure, the following definition will be sufficient for our purposes.

Given a pair of disjoint set $E,\, F\subset\Omega$, define the $\mathscr{W}^{1,p}$-capacity as

$$\mathrm{Cap}_{\mathscr{W}^{1,\,p}}(E,\,F,\,\Omega)=\inf\{\|u\|^p_{\mathscr{W}^{1,\,p}(\Omega)}:\ u\in\Delta(E,\,F,\,\Omega)\},\tag{5.1}$$

where $\Delta(E,\,F,\,\Omega)$ denotes the class of all functions $u\in\mathscr{W}^{1,p}(\Omega)$ such that $0\le u\le 1$ on Ω, $u=0$ a.e. on E, and $u=1$ a.e. on F. Obviously, if $\widetilde{E},\,\widetilde{F}\subset\mathbb{R}^n$ are disjoint sets, $E\subset\widetilde{E}$ and $F\subset\widetilde{F}$, then

$$\mathrm{Cap}_{\mathscr{W}^{1,\,p}}(E,\,F,\,\Omega)\le\mathrm{Cap}_{\mathscr{W}^{1,\,p}}(\widetilde{E},\,\widetilde{F},\,\mathbb{R}^n).$$

Moreover, we have the following trivial estimates.

Lemma 5.1. *If Ω is a $\mathscr{W}^{1,p}$-extension domain with $p\in[1,\,n)$, then there exists a positive constant C such that for every pair of disjoint sets $E,\,F\subset\Omega$,*

$$\mathrm{Cap}_{\mathscr{W}^{1,\,p}}(E,\,F,\,\Omega)\le\mathrm{Cap}_{\mathscr{W}^{1,\,p}}(E,\,F,\,\mathbb{R}^n)\le C\mathrm{Cap}_{\mathscr{W}^{1,\,p}}(E,\,F,\,\Omega).$$

Lemma 5.2. *Let Ω is a $\mathscr{W}^{1,p}$-extension domain for $p\in[1,\,n)$ and $\delta>0$. There exists a positive constant C depending only on n, δ and p such that for every pair of disjoint sets $E,\,F\subset\Omega\cap B(x_0,\,r)$ for some $r>0$, if $\min\{|E|,\,|F|\}\ge\delta r^n$ then*

$$\mathrm{Cap}_{\mathscr{W}^{1,\,p}}(E,\,F,\,\Omega)\ge C.$$

Proof. By Lemma 5.1, we may assume that $\Omega=\mathbb{R}^n$. Let $u\in\Delta(E,F,\mathbb{R}^n)$. Assume first that $u_{B(x_0,r)}\ge 1/2$. Then, by the Poincaré inequality,

$$\frac{|E|}{2|B(x_0,r)|}\le\fint_{B(x_0,r)}|u-u_{B(x_0,r)}|\le Cr\left(\fint_{B(x_0,r)}|\nabla u|^p\right)^{1/p},$$

and the desired estimate follows. If $u_{B(x_0,r)}<1/2$, we obtain our estimate by replacing E in the above string of inequalities by F. □

Define the Sobolev $W^{1,p}$-capacity $\mathrm{Cap}_{W^{1,p}}(E,F,\Omega)$ similarly to (5.1). We have the following well-known consequence of the above proof.

Lemma 5.3. *Let Ω is a $W^{1,p}$-extension domain for $p\in[1,\,n)$ and $\delta>0$. There exists a positive constant C depending only on n, δ and p such that for every pair of disjoint sets $E,\,F\subset\Omega\cap B(x_0,\,r)$ for some $r>0$, if $\min\{|E|,\,|F|\}\ge\delta r^n$ then*

$$\mathrm{Cap}_{W^{1,\,p}}(E,\,F,\,\Omega)\ge Cr^{n-p}.$$

6 Proofs of Proposition 1.3 and Theorem 1.5 (ii)

Proof of Theorem 1.5 (ii). Case LLC(2). Let $x_1,\, x_2 \in \Omega \setminus B(x_0,\, r)$ for some $x_0 \in \Omega$ and $r > 0$. Without loss of generality, we may assume that $x_1, x_2 \in \partial B(x_0,\, r) \cap \Omega$. Indeed, the general case can easily be reduced to this one since Ω is a domain and hence pathwise connected.

Assume that x_1 and x_2 are not in the same component of $\Omega \setminus \overline{B(x_0,\, br)}$ for some $b \in (0,\, 1)$. We may assume that $b < 1/16$. It then suffices to prove that b is bounded away from zero, independent of $x_1,\, x_2,\, x_0,\, r$. Denote by Ω_i the component of $\Omega \setminus \overline{B(x_0,\, br)}$ containing x_i for $i = 1, 2$. Then $\Omega_1 \cap \Omega_2 = \emptyset$.

Let $F_i = \Omega_i \cap B(x_i,\, r/2)$ for $i = 1,\, 2$. Obviously, $F_1,\, F_2 \subset B(x_0,\, 2r)$, $F_1 \cap F_2 = \emptyset$,

$$\operatorname{dist}(x_i, \partial\Omega_i \setminus \partial\Omega) \geq r - br > r/2$$

and $F_i \cap B(x_0, r/2) = \emptyset$ for $i = 1,\, 2$. By Theorem 4.1, $|F_i| \gtrsim r^n$ for $i = 1,\, 2$. Applying Lemma 5.2, we obtain

$$\operatorname{Cap}_{\mathscr{W}^{1,\,p}}(F_1,\, F_2,\, \Omega) \gtrsim 1. \tag{6.1}$$

The lower bounded on b then will follow from the upper bound

$$\operatorname{Cap}_{\mathscr{W}^{1,\,p}}(F_1,\, F_2,\, \Omega) \lesssim \left(\log \frac{1}{2b}\right)^{(1-n)/n}. \tag{6.2}$$

Indeed, if this holds, then we have $\left(\log \frac{1}{b}\right)^{\frac{1}{n}-1} \gtrsim 1$, which gives $b \gtrsim 1$.

To obtain (6.2), for all $x \in \mathbb{R}^n$, define

$$u(x) \equiv \begin{cases} 1, & x \in \Omega_2 \setminus B(x_0,\, r/2); \\ \frac{1}{\log \frac{1}{2b}} \log \frac{|x-x_0|}{br}, & x \in \Omega_2 \cap (B(x_0,\, r/2) \setminus B(x_0,\, br)); \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

Obviously this function is well defined. Then

$$|\nabla u| = \left(\log \frac{1}{2b}\right)^{-1} |x - x_0|^{-1} \chi_{\Omega_2 \cap (B(x_0,\, r/2) \setminus B(x_0,\, br))}$$

and hence $u \in \mathscr{W}^{1,\,p}(\Omega)$ and

$$\begin{aligned} \|u\|_{\mathscr{W}^{1,\,p}(\Omega)} &\leq \|u\|_{W^{1,\,n}(\Omega)} \\ &\lesssim \left(\log \frac{1}{2b}\right)^{-1} \left\{ \int_{B(x_0,\, r/2) \setminus B(x_0,\, br)} |z - x_0|^{-n}\, dz \right\}^{1/n} \\ &\lesssim \left(\log \frac{1}{2b}\right)^{1/n-1}. \end{aligned}$$

Obviously, $u \in \Delta(F_1,\, F_2,\, \Omega)$, and hence (6.2) holds as desired.

Case LLC(1). Let x_1, x_2 be a pair of points in Ω and $x_1, x_2 \in B(z, r)$ for some $z \in \mathbb{R}^n$ and $r > 0$. We are going to show the existence of a rectifiable curve $\gamma \subset \Omega \cap B(z, r/b)$ joining x_1 and x_2 for some constant $b < 1$.

If $|x_1 - x_2| \le \max\{d(x_1, \Omega^\complement),\, d(x_2, \Omega^\complement)\}$, then the line segment joining x_1 and x_2 gives the desired curve. Below we assume that $|x_1 - x_2| > \max\{d(x_1, \Omega^\complement),\, d(x_2, \Omega^\complement)\}$.

Let Ω_i be the component of

$$\Omega \cap B(x_i,\, |x_1 - x_2|/8) = \Omega \setminus \{B(x_i,\, |x_1 - x_2|/8)\}^\complement$$

containing x_i for $i = 1, 2$. By Theorem 4.1,

$$|\Omega_i| \gtrsim |x_1 - x_2|^n$$

for $i = 1, 2$. Obviously, $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1, \Omega_2 \subset B(x_1, 2|x_1 - x_2|)$. Applying Lemma 5.2, we obtain

$$\mathrm{Cap}_{\mathscr{W}^{1,p}}(\Omega_1, \Omega_2, \Omega) \gtrsim 1.$$

Now we claim that there exists a positive constant $N_0 \ge 2$ independent of x_1, x_2, Ω_1, Ω_2 such that Ω_1, Ω_2 are in the same component of $\Omega \cap B(x_1, N_0|x_1 - x_2|)$. To see this, assume that Ω_1, Ω_2 are not in the same component of $\Omega \cap B(x_1, N|x_1 - x_2|)$ for some $N > 2$, say, in components $\widetilde{\Omega}_1$, $\widetilde{\Omega}_2$ of $\Omega \cap B(x_1, N|x_1 - x_2|)$ respectively. Define a function u on Ω by setting

$$u(y) \equiv \begin{cases} 1, & y \in B(x_1, |x_1 - x_2|/8) \cap \widetilde{\Omega}_1, \\ \frac{1}{\log 8N} \log \frac{N|x_1 - x_2|}{|y - x_1|}, & y \in (B(x_1, N|x_1 - x_2|) \setminus B(x_1, |x_1 - x_2|/8)) \cap \widetilde{\Omega}_1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\|u\|_{\mathscr{W}^{1,p}(\Omega)} \le \|u\|_{W^{1,n}(\Omega)} \lesssim (\log N)^{(1-n)/n}.$$

Observing that $u = 1$ on Ω_1 and $= 0$ on Ω_2, we have

$$\mathrm{Cap}_{\mathscr{W}^{1,p}}(\Omega_1, \Omega_2, \Omega) \lesssim (\log N)^{(1-n)/n}$$

which means $N \lesssim 1$ and hence shows the existence of N_0 as claimed.

Therefore there exists a rectifiable curve $\gamma_0 \subset \Omega \cap B(x_1, N_0|x_1 - x_2|)$ joining Ω_1 and Ω_2. Let $\widetilde{x}_i \in \Omega_i \cap \gamma_0$ for $i = 1, 2$. Since $x_i, \widetilde{x}_i \in \Omega_i$ and Ω_i is connected, we can find a rectifiable curve $\gamma_i \subset \Omega_i \subset \Omega \cap B(x_1, N_0|x_1 - x_2|)$ joining x_i and $\widetilde{x}_i$ for $i = 1, 2$. Set $\gamma = \gamma_0 \cup \gamma_1 \cup \gamma_2$. Then γ joins x_1 and x_2 and $\gamma \subset \Omega \cap B(x_1, N_1|x_1 - x_2|)$. Observing that

$$B(x_1, N_0|x_1 - x_2|) \subset B(z, N_0|x_1 - x_2| + |z - x_1|) \subset B(z, (N_0 + 1)|x_1 - x_2|) \subset B(z, 2(N_0 + 1)r),$$

we have $\gamma \subset \Omega \cap B(z, r/b)$ with $b = 1/2(N_0 + 1)$ as desired. $\square$

Proof of Proposition 1.3. The proof is similar to the Case LLC(2) in the proof of Theorem 1.5(ii). Indeed, assume that that x_1 and x_2 are not in the same component of $\Omega \setminus \overline{B(x_0, br)}$ for some $0 < b < 1/16$. It then suffices to prove that b is bounded away from zero, independently of x_1, x_2, x_0, r.

Let Ω_i be the component of $\Omega \setminus \overline{B(x_0,\, br)}$ containing x_i and let $F_i = \Omega_i \cap B(x_i,\, r/2)$ for $i = 1,\, 2$. Since Ω is pathwise connected, Theorem 4.4 easily implies that $|F_i| \gtrsim r^n$ for $i = 1,\, 2$. Applying Lemma 5.3, similarly to (6.1), we obtain

$$\mathrm{Cap}_{W^{1,\,p}}(F_1,\, F_2,\, \Omega) \gtrsim r^{n-p}.$$

The lower bounded of b then will follow from the upper bound

$$\mathrm{Cap}_{W^{1,\,p}}(F_1,\, F_2,\, \Omega) \lesssim r^{n-p} b^{n-p} \log \frac{1}{2b} \tag{6.4}$$

Indeed, if this holds, then we have $b^{n-p} \log \frac{1}{2b} \gtrsim 1$, which gives $b \gtrsim 1$.

To show (6.4), we set

$$u(x) \equiv \begin{cases} 1, & x \in \Omega_2 \setminus B(x_0,\, r/2); \\ \frac{(br)^{-n/p+1} - |x-x_0|^{-n/p+1}}{(br)^{-n/p+1} - (r/2)^{-n/p+1}}, & x \in \Omega_2 \cap (B(x_0,\, r/2) \setminus B(x_0,\, br)); \\ 0, & \text{otherwise.} \end{cases}$$

Notice that here u is different from that in (6.3). Since $p < n$, we have $u \in W^{1,\,p}(\Omega)$ with

$$\begin{aligned} \|u\|_{W^{1,\,p}(\Omega)} &\lesssim \frac{1}{(br)^{-n/p+1} - (r/2)^{-n/p+1}} \left\{ \int_{B(x_0,\, r/2) \setminus B(x_0,\, br)} |x - x_0|^{-n}\, dz \right\}^{1/p} \\ &\lesssim r^{n/p-1} b^{n/p-1} (\log \frac{1}{2b})^{1/p}. \end{aligned}$$

Obviously, $u \in \Delta(F_1,\, F_2,\, \Omega)$, and hence (6.4) holds as desired. □

Remark 6.1. Finally, if $p \in [1,\, n)$, one can not prove the condition LLC(1) for $W^{1,\,p}$-extension domains by using the argument in the proof of the case LLC(1), Theorem 1.5 (ii). The point is that if $p \in [1,\, n)$, the seminorm of $W^{1,\,p}$ is not scaling invariant, while the seminorms for $W^{1,\,n}$ and $\mathscr{W}^{1,\,p}$ are. With the aid of scaling property, $\mathscr{W}^{1,\,p}$ is imbedded into the Sobolev space $W^{1,n}$ and one has the estimate

$$\mathrm{Cap}_{\mathscr{W}^{1,\,p}}(\Omega_1,\, \Omega_2,\, \Omega) \gtrsim 1.$$

whenever $\Omega_1, \Omega_2 \subset \Omega$ satisfy $|\Omega_1| \sim |\Omega_2| \sim r^n$ and $\mathrm{dist}\,(\Omega_1, \Omega_2) \gtrsim r^n$. In fact, for $p \in [1,\, n)$, a $W^{1,\,p}$-extension domain may fail to satisfy LLC(1), see e.g. [7, 9, 10].

References

[1] B. Bojarski, Remarks on Sobolev imbedding inequalities, Complex analysis, Joensuu 1987, 52-68, Lecture Notes in Math., 1351, Springer, Berlin, 1988.

[2] F. W. Gehring and O. Martio, Quasiextremal distance domains and extension of quasiconformal mappings, J. Analyse Math. 45 (1985), 181-206.

[3] V.M. Gol'dshtein, Extension of functions with first generalized derivatives from plane domains (Russian), Doklady Acad. Nauk SSSR, 1981, 257, 2, 268-271. (Engl. transl.: Soviet Math. Dokl. 257 (1981).2, 255-257).

[4] P. Hajłasz and P. Koskela and Heli Tuominen, Sobolev embeddings, extensions and measure density condition, J. Funct. Anal. 254 (2008), no. 5, 1217-234.

[5] D.A. Herron and P. Koskela, Uniform, Sobolev extension and quasiconformal circle domains, J. Anal. Math. 57 (1991), 172–202.

[6] P. W. Jones, Quasiconformal mappings and extendability of functions in Sobolev spaces, Acta Math. 147 (1981), no. 1-2, 71-88.

[7] P. Koskela, Capacity extension domains, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes No. 73 (1990), 42 pp.

[8] P. Koskela, Extensions and imbeddings, J. Funct. Anal. 159 (1998), 369-384.

[9] P. Koskela, D. Yang and Y. Zhou, A Jordan Sobolev extension domain, Ann. Acad. Sci. Fenn. Math. 35 (2010), 309-320.

[10] V.G. Maz'ya, On the extension of functions belonging to S. L. Sobolev spaces (Russian), Investigations on linear operators and the theory of functions, XI. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 113 (1981), 231-236.

[11] P. Shvartsman, On Sobolev extension domains in R^n, J. Funct. Anal. 258 (2010), 2205-2245.

[12] D.A. Trotsenko, Approximation of space mappings with bounded distortion by similarities (Russian), Sibirsk. Math. Zh. 27 (1986), 196-205 (Engl. transl.: Siberian Math. J. 27 (1986), 946-954).

[13] S. K. Vodop'janov, V. M. Gol'dšteĭn and T. G. Latfullin, A criterion for the extension of functions of the class L_2^1 from unbounded plane domains (Russian), Sibirsk. Mat. Zh. 20 (1979), 416-419 (English transl. Siberian Math. J. 20 (1979), 298-301).

[14] S. K. Vodop'janov, Existence conditions for extendability of differentiable functions and bounds for the norm of the extension operator, - A. Haar. Mem. Conf., Budapest/Hung. 1985, Colloq. Math. Soc. János Bolyai vol. 49, pp.957-973, North-Holland Publishing Company, Amsterdam-New York, 1987.

[15] Y. Zhang and Y. Zhou, A quasiconformal composition problem for Morrey-Sobolev spaces, preprint.

Pekka Koskela
Department of Mathematics and Statistics, P. O. Box 35 (MaD), FI-40014, University of Jyväskylä, Finland

E-mail : pkoskela@maths.jyu.fi

Yi Zhang
Department of Mathematics, Beijing University of Aeronautics and Astronautics, Beijing 100191, P. R. China

E-mail: ddfx023zy@gmail.com

Yuan Zhou
Department of Mathematics, Beijing University of Aeronautics and Astronautics, Beijing 100191, P. R. China
E-mail : yuanzhou@buaa.edu.cn