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INJECTIVITY RESULTS FOR THE GEODESIC RAY TRANSFORM

JERE LEHTONEN

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Finally, I want to thank my family and friends.

Jyväskylä, August 2017

Jere Lehtonen

List of included articles

This dissertation consists of an introductory part and the following articles:

References

[A] J. Lehtonen. The geodesic ray transform on two-dimensional Cartan-Hadamard manifolds.

[B] J. Ilmavirta, J. Lehtonen and M. Salo. Geodesic X-ray tomography for piecewise constant functions on nontrapping manifolds.

[C] J. Lehtonen, J. Railo, and M. Salo. Tensor tomography on Cartan-Hadamard manifolds.

In the introduction these articles will be referred to as [A], [B], and [C], whereas other references will be referred as [AR97], [BGL02].

The author of this dissertation has actively taken part in the research of the joint articles [B], [C].

1. Introduction

This thesis studies the inverse problem for the geodesic ray transform. In this introduction we give a brief overview of the field and the results of the three articles contained in the thesis. We explain the main ideas behind proofs, but omit many technical details.

Consider the following question:

Question 1. *Suppose f is an unknown continuous function on the plane, but we know the line integrals of f over every line. Can we recover the function f?*

This is an inverse problem for an integral transform taking a function to its line integrals. The direct problem would be just to calculate all the line integrals of a given function and that is usually a straightforward task. The knowledge of all the integrals is also referred to as the data of the inverse problem.

This problem was considered by Radon in his paper [Rad17] from the year 1917. Radon proved that a smooth compactly supported function can be determined from its line integrals and provided a reconstruction formula. However, it seems that there was no great interest in this problem or similar ones during the following decades.

An injectivity result and a reconstruction formula for functions in Schwartz space, i.e. the space of rapidly decreasing functions, was provided by Helgason [Hel64], [Hel65]. Nonetheless, in general the answer to Question 1 for non-compactly supported functions depends on the decay assumptions we impose on the functions and it is not necessarily positive: Zalcman [Zal82] gave an example of a non-zero function with vanishing line integrals.

Radon was not the first one to consider the problem of determining a function from integrals over curves. Funk [Fun15] proved that a symmetric function on S^2, the unit sphere in $\mathbb{R}^3$, can be recovered from the knowledge of integrals over great circles.

Question 1 can be generalized in many ways. One way is to change the geometry of the space. Instead of the Eucliden plane and straight lines we can consider Riemannian manifolds and integrals over geodesics. Another way would be for example to consider $\mathbb{R}^n$ and integrals over hyperplanes. From Section 2 onwards we will concentrate on the first option.

In general the main questions of interest can be stated as follows:

(a) Injectivity: Does the data determine the function uniquely?
(b) Reconstruction: How can we reconstruct the unknown function from the data?
(c) Stability: Can small differences in the data mean large differences in the reconstructed function?

In this thesis we are concentrating on the question of injectivity.

In two dimensions the integral transform taking a function to its line integral is known as the Radon transform and also as the X-ray transform.

The latter name stems from the connection with X-ray tomography: An unknown function f represents the attenuation coefficients of a human body. A X-ray, which is modeled as a straight line l, is shot through the body. Different tissues, bones and organs attenuate the ray depending on the density. The loss of intensity of the X-ray as it travels through the body is then related to the integral of f over the line l. This formulation leads to Question 1. Cormack [Cor63], unaware of the work of Radon, proved a reconstruction formula when developing the theoretical basis of a modern computer tomography (CT). For this work he was awarded the 1979 Nobel Prize in Physiology or Medicine together with Hounsfield. For more about applications related to computerized tomography see for example the book by Natterer [Nat86].

The generalized problem on Riemannian manifolds has connections to applications in for example seismic imaging and electrical impedance tomography. In the former the idea is to use seismic data to obtain knowledge about Earth's interior structure. For more information see for example [dH03]. The latter considers the problem of determining the conductivity of an object using surface electrode measurements. For more information about the geodesic ray transform in electrical imaging see for example [DSFKSU09], [DSFKLS16].

2. The geodesic ray transform

In this section we discuss the inverse problem for the geodesic ray transform on Riemannian manifolds. Let (M, g) be a compact Riemannian manifold with smooth boundary. The unit sphere bundle is defined as

$$SM := \bigcup_{x \in M} S_x M, \quad S_x M := \{(x, v) \in TM \,; |v|_g = 1\}.$$

For $(x, v) \in SM$ we denote by $\gamma_{x,v}$ the unique geodesic so that $\gamma_{x,v}(0) = x$ and $\dot{\gamma}_{x,v}(0) = v$. We assume all geodesics to be parametrized to have unit speed. Furthermore, when speaking of geodesics we assume them to be maximal. By $\tau(x, v) \geq 0$ we denote the first time for which the geodesic $\gamma_{x,v}$ exits the manifold M. Obviously for some geodesics this may not happen at all and those geodesics are called trapped. The manifold M is called non-trapping if there is no trapped geodesics.

The boundary ∂SM of SM is defined as

$$\partial SM := \{(x, v) \in SM \,; x \in \partial M\}.$$

Let ν denote the inward pointing unit normal vector field for ∂M. We say a tangent vector v at $x \in \partial M$ is inward pointing if $\langle v, \nu(x)\rangle \geq 0$. The set of inward pointing vectors is denoted ∂SM_+. We remark that if $\langle v, \nu(x)\rangle < 0$ then $\tau(x, v) = 0$.

We define the geodesic ray transform $If\colon SM \to \mathbb{R}$ of a real-valued function $f \in C(M)$ by

$$If(x,v) := \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\,\mathrm{d}t.$$

This is well-defined for $(x,v) \in SM$ if $\gamma_{x,v}$ is not trapped.

We are ready to discuss the inverse problem for the geodesic ray transform, which in general form asks what can we say about an unknown function on M if we know its integrals over all geodesics joining boundary points. Here we are interested in the question of injectivity, which can be formulated as follows:

Question 2. *Let $f, g \in C^\infty(M)$ and suppose that $If\big|_{\partial SM_+} = Ig\big|_{\partial SM_+}$. Does it follow that $f = g$ on M?*

Since I is a linear operator it is equivalent to ask whether $If(x,v) = 0$ for all $(x,v) \in \partial SM_+$ implies that $f(x) = 0$ for all $x \in M$. We assume smoothness for convenience.

It comes as no suprise that the answer to this question depends on the geometry of the manifold M. A Riemannian manifold (M,g) is said to be simple if for every point $x \in M$ the exponential map $\exp_x$ is a diffeomorphism from its maximal domain to M and the boundary ∂M is strictly convex in the sense of the second fundamental form. A positive answer to Question 2 for simple surfaces was provided by Mukhometov [Muk77]. For more information and recent results we refer to [UV16] and the survey [PSU14].

There are also known examples of manifolds for which the answer is negative. We will describe one next: Suppose M is a sphere but a small spherical cap is cut out at the south pole. The metric is the restriction of the usual metric of the sphere, so geodesics are just restrictions of the great circles of the sphere. Take a continuous and antipodal antiZsymmetric function with support contained in a neighborhood of the equator. It is easy to see that all integrals of this function over maximal geodesics having end points on the boundary equal to zero.

We point out that the above geometry has a set of trapped geodesics near the equator. On the other hand, simple manifolds do not have trapped geodesics at all. It seems that the existence of trapped geodesics is connected to the injectivity of the geodesic ray transform. The following conjecture was stated in [PSU13] for surfaces:

Conjecture 1. *The geodesic ray transform is injective on compact non-trapping manifolds with strictly convex boundary.*

In the article [B] the conjecture is confirmed for two-dimensional manifolds in the case of piecewise constant functions.

In articles [A], [C] the question of injectivity is studied on non-compact manifolds without boundary. More specifically the manifolds are assumed to be Cartan-Hadamard manifolds, i.e. complete and simply connected with everywhere non-positive sectional curvature. There has been earlier work on non-compact symmetric and homogenous spaces, including injectivity results. For these we refer to Helgason's book [Hel99], see also [Hel13]. The injectivity results of [A] and [C] seem to be among the first works that study the geodesic ray transform on non-compact Riemannian manifolds without symmetry or homogeneity asumptions. The questions of reconstruction and stability in this case seem to be mostly unanswered at the moment.

On a Riemannian manifold the geodesic ray transform can be defined also for symmetric covariant m-tensor fields: Let f be such a tensor field on M. Then in local coordinates (using the Einstein summation convention)

$$f = f_{i_1 \dots i_m}(x) dx^{i_1} \otimes \cdots \otimes dx^{i_m}.$$

The tensor field f induces a function f_m on SM by $f_m(x, v) = f_x(v, \dots, v)$, which in local coordinates is

$$f_m(x, v) = f_{i_1 \dots i_m}(x) v^{i_1} \dots v^{i_m}.$$

The geodesic ray transform $I_m f \colon SM \to \mathbb{R}$ of the tensor field f is then defined by $I_m f := I f_m$.

For $m \geq 1$ the geodesic ray transform is not injective: Let ∇ denote the covariant derivative and σ the symmetrization of a tensor field. If h is a symmetric covariant $(m-1)$-tensor field on M with $h\big|_{\partial SM} = 0$ (if M is compact with boundary), then $I_m(\sigma \nabla h) = 0$. So instead of asking whether $I_m f = 0$ on ∂SM_+ implies $f = 0$ we ask if $I_m f = 0$ on ∂SM_+ implies that $f = \sigma \nabla h$ for some symmetric covariant $(m-1)$-tensor field. This question is known as solenoidal injectivity of I_m.

The solenoidal injectivity on simple manifolds for $m = 1$ was proved by Anikonov and Romanov [AR97] for dimensions $n \geq 2$ and for all m by Paternain, Salo and Uhlmann [PSU13] if $n = 2$. For manifolds of non-positive sectional curvature is was proved for all m by Pestov and Sharafutdinov [PS88]. For more details and a survey of results we refer to [PSU14]. In the article [C] we prove solenoidal injectivity for all m on Cartan-Hadamard manifolds for a set of tensor fields fulfilling certain decay conditions. We are unaware of any previous injectivity results for tensor fields on non-compact manifolds without symmetries.

The inverse problem for tensor fields has a connection to the boundary rigidity problem on a Riemannian manifold (M, g) with boundary. The problem asks if the knowledge of the distance mapping $d \colon M \times M \to \mathbb{R}$ restricted to the boundary of the manifold, that is $\partial M \times \partial M$, is enough to determine the metric tensor (up to a diffeomorphism). Linearization of this problem leads to the question of injectivity of the geodesic ray transform for tensor fields.

3. Injectivity of the geodesic ray transform on Cartan-Hadamard manifolds

In this section we formulate the results of papers [A], [C], and explain the main ideas and methods. In those papers the injectivity of the geodesic ray transform is studied on Cartan-Hadamard manifolds. Since these are manifolds without boundary, the geodesic ray transform of a function $f \in C(M)$ is defined as

$$If(x,v) := \int_{-\infty}^{\infty} f(\gamma_{x,v}(t))\,\mathrm{d}t.$$

Since we are integrating over an infinite interval we must impose additional decay conditions for the function f in order to make the transform well defined. Let us fix $o \in M$. We define, for $\eta > 0$,

$$\begin{aligned} P_\eta(M) &= \{f \in C(M)\,;\,|f(x)| \le C(1+d(x,o))^{-\eta}\},\\ P^1_\eta(M) &= \{f \in C^1(M)\,;\,|f(x)| \le C(1+d(x,o))^{-\eta} \text{ and}\\ &\qquad\qquad |\nabla f(x)|_g \le C(1+d(x,o))^{-\eta-1} \text{ for some } C>0\}, \end{aligned}$$

and

$$\begin{aligned} E_\eta(M) &= \{f \in C(M)\,;\,\mathrm{e}^{-\eta d(x,o)} \le Ce^{-\eta d(x,o)} \text{ for some } C>0\},\\ E^1_\eta(M) &= \{f \in C^1(M)\,;\,|f(x)| + |\nabla f(x)|_g \le Ce^{-\eta d(x,o)} \text{ for some } C>0\}. \end{aligned}$$

The reason for the decay condition for the gradient will become clear from the proofs.

We will denote the sectional curvature of a two-plane $\Pi \subset T_xM$ by $K_x(\Pi)$, and we write $-K_0 \le K \le 0$ if $-K_0 \le K_x(\Pi) \le 0$ for all $x \in M$ and for all two-planes $\Pi \subset T_xM$. It turns out that the curvature of the manifold and the required decay rate for the functions are connected. Our first injectivity theorem is in the case of bounded curvature:

Theorem 3.1. *Suppose (M,g) is a Cartan-Hadamard manifold whose sectional curvatures satisfy $0 \ge K \ge -K_0$ for some $K_0 > 0$. If $f \in E^1_\eta(M), \eta > \frac{n+1}{2}\sqrt{K_0}$ and if $If = 0$, then $f = 0$.*

In our second injectivity theorem we assume that the sectional curvature K satisfies a decay condition

$$\sup_{\Pi \subset T_xM} |K_x(\Pi)| \le \frac{C}{(1+d_g(x,o))^{2+\varepsilon}} \tag{3.1}$$

for some $C > 0$ and $\varepsilon > 0$. The idea behind this is that if the curvature decays rapidly enough the manifold will asymptotically resemble a flat space and hence we are able to relax the decay condition imposed for the functions.

Theorem 3.2. *Suppose (M,g) is a Cartan-Hadamard manifold whose sectional curvature satisfies (3.1). If $f \in P^1_\eta(M), \eta > \frac{n+2}{2}$ and if $If = 0$, then $f = 0$.*

The two-dimensional versions of Theorems 3.1 and 3.2 were proved in [A] and the higher dimensional cases in [C]. A way to construct examples of two-dimensional manifolds satisfying the assumptions of Theorem 3.2 was given in [C]. In higher dimensions the decay condition for curvature turns out to be very restrictive: By results of Greene and Wu [GW82] all higher dimensional Cartan-Hadamard manifolds satisfying the condition (3.1) are isometric with the Euclidean space.

In [C] we also consider the question of the solenoidal injectivity. Suppose f is a C^1-smooth symmetric m-tensor field on m. Then $I_m f$ is defined as in Section 2. We say that a symmetric covariant m-tensor field f is in $E^1_\eta(M)$ if $|f|_g$ and $|\nabla f|_g$ are in $E^1_\eta(M)$. Respectively we say that f is in $P^1_\eta(M)$ if $|f|_g$ is in $P_\eta(M)$ and $|\nabla f|_g$ is in $P_{\eta+1}(M)$. We have the following results for all $m \geq 1$:

Theorem 3.3. *Suppose (M, g) is a Cartan-Hadamard manifold whose sectional curvatures satisfy $0 \geq K \geq -K_0$ for some $K_0 > 0$. If f is a symmetric covariant m-tensor field in $E^1_\eta(M), \eta > \frac{n+1}{2}\sqrt{K_0}$ and if $I_m f = 0$, then $f = \sigma\nabla h$ for some symmetric covariant $(m-1)$-tensor field h.*

Theorem 3.4. *Suppose (M, g) is a Cartan-Hadamard manifold whose sectional curvature satisfies (3.1). If f is a symmetric covariant m-tensor field in $P^1_\eta(M), \eta > \frac{n+2}{2}$ and if $I_m f = 0$, then $f = \sigma\nabla h$ for some symmetric covariant $(m-1)$-tensor field h.*

Next we will briefly explain the main ideas and tools used to prove our results.

3.1. **Two-dimensional manifolds.** The methods used in [A] are based on those used in the work of Paternain, Salo and Uhlmann [PSU13]. On SM we have the geodesic vector field, which we denote by X. The vertical vector field V is the infinitesimal generator of a circle action on the fibers of SM. Heuristically, if $u \in C^1(SM)$ then $Xu(x, v)$ is the gradient in x of the function u in the direction of v and $Vu(x, v)$ is the derivative with respect to v.

The unit sphere bundle SM can be endowed with the Sasaki metric to make it a 3-dimensional Riemannian manifold. We can then complete $\{X, V\}$ to an orthonormal frame of $T(SM)$ by defining $X_\perp := [X, V]$. In addition we have commutator formulas $[V, X_\perp] = X$ and $[X, X_\perp] = -KV$.

The main tool in the proof is the Pestov identity, which can be seen as an energy identity for the operator VX. Pestov identities have been earlier used to prove injectivity, see for example [PS88], [Sha94], [Kni02], [PSU14]. In [A] it was used in the following form:

Lemma 3.1 (Pestov identity)**.** *Suppose (M, g) is a compact Riemannian manifold with boundary and $u \in C^2(SM)$. Then*

$$\|VXu\|^2_{L^2(SM)} = \|XVu\|^2_{L^2(SM)} - \langle KVu, Vu\rangle_{SM} + \|Xu\|^2_{L^2(SM)} + \langle Xu, \langle v_\perp, \nu\rangle Vu\rangle_{\partial SM} - \langle X_\perp u, \langle v, \nu\rangle Vu\rangle_{\partial SM}.$$

Here $v_\perp$ is an unit vector obtained by rotating the vector v by an angle $-\pi/2$. Suppose $f \in P_\eta(M)$ for some $\eta > 1$. We define a function $u^f : SM \to \mathbb{R}$,

$$u^f(x,v) := \int_0^\infty f(\gamma_{x,v}(t))\,\mathrm{d}t. \tag{3.2}$$

A straigthforward calculation shows that $Xu^f = -f$, and because f depends only on the variable $x \in M$ we get $VXu^f = 0$.

We denote by M_r a geodesic ball of radius r centered at o, which is a submanifold of M with a strictly convex boundary. If $u^f \in C^2(SM)$, then by applying the Pestov identity to u^f on M_r we get that

$$\begin{aligned} 0 = \|XVu^f\|^2_{L^2(SM_r)} - \langle KVu^f, Vu^f\rangle_{SM_r} + \|f\|^2_{L^2(SM_r)} \\ + \langle Xu^f, Vu^f\rangle_{\partial SM_r} - \langle X_\perp u^f, Vu^f\rangle_{\partial SM_r} \end{aligned}$$

Since K is non-positive the first three terms on the right-hand side are non-negative. It is then sufficient to show that the two boundary terms tend to zero as r goes to infinity to conclude that

$$\|f\|^2_{L^2(SM)} = 0$$

and hence $f = 0$ on M.

In article [A] the Pestov identity is not applied directly to the function u^f. We use an approximating sequence of functions for which we can use the Pestov identity. The assumption that f integrates to zero over all geodesics together with growth estimates for Jacobi fields are needed to show the convergence for the boundary terms.

Remark. In article [A] Lemma 4.6 states that if $f \in C^2(M)$ is compactly supported, then $u^f \in C^2(SM)$. This is not proved and for that part proofs of the main theorems are incomplete. However, results of the article [C] contain those of the article [A] with a small exception. In [A, Theorem 2] we assume that $\eta > \frac{3}{2}$. In [C] it is shown that this theorem holds for $\eta > 2$.

3.2. **Higher dimensional manifolds and solenoidal injectivity.** The methods used in [C] are generalizations of those used in two-dimensional case and are based on [PSU15]. In our results we need to assume only C^1-regularity for the functions, but in the following exposition we assume everything to be smooth for simplicity. The differences are merely technical.

As in the two-dimensional case we endow the unit sphere bundle SM with the Sasaki metric, which makes it a Riemannian manifold of dimension $2n-1$. On SM we can define a vertical Laplacian $\Delta\colon C^\infty(SM) \to C^\infty(SM)$. For fixed $x \in M$ it coincides with the Laplacian of (S_xM, g_x). The vertical Laplacian has eigenvalues $\lambda_k = k(n+k-2)$ for $k = 0,1,2\ldots.$ and the corresponding eigenspaces are denoted by $H_k(SM)$. The eigenfunctions are homogeneous polynomials with respect to variable v.

We obtain an orthogonal decomposition

$$L^2(SM) = \bigoplus_{k\geq 0} H_k(SM). \tag{3.3}$$

If $u \in L^2(SM)$ this decomposition will be written as

$$u = \sum_{k=0}^{\infty} u_k.$$

We say that the degree of u is m, if $u_k = 0$ for $k > m$.

We define $\Omega_k = H_k(SM) \cap C^\infty(SM)$. If u is a smooth function in $H_k(SM)$ then $Xu \in H_{k-1}(SM) \oplus H_{k+1}(SM)$ and hence we can write $X = X_+ + X_-$, where $X_+ \colon \Omega_k \to \Omega_{k+1}$ and $X_- \colon \Omega_k \to \Omega_{k-1}$. The following lemma was proved in [PSU15]:

Lemma 3.2. *Suppose $u \in \Omega_k, k \geq 1$, is compactly supported. Then*

$$\|X_- u\|_{L^2(SM)} \leq D_n(k)\|X_+ u\|_{L^2(SM)}$$

where

$$D_2(k) = \begin{cases} \sqrt{2}, & k = 1 \\ 1, & k \geq 2, \end{cases}$$

$$D_3(k) = \left[1 + \frac{1}{(k+1)^2(2k-1)}\right]^{1/2}$$

$$D_n(k) \leq 1, \quad \text{for } n \geq 4.$$

Suppose that f is as assumed in Theorem 3.3 or Theorem 3.4. We have $Xu^f = -f_m$ where u^f is defined in a similar fashion as (3.2). The function f_m is of degree m and by using decompositions (3.3) of f_m and u^f we get that

$$X_+(u^f)_k + X_-(u^f)_{k+2} = 0 \tag{3.4}$$

for $k \geq m$. By using Lemma 3.2 and equation (3.4) iteratively we can show that $(u^f)_k = 0$ for $k \geq m$, so u^f is at most of degree $m-1$. Then from u^f we can construct a $(m-1)$-tensor field h so that $\sigma\nabla h = f$.

In [C] Lemma 3.2 is extended for functions in $H_k(SM) \cap H^1(SM)$. If f is C^1 smooth and not compactly supported we need to do some work in order to show that u^f and $(u^f)_k$ are in $H^1(SM)$ for all $k \geq 0$. For that we need the decay assumptions.

4. Article [B], Piecewise constant functions

In article [B] the Conjecture 1 is confirmed in the case of piecewise constant functions on two-dimensional manifolds. Before stating the main result we must define what we mean by a piecewise constant function on a Riemannian manifold (M, g).

A regular n-simplex is an injective C^1-image of the standard n-simplex of $\mathbb{R}^{n+1}$, that is, the convex hull of $n+1$ coordinate vectors. The boundary of an

n-simplex consists of $(n-1)$-simplices, which all have boundaries consisting of $(n-2)$-simplices and so forth. These are all referred to as a boundary simplices of the original simplex and we say that a point x in a simplex has a depth k if it belongs in the interior of a $(n-k)$-boundary simplex.

We define a regular tiling to be a collection of regular n-simplices $\Delta_i, i \in I$ so that the following hold:

(1) The collection is locally finite: for any compact set $K \subset M$ the index set $\{i \in I\,; \Delta_i \cap K \neq \emptyset\}$ is finite.
(2) $M = \cup_{i \in I} \Delta_i$.
(3) $\mathrm{int}(\Delta_i) \cap \mathrm{int}(\Delta_j) \neq \emptyset$ when $i \neq j$.
(4) If $x \in \Delta_i \cap \Delta_j$, x then x has the same depth in both Δ_i and Δ_j.

We then say that $f\colon M \to \mathbb{R}$ is piecewise constant if there is a regular tiling $\{\Delta_i\,; i \in I\}$ of M so that f is constant in the interior of each Δ_i and vanishes on $\cup_{i \in I} \partial \Delta_i$. The latter condition is just for convenience.

We can now formulate the main result of [B]:

Theorem 4.1. *Let (M, g) be a compact non-trapping Riemannian manifold with strictly convex smooth boundary, and let $f\colon M \to \mathbb{R}$ be a piecewise constant function. Let either*

(a) $\dim(M) = 2$*,*
(b) $\dim(M) \geq 3$ *and (M, g) admits a smooth strictly convex function.*

If f integrates to zero over all geodesics joining boundary points, then $f = 0$.

The result for $\dim(M) \geq 3$ was already proved for functions in $L^2(M)$ by Uhlmann and Vasy [UV16]. The methods we use are very different from theirs and our proof is rather elementary. We point out that this result does not prove injectivity for piecewise constant functions, since they do not form a vector space. However, we get injectivity on the set of piecewise constant functions sharing the same tiling.

The assumption of admitting a smooth strictly convex function is also known as a foliation condition [PSUZ16]. It was proved by Betelu, Gulliver and Littman [BGL02] that a two-dimensional non-trapping manifold with strictly convex smooth boundary admits a smooth strictly convex function and hence in Theorem 4.1 we do not need to make the additional assumption to guarantee the existence of a such function. In higher dimensions it is still unknown if the same is true.

The key ingredient in the proof of the main theorem is the following local injectivity result.

Lemma 4.1. *Let (M, g) be a C^2-smooth Riemannian manifold with strictly convex boundary and $f\colon M \to \mathbb{R}$ be a piecewise smooth function. Suppose that $x \in \partial M$. If U is a neighborhood of x so that U intersects only simplices containing x and f integrates to zero over every maximal geodesic having endpoints on ∂M, then $f = 0$ on U.*

Here the lemma is stated in a simpler form than is actually needed to prove Theorem 4.1. The idea is to iterate this local result by using a foliation of the manifold by strictly convex hypersurfaces. Thus the local result combined with existence of the foliation makes it possible to attain a global result.

References

[AR97] Yu. E. Anikonov and V. G. Romanov. On uniqueness of determination of a form of first degree by its integrals along geodesics. *J. Inverse Ill-Posed Probl.*, 5(6):487–490 (1998), 1997.

[BGL02] Santiago Betelú, Robert Gulliver, and Walter Littman. Boundary control of PDEs via curvature flows: the view from the boundary. II. *Appl. Math. Optim.*, 46(2-3):167–178, 2002. Special issue dedicated to the memory of Jacques-Louis Lions.

[Cor63] A. M. Cormack. Representation of a function by its line integrals, with some radiological applications. *Journal of Applied Physics*, 34(9):2722–2727, 1963.

[dH03] Maarten V. de Hoop. Microlocal analysis of seismic inverse scattering. In *Inside out: inverse problems and applications*, volume 47 of *Math. Sci. Res. Inst. Publ.*, pages 219–296. Cambridge Univ. Press, Cambridge, 2003.

[DSFKLS16] David Dos Santos Ferreira, Yaroslav Kurylev, Matti Lassas, and Mikko Salo. The Calderón problem in transversally anisotropic geometries. *J. Eur. Math. Soc. (JEMS)*, 18(11):2579–2626, 2016.

[DSFKSU09] David Dos Santos Ferreira, Carlos E. Kenig, Mikko Salo, and Gunther Uhlmann. Limiting Carleman weights and anisotropic inverse problems. *Invent. Math.*, 178(1):119–171, 2009.

[Fun15] Paul Funk. über eine geometrische Anwendung der Abelschen Integralgleichung. *Math. Ann.*, 77(1):129–135, 1915.

[GW82] R. E. Greene and H. Wu. Gap theorems for noncompact Riemannian manifolds. *Duke Math. J.*, 49(3):731–756, 1982.

[Hel64] Sigurdur Helgason. A duality in integral geometry; some generalizations of the Radon transform. *Bull. Amer. Math. Soc.*, 70:435–446, 1964.

[Hel65] Sigurđur Helgason. The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassmann manifolds. *Acta Math.*, 113:153–180, 1965.

[Hel99] Sigurdur Helgason. *The Radon transform*, volume 5 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 1999.

[Hel13] Sigurdur Helgason. Some personal remarks on the Radon transform. In *Geometric analysis and integral geometry*, volume 598 of *Contemp. Math.*, pages 3–19. Amer. Math. Soc., Providence, RI, 2013.

[Kni02] Gerhard Knieper. Hyperbolic dynamics and Riemannian geometry. In *Handbook of dynamical systems, Vol. 1A*, pages 453–545. North-Holland, Amsterdam, 2002.

[Muk77] R.G. Mukhometov. The problem of recovery of a two-dimensional Riemannian metric and integral geometry. *Sov. Math., Dokl.*, 18:27–31, 1977.

[Nat86] F. Natterer. *The mathematics of computerized tomography.* B. G. Teubner, Stuttgart; John Wiley & Sons, Ltd., Chichester, 1986.

[PS88] L. N. Pestov and V. A. Sharafutdinov. Integral geometry of tensor fields on a manifold of negative curvature. *Sibirsk. Mat. Zh.*, 29(3):114–130, 221, 1988.

[PSU13] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Tensor tomography on surfaces. *Invent. Math.*, 193(1):229–247, 2013.

[PSU14] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Tensor tomography: progress and challenges. *Chin. Ann. Math. Ser. B*, 35(3):399–428, 2014.

[PSU15] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Invariant distributions, Beurling transforms and tensor tomography in higher dimensions. *Math. Ann.*, 363(1-2):305–362, 2015.

[PSUZ16] G. P. Paternain, M. Salo, G. Uhlmann, and H. Zhou. The geodesic X-ray transform with matrix weights. *ArXiv e-prints*, May 2016.

[Rad17] J. Radon. Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. Leipz. Ber. 69, 262-277 (1917)., 1917.

[Sha94] V. A. Sharafutdinov. *Integral geometry of tensor fields*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.

[UV16] Gunther Uhlmann and András Vasy. The inverse problem for the local geodesic ray transform. *Invent. Math.*, 205(1):83–120, 2016.

[Zal82] Lawrence Zalcman. Uniqueness and nonuniqueness for the Radon transform. *Bull. London Math. Soc.*, 14(3):241–245, 1982.

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THE GEODESIC RAY TRANSFORM ON TWO-DIMENSIONAL CARTAN-HADAMARD MANIFOLDS

J. LEHTONEN

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Abstract. We prove two injectivity theorems for the geodesic ray transform on two-dimensional, complete, simply connected Riemannian manifolds with non-positive Gaussian curvature, also known as Cartan-Hadamard manifolds. The first theorem is concerned with bounded non-positive curvature and the second with decaying non-positive curvature.

1. Introduction and statement of main results

In [Hel99] Helgason presents the following result: Suppose that f is a continuous function in $\mathbb{R}^2$, $|f(x)| \le C(1+|x|)^{-\eta}$ for some $\eta > 2$, and $Rf = 0$ where Rf is the Radon transform defined by

$$Rf(x,\omega) := \int_{\mathbb{R}} f(x+t\omega)\,dt$$

for $x \in \mathbb{R}^2$ and $\omega \in S^1$. Then $f = 0$. Since the operator R is linear this corresponds to the injectivity of the operator. This result was later improved by Jensen [Jen04] requiring that $f = O(|x|^{-\eta}), \eta > 1$.

In [Hel94] Helgason presents a similar injectivity result for the hyperbolic 2-space H^2: Suppose f is a continuous function on H^2 such that $|f(x)| \le C\mathrm{e}^{-d_g(x,o)}$, where o is a fixed point in H^2, and

$$\int_\gamma f\,ds = 0$$

for every geodesic γ of H^2. Then $f = 0$.

The previous results are concerned with constant curvature spaces. There are many related results for Radon type transforms on constant curvature spaces and noncompact homogeneous spaces, see [Hel99],[Hel13]. These types of spaces possess many symmetries. On the other hand, there is also a substantial literature related to geodesic ray transforms on Riemannian manifolds, see e.g. [Muk77], [Sha94], [PSU14]. Here the symmetry assumptions are replaced by curvature or conjugate points conditions, but the spaces are required to be compact with boundary.

In this paper we present injectivity results on two-dimensional, complete, simply connected Riemannian manifolds with non-positive Gaussian curvature. Such manifolds are called Cartan-Hadamard manifolds, and they are diffeomorphic to $\mathbb{R}^2$ (hence non-compact) but do not necessarily have symmetries. In order to prove our results we extend energy estimate methods used in [PSU13] to the non-compact case.

Suppose (M, g) is such a manifold and we have a continuous function $f\colon M \to \mathbb{R}$. We define the geodesic ray transform $If\colon SM \to \mathbb{R}$ of the function f as

$$If(x,v) := \int_{-\infty}^{\infty} f(\gamma_{x,v}(t))\, dt,$$

where the unit tangent bundle SM is defined as

$$SM := \{(x,v) \in TM\colon |v|_g = 1\}$$

and $\gamma_{x,v}$ is the unit speed geodesic with $\gamma_{x,v}(0) = x$ and $\gamma'_{x,v}(0) = v$. Since we are working on non-compact manifolds the geodesic ray transform is not well defined for all continuous functions. We need to impose decay requirements for the functions under consideration. Because of the techniques used we will also impose decay requirements for the first derivatives of the function.

We denote by $C_0(M)$ the set of functions $f \in C(M)$ such that for some $p \in M$ one has $f(x) \to 0$ as $d(p,x) \to \infty$. Suppose $p \in M$ and $\eta \in \mathbb{R}$. We define

$$P_\eta(p,M) := \{f \in C(M)\colon |f(x)| \le C(1 + d_g(x,p))^{-\eta} \text{ for all } x \in M\},$$
$$P^1_\eta(p,M) := \{f \in C^1(M)\colon |\nabla f|_g \in P_{\eta+1}(p,M)\} \cap C_0(M).$$

and similarly

$$E_\eta(p,M) := \{f \in C(M)\colon |f(x)| \le C\mathrm{e}^{-\eta d_g(x,p)} \text{ for all } x \in M\},$$
$$E^1_\eta(p,M) := \{f \in C^1(M)\colon |\nabla f|_g \in E_\eta(p,M)\} \cap C_0(M).$$

For all $\eta > 0$ we have inclusions

$$P^1_\eta(p,M) \subset P_\eta(p,M)$$

and

$$E^1_\eta(p,M) \subset E_\eta(p,M),$$

which can be seen by using Lemma 2.1, equation (2.1) and the fundamental theorem of calculus. In addition

$$E_{\eta_1}(p,M) \subset P_{\eta_2}(p,M)$$

for all $\eta_1, \eta_2 > 0$.

We can now state our first injectivity theorem.

Theorem 1. *Suppose (M, g) is a two-dimensional, complete, simply connected Riemannian manifold whose Gaussian curvature satisfies $-K_0 \le K(x) \le 0$ for some K_0. Then the geodesic ray transform is injective on the set $E^1_\eta(M) \cap C^2(M)$ for $\eta > \frac{5}{2}\sqrt{K_0}$.*

The second theorem considers the case of suitably decaying Gaussian curvature. By imposing decay requirements for the Gaussian curvature we are able to relax the decay requirements of the functions we are considering.

Theorem 2. *Suppose (M, g) is a two-dimensional, complete, simply connected Riemannian manifold of non-positive Gaussian curvature K such that $K \in P_{\tilde{\eta}}(p, M)$ for some $\tilde{\eta} > 2$ and $p \in M$. Then the geodesic ray transform is injective on set $P^1_\eta(p, M) \cap C^2(M)$ for $\eta > \frac{3}{2}$.*

One question arising is of course the existence of manifolds satisfying the restrictions of the theorems. By the Cartan-Hadamard theorem such manifolds are always diffeomorphic with the plane $\mathbb{R}^2$ so the question is what kind of Gaussian curvatures we can have on $\mathbb{R}^2$ endowed with a complete Riemamnian metric? The following theorem by Kazdan and Warner [KW74] answers this:

Theorem. *Let $K \in C^\infty(\mathbb{R}^2)$. A necessary and sufficient condition for there to exist a complete Riemannian metric on $\mathbb{R}^2$ with Gaussian curvature K is that*

$$\lim_{r\to\infty} \inf_{|x|\geq r} K(x) \leq 0.$$

Especially for every non-positive function $K \in C^\infty(\mathbb{R}^2)$ there exists a metric on $\mathbb{R}^2$ with Gaussian curvature K.

The case where the metric g differs from the euclidean metric g_0 only in some compact set and the Gaussian curvature is everywhere non-positive is not interesting from the geometric point of view. By a theorem of Green and Gulliver [GG85] if the metric g differs from the euclidean metric g_0 at most on a compact set and there are no conjugate points, then the manifold is isometric to $(\mathbb{R}^2, g_0)$. Since non-positively curved manifolds can not contain conjugate points this would be the case.

The problem of recovering a function from its integrals over all lines in the plane goes back to Radon [Rad17]. He proved the injectivity of the integral transform nowadays known as the Radon transform and provided a reconstruction formula.

It is also worth mentioning a counterexample for injectivity of the Radon transform provided by Zalcman [Zal82] He showed that on $\mathbb{R}^2$ there exists a non-zero continuous function which is $O(|x|^{-2})$ along every line and integrates to zero over any line. See also [AG93],[Arm94].

This work is organized as follows. In the second section we describe the geometrical setting of this work and present some results mostly concerning behaviour of geodesics. The third section is about the geodesic ray transform. In the fourth section we derive estimates for the growth of Jacobi fields in our setting and use those to prove useful decay estimates. The fifth section contains the proofs of our main theorems. As mentioned, those rely on certain type of Pestov identity, which is presented there too.

Notational convention. *Throughout this work we denote by $C(a, b, \dots)$ (with a possible subscript) a constant depending on $a, b, \dots$ The value of the constant may vary from line to line.*

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2. The setting of this work and preliminaries

Throughout this paper we assume (M, g) to be a two-dimensional, complete, simply connected manifold with non-positive Gaussian curvature K. By the Cartan-Hadamard theorem the exponential map $\exp_x : T_xM \to M$ is a diffeomorphism for every point $x \in M$. Thereby we have global normal coordinates centered at any point and we could equivalently work with $(\mathbb{R}^2, \tilde{g})$ where $\tilde{g}$ is pullback of the metric g by exponential map, but we choose to present this work in the general setting of (M, g).

We make the standing assumption of unit-speed parametrization for geodesics. If $x \in M$ and $v \in T_xM$ is such that $|v|_g = 1$ we denote by $\gamma_{x,v} : \mathbb{R} \to M$ the geodesic with $\gamma_{x,v}(0) = x$ and $\gamma'_{x,v}(0) = v$.

The fact that for every point the exponential map is a diffeomorphism implies that every pair of distinct points can be joined by an unique geodesic. Furthermore, by using the triangle inequality, we have

$$d_g(\gamma_{x,v}(t), p) \geq d_g(\gamma_{x,v}(t), x) - d_g(x, p) = |t| - d_g(x, p) \tag{2.1}$$

for every $p \in M$ and $(x, v) \in SM$.

Because of the everywhere non-positive Gaussian curvature, the function $t \mapsto d_g(\gamma(t), p)$ is convex on $\mathbb{R}$ and the function $t \mapsto d_g(\gamma(t), p)^2$ is strictly convex on $\mathbb{R}$ for every geodesic γ and point $p \in M$ (see e.g. [Pet98]).

We say that the geodesic $\gamma_{x,v}$ is escaping with respect to point p if function $t \mapsto d_g(\gamma_{x,v}(t), p)$ is strictly increasing on the interval $[0, \infty)$. The set of such geodesics is denoted by $\mathcal{E}_p(M)$.

Lemma 2.1. *Let $p \in M$ and $(x, v) \in SM$. At least one of geodesics $\gamma_{x,v}$ and $\gamma_{x,-v}$ is in set $\mathcal{E}_p(M)$.*

Proof. The function $t \mapsto d_g(\gamma(t), p)^2$ is strictly convex on $\mathbb{R}$ so it has a strict global minimum. Therefore the function $t \mapsto d_g(\gamma(t), p)$ also has a strict global minimum, which implies that at least one of functions $t \mapsto d_g(\gamma_{x,v}(t), p)$ and $t \mapsto d_g(\gamma_{x,-v}(t), p)$ is strictly increasing on the interval $[0, \infty)$. □

If the geodesic $\gamma_{x,v}$ belongs to $\mathcal{E}_p(M)$ equation (2.1) implies the estimate

$$d_g(\gamma_{x,v}(t), p) \geq \begin{cases} d_g(x, p), & \text{if } 0 \leq t \leq 2d_g(x, p), \\ t - d_g(x, p), & \text{if } 2d_g(x, p) < t. \end{cases} \tag{2.2}$$

The manifold M is two-dimensional and oriented and so is also the tangent space T_xM for every $x \in M$. Thus given $v \in T_xM$ we can define $e^{it}v \in$

$T_xM, t \in \mathbb{R}$, to be the unit vector obtained by rotating the vector v by an angle t. We will use the shorthand notation $v_\perp := \mathrm{e}^{-i\pi/2}v$.

The unit tangent bundle SM is a 3-dimensional manifold and there is a natural Riemannian metric on it, namely the Sasaki metric [Pat99]. The volume form given by this metric is denoted by $d\Sigma^3$.

On the manifold SM we have the geodesic flow $\varphi_t \colon SM \to SM$ defined by

$$\varphi_t(x, v) = (\gamma_{x,v}(t), \gamma'_{x,v}(t)).$$

We denote by X the vector field associated with this flow. We define flows $p_t, h_t \colon SM \to SM$ as

$$\begin{aligned} p_t(x, v) &:= (x, \mathrm{e}^{it}v), \\ h_t(x, v) &:= (\gamma_{x,v_\perp}(t), Z(t)), \end{aligned}$$

where $Z(t)$ is the parallel transport of the vector v along the geodesic $\gamma_{x,v_\perp}$, and denote the associated vector fields by V and $X_\perp$.

These three vector fields form a global orthonormal frame for $T(SM)$ and we have following structural equations (see [PSU13])

$$\begin{aligned} [X, V] &= X_\perp, \\ [V, X_\perp] &= X, \\ [X, X_\perp] &= -KV, \end{aligned}$$

where K is the Gaussian curvature of the manifold M.

Let $f \colon U \subset M \to \mathbb{R}$ be such that $|\nabla f|_g = 1$. Then level sets of the function f are submanifolds of M. The second fundamental form $\mathbb{I}$ on such a level set is defined as

$$\mathbb{I}(v, w) := \mathrm{Hess}(f)(v, w),$$

where $v, w \perp \nabla f$ and $\mathrm{Hess}(f)$ is the covariant Hessian (see [Pet98]).

Suppose that $p \in M$. Denote by $B_p(r)$ the open geodesic ball with radius r, and by $S_p(r)$ its boundary.

Lemma 2.2. *For every $p \in M$ and $r > 0$ the geodesic ball $B_p(r)$ has a strictly convex boundary, i.e. the second fundamental form of $S_p(r)$ is positive definite.*

Proof. Suppose $x \in S_p(r)$ and v is tangent to $S_p(r)$ at x. Denote $f(y) = d_p(y, p)$. We have

$$\mathrm{Hess}(f^2)(x) = 2f(x)\,\mathrm{Hess}\, f(x) + 2\, d_x f \otimes d_x f$$

and thus

$$\mathrm{Hess}(f^2)(x)(v, v) = 2f(x)\,\mathrm{Hess}\, f(x)(v, v)$$

since $d_x f(v) = \langle \nabla f(x), v \rangle_g = 0$.

Since the function $t \to d(\gamma_{x,v}(t), p)^2$ is strictly convex we get

$$\mathrm{Hess}(f^2)(x)(v, v) = \frac{d^2}{dt^2}((f^2 \circ \gamma_{x,v})(t))\Big|_{t=0} > 0.$$

Therefore $\mathrm{Hess}(f^2)$ is positive definite in tangential directions and so is also $\mathrm{Hess}\, f$ □

Equivalently, the boundary of $B_p(r)$ is strictly convex if and only if every geodesics starting from a boundary point in a direction tangent to boundary stays outside $B_p(r)$ for small positive and negative times and has a second order contact at time $t = 0$. From this we see that if $x \in M$ and v is tangent to $S_p(d_g(x,p))$ then function $t \mapsto d_g(\gamma_{x,v}(t), p)^2$ has a global minimum at $t = 0$.

Lemma 2.3. *Suppose $p \in M$ and $(x, v) \in SM$ is such that $\gamma_{x,v} \in \mathcal{E}_p(M)$ and v is not tangent to $S_p(d(x,p))$. Then $\gamma_{p_s(x,v)} \in \mathcal{E}_p(M)$ for small s.*

If v is tangent then $\gamma_{p_t(x,v)} \in \mathcal{E}_p(M)$ for either small $t > 0$ or small $t < 0$.

Proof. Suppose first that v is not tangent to $S_p(d(x,p))$. Then it must be that

$$\frac{d}{dt} d_g(\gamma_{x,v}(t), p)^2 \Big|_{t=0} > 0.$$

The function $s \mapsto \frac{d}{dt} d(\gamma_{p_s(x,v)}(t), p)^2$ is continuous and hence

$$\frac{d}{dt} d_g(\gamma_{p_s(x,v)}(t), p)^2 \Big|_{t=0} > 0 \tag{2.3}$$

for small s. Thus $\gamma_{p_s(x,v)} \in \mathcal{E}_p(M)$.

If v is tangent to $S_p(d(x,p))$ then

$$\frac{d}{dt} d_g(\gamma_{x,v}(t), p)^2 \Big|_{t=0} = 0$$

and (2.3) holds either for small positive s or for small negative s. □

Lemma 2.4. *Suppose $p \in M$ and $(x, v) \in SM$ is such that $\gamma_{x,v} \in \mathcal{E}_p(M)$. Then $\gamma_{h_s(x,v)} \in \mathcal{E}_p(M)$ for small s.*

Proof. If v is not tangent to $S_p(d_g(x,p))$ then proof is as for the flow p_s. If v is tangent to $S_p(d_g(x,p))$ then $\gamma_{h_s(x,v)}(0)$ is tangent to $S_p(d_g(x,p) + s)$ or $S_p(d_g(x,p) - s)$ and thus $\gamma_{h_s(x,v)} \in \mathcal{E}_p(M)$. □

The next lemma is equation (2.2) for γ_{h_s} and γ_{p_s}.

Lemma 2.5. *For all s such that $\gamma_{h_s(x,v)} \in \mathcal{E}_p(M)$ we have*

$$d_g(\gamma_{h_s(x,v)}(t), p) \geq \begin{cases} d_g(x,p) - s, & 0 \leq t \leq 2d_g(x,p), \\ t - d_g(x,p) - s, & t > 2d_g(x,p). \end{cases}$$

For all s such that $\gamma_{p_s(x,v)} \in \mathcal{E}_p(M)$ we have

$$d_g(\gamma_{p_s(x,v)}(t), p) \geq \begin{cases} d_g(x,p), & 0 \leq t \leq 2d_g(x,p), \\ t - d_g(x,p), & t > 2d_g(x,p). \end{cases}$$

Proof. We have for $\gamma_{h_s(x,v)}$ by triangle inequality

$$d_g(\gamma_{h_s(x,v)}(0), p) \leq d_g(\gamma_{h_s(x,v)}(0), x) + d_g(x, p) = s + d_g(x, p)$$

and furthermore

$$\begin{aligned} d_g(\gamma_{h_s(x,v)}(t), \gamma_{h_s(x,v)}(0)) &\leq d_g(\gamma_{h_s(x,v)}(t), p) + d_g(\gamma_{h_s(x,v)}(0), p) \\ &\leq d_g(\gamma_{h_s(x,v)}(t), p) + s + d_g(x, p). \end{aligned}$$

so

$$t - s - d_g(x, p) \leq d_g(\gamma_{h_s(x,v)}(t), p).$$

By triangle inequality

$$d_g(x, p) \leq d_g(\gamma_{h_s(x,v)}(0), p) + d_g(\gamma_{h_s(x,v)}(0), x) = d_g(\gamma_{h_s(x,v)}(0), p) + s.$$

Because $\gamma_{h_s(x,v)}$ is in $\mathcal{E}_p(M)$ we get for $t \geq 0$

$$d_g(\gamma_{h_s(x,v)}(t), p) \geq d_g(\gamma_{h_s(x,v)}(0), p) \geq d_g(x, p) - s.$$

The result for $\gamma_{h_s(x,v)}$ follows by combining these estimates. For $\gamma_{p_s(x,v)}$ proof is similar, but we have $d_g(\gamma_{p_s(x,v)}(0), x) = 0$. □

3. The geodesic ray transform

As mentioned in the introduction the geodesic ray transform $If \colon SM \to \mathbb{R}$ of a function $f \colon SM \to \mathbb{R}$ is defined by

$$If(x, v) := \int_{-\infty}^{\infty} f(\gamma_{x,v}(t))\, dt.$$

Lemma 3.1. *The geodesic ray transform is well defined for $f \in P_\eta(p, M)$ for $\eta > 1$.*

Proof. Let $(x, v) \in SM$. Since $If(\gamma_{x,v}(t), \gamma'_{x,v}(t)) = If(x, v)$ for all $t \in \mathbb{R}$, we can assume x to be such that

$$\min_{t \in \mathbb{R}} d_g(\gamma_{x,v}(t), p) = d_g(x, p).$$

Such a point always exists on any geodesic γ since the mapping $t \mapsto d_g(\gamma(t), p)^2$ is strictly convex.

By (2.1) we then have

$$d_g(\gamma_{x,v}(t), p) \geq \begin{cases} d_g(x, p), & \text{if } |t| \leq 2d_g(x, p), \\ |t| - d_g(x, p), & \text{if } 2d_g(x, p) < |t|. \end{cases}$$

Hence for $f \in P_\eta(p,M), \eta > 1$,

$$\begin{aligned}|If(x,v)| &\le \int_{-\infty}^{\infty} |f(\gamma_{x,v}(t))|\,dt \le \int_{-\infty}^{\infty} \frac{C}{(1+d_g(\gamma_{x,v}(t),p))^\eta}\,dt \\ &\le C\left(\int_0^{2d_g(x,p)} \frac{1}{(1+d_g(x,p))^\eta}\,dt + \int_{2d_g(x,p)}^{\infty} \frac{1}{(1+t-d_g(x,p))^\eta}\,dt\right) \\ &\le C\left(\frac{2d_g(x,p)}{(1+d_g(x,p))^\eta} + \frac{1}{(\eta-1)(1+d_g(x,p))^{\eta-1}}\right) \\ &\le \frac{C(\eta)}{(1+d_g(x,p))^{\eta-1}}.\end{aligned}$$

□

Given a function f on M we define the function $u^f : SM \to \mathbb{R}$ by

$$u^f(x,v) = \int_0^\infty f(\gamma_{x,v}(t))\,dt.$$

We observe that

$$If(x,v) = u^f(x,v) + u^f(x,-v)$$

for all $(x,v) \in SM$ whenever all the functions are well defined.

In the next lemma we assume that f is such that $If \equiv 0$ since those functions are in our interest.

Lemma 3.2. *Suppose $p \in M$ and f is a function on M such that $If \equiv 0$.*

(1) *If $f \in E_\eta(p,M)$ for some $\eta > 0$, then*

$$|u^f(x,v)| \le C(\eta)(1+d_g(x,p))e^{-\eta d_g(x,p)}.$$

(2) *If $f \in P_\eta(p,M)$ for some $\eta > 1$, then*

$$|u^f(x,v)| \le \frac{C(\eta)}{(1+d(x,p))^{\eta-1}}.$$

Proof. Since $If(x,v) = 0$ we have $|u^f(x,v)| = |u^f(x,-v)|$ for all $(x,v) \in SM$. Thus, by Lemma 2.1, we can assume (x,v) to be such that $\gamma_{x,v} \in \mathcal{E}_p(M)$.

If $f \in P_\eta(p,M), \eta > 1$, using the estimate (2.2) we obtain

$$\begin{aligned}|u^f(x,v)| \le C\Bigg(&\int_0^{2d_g(x,p)} \frac{1}{(1+d_g(\gamma_{x,v}(t),p))^\eta}\,dt \\ &+ \int_{2d_g(x,p)}^{\infty} \frac{1}{(1+d_g(\gamma_{x,v}(t),p))^\eta}\,dt\Bigg) \\ \le\ & \frac{C(\eta)}{(1+d_g(x,p))^{\eta-1}}.\end{aligned}$$

Similarly for $f \in E_\eta(p, M), \eta > 0$, we get

$$\begin{aligned}|u^f(x,v)| &\leq C\left(\int_0^{2d_g(x,p)} \mathrm{e}^{-\eta d_g(x,p)}\,dt + \int_{2d_g(x,p)}^\infty \mathrm{e}^{-\eta(t-d_g(x,p))}\,dt\right)\\ &\leq C(\eta)(1+d_g(x,p))\mathrm{e}^{-\eta d_g(x,p)}.\end{aligned}$$

□

Next we prove that $Xu^f = -f$, which can be seen as a reduction to transport equation. This idea is explained in details in [PSU13].

Lemma 3.3. *Suppose $f \in P^1_\eta(p, M)$ for some $\eta > 1$ and $If = 0$. Then $Xu^f(x, v) = -f(x)$ for every $(x, v) \in SM$.*

Proof. We begin by observing that

$$X(If(x,v)) = Xu^f(x,v) + X(u^f(x,-v)) = 0$$

so $Xu^f(x, v) = -X(u^f(x, -v))$. Hence we can assume the geodesic $\gamma_{x,v}$ to be in $\mathcal{E}_p(M)$ by Lemma 2.1.

We have

$$\begin{aligned}Xu^f(x,v) &= \frac{d}{ds}u^f(\varphi_s(x,v))\Big|_{s=0} = \frac{d}{ds}\int_0^\infty f(\gamma_{\varphi_s(x,v)}(t))\,dt\Big|_{s=0}\\ &= \int_0^\infty \frac{d}{ds}f(\gamma_{x,v}(s+t))\Big|_{s=0}\,dt\end{aligned}$$

where the last step needs to be justified.

Since we assumed our geodesic to be in $\mathcal{E}_p(M)$, for $t, s \geq 0$ it holds

$$\begin{aligned}|\frac{d}{ds}f(\gamma_{x,v}(t+s))| &= |d_{\gamma_{x,v}(t+s)}f(\gamma'_{x,v}(t+s))|\\ &\leq \frac{C}{(1+d_g(\gamma_{x,v}(t+s),p))^{\eta+1}}\\ &\leq \frac{C}{(1+d_g(\gamma_{x,v}(t),p))^{\eta+1}}.\end{aligned}$$

Using estimate (2.1) as in the earlier proofs we obtain

$$\begin{aligned}\int_0^\infty |\frac{d}{ds}f(\gamma_{x,v}(s+t))|\,dt &\leq \int_0^\infty \frac{C}{(1+d_g(\gamma_{x,v}(t),p))^{\eta+1}}\,dt\\ &\leq \frac{C(\eta)}{(1+d_g(x,p))^\eta},\end{aligned}$$

which shows that the last step earlier is justified by the dominated convergence theorem.

Since

$$\frac{d}{ds}f(\gamma_{x,v}(t+s))\Big|_{s=0} = \frac{d}{dt}f(\gamma_{x,v}(t))$$

and $f(\gamma_{x,v}(t)) \to 0$ as $t \to \infty$ we have

$$\int_0^\infty \frac{d}{ds}f(\gamma_{x,v}(s+t))\Big|_{s=0}\,dt = -f(x)$$

by the fundamental theorem of calculus. □

4. Regularity and decay of u^f

In order to prove our main theorems we need to prove C^1-regularity for u^f given that the function f has suitable regularity and decay properties. For that we derive estimates for functions $X_\perp u^f$ and Vu^f. To prove the estimates for functions $X_\perp u^f$ and Vu^f we will proceed as in the case of $Xu = -f$ (Lemma 3.3). In the proof we calculated

$$\frac{d}{ds} f(\gamma_{\varphi_s(x,v)}(t))\Big|_{s=0} = d_{\gamma_{\varphi_s(x,v)}(t)} f(\frac{d}{ds}\gamma_{\varphi_s(x,v)}(t)\Big|_{s=0}).$$

We can interpret $\frac{d}{ds}\gamma_{\varphi_s(x,v)}(t)\big|_{s=0}$ as a Jacobi field along the geodesic $\gamma_{x,v}$ since it is just the tangent vector field. For $X_\perp u^f$ and Vu^f we proceed in a similar manner, the difference being that the geodesic flow φ_t is replaced with the flows h_t and p_t respectively.

Given geodesic $\gamma_{x,v}$ we denote

$$J_{\gamma_{x,v},h}(s,t) = \frac{d}{dr}\gamma_{h_r(x,v)}(t)\Big|_{r=s}$$

and $J_{\gamma_{x,v},p}$ similarly. Then $J_{\gamma_{x,v},h}(s,t)$ is a Jacobi field along geodesic $\gamma_{h_s(x,v)}$ for fixed s. We will write $J_h(s,t)$ when it is clear from the context what the undelying geodesic is. We will also use shorthand notation $J_h(t) = J_h(0,t)$ and $J_p(t) = J_p(0,t)$.

The Jacobi fields obtained in this manner turn out to be normal with initial data (see [PU04])

$$J_h(s,0) = 1, \quad D_t J_h(s,0) = 0,$$
$$J_p(s,0) = 0, \quad D_t J_p(s,0) = 1.$$

We need to have estimates for the growth of these two Jacobi fields in particular. The first lemma giving estimates for the growth is based on comparison theorems for Jacobi fields. See for example [Jos08, Theorem 4.5.2].

Lemma 4.1. *Suppose $|K(x)| \leq K_0$ and γ is a geodesic. Then for Jacobi fields J_p and J_h along a geodesic γ it holds that*

$$|J_p(t)| \leq C(K_0)e^{\sqrt{K_0}t},$$
$$|J_h(t)| \leq C(K_0)e^{\sqrt{K_0}t},$$

for $t \geq 0$.

This lemma tells us that these Jacobi fields will grow at most exponentially in presence of bounded curvature. If the curvature happens to decay suitably we will see that these Jacobi fields will grow only at a polynomial rate.

If $J(t)$ is a normal Jacobi field along a geodesic γ then we can write $J(t) = u(t)E(t)$ where u is a real valued function and $E(t)$ is a unit normal vector field along γ. From the Jacobi equation it follows that u is a solution to

$$u''(t) + K(\gamma(t))u(t) = 0$$

for $t \geq 0$ with initial values $u(0) = \pm|J(t)|$ and $u'(0) = \pm|D_t J(t)|$.

This leads us to consider an ordinary differential equation

$$\begin{cases} u''(t) + K(t)u(t) = 0, \quad t \geq 0, \\ u(0) = c_1, \\ u'(0) = c_2, \end{cases} \tag{4.1}$$

for continuous K, where $c_1, c_2 \in \mathbb{R}$. Note that for J_h and J_p the constants c_1 and c_2 are either 0 or ± 1.

Waltman [Wal64] proved that if u is a solution to (4.1) with K such that

$$\int_0^\infty t|K(t)|\,ds < \infty$$

then $\lim_{t\to\infty} u(t)/t$ exists. We reproduce essential parts of the proof in order to obtain a more quantitative estimate for the growth of the solution u.

Lemma 4.2. *Suppose u is a solution to* (4.1) *with*

$$M_K := \int_0^\infty s|K(s)|\,ds < \infty.$$

and $c_1 = 1, c_2 = 0$ or other way around. Then

$$|u(t)| \leq C_1 t + C_2$$

for all $t \geq 0$ where $C_1, C_2 \geq 0$.

Proof. We define $A(t) = u'(t)$ and $B(t) = u(t) - tu'(t)$ so $u(t) = A(t)t + B(t)$. Fix $t_0 > 0$. For all $t > t_0$ it holds

$$A(t) = A(t_0) - \int_{t_0}^t K(s)s\left(A(s) + \frac{B(s)}{s}\right) ds,$$
$$B(t) = B(t_0) + \int_{t_0}^t K(s)s^2\left(A(s) + \frac{B(s)}{s}\right) ds.$$

If we define $|v(t)| = |A(t)| + |B(t)/t|$ we have

$$|v(t)| \leq |v(t_0)| + 2\int_{t_0}^t s|K(s)||v(s)|\,ds.$$

By a theorem of Viswanatham [Vis63] it holds $|v(t)| \leq \psi(t)$ on $[t_0, \infty)$ where ψ is a solution to

$$\psi'(t) = 2t|K(t)|\psi(t)$$

with $\psi(t_0) = |v(t_0)|$. Hence

$$\psi(t) = |v(t_0)|\mathrm{e}^{2\int_{t_0}^t s|K(s)|\,ds} \leq |v(t_0)|\mathrm{e}^{2M_K}$$

and furthermore

$$|u(t)| = |tv(t)| \leq t\mathrm{e}^{2M_K}|v(t_0)|$$

for $t \geq t_0$.

Then we need to estimate $|v(t_0)|$. In order to do so we need estimates for $|u(t_0)|$ and $|u'(t_0)|$. We can apply Lemma 4.1 to get

$$|u(t)| \leq C(K_0)\mathrm{e}^{\sqrt{K_0}t_0}$$

on interval $[0, t_0]$ where we have denoted $K_0 = \sup_{t\in[0,t_0]} |K(t)|$. By integrating equation (4.1) we obtain

$$\begin{aligned} |u'(t_0)| &\leq |u'(0)| + \int_0^{t_0} |K(t)||u(t)| \, ds \\ &\leq |c_2| + |\sup_{t\in[0,t_0]} u(t)|K_0 t_0 \end{aligned}$$

Thus

$$\begin{aligned} |v(t_0)| &\leq |A(t_0)| + |B(t_0)/t_0| \leq |u(t_0)/t_0| + 2|u'(t_0)| \\ &\leq C(K_0)(\frac{1}{t_0} + 2K_0 t_0)\mathrm{e}^{\sqrt{K_0}t_0} + 2. \end{aligned}$$

By combining the estimates for intervals $[0, t_0]$ and $[t_0, \infty)$ and setting $t_0 = 1$ we obtain that

$$|u(t)| \leq t\mathrm{e}^{2M_K}|v(1)| + C(K_0)\mathrm{e}^{\sqrt{K_0}}$$

for $t \geq 0$. □

Lemma 4.3. *Suppose $|K(x)| \leq K_0$ and that G is a set of geodesics such that*

$$M_G := \sup_{\gamma\in G} \int_0^\infty t|K(t)| \, dt < \infty.$$

Let $\gamma \in G$. Then for Jacobi fields J_p and J_h along geodesic γ holds

$$\begin{aligned} |J_p(t)| &\leq C(M_G)t, \\ |J_h(t)| &\leq C(M_G)(t+1). \end{aligned}$$

for all $t \geq 0$. Especially the constants do not depend on the geodesic γ.

Proof. Suppose geodesic $\gamma_{x,v}$ is in G. By Lemma 4.2 we obtain

$$\begin{aligned} |J_h(t)| &\leq C_1 t + C_2, \\ |J_p(t)| &\leq C_1 t + C_2. \end{aligned}$$

From the proof of that lemma we see that constants C_1 and C_2 above depend on the lower bound for K and the quantity

$$\int_0^\infty -tK(\gamma_{x,v}(t)) \, dt.$$

Since this quantity is bounded from above by M_G we can estimate constants C_1 and C_2 by above and get rid of the dependence on the geodesic $\gamma_{x,v}$. So the constants depend only on the Gaussian curvature K and the initial conditions.

Furthermore, since $|J_p(0)| = 0$ we can drop the constant C_2 in the estimate for $J_p(t)$ by making C_1 accordingly larger. □

Next lemma is a straightforward corollary of the preceding lemma.

Lemma 4.4. *Suppose $K \in P_\eta(p, M)$ for some $\eta > 2$. If $\gamma \in \mathcal{E}_p(M)$ then for Jacobi fields J_p and J_h along geodesic γ one has*

$$|J_p(t)| \le Ct,$$
$$|J_h(t)| \le C(t+1),$$

for all $t \ge 0$, where the constants do not depend on the geodesic γ.

Proof. Since $K \in P_\eta(p, M), \eta > 2$, we have

$$\sup_{\gamma \in \mathcal{E}_p(M)} \int_0^\infty -K(\gamma(t))t\, dt < \infty.$$ □

With Lemmas 4.1 and 4.4 we can derive estimates for $X_\perp u^f$ and Vu^f.

Lemma 4.5. *Let $f \in C(M)$ be such that $If = 0$.*

(1) *If $|K(x)| \le K_0$ and $f \in E^1_\eta(p, M)$ for some $\eta > \sqrt{K_0}$, then*
$$|X_\perp u^f(x, v)| \le C(\eta, K_0)e^{(2\sqrt{K_0}-\eta)d_g(x,p)}$$
for all $(x, v) \in SM$.

(2) *If $f \in P^1_\eta(p, M)$ for some $\eta > 1$ and $K \in P_{\tilde{\eta}}(p, M)$ for some $\tilde{\eta} > 2$, then*
$$|X_\perp u^f(x, v)| \le \frac{C(\eta)}{(1 + d_g(x, p))^{\eta-1}}$$
for all $(x, v) \in SM$.

Both estimates hold also if $X_\perp$ is replaced by V.

Proof. Let us first notice that since $If = 0$, it holds $|X_\perp u^f(x, -v)| = |X_\perp u^f(x, v)|$ for all $(x, v) \in SM$. Thus we will assume that v is such that $\gamma_{x,v} \in \mathcal{E}_p(M)$.

Firts we note that

$$\frac{d}{ds} f(\gamma_{h_s(x,v)}(t)) = d_{\gamma_{h_s(x,v)}(t)} f(J_h(s, t)).$$

By definition

$$\begin{aligned} X_\perp u^f(x, v) &= \frac{d}{ds} \int_0^\infty f(\gamma_{h_s(x,v)}(t))\, dt \Big|_{s=0} = \int_0^\infty \frac{d}{ds} f(\gamma_{h_s(x,v)}(t))\, dt \Big|_{s=0} \\ &= \int_0^\infty d_{\gamma_{x,v}(t)} f(J_h(s, t))\, dt \end{aligned}$$

where the second equality holds by the dominated convergence theorem provided that there exists function $F \in L^1([0, \infty))$ such that

$$\left| \frac{d}{ds} f(\gamma_{h_s(x,v)}(t)) \right| \le F(t) \tag{4.2}$$

for all $t \geq 0$ and for small non-negative s.

Lemma 2.4 states that for small s it holds that $\gamma_{h_s(x,v)} \in \mathcal{E}_p(M)$. Hence in the first case using Lemmas 2.5 and 4.1 we get

$$\begin{aligned}
|d_{\gamma_{h_s(x,v)}(t)} f(J_h(s,t))| &\leq C(K_0)\mathrm{e}^{\sqrt{K_0}t}\mathrm{e}^{-\eta d_g(\gamma_{h_s(x,v)}(t),p)} \\
&\leq \begin{cases} C(K_0)\mathrm{e}^{\eta s}\mathrm{e}^{\sqrt{K_0}t}\mathrm{e}^{-\eta d_g(x,p)}, & 0 \leq t \leq 2d_g(x,p), \\ C(K_0)\mathrm{e}^{\eta s}\mathrm{e}^{\sqrt{K_0}t}\mathrm{e}^{-\eta(t-d_g(x,p))}, & t > 2d_g(x,p), \end{cases}
\end{aligned}$$

and thus

$$\int_0^\infty |d_{\gamma_{h_s(x,v)}(t)} f(J_h(s,t))| \leq C(\eta, K_0)\mathrm{e}^{\eta s}\mathrm{e}^{(2\sqrt{K_0}-\eta)d_g(x,p)}.$$

In the second case we obtain

$$|d_{\gamma_{h_s(x,v)}(t)} f(J_h(s,t))| \leq \begin{cases} \frac{C(t+1)}{(1+d_g(x,p)-s)^{\eta+1}}, & 0 \leq t \leq 2d_g(x,p), \\ \frac{C(t+1)}{(1+t-d_g(x,p)-s)^{\eta+1}}, & t > 2d_g(x,p). \end{cases}$$

Therefore

$$\int_0^\infty |d_{\gamma_{h_s(x,v)}(t)} f(J_h(s,t))|\, dt \leq \frac{C(\eta)}{(1-s+d_g(x,p))^{\eta-1}}.$$

From these estimates we see that such a function F exists in both cases. Setting $s = 0$ gives the estimates for $|X_\perp u^f(x,v)|$.

In case of V instead of $X_\perp$ we proceed in the same manner. First we notice that $|Vu^f(x,-v)| = |Vu^f(x,v)|$ for all $(x,v) \in SM$. Thus we will assume that v is such that $\gamma_{x,v} \in \mathcal{E}_p(M)$. In addition we will assume v to be such that $\gamma_{p_s(x,v)} \in \mathcal{E}_p(M)$ for small non-negative s, this can be done by Lemma 2.3. The rest of the proof is then similar. □

From this result we see that if f is a C^1-function with suitable decay properties then u^f is in $C^1(SM)$. Later we will approximate u^f with functions $u^{f_k} \in C^2(SM)$ where functions f_k are compactly supported C^2-functions on M. The following lemma shows that functions u^{f_k} are indeed in $C^2(SM)$.

Lemma 4.6. *Suppose that $f \in C^2(M)$ is compactly supported. Then $u^f \in C^2(SM)$.*

Proof. Since f is compactly supported we have

$$\begin{aligned}
Xu^f(x,v) &= -f(x), \\
X_\perp u^f(x,v) &= \int_0^\infty d_{\gamma_{x,v}(t)} f(J_h(t))\, dt, \\
Vu^f(x,v) &= \int_0^\infty d_{\gamma_{x,v}(t)} f(J_p(t))\, dt.
\end{aligned}$$

From the structural equations and the knowledge that $Xu^f = -f$ we can deduce that $VXu^f, XVu^f, X_\perp Xu^f, XX_\perp u^f$ and X^2u^f exist.

With other means we have to check that $V^2u^f, X_\perp^2 u^f$ and $VX_\perp u^f$ (or equivalently $X_\perp Vu^f$) exist.

Let us calculate a formula for $VX_\perp u^f(x,v)$ and from that we see the existence. By definition

$$\begin{aligned} VX_\perp u^f(x,v) &= \frac{d}{ds} X_\perp u^f(p_s(x,v))\Big|_{s=0} \\ &= \frac{d}{ds} \int_0^\infty d_{\gamma_{p_s(x,v)}(t)} f(J_{\gamma_{p_s(x,v)},h}(t))\,dt\Big|_{s=0}. \end{aligned}$$

We write

$$d_{\gamma_{p_s(x,v)}(t)} f(J_{\gamma_{p_s(x,v)},h}(t)) = \langle \nabla f(\gamma_{p_s(x,v)}(t)), J_{\gamma_{p_s(x,v)},h}(t)\rangle.$$

Since

$$\langle D_s \nabla f(\gamma_{p_s(x,v)}(t)), J_{\gamma_{p_s(x,v)},h}(t)\rangle = \operatorname{Hess} f(\gamma_{p_s(x,v)})(J_p(s,t), J_{\gamma_{p_s(x,v)},h}(t))$$

we have

$$\begin{aligned} \frac{d}{ds} d_{\gamma_{p_s(x,v)}(t)} f(J_{\gamma_{p_s(x,v)},h}(t)) &= \operatorname{Hess} f(\gamma_{p_s(x,v)})(J_p(s,t), J_{\gamma_{p_s(x,v)},h}(t)) \\ &\quad + \langle \nabla f(\gamma_{p_s(x,v)})(t), D_s J_{\gamma_{p_s(x,v)},h}(t)\rangle. \end{aligned}$$

Since $\operatorname{Hess} f$ and ∇f are compactly supported we can move derivative $\frac{d}{ds}$ into integral and deduce that $VX_\perp u^f(x,v)$ exists for all $(x,v) \in SM$.

Proofs for $V^2 u^f$ and $X_\perp^2 u^f$ are once again similar. □

As a last application of Lemmas 4.1 and 4.4 we derive an estimate for the volumes of spheres in our setting.

Lemma 4.7. *Suppose $|K| \leq K_0$ and $p \in M$. Then*

$$\operatorname{Vol} S_p(r) \leq C(K_0) e^{\sqrt{K_0} r}.$$

If $K \in P_\eta(p,M)$ for some $\eta > 2$, then

$$\operatorname{Vol} S_p(r) \leq Ct.$$

Proof. We use polar coordinates centered at point p. Fix a tangent vector $v \in S_pM$ and define mapping $f\colon [0,\infty) \times (0,2\pi) \to M$ by $f(r,\theta) = \exp_p(re^{i\theta}v)$. This gives the usual polar coordinates in which the metric g takes form

$$g(r,\theta) = dr^2 + \left|\frac{df}{d\theta}\right|^2 d\theta^2$$

and the corresponding volume form is

$$dV_g(r,\theta) = \left|\frac{df}{d\theta}\right| dr \wedge d\theta.$$

Since $\exp_p(re^{i\theta}v) = \gamma_{p_\theta(p,v)}(r)$ we have

$$\frac{df}{d\theta}(r,\theta) = \frac{d}{dt}\gamma_{p_\theta(p,v)}(r) = J_p(r,\theta)$$

and hence the volume form on $S_p(r)$ is given by

$$\iota_{\partial_r} dV_g(r,\theta) = \left|\frac{df}{d\theta}\right| d\theta = J_p(r,\theta)d\theta.$$

By Lemma 4.1

$$\operatorname{Vol} S_p(r) \leq \int_0^{2\pi} C(K_0)\mathrm{e}^{\sqrt{K_0}r}\, d\theta = C(K_0)\mathrm{e}^{\sqrt{K_0}r}.$$

In the presence of the additional assumption for the Gaussian curvauture Lemma 4.4 yields

$$\operatorname{Vol} S_p(r) \leq Ct. \qquad \square$$

5. Pestov identity and C^2-approximation

In this section we prove our main theorems. The proofs are based on a certain kind of energy estimate for the operator $P = VX$ called the Pestov identity. We will use Pestov identity with boundary terms on submanifolds of (M,g). Througout this section we denote $M_{p,r} = B_p(r) \subset M$, a submanifold of M with boundary $S_p(r)$.

The following form of Pestov identity constitutes the main argument for our proofs of the main theorems.

Lemma 5.1 ([IS16]). *For $u \in C^2(SM)$ it holds*

$$\begin{aligned}\|VXu\|^2_{L^2(SM_{p,r})} &= \|XVu\|^2_{L^2(SM_{p,r})} + \|Xu\|^2_{L^2(SM_{p,r})} - \langle KVu, Vu\rangle_{SM_{p,r}} \\ &\quad - \langle\langle v,\nu\rangle Vu, X_\perp u\rangle_{\partial SM_{p,r}} + \langle\langle v_\perp,\nu\rangle Vu, Xu\rangle_{\partial SM_{p,r}}\end{aligned}$$

By using approximating sequences we can relax the regularity assumptions for the Pestov identity. Especially the Pestov identity holds for u^f with suitable f.

Lemma 5.2. *Suppose either one of the following:*

1. *$|K(x)| \leq K_0$ and $f \in E^1_\eta(p,M) \cap C^2(M)$ for some $\eta > \sqrt{K_0}$.*
2. *$f \in P^1_\eta(p,M) \cap C^2(M)$ for some $\eta > 1$ and $K \in P_{\tilde\eta}(p,M)$ for some $\tilde\eta > 2$.*

If $If = 0$, then the Pestov identity in Lemma 5.1 holds for u^f.

Proof. Lemmas 3.3 and 4.5 ensure that all terms of the Pestov identity are finite.

We define $u_k = u^{\varphi_k f}$ where $\varphi_k \colon M \to \mathbb{R}$ is a smooth cutoff function such that

1. $0 \leq \varphi_r(x) \leq 1$ for all $x \in M$.
2. $\varphi_k(x) = 1$ for $x \in B_p(k)$.
3. $\varphi_k(x) = 0$ for $x \notin B_p(2k)$.
4. $|\nabla\varphi|_g \leq C/k$ for all $x \in M$ and $v \in T_xM$.

Such a function can be defined by

$$\varphi_k(x) := \varphi\left(\frac{d_g(x,p)}{k}\right)$$

where φ is a suitable smooth cutoff function on $\mathbb{R}$. Since functions φ_k are smooth and compactly supported, we have $u_k \in C^2(SM)$ by Lemma 4.6.

Let us move on to prove the convergence. First we observe that

$$Xu_k(x,v)\big|_{SM_{p,r}} = -f(x)$$

for large k. Therefore we have convergence in L^2-norm for the term Xu_k.

Next we prove convergence for XVu_k under the assumption that $f \in P^1_\eta(p,M) \cap C^2(M)$ for some $\eta > 1$ and $K \in P_{\tilde{\eta}}(p,M)$ for some $\tilde{\eta} > 2$. First we notice that

$$XVu_k = VXu_k + X_\perp u_k = X_\perp u_k$$

for large k. Similarly $XVu^f = X_\perp u^f$ so it is enough to prove that $X_\perp u_k$ converges to $X_\perp u^f$. Furthermore since $SM_{p,r}$ has finite volume it is enough to prove that $X_\perp u_k \to X_\perp u^f$ in L^∞-norm.

Let us denote $G = \{\gamma_{x,v} \colon (x,v) \in SM_{p,r}\}$. The set G fulfills the assumption of Lemma 4.3. Suppose $(x,v) \in SM_r$. We have

$$\begin{aligned} X_\perp u_k(x,v) - X_\perp u^f(x,v) &= \int_0^\infty d_{\gamma_{x,v}(t)}(\varphi_k f)(J_h(t))\,dt \\ &\quad - \int_0^\infty d_{\gamma_{x,v}(t)} f(J_h(t))\,dt \\ &= \int_0^\infty (\varphi_k(\gamma_{x,v}(t)) - 1) d_{\gamma_{x,v}(t)} f(J_h(t))\,dt \\ &\quad + \int_0^\infty f(\gamma_{x,v}(t)) d_{\gamma_{x,v}(t)} \varphi_k(J_h(t))\,dt. \end{aligned}$$

For $t \geq 0$ holds

$$d_g(\gamma_{x,v}(t), p) \geq t - d_g(x,p) \geq t - r.$$

Also

$$(1 - \varphi_k(\gamma_{x,v}(t)) = 0$$

at least for $0 \leq t \leq k - r$ and $d_{\gamma_{x,v}(t)}\varphi_k$ can be non-zero only in interval $[k-r, 2k+r]$, which can be seen using triangle inequality.

Hence we can estimate, with help of Lemma 4.3, that

$$\begin{aligned}
|X_\perp u_k(x,v) - X_\perp u^f(x,v)| &\leq \int_{k-r}^{\infty} |d_{\gamma_{x,v}(t)} f(J_h(t))|\, dt \\
&+ \int_{k-r}^{2k+r} |f(\gamma_{x,v}(t)) d_{\gamma_{x,v}(t)} \varphi_k(J_h(t))|\, dt \\
&\leq C_1 \int_{k-r}^{\infty} \frac{t}{(1+d(\gamma_{x,v}(t),p))^{\eta+1}}\, dt \\
&+ \frac{C_2}{k} \int_{k-r}^{2k+r} \frac{t}{(1+d(\gamma_{x,v}(t),p))^{\eta+1}}\, dt \\
&\leq C_1 \int_{k-r}^{\infty} \frac{t}{(1+t-r)^{\eta+1}}\, dt \\
&+ \frac{C_2}{k} \int_{k-r}^{2k+r} \frac{t}{(1+t-r)^{\eta+1}}\, dt.
\end{aligned}$$

The last two integrals do not depend on (x,v) and they also tend to zero as $k \to \infty$, which proves the L^∞-convergence. In similar manner we can prove convergence for Vu_k.

Convergence for the boundary terms follows also from the L^∞-convergence because the boundary $\partial SM_{p,r}$ has a finite volume.

In the other case we proceed similarly but use Lemma 4.1 instead of Lemma 4.3. □

We are ready to prove our main theorems.

Proof of Theorem 1. Since the geodesic ray transform is linear it is enough to show that $If = 0$ implies $f = 0$.

Let us assume $f \in E^1_\eta(p,M) \cap C^2(M)$, $\eta > \frac{5}{2}\sqrt{K_0}$, is such that $If = 0$. Lemma 5.2 tell us that Pestov identity holds for u^f. We will apply it on submanifold $SM_{p,r}$.

Since $Xu^f = -f$, the term on the left hand side of the Pestov identity is zero. Because we assume Gaussian curvature to be non-positive we have

$$-\langle KVu^f, Vu^f \rangle_{SM_{p,r}} \geq 0.$$

Thus if we can show that the two boundary terms tend to zero as $r \to \infty$, it must be that

$$\lim_{r\to 0} \|Xu^f\|_{L^2(SM_{p,r})} = \lim_{r \to 0} \|f\|_{L^2(SM_{p,r})} = 0$$

which proves the injectivity.

Using Lemma 4.5 together with Lemma 4.7 gives

$$\begin{aligned}
\left|\langle\langle v,\nu\rangle Vu, X_\perp u\rangle_{\partial SM_{p,r}}\right| &\le \int_{\partial SM_{p,r}} |Vu^f||X_\perp u^f|\, d\Sigma^2 \\
&\le C(\eta,K_0)\int_{\partial M_{p,r}}\int_{S_xM} \mathrm{e}^{2(2\sqrt{K_0}-\eta)d_g(x,p)}\, dS\, dV_g \\
&\le C(\eta,K_0)\int_{\partial SM_{p,r}} \mathrm{e}^{2(2\sqrt{K_0}-\eta)r}\, dV_g \\
&\le C(\eta,K_0)\int_{\partial M_{p,r}} \mathrm{e}^{2(2\sqrt{K_0}-\eta)r}\, dV_g \\
&\le C(\eta,K_0)\mathrm{e}^{2(2\sqrt{K_0}-\eta)r}\operatorname{Vol} S_p(r) \\
&\le C(\eta,K_0)\mathrm{e}^{(5\sqrt{K_0}-2\eta)r},
\end{aligned}$$

which indeed tends to zero as $r\to\infty$.

Similarly we obtain

$$\left|\int_{\partial SM_{p,r}} \langle v_\perp,\nu\rangle (Vu^f)(Xu^f)d\Sigma^2\right| \le C(\eta,K_0)\mathrm{e}^{(3\sqrt{K_0}-2\eta)r}.$$

which also tends to zero as $r\to\infty$. □

Proof of Theorem 2. The proof is as for the Theorem 1, just using the other estimates provided by Lemmas 4.5 and 4.7. □

References

[AG93] D. H. Armitage and M. Goldstein. Nonuniqueness for the Radon transform. *Proc. Amer. Math. Soc.*, 117(1):175–178, 1993.

[Arm94] D. H. Armitage. A non-constant continuous function on the plane whose integral on every line is zero. *Amer. Math. Monthly*, 101(9):892–894, 1994.

[GG85] Leon Green and Robert Gulliver. Planes without conjugate points. *J. Differential Geom.*, 22(1):43–47, 1985.

[Hel94] Sigurdur Helgason. *Geometric analysis on symmetric spaces*, volume 39 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1994.

[Hel99] Sigurdur Helgason. *The Radon transform*, volume 5 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 1999.

[Hel13] Sigurdur Helgason. Some personal remarks on the Radon transform. In *Geometric analysis and integral geometry*, volume 598 of *Contemp. Math.*, pages 3–19. Amer. Math. Soc., Providence, RI, 2013.

[IS16] Joonas Ilmavirta and Mikko Salo. Broken ray transform on a Riemann surface with a convex obstacle. *Comm. Anal. Geom.*, 24(2):379–408, 2016.

[Jen04] Sine R. Jensen. Sufficient conditions for the inversion formula for the k-plane Radon transform in $\mathbf{R}^n$. *Math. Scand.*, 94(2):207–226, 2004.

[Jos08] Jürgen Jost. *Riemannian geometry and geometric analysis*. Universitext. Springer-Verlag, Berlin, fifth edition, 2008.

[KW74] Jerry L. Kazdan and F. W. Warner. Curvature functions for open 2-manifolds. *Ann. of Math. (2)*, 99:203–219, 1974.

[Muk77] R.G. Mukhometov. The problem of recovery of a two-dimensional Riemannian metric and integral geometry. *Sov. Math., Dokl.*, 18:27–31, 1977.

[Pat99] Gabriel P. Paternain. *Geodesic flows*, volume 180 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1999.

[Pet98] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[PSU13] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Tensor tomography on surfaces. *Invent. Math.*, 193(1):229–247, 2013.

[PSU14] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Tensor tomography: progress and challenges. *Chin. Ann. Math. Ser. B*, 35(3):399–428, 2014.

[PU04] Leonid Pestov and Gunther Uhlmann. On characterization of the range and inversion formulas for the geodesic X-ray transform. *Int. Math. Res. Not.*, (80):4331–4347, 2004.

[Rad17] J. Radon. Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. Leipz. Ber. 69, 262-277 (1917)., 1917.

[Sha94] V. A. Sharafutdinov. *Integral geometry of tensor fields*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.

[Vis63] B. Viswanatham. A generalisation of Bellman's Lemma. *Proc. Amer. Math. Soc.*, 14:15–18, 1963.

[Wal64] Paul Waltman. On the asymptotic behavior of solutions of a nonlinear equation. *Proc. Amer. Math. Soc.*, 15:918–923, 1964.

[Zal82] Lawrence Zalcman. Uniqueness and nonuniqueness for the Radon transform. *Bull. London Math. Soc.*, 14(3):241–245, 1982.

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[B]

GEODESIC X-RAY TOMOGRAPHY FOR PIECEWISE CONSTANT FUNCTIONS ON NONTRAPPING MANIFOLDS

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GEODESIC X-RAY TOMOGRAPHY FOR PIECEWISE CONSTANT FUNCTIONS ON NONTRAPPING MANIFOLDS

JOONAS ILMAVIRTA, JERE LEHTONEN, AND MIKKO SALO

Abstract. We show that on a two-dimensional compact nontrapping manifold with strictly convex boundary, a piecewise constant function is determined by its integrals over geodesics. In higher dimensions, we obtain a similar result if the manifold satisfies a foliation condition. These theorems are based on iterating a local uniqueness result. Our proofs are elementary.

1. Introduction

If (M, g) is a compact Riemannian manifold with boundary and f is a function on M, the geodesic X-ray transform of f encodes the integrals of f over all geodesics between boundary points. This transform generalizes the classical X-ray transform that encodes the integrals of a function in Euclidean space over all lines. The geodesic X-ray transform is a central object in geometric inverse problems and it arises in seismic imaging applications, inverse problems for PDEs such as the Calderón problem, and geometric rigidity questions including boundary and scattering rigidity (see the surveys [6, 8]).

It has been conjectured that the geodesic X-ray transform on compact nontrapping Riemannian manifolds with strictly convex boundary is injective [5]. Here, we say that a Riemannian manifold (M, g) has *strictly convex boundary* if the second fundamental form of ∂M in M is positive definite, and is *nontrapping* if for any $x \in M$ and any $v \in T_xM \setminus \{0\}$ the geodesic starting at x in direction v meets the boundary in finite time.

In this work we prove the following result, which verifies this conjecture on two-dimensional nontrapping manifolds if we restrict our attention to piecewise constant functions (see definition 2.5).

Theorem 1.1. *Let (M, g) be a compact nontrapping Riemannian manifold with strictly convex smooth boundary, and let $f\colon M \to \mathbb{R}$ be a piecewise constant function. Let either*

(a) $\dim(M) = 2$, *or*

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(b) $\dim(M) \geq 3$ *and* (M, g) *admits a smooth strictly convex function.*

If f *integrates to zero over all geodesics joining boundary points, then* $f \equiv 0$.

This result follows from theorem 6.4; for other corollaries, see section 6.3. We point out that theorem 1.1 implies that a piecewise constant function is uniquely determined by the data if the tiling of the domain is known a priori, but not in general; see remark 2.7.

The result for $\dim(M) = 2$ appears to be new, but when $\dim(M) \geq 3$ this is a special case of the much more general result [9] that applies to functions in $L^2(M)$ (see the survey [6] for further results on the injectivity of the geodesic X-ray transform also in two dimensions). However, our proof in the case of piecewise constant functions is elementary. We first prove a local injectivity result (lemmas 5.1 and 6.2) showing that if a piecewise constant function integrates to zero over all short geodesics near a point where the boundary is strictly convex, then the function has to vanish near that boundary point. We then iterate the local result by using a foliation of the manifold by strictly convex hypersurfaces as in [9, 7], given by the existence of a strictly convex function. The two-dimensional result follows since for $\dim(M) = 2$, the nontrapping condition implies the existence of a strictly convex function [1]. See [7] for further discussion on strictly convex functions and foliations.

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2. Preliminaries

In this section we will define what we mean by regular simplices, piecewise constant functions and foliations.

2.1. **Regular tilings.**

Recall that the standard n-simplex is the convex hull of the $n + 1$ coordinate unit vectors in $\mathbb{R}^{n+1}$.

Definition 2.1 (Regular simplex)**.** Let M be a manifold with or without boundary, with a C^1-structure. A regular n-simplex on M is an injective C^1-image of the standard n-simplex in $\mathbb{R}^{n+1}$. The embedding is assumed C^1 up to the boundary of the standard simplex.

The boundary of a regular n-simplex is a union of $n + 1$ different regular $(n - 1)$-simplices. The boundary is C^1-smooth except where the $(n - 1)$-simplices intersect. When discussing interiors and boundaries of simplices, we refer to the natural structure of simplices, not to the toplogy of the underlying manifold. Every regular n-simplex is a manifold with corners in the sense of [3].

Definition 2.2 (Depth of a point in a regular simplex)**.** We associate to each point in a regular simplex an integer which we call the depth of the point. Interior points have depth 0, the interiors of the regular $(n-1)$-simplices making up the boundary have depth 1, the interiors of their boundary simplices of dimension $n-2$ have depth 2, and so on. Finally the $n+1$ corner points have depth n.

Definition 2.3 (Regular tiling)**.** Let M be a manifold with or without boundary, with a C^1-structure. A regular tiling of M is a collection of regular n-simplices Δ_i, $i \in I$, so that the following hold:

(1) The collection is locally finite: for any compact set $K \subset M$ the index set $\{i \in I; \Delta_i \cap K \neq \emptyset\}$ is finite. (Consequently, if M is compact, the index set I itself is necessarily finite.)
(2) $M = \bigcup_{i \in I} \Delta_i$.
(3) $\operatorname{int}(\Delta_i) \cap \operatorname{int}(\Delta_j) = \emptyset$ when $i \neq j$.
(4) If $x \in \Delta_i \cap \Delta_j$, then x has the same depth in both Δ_i and Δ_j.

The simplices of a tiling have boundary simplices, the boundary simplices have boundary simplices, and so forth. We refer to all these as the boundary simplices of the tiling.

2.2. **Tangent cones of simplices.** We will need tangent spaces of simplices, and at corners these are more naturally conical subsets than vector subspaces of the tangent spaces of the underlying manifold.

Definition 2.4 (Tangent cone and tangent space of a regular simplex)**.** Let M be a manifold with or without boundary, with a C^1-structure. Consider a regular m-simplex Δ in M with $0 \leq m \leq n = \dim(M)$. Let $x \in \Delta$, and let $\Gamma = \Gamma(x, \Delta)$ be the set of all C^1-curves starting at x and staying in Δ. The tangent cone of Δ at x, denoted by $C_x\Delta$, is the set

$$\{\dot{\gamma}(0); \gamma \in \Gamma\} \subset T_xM. \tag{1}$$

The tangent space of Δ at x, denoted by $T_x\Delta$, is the vector space spanned by $C_x\Delta \subset T_xM$.

One can easily verify that for any m-dimensional regular simplex Δ the tangent cone $C_x\Delta$ is indeed a closed subset of T_xM and $T_x\Delta$ is the tangent space in the usual sense. The cone is scaling invariant but it need not be convex. If x is an interior point of Δ, then $C_x\Delta = T_x\Delta$ and they coincide with T_xM if $m = n$.

2.3. **Piecewise constant functions.** We are now ready to give a definition of piecewise constant functions and their tangent functions.

Definition 2.5 (Piecewise constant function)**.** Let M be a manifold with or without boundary, with a C^1-structure. We say that a function $f\colon M \to \mathbb{R}$ is piecewise constant if there is a regular tiling $\{\Delta_i; i \in I\}$

of M so that f is constant in the interior of each regular n-simplex Δ_i and vanishes on $\bigcup_{i\in I} \partial\Delta_i$.

The assumption of vanishing on the union of lower dimensional submanifolds is not important; we choose it for convenience. The values of the function in this small set play no role in our results.

In dimension two one can essentially equivalently define piecewise constant functions via tilings by curvilinear polygons. The only difference is in values on lower dimensional manifolds.

Definition 2.6 (Tangent function)**.** Let M be a manifold with or without boundary, with a C^1-structure. Let f be a piecewise constant function on it and x any point in M. Let $\Delta_1, \dots, \Delta_N$ be the simplices of the tiling that contain x. Denote by $a_1, \dots, a_N$ the constant values of f in the interiors of these simplices. The tangent function $T_x f\colon T_xM \to \mathbb{R}$ of f at x is defined so that for each $i \in \{1, \dots, N\}$ the function $T_x f$ takes the constant value a_i in the interior of the tangent cone $C_x\Delta_i$ (see definition 2.4). The tangent function takes the value zero outside $\bigcup_{i=1}^N \operatorname{int}(C_x\Delta_i)$.

If x is an interior point of a regular simplex in the tiling (it has depth zero in the sense of definition 2.2), then the tangent function is a constant function with the constant value of the ambient simplex.

Remark 2.7. The set of all piecewise constant functions $M \to \mathbb{R}$ is not a vector space, since the intersection of two regular simplices is not always a union of regular simplices. Thus, if f_1 and f_2 are piecewise constant functions that have the same integrals over geodesics, then theorem 1.1 shows that $f_1 = f_2$ in M whenever f_1 and f_2 are adapted to a common regular tiling but not in general.

2.4. **Foliations.**

For foliations we use the definition given in [7]:

Definition 2.8 (Strictly convex foliation)**.** Let M be a smooth Riemannian manifold with boundary.

(1) The manifold M satisfies the foliation condition if there is a smooth strictly convex function $\varphi\colon M \to \mathbb{R}$.
(2) A connected open subset U of M satisfies the foliation condition if there is a smooth strictly convex exhaustion function $\varphi\colon U \to \mathbb{R}$, in the sense that the set $\{x \in U; \varphi(x) \geq c\}$ is compact for any $c > \inf_U \varphi$.

If (2) is satisfied, then $U \cap \partial M \neq \emptyset$, the level sets of φ provide a foliation of U by smooth strictly convex hypersurfaces (except possibly at the minimum point of φ if $U = M$), and the fact that φ is an exhaustion function ensures that one can iterate a local uniqueness result to obtain uniqueness in all of U by a layer stripping argument.

Furthermore, since the set $\{x \in U; \varphi(x) \geq c\}$ is compact for any $c > \inf_U \varphi$, it follows that, intuitively speaking, the level sets $\{\varphi = c\}$

extend all the way from ∂M to ∂M without terminating at ∂U. For more details, see [7].

3. X-ray tomography of conical functions in the Euclidean plane

Let us denote the upper half plane by $H_+ = \{(x,y) \in \mathbb{R}; y > 0\}$. Let $\alpha_1 > \alpha_2 > \cdots > \alpha_N > \alpha_{N+1}$ and $a_1, \dots, a_N$ be any real numbers. Consider the function $f \colon H_+ \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} a_1, & \alpha_1 y > x > \alpha_2 y \\ a_2, & \alpha_2 y > x > \alpha_3 y \\ \vdots & \\ a_N, & \alpha_N y > x > \alpha_{N+1} y, \end{cases} \tag{2}$$

and $f(x,y) = 0$ for other $(x,y) \in H_+$. This is an example of a tangent function (see definition 2.6) of a piecewise constant function. The analysis of this archetypical example is crucial to the proof of our main result in all dimensions.

Fix some $h > 0$ and consider the lines ℓ_t given by $y = h + tx$.

Lemma 3.1. *Fix any $h > 0$, an integer $N \geq 1$ and real numbers $\alpha_1 > \alpha_2 > \cdots > \alpha_{N+1}$. Let $f \colon H_+ \to \mathbb{R}$ be as in* (2) *above. Then the numbers $a_1, \dots, a_N$ are uniquely determined by the integrals of f over the lines ℓ_t where t ranges in any neighborhood of zero.*

Proof. It suffices to show that if f integrates to zero over all these lines, then all the constants $a_1, \dots, a_N$ are zero.

The lines ℓ_t and $x = \alpha_i y$ intersect at a point whose first coordinate is $\frac{h\alpha_i}{1-\alpha_i t} =: hz_i^t$. The integral of f over the line ℓ_t is

$$\int_{\ell_t} f \mathrm{d}s = h\sqrt{1+t^2}[(z_1^t - z_2^t)a_1 + (z_2^t - z_3^t)a_2 + \dots + (z_N^t - z_{N+1}^t)a_N]. \tag{3}$$

Since this vanishes for all t near zero, we get

$$c_1^t a_1 + c_2^t a_2 + \cdots + c_N^t a_N = 0 \tag{4}$$

for all t, where $c_i^t = z_i^t - z_{i+1}^t$.

Let D_t^k denote the kth order derivative with respect to t. Differentiating with respect to t gives the system of equations

$$\begin{cases} c_1^t a_1 + \cdots + c_N^t a_N & = 0 \\ D_t c_1^t a_1 + \cdots + D_t c_N^t a_N & = 0 \\ \frac{1}{2} D_t^2 c_1^t a_1 + \cdots + \frac{1}{2} D_t^2 c_N^t a_N & = 0 \\ \vdots & \\ \frac{1}{(N-1)!} D_t^{N-1} c_1^t a_1 + \cdots + \frac{1}{(N-1)!} D_t^{N-1} c_N^t a_N & = 0. \end{cases} \tag{5}$$

We will show that this system is uniquely solvable for the numbers a_i at $t = 0$.

Since

$$z_i^t = \frac{\alpha_i}{1-\alpha_i t} = \alpha_i \left[1 + \alpha_i t + (\alpha_i t)^2 + (\alpha_i t)^3 + \dots\right], \tag{6}$$

we easily observe

$$\frac{1}{k!} D_t^k c_i^t \Big|_{t=0} = \alpha_i^{k+1} - \alpha_{i+1}^{k+1}. \tag{7}$$

Therefore our system of equations at $t = 0$ takes the form

$$A \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} = 0, \tag{8}$$

where

$$A = \begin{pmatrix} \alpha_1 - \alpha_2 & \alpha_2 - \alpha_3 & \cdots & \alpha_N - \alpha_{N+1} \\ \alpha_1^2 - \alpha_2^2 & \alpha_2^2 - \alpha_3^2 & \cdots & \alpha_N^2 - \alpha_{N+1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^N - \alpha_2^N & \alpha_2^N - \alpha_3^N & \cdots & \alpha_N^N - \alpha_{N+1}^N \end{pmatrix}. \tag{9}$$

We need to show now that the matrix A is invertible. This will be proven in lemma 3.2 below, and that concludes the proof of the present lemma. □

Lemma 3.2. *The determinant of the matrix A in (9) is*

$$(-1)^N \prod_{1 \le i < j \le N+1} (\alpha_j - \alpha_i), \tag{10}$$

which is non-zero if all numbers α_i are distinct.

Proof. We modify the matrix following these steps:

- Add the last column to the second last one, then the second last one to the third last one, and so forth. Finally add the second column to the first one. The determinant is not changed.
- Make the transformation

$$A \mapsto \begin{pmatrix} 0 & 1 \\ A & v \end{pmatrix}. \tag{11}$$

  The determinant is multiplied by $(-1)^N$. We can choose the vector v freely, and we pick $v = (\alpha_{N+1}, \alpha_{N+1}^2, \dots, \alpha_{N+1}^N)$.
- Add the last column to all other columns. The determinant is unchanged.

We end up with the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^N & \alpha_2^N & \cdots & \alpha_{N+1}^N \end{pmatrix}. \tag{12}$$

This is a Vandermonde matrix and the determinant is well known to be the product of differences. □

4. Limits of geodesics at corners in two dimensions

4.1. Geodesics on manifolds and tangent spaces. Let M be a C^2-smooth Riemannian surface with C^2-boundary. Suppose the boundary ∂M is strictly convex at $x \in \partial M$. Let γ_i, $i = 1, 2$, be two unit speed C^1-curves in M starting from x so that the initial velocities $\dot\gamma_i(0)$ are distinct and non-tangential.

Let $r > 0$ be such that $B(x, r) \subset M$ is split by the curves into three parts. Let A be the middle one.

Let σ_i, $i = 1, 2$, be the curves on T_xM with constant speed $\dot\gamma_i(0)$, respectively. Let S be the sector in T_xM lying between σ_1 and σ_2, corresponding to A.

If $h > 0$ and if $v \in T_xM$ is an inward pointing unit vector, let the geodesic γ_v^h be constructed as follows: Take a unit vector w normal to v at x and let $w(h)$ be its parallel transport along the geodesic through v by distance h. Then take the maximal geodesic in this direction. For sufficiently small h and v sufficiently close to normal the geodesic γ_v^h has endpoints near x since the boundary is strictly convex.

Moreover, let σ_v^h be the similarly constructed line on T_xM. Varying h translates the line and varying v rotates it.

4.2. Limits in two dimensions. We are now ready to present our key lemma 4.1 which allows us to reduce the problem into a Euclidean one.

Lemma 4.1. *Let M be a C^2-smooth Riemannian surface with C^2-boundary, and assume the boundary is strictly convex at $x \in \partial M$. For an open set of vectors v in a neighborhood of the inward unit normal, we have*

$$\lim_{h\to 0} \frac{1}{h} \int_{\gamma_v^h \cap A} \mathrm{d}s = \int_{\sigma_v^1 \cap S} \mathrm{d}s, \tag{13}$$

where we have used the notation of section 4.1.

Notice that $\frac{1}{h} \int_{\sigma_v^h \cap S} \mathrm{d}s$ is independent of $h > 0$ by scaling invariance.

Proof of lemma 4.1. We will prove the claim for $v = \nu$, the inward pointing unit normal to the boundary. The same argument will work if v is sufficiently close to normal (so that neither $\dot\gamma_i(0)$ is parallel to w).

Since the statement is local we can assume that we are working in $\mathbb{R}^2$ near the boundary point 0, and the metric g is extended smoothly outside M. We wish to express the left hand side of (13) in terms of the intersection points of γ_i with γ_v^h. To do this, consider the map $F\colon B \to \mathbb{R}^{2+2}$,

$$F(h,s) = (\gamma_1(s_1) - \gamma_v^h(s_3), \gamma_2(s_2) - \gamma_v^h(s_4)), \tag{14}$$

where $B \subset \mathbb{R}^{1+4}$ is a small neighborhood of the origin and $s = (s_1, s_2, s_3, s_4)$. The map F is C^1 and satisfies $F(0,0) = 0$. The s-derivatives are given by

$$D_s F(0,0) = \begin{pmatrix} \dot\gamma_1(0) & 0 & -w & 0 \\ 0 & \dot\gamma_2(0) & 0 & -w \end{pmatrix} \tag{15}$$

and this matrix is invertible since $\dot\gamma_i(0)$ are nontangential. By the implicit function theorem, there is a C^1-function $s(h)$ near 0 so that

$$F(h,s) = 0 \text{ for } (h,s) \text{ near } 0 \quad \iff \quad s = s(h). \tag{16}$$

We may now express the quantity of interest as

$$\lim_{h\to 0} \frac{1}{h} \int_{\gamma_v^h \cap A} \mathrm{d}s = \lim_{h \to 0} \frac{s_4(h) - s_3(h)}{h} = s_4'(0) - s_3'(0). \tag{17}$$

It remains to compute $s_3'(0)$ and $s_4'(0)$. Differentiating the equation $F(h, s(h)) = 0$ with respect to h gives that

$$s'(0) = -D_s F(0,0)^{-1} \partial_h F(0,0). \tag{18}$$

Since $\gamma_v^h(0) = \eta(h)$ where $\eta(h)$ is the geodesic through v we have $\partial_h \gamma_v^h(0)|_{h=0} = v$ and $\partial_h F(0,0) = (-v,-v)$. Then by direct computations

$$s'(0) = \begin{pmatrix} \frac{w^2v^1 - w^1v^2}{\dot\gamma_1^1(0)w^2 - \dot\gamma_1^2(0)w^1} \\ \frac{w^2v^1 - w^1v^2}{\dot\gamma_2^1(0)w^2 - \dot\gamma_2^2(0)w^1} \\ \frac{\dot\gamma_1^2(0)v^1 - \dot\gamma_1^1(0)v^2}{\dot\gamma_1^1(0)w^2 - \dot\gamma_1^2(0)w^1} \\ \frac{\dot\gamma_2^2(0)v^1 - \dot\gamma_2^1(0)v^2}{\dot\gamma_2^1(0)w^2 - \dot\gamma_2^2(0)w^1} \end{pmatrix}, \tag{19}$$

where $v = (v^1, v^2)$, $w = (w^1, w^2)$, and $\dot\gamma_i(0) = (\dot\gamma_i^1(0), \dot\gamma_i^2(0))$.

We assume that ν is between $\dot\gamma_1(0)$ and $\dot\gamma_2(0)$ (similar reasoning works also in the other cases). Notice that

$$s_3'(0) = \frac{-\sin(\angle(\dot\gamma_1(0), v))}{\sin(\angle(\dot\gamma_1(0), w))} \quad \text{and} \quad s_4'(0) = \frac{-\sin(\angle(\dot\gamma_2(0), v))}{\sin(\angle(\dot\gamma_2(0), w))}. \tag{20}$$

Next we calculate $\int_{\sigma_v^1 \cap S} \mathrm{d}s$ for $v = \nu$. We denote by $z_i \in T_xM$ the intersection point of the curves σ_i and σ_ν^1. Since ν is between $\dot\sigma_1(0) = \dot\gamma_1(0)$ and $\dot\sigma_2(0) = \dot\gamma_2(0)$, we find $|z_1 - \nu| = -s_3'(0)$ and $|\nu - z_2| = s_4'(0)$. Thus

$$\int_{\sigma_v^1 \cap S} \mathrm{d}s = |z_1 - \nu| + |\nu - z_2| = s_4'(0) - s_3'(0). \tag{21}$$

$\square$

We can extend lemma 4.1 to piecewise constant functions and their tangent functions.

Lemma 4.2. *Let M be a C^2-smooth Riemannian surface with C^2-boundary, and assume the boundary is strictly convex at $x \in \partial M$. Let $\tilde{M} \supset M$ be an extension so that x is an interior point of $\tilde{M}$. Let $f\colon \tilde{M} \to \mathbb{R}$ be a piecewise constant function and assume that $T_x f$ is supported in an inward-pointing cone which meets $T_x \partial M$ only at $0 \in T_x M$.*

For all vectors v in some neighborhood of the inward unit normal $\nu \in T_x M$, we have

$$\lim_{h \to 0} \frac{1}{h} \int_{\gamma_v^h} f \mathrm{d}s = \int_{\sigma_v^1} T_x f \mathrm{d}s, \tag{22}$$

where we have used the notation of section 4.1 and definition 2.6. Here the geodesics γ_v^h are geodesics of M and do not extend into $\tilde{M} \setminus M$.

Proof. It suffices to apply lemma 4.1 to each cone $C_x\Delta$ of a simplex Δ containing x separately. □

We remark that the extension $\tilde{M}$ only plays a role in the tiling, not in the geodesics.

5. X-ray tomography in two dimensions

5.1. **Local result.** Let us suppose that M is a Riemannian surface and x is a point in its interior. Suppose Σ is a hypersurface going through the point x and that Σ is strictly convex in a neighborhood of x. If V is a sufficiently small neighborhood of x then $V \setminus \Sigma$ consists of two open sets which we denote by V_+ and V_-. Here V_+ is the open set for which the part of the boundary coinciding with Σ is strictly convex.

Lemma 5.1. *Let M be a C^2-smooth Riemannian surface and assume that $f\colon M \to \mathbb{R}$ is a piecewise constant function in the sense of definition 2.5. Fix $x \in \operatorname{int}(M)$ and let Σ be a 1-dimensional hypersurface (curve) through x. Suppose that V is a neighborhood of x so that*

- *V intersects only simplices containing x,*
- *Σ is strictly convex in V,*
- *$f|_{V_-} = 0$, and*
- *f integrates to zero over every maximal geodesic in V having endpoints on Σ.*

Then $f|_V = 0$.

Remark 5.2. We will also use the lemma in the case where $x \in \partial M$, the boundary is strictly convex at x, and $\Sigma = \partial M$ (the set V_- is not needed then). The proof of the lemma is valid also in this situation.

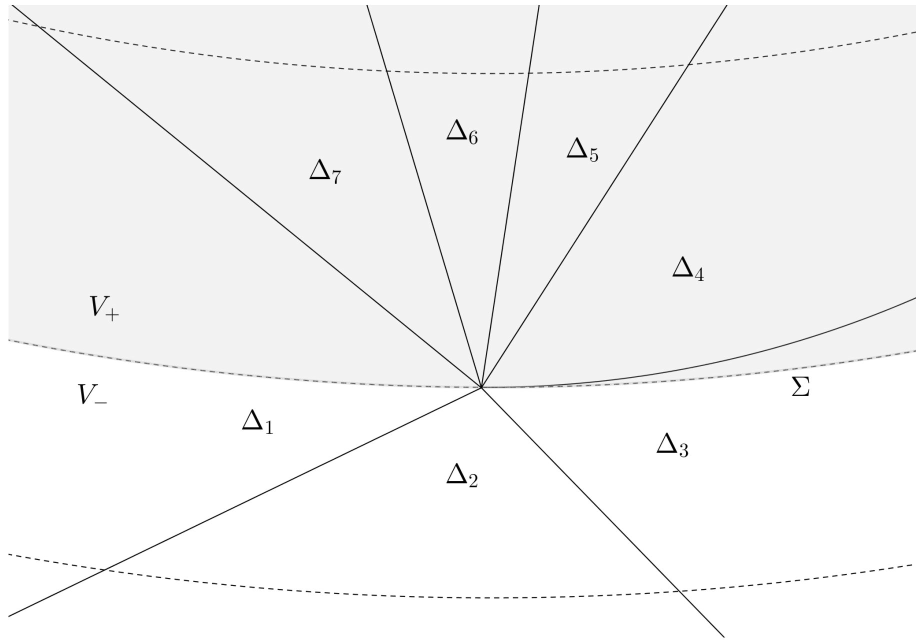

FIGURE 1. The geometric setting of lemma 5.1. Simplices $\Delta_1, \Delta_2, \Delta_3$ are of the first type, simplex Δ_4 is of the second type, and simplices $\Delta_5, \Delta_6, \Delta_7$ are of the third type. Later, this lemma will be applied in the case where Σ is the level of an exhaustion function, and the dashed lines represent level sets of this function.

Proof of lemma 5.1. We denote by $\Delta_1, \dots, \Delta_N$ the simplices containing the point x. The case $N = 1$ is trivial, so we suppose that $N > 1$. Assume that ν is the normal of Σ at x pointing into V_+. We denote $H_\pm = \{w \in T_xM; \pm\langle \nu, w\rangle > 0\}$ and $H_0 = T_x\Sigma \subset T_xM$.

We divide simplices $\Delta_1, \dots, \Delta_N$ into three mutually exclusive types as follows:

(1) simplices Δ so that $C_x\Delta \cap H_- \neq \emptyset$,
(2) simplices Δ so that $C_x\Delta \subset H_+ \cup H_0$ and $C_x\Delta \cap H_0 \neq \{0\}$, and
(3) simplices Δ so that $C_x\Delta \subset H_+ \cup \{0\}$.

Different types of simplices are illustrated in figure 1.

Let us first suppose that simplex Δ is of the first type. Since $C_x\Delta$ has a non-empty interior, the set $C_x\Delta \cap H_-$ has a non-empty interior. It must be that $\Delta \cap V_- \neq \emptyset$ and thus f vanishes on Δ.

Suppose then that Δ is a simplex of the second type. We take $v \in C_x\Delta \cap H_0$, with $v \neq 0$, and define γ_v^ε to be the geodesic with initial data $\gamma_v^\varepsilon(0) = x$ and $\dot\gamma_v^\varepsilon(0) = v + \varepsilon\nu$ where $\varepsilon > 0$ and ν points into V_+. Since Σ is strictly convex near x, we can take ε to be small enough so that the geodesic has endpoints on $\Sigma \cap V$ and it is contained in V.

For small positive t we have $\gamma_v^\varepsilon(t) \in \Delta$. If γ_v^ε is completely contained in Δ we immediately get that $f|_\Delta = 0$. If this is not the case then there is a unique simplex $\tilde{\Delta}$ so that $C_x\Delta \cap C_x\tilde{\Delta} = C_x\Delta \cap H_0 = C_x\tilde{\Delta} \cap H_0$, in other words simplices Δ and $\tilde{\Delta}$ have a common boundary simplex which is tangent to Σ at x. Thus for small ε we have $\gamma_v^\varepsilon \subset \Delta \cup \tilde{\Delta}$. Since tangent cones of simplices have non-empty interior it must be that $\tilde{\Delta} \cap V_- \neq \emptyset$, which implies that $f|_{\tilde{\Delta}} = 0$. By our assumptions f integrates to zero over γ_v^ε, hence $f|_\Delta = 0$.

We are left with simplices of the third type. For those we can apply lemma 4.2 combined with lemma 3.1: The geodesics γ_v^h introduced in section 4.1 are contained in V for small h and v close enough to normal. By lemma 4.2 integrals of $T_x f$ are zero over the lines σ_v^1 for v close enough to normal. Since f can be non-zero only in simplices of the third type we can apply lemma 3.1 to conclude that $T_x f = 0$ which implies that f vanishes on those simplices too. □

5.2. **Global result.** The local result of lemma 5.1 combined with the foliation condition allows us to obtain a global result:

Theorem 5.3. *Let M be a smooth Riemannian surface with strictly convex boundary. Suppose there is a strictly convex foliation of an open subset $U \subset M$ in the sense of definition 2.8. Let $f\colon M \to \mathbb{R}$ be a piecewise constant function in the sense of definition 2.5. If f integrates to zero over all geodesics in U, then $f|_U = 0$.*

Remark 5.4. In the previous result, it is not required to assume that ∂M is strictly convex (it would actually follow from the foliation condition that the foliation starts at a strictly convex boundary point). However, we have made this assumption for convenience.

Proof of theorem 5.3. Denote $T = \max_U \varphi$. The sets $U_t = \{\varphi \geq t\}$ are compact for every $t > \inf_U \varphi$ by assumption, and $\bigcup_{t>\inf_U \varphi} U_t = U$. It suffices to show that $f|_{U_t} = 0$ for any $t > \inf_U \varphi$.

Fix any such t. The set U_t meets only finitely many regular simplices $\Delta_1, \dots, \Delta_N$ of the tiling corresponding to f. Denote by $T_1 > T_2 > \cdots > T_k$ the distinct elements of the set $\{\max_{\Delta_i} \varphi; 1 \leq i \leq N\}$. Note that $T_1 = T$.

First, take any point $x \in U \cap \partial M$ for which $\varphi(x) = T$. By remark 5.2 the function f vanishes on a neighborhood of x. Therefore f vanishes on all simplices Δ_i for which $\max_{\Delta_i} \varphi = T = T_1$, and hence in the set $\{\varphi > T_2\}$.

We wish to continue this argument at points of the level set $\{\varphi = T_2\}$. We can apply lemma 5.1 at all points of $\{\varphi = T_2\}$ which are in $\operatorname{int}(M)$ to show that f vanishes near these points. This uses the fact that f will integrate to zero along short geodesics in $\{\varphi \leq T_2\}$ near such points, since the foliation condition implies that the maximal extensions of such short geodesics reach ∂M in finite time by [7, Lemma 6.1] and since f integrates to zero over geodesics in U. The points in $\{\varphi = T_2\}$ which are not in $\operatorname{int}(M)$ are handled similarly by using remark 5.2

(such points are on ∂M, since the set $\{\varphi \geq T_2\}$ cannot intersect the boundary of U except at the boundary of M). We find that f vanishes on all simplices Δ_i for which $\max_{\Delta_i} \varphi = T_2$. Continuing iteratively we reach the index k and conclude that f vanishes on all simplices $\Delta_1, \dots, \Delta_N$. □

We remark that the two-dimensional versions of the corollaries presented in section 6.3 follow from theorem 5.3.

6. Higher dimensions

6.1. **Local result.**

As in the two-dimensional case we begin by proving a local result. First we need a technical lemma.

Lemma 6.1. *Let M be an n-dimensional C^1-smooth Riemannian manifold with or without boundary and $\{\Delta_i; i \in I\}$ a regular tiling of it. Take $x \in M$ of depth at least 1 and Σ a hypersurface through it. Let $\Delta_1, \dots, \Delta_N$ be the simplices meeting x. For any $i \in \{1, \dots, N\}$ there is a 2-plane $P \subset T_x M$ so that the following hold:*

1. $P \cap \operatorname{int}(C_x \Delta_i) \neq \emptyset$,
2. *for any boundary simplex δ of dimension $n-2$ or lower in the tiling, we have $P \cap C_x \delta = \{0\}$, and*
3. $\dim(P \cap T_x \Sigma) = 1$.

Proof. We begin by picking a vector v from a small neighborhood of ν, where ν is any unit normal of Σ at x, so that $v \in \operatorname{int}(C_x \Delta_j)$ for some j. Let $\pi_v \colon T_x M \to T_x \Sigma$ be the projection down to $T_x \Sigma$ in direction of v, i.e., $\pi_v(av + z) = z$ whenever $a \in \mathbb{R}$ and $z \in T_x \Sigma$. Let δ be any regular m-simplex for $m \leq n-2$ contained in the tiling as a boundary simplex. The projection $\pi_v(T_x \delta)$ has dimension m (since $v \in \operatorname{int}(C_x \Delta_j)$) and therefore codimension $n - 1 - m \geq 1$ in $T_x \Sigma$. There are finitely many such simplices δ, so there is an open dense subset of vectors in $\pi_v(C_x \Delta_i)$ that do not belong to the π_v-projection of the tangent cone of any boundary simplex of dimension $n-2$ or lower. We pick a vector w from that set and take P to be the plane spanned by v and w. This P satisfies all the requirements. □

The next lemma is a higher dimensional analogue of lemma 5.1. Here V_+ and V_- are defined similarly as in the beginning of section 5.1.

Lemma 6.2. *Let M be a C^2-smooth Riemannian manifold and $f \colon M \to \mathbb{R}$ be a piecewise constant function in the sense of definition 2.5. Fix $x \in \operatorname{int}(M)$ and let Σ be a $(n-1)$-dimensional hypersurface through x. Suppose that V is a neighborhood of x so that*

- *V intersects only simplices containing x,*
- *Σ is strictly convex in V,*
- *$f|_{V_-} = 0$, and*

- *f integrates to zero over every maximal geodesic in V having endpoints on Σ.*

Then $f|_V = 0$.

Remark 6.3. As in the two-dimensional case this lemma holds also for points of the boundary with minor modifications. See remark 5.2.

Proof of lemma 6.2. We denote by $\Delta_1, \ldots, \Delta_N$ the simplices containing the point x. The case $N = 1$ is trivial, so we suppose that $N > 1$. We denote by H_+ and H_- the open upper half-plane and the open lower half-plane in T_xM corresponding to V_+ and V_-. Furthermore we denote $H_0 = T_xM \setminus (H_+ \cup H_-)$.

As in the two-dimensional case (lemma 5.1) we divide simplices $\Delta_1, \ldots, \Delta_N$ into three mutually exclusive types resembling those of the two-dimensional case:

(1) simplices Δ so that $C_x\Delta \cap H_- \neq \emptyset$,
(2) simplices Δ so that $C_x\Delta \subset H_+ \cup H_0$ and $\dim(C_x\Delta \cap H_0) = n-1$ (i.e. $C_x\Delta \cap H_0$ contains an open set of H_0), and
(3) simplices Δ so that $C_x\Delta \subset H_+ \cup H_0$ and $\dim(C_x\Delta \cap H_0) \leq n-2$.

The proof that f vanishes on simplices of first and second type is the same as in the two-dimensional case.

Let Δ be a simplex of the third type and P be a plane given by lemma 6.1 so that $P \cap \operatorname{int}(C_x\Delta) \neq \emptyset$. We define $P_+ = P \cap H_+$, and P_- and P_0 similarly.

We wish to show that $T_xf|_P = 0$. This implies that $f|_\Delta = 0$. Since Δ is arbitrary we can conclude that $T_xf = 0$ or, equivalently, $f|_V = 0$. To prove $T_xf|_P = 0$, we use a two-dimensional argument similar to that used in the proof of lemma 5.1. In order to do that we must first show that $T_xf|_P$ vanishes outside some closed conical set in P_+.

Suppose that $\tilde{\Delta}$ is a simplex so that $C_x\tilde{\Delta} \cap (P_0 \cup P_-) \neq \{0\}$. We want to show that it is of the first or second type. If $C_x\tilde{\Delta} \cap H_- \neq \emptyset$, meaning that the simplex is of the first type, we have $f|_{\tilde{\Delta}} = 0$. If this is not the case, then $C_x\tilde{\Delta} \subset H_+ \cup H_0$ and $C_x\tilde{\Delta} \cap P_0 \neq \{0\}$. Furthermore we know that for any boundary simplex δ of $\tilde{\Delta}$ for which $\dim(\delta) \leq n-2$ we have $P \cap \delta = \{0\}$. Thus every vector in $P_0 \cap C_x\tilde{\Delta}$ must be in the interior of a tangent cone of some $(n-1)$-dimensional boundary simplex. Therefore $\dim(C_x\tilde{\Delta} \cap H_0) = n-1$, so the simplex $\tilde{\Delta}$ is of the second type and hence $f|_{\tilde{\Delta}} = 0$.

The remaining case is where $C_x\tilde{\Delta} \subset P_+ \cup \{0\}$. To deal with this case, we will apply lemma 4.2 on suitably chosen submanifolds. We choose $w_0 \in T_x\Sigma \cap P = P_0$ and $v_0 \in P_+$ orthogonal to w_0. Then P is spanned by v_0 and w_0. Suppose $v \in P$ is in a neighborhood of v_0 and $w \in P$ is perpendicular to v. We discussed the geodesics γ_v^h on M and σ_v^h on T_xM in section 4.1. In two dimensions they did not depend on the choice of w (except for sign). In higher dimensions they do, and we will denote the corresponding curves in $V_+ \cup \Sigma$ by $\gamma^h_{v,w}$ and the tangent space curves by $\sigma^h_{v,w}$ instead.

The family of geodesics $\{\gamma^h_{v,w}; h \in [0,h_0)\}$ foliates a smooth two-dimensional manifold $M_{v,w} \subset V_+ \cup \Sigma$, for a small enough $h_0 > 0$. The geodesics $\gamma^h_{v,w}$ are also geodesics on the manifold $M_{v,w}$ although the submanifold may not be totally geodesic. Note that if M_{v_0,w_0} happens to be totally geodesic, which is always the case when $n = 2$, then $M_{v,w} = M_{v_0,w_0}$ for all such v and w.

The foliated manifold $M_{v,w}$ has four essential properties: its boundary near x (which is a subset of Σ) is strictly convex, we have $T_x M_{v,w} = P$, the vector v_0 is the inward pointing boundary normal at x, and for small enough h_0 the tiling of M induces a proper tiling for $M_{v,w}$ near x. To see this, observe that the plane P meets all $(n-1)$-dimensional boundary simplices transversally and does not meet lower dimensional simplices outside the origin, so the surface $M_{v,w}$ will locally do the same due to the implicit function theorem.

We construct $\sigma^h_{v,w}$ on P as in section 4.1. Now we can apply lemma 4.2 on manifold $M_{v,w}$ to get $\int_{\sigma^1_{v,w}} T_x f|_P \, \mathrm{d}s = 0$. By varying v and w, and hence varying $M_{v,w}$ too, we are able to reduce the problem to a Euclidean one on the tangent space P. We can apply lemma 3.1 to deduce that $T_x f|_P = 0$. Especially $f|_\Delta = 0$. □

6.2. **The key theorem.** We next present our key theorem in all dimensions. Theorem 6.4 contains theorem 5.3.

Theorem 6.4. *Let M be a smooth Riemannian manifold with strictly convex boundary. Assume* $\dim(M) \geq 2$. *Suppose there is a strictly convex foliation of an open subset $U \subset M$ in the sense of definition 2.8. Let $f\colon M \to \mathbb{R}$ be a piecewise constant function in the sense of definition 2.5. If f integrates to zero over all geodesics in U, then $f|_U = 0$.*

Proof. The proof is very similar to the two-dimensional case given in theorem 5.3. The analogue of lemma 5.1 for higher dimensions is provided by lemma 6.2. The rest of the proof is unchanged. □

6.3. **Corollaries.** Let us discuss some consequences of theorem 6.4.

The theorem can be seen as a support theorem. In particular, the classical support theorem of Helgason [2] for the X-ray transform in the case of piecewise constant functions follows easily from our theorem.

Corollaries 6.5 and 6.6 are easy to prove from theorem 6.4.

Corollary 6.5. *Let M be a smooth Riemannian manifold with strictly convex boundary. Let* $\dim(M) \geq 2$. *Suppose open subsets $U_1, \dots, U_N \subset M$ each have a foliation in the sense of definition 2.8. If a piecewise constant function $f\colon M \to \mathbb{R}$ has zero X-ray transform, then it vanishes in $\bigcup_{i=1}^N U_i$.*

In particular, in the case $N = 1$ and $U_1 = M$ we find:

Corollary 6.6. *Let M be a smooth Riemannian manifold with strictly convex boundary and with a strictly convex foliation. Assume* $\dim(M) \geq 2$. *If a piecewise constant function $f\colon M \to \mathbb{R}$ has zero X-ray transform, then it vanishes everywhere.*

If we do not assume the foliation condition, but instead just assume the boundary to be strictly convex locally, lemma 6.2 implies the following result, which can be seen as a local support theorem:

Corollary 6.7. *Let M be a C^2-smooth Riemannian manifold with boundary. Assume* $\dim(M) \geq 2$ *and let $f\colon M \to \mathbb{R}$ be a piecewise constant function. Suppose that $x \in \partial M$ is such that the boundary of M is strictly convex at x. If f integrates to zero over geodesics having endpoints near x, then f vanishes in a neighborhood of x.*

Especially if the boundary of M is strictly convex and f integrates to zero over all geodesic contained in a neighborhood of the boundary, then f vanishes near the boundary.

If $\dim(M) \geq 3$ the previous result is a special case of the local support theorem for L^2-functions proven in [9]. For $\dim(M) = 2$, a similar support theorem is known on real-analytic simple surfaces [4].

6.4. **Proof of theorem 1.1.** Part (a) follows from corollary 6.6 and the fact that a compact nontrapping Riemannian surface with strictly convex boundary always has a strictly convex foliation, see [1, 7]. Part (b) is implied by corollary 6.6. The theorem is proven.

References

[1] S. Betelú, R. Gulliver, and W. Littman. Boundary control of PDEs via curvature flows: the view from the boundary. II. *Appl. Math. Optim.*, 46(2-3):167–178, 2002. Special issue dedicated to the memory of Jacques-Louis Lions.

[2] S. Helgason. *The Radon transform*, volume 5 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 1999.

[3] D. Joyce. On manifolds with corners. In *Advances in geometric analysis*, volume 21 of *Adv. Lect. Math. (ALM)*, pages 225–258. Int. Press, Somerville, MA, 2012.

[4] V. P. Krishnan. A Support Theorem for the Geodesic Ray Transform on Functions. *Journal of Fourier Analysis and Applications*, 15:515–520, 2009.

[5] G. P. Paternain, M. Salo, and G. Uhlmann. Tensor tomography on surfaces. *Invent. Math.*, 193(1):229–247, 2013.

[6] G. P. Paternain, M. Salo, and G. Uhlmann. Tensor tomography: progress and challenges. *Chin. Ann. Math. Ser. B*, 35(3):399–428, 2014.

[7] G. P. Paternain, M. Salo, G. Uhlmann, and H. Zhou. The geodesic X-ray transform with matrix weights. May 2016. Preprint, arXiv: 1605.07894.

[8] G. Uhlmann. Inverse problems: seeing the unseen. *Bull. Math. Sci.*, 4(2):209–279, 2014.

[9] G. Uhlmann and A. Vasy. The inverse problem for the local geodesic ray transform. *Invent. Math.*, 205(1):83–120, 2016.

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TENSOR TOMOGRAPHY ON CARTAN-HADAMARD MANIFOLDS

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TENSOR TOMOGRAPHY ON CARTAN-HADAMARD MANIFOLDS

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Abstract. We study the geodesic X-ray transform on Cartan-Hadamard manifolds, generalizing the X-ray transforms on Euclidean and hyperbolic spaces that arise in medical and seismic imaging. We prove solenoidal injectivity of this transform acting on functions and tensor fields of any order. The functions are assumed to be exponentially decaying if the sectional curvature is bounded, and polynomially decaying if the sectional curvature decays at infinity. This work extends the results of [Leh16] to dimensions $n \geq 3$ and to the case of tensor fields of any order.

1. Introduction

1.1. **Motivation.** This article considers the geodesic X-ray transform on noncompact Riemannian manifolds. This transform encodes the integrals of a function f, where f satisfies suitable decay conditions at infinity, over all geodesics. In the case of Euclidean space the geodesic X-ray transform is just the usual X-ray transform involving integrals over all lines, and in two dimensions it coincides with the Radon transform introduced in the seminal work of Radon in 1917 [Rad17].

X-ray and Radon type transforms in Euclidean space are widely used as mathematical models for medical and industrial imaging methods, such as CT, PET, SPECT and MRI (see [Nat01]). In these applications one is interested in reconstructing unknown coefficients in a bounded region. However, it is often convenient to model the problems in terms of compactly supported functions in the noncompact space $\mathbb{R}^n$, which makes it possible to use Fourier transform based methods for instance.

Another important class of imaging problems arises in geophysics, when determining interior properties of the Earth from acoustic scattering or earthquake measurements. In these problems one encounters X-ray transforms over general families of curves, which often correspond to geodesic curves of a sound speed profile within the Earth. Moreover, if the sound speed is anisotropic (depends on direction), then one needs to consider geodesic X-ray transforms of tensor fields [Sha94]. A typical feature is that rays originating near the Earth surface eventually curve back to the surface. A simple mathematical model, which has been used as a first approximation for this behaviour, is to think of the domain as embedded in hyperbolic space $\mathbb{H}^n$ and to consider the geodesic X-ray transform in $\mathbb{H}^n$ [Bal05]. The hyperbolic geodesic X-ray transform also appears in Electrical Impedance Tomography in connection with the method of Barber and Brown [BCT96] and in partial data problems [KS14].

Another setting where X-ray transforms on noncompact manifolds appear is inverse scattering theory (for instance in quantum mechanics, acoustics, or electromagnetics). The connection between scattering theory and Radon type transforms goes back at least to Lax and Phillips [LP89], and the X-ray transform of a scattering potential can be determined from measurements of the full scattering amplitude at high frequencies (see e.g. [Wed14]). The X-ray transforms that appear in these contexts are often Euclidean. However, in inverse scattering applications related to general relativity and black holes one encounters more general manifolds that resemble asymptotically hyperbolic ones [JSB00], and in recent results on phaseless inverse scattering problems more general geodesic X-ray transforms also arise (see [Kli17] and references therein). We remark that both in quantum mechanics and general relativity, the functions that one would like to reconstruct are often not compactly supported and thus it is important to deal with noncompact manifolds.

In this article we will study the invertibility of geodesic X-ray transforms on noncompact Riemannian manifolds. Our results will include Euclidean and hyperbolic space as special cases, but will apply to more general manifolds with nonpositive curvature (Cartan-Hadamard manifolds). This work also follows the long tradition of integral geometry problems as discussed for instance in [GGG03, Hel99, Hel13]. Here one of the main points is that our results apply to manifolds that do not need to have special symmetries.

1.2. **Results.** For Euclidean or hyperbolic space in dimensions $n \geq 2$, one has the following basic theorems on the injectivity of this transform (see [Hel99], [Jen04], [Hel94]):

Theorem A. *If f is a continuous function in $\mathbb{R}^n$ satisfying $|f(x)| \leq C(1+|x|)^{-\eta}$ for some $\eta > 1$, and if f integrates to zero over all lines in $\mathbb{R}^n$, then $f \equiv 0$.*

Theorem B. *If f is a continuous function in the hyperbolic space $\mathbb{H}^n$ satisfying $|f(x)| \leq C e^{-d(x,o)}$, where $o \in \mathbb{H}^n$ is some fixed point, and if f integrates to zero over all geodesics in $\mathbb{H}^n$, then $f \equiv 0$.*

We remark that some decay conditions for the function f are required, since there are examples of nontrivial functions in $\mathbb{R}^2$ which decay like $|x|^{-2}$ on every line and whose X-ray transform vanishes [Zal82], [Arm94]. Related results on the invertibility of Radon type transforms on constant curvature spaces or noncompact homogeneous spaces may be found in [Hel99], [Hel13].

The purpose of this article is to give analogues of the above theorems on more general, not necessarily symmetric Riemannian manifolds. We will work in the setting of Cartan-Hadamard manifolds, i.e. complete simply connected Riemannian manifolds with nonpositive sectional curvature. Euclidean and hyperbolic spaces are special cases of Cartan-Hadamard manifolds, and further explicit examples are recalled in Section 2. It is well known that any Cartan-Hadamard manifold is diffeomorphic to $\mathbb{R}^n$, the exponential map at any point is a diffeomorphism, and the map $x \mapsto d(x,p)^2$ is strictly convex for any $p \in M$ (see e.g. [Pet06]).

Definition. Let (M,g) be a Cartan-Hadamard manifold, and fix a point $o \in M$. If $\eta > 0$, define the spaces of exponentially and polynomially decaying continuous functions by

$$E_\eta(M) = \{f \in C(M)\,;\, |f(x)| \leq C e^{-\eta d(x,o)} \text{ for some } C > 0\},$$
$$P_\eta(M) = \{f \in C(M)\,;\, |f(x)| \leq C(1+d(x,o))^{-\eta} \text{ for some } C > 0\}.$$

Also define the spaces

$$E^1_\eta(M) = \{f \in C^1(M)\,;\, |f(x)| + |\nabla f(x)| \leq C e^{-\eta d(x,o)} \text{ for some } C > 0\},$$
$$P^1_\eta(M) = \{f \in C^1(M)\,;\, |f(x)| \leq C(1+d(x,o))^{-\eta} \text{ and } |\nabla f(x)| \leq C(1+d(x,o))^{-\eta-1} \text{ for some } C > 0\}.$$

Here $\nabla = \nabla_g$ is the total covariant derivative in (M,g) and $|\cdot| = |\cdot|_g$ is the g-norm on tensors.

It follows from Lemma 4.1 that if $f \in P_\eta(M)$ for some $\eta > 1$, then the integral of f over any maximal geodesic in M is finite. For such functions f we may define the geodesic X-ray transform $I_0 f$ of f by

$$I_0 f(\gamma) = \int_{-\infty}^{\infty} f(\gamma(t))\,dt, \qquad \gamma \text{ is a geodesic.}$$

The inverse problem for the geodesic X-ray transform is to determine f from the knowledge of $I_0 f$. By linearity, uniqueness for this inverse problem reduces to showing that $I_0 f = 0$ implies $f = 0$.

More generally, suppose that f is a C^1-smooth symmetric covariant m-tensor field on M, written in local coordinates (using the Einstein summation convention) as

$$f = f_{j_1 \cdots j_m}(x)\, dx^{j_1} \otimes \cdots \otimes dx^{j_m}.$$

We say that $f \in P_\eta(M)$ if $|f|_g \in P_\eta(M)$, and $f \in P^1_\eta(M)$ if $|f|_g \in P_\eta(M)$ and $|\nabla f|_g \in P_{\eta+1}(M)$, etc. Now if $f \in P_\eta(M)$ for some $\eta > 1$, then the geodesic X-ray transform $I_m f$ of f is well defined by the formula

$$I_m f(\gamma) = \int_{-\infty}^{\infty} f_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t))\, dt, \qquad \gamma \text{ is a geodesic.}$$

This transform always has a kernel when $m \geq 1$: if h is a symmetric $(m-1)$-tensor field satisfying $h \in P^1_\eta(M)$ for some $\eta > 0$, then $I_m(\sigma\nabla h) = 0$ where σ denotes symmetrization of a tensor field (see Section 3.3). We say that I_m is solenoidal injective if $I_m f = 0$ implies $f = \sigma\nabla h$ for some $(m-1)$-tensor field h.

Our first theorem proves solenoidal injectivity of I_m for any $m \geq 0$ on Cartan-Hadamard manifolds with bounded sectional curvature, assuming exponential decay of the tensor field and its first derivatives. We will denote the sectional curvature of a two-plane $\Pi \subset T_xM$ by $K_x(\Pi)$, and we write $-K_0 \leq K \leq 0$ if $-K_0 \leq K_x(\Pi) \leq 0$ for all $x \in M$ and for all two-planes $\Pi \subset T_xM$.

Theorem 1.1. *Let (M, g) be a Cartan-Hadamard manifold of dimension $n \geq 2$, and assume that*

$$-K_0 \leq K \leq 0, \qquad \textit{for some } K_0 > 0.$$

If f is a symmetric m-tensor field in $E^1_\eta(M)$ for some $\eta > \frac{n+1}{2}\sqrt{K_0}$, and if $I_m f = 0$, then $f = \sigma\nabla h$ for some symmetric $(m-1)$-tensor field h. (If $m = 0$, then $f \equiv 0$.)

The second theorem considers the case where the sectional curvature decays polynomially at infinity, and proves solenoidal injectivity if the tensor field and its first derivatives also decay polynomially.

Theorem 1.2. *Let (M, g) be a Cartan-Hadamard manifold of dimension $n \geq 2$, and assume that the function*

$$\mathcal{K}(x) = \sup\{|K_x(\Pi)|\,;\, \Pi \subset T_xM \text{ is a two-plane}\}$$

satisfies $\mathcal{K} \in P_\kappa(M)$ for some $\kappa > 2$. If f is a symmetric m-tensor field in $P^1_\eta(M)$ for some $\eta > \frac{n+2}{2}$, and if $I_m f = 0$, then $f = \sigma\nabla h$ for some symmetric $(m-1)$-tensor field h. (If $m = 0$, then $f \equiv 0$.)

The second theorem is mostly of interest in two dimensions because of the following rigidity phenomenon: any manifold of dimension ≥ 3 that satisfies the conditions of the theorem is isometric to Euclidean space [GW82]. See Section 2 for a discussion. We will give the proof in any dimension since this may be useful in subsequent work.

We remark that Theorems 1.1–1.2 correspond to Theorems A and B above, but the manifolds considered in Theorems 1.1–1.2 can be much more general and include many examples with non-constant curvature (see Section 2). The results will be proved by using energy methods based on Pestov identities, which have been studied extensively in the case of compact manifolds with strictly convex boundary. We refer to [Muk77], [PS88], [Sha94], [Kni02], [PSU14] for some earlier results. In fact, Theorems 1.1–1.2 can be viewed as an extension of the tensor tomography results in [PS88] from the case of compact nonpositively curved manifolds with boundary to the case of certain noncompact manifolds. We remark that one of the main points in our theorems is that the functions and tensor fields are not compactly supported (indeed, the compactly supported case would reduce to known results on compact manifolds with boundary).

More recently, the work [PSU13] gave a particularly simple derivation of the basic Pestov identity for X-ray transforms and proved solenoidal injectivity of I_m on simple two-dimensional manifolds. Some of these methods were extended to all dimensions in [PSU15] and to the case of attenuated X-ray transforms in [GPSU16]. Following some ideas in [PSU13], the work [Leh16] proved versions of Theorems 1.1–1.2 for the case of two-dimensional Cartan-Hadamard manifolds.

In this paper we combine the main ideas in [Leh16] with the methods of [PSU15] and prove solenoidal injectivity results on Cartan-Hadamard manifolds in any dimension $n \geq 2$. However, instead of using the Pestov identity in its standard form (which requires two derivatives of the functions involved), we will use a different argument from [PSU15] related to the L^2 contraction property of a Beurling transform on nonpositively curved manifolds. This argument dates back to [GK80a, GK80b], it only involves first order derivatives and immediately applies to tensor fields of arbitrary order. The C^1 assumption in Theorems 1.1–1.2 is due to this method of proof, and the decay assumptions are related to the growth of Jacobi fields. We mention that Theorems 1.1–1.2 also extend the two-dimensional results of [Leh16] by assuming slightly weaker conditions.

This article is organized as follows. Section 1 is the introduction, and Section 2 contains examples of Cartan-Hadamard manifolds. In Section 3 we review basic facts related to geodesics on Cartan-Hadamard manifolds, geometry of the sphere bundle and symmetric covariant tensors fields, following [Leh16], [PSU15], [DS10]. Section 4 collects some estimates concerning the growth of Jacobi fields and related decay properties for solutions of transport equations. Finally, Section 5 includes the proofs of the main theorems based on L^2 inequalities for Fourier coefficients.

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2. Examples of Cartan-Hadamard manifolds

In this section we recall some facts and examples related to Cartan-Hadamard manifolds. Most of the details can be found in [BO69], [KW74], [GW79], [GW82], [Pet06]. We first discuss the case of two-dimensional manifolds, which is quite different compared to manifolds of higher dimensions.

2.1. **Dimension two.** Let $K \in C^\infty(\mathbb{R}^2)$. A theorem of Kazdan and Warner [KW74] states that a necessary and sufficient condition for existence of a complete Riemannian metric on $\mathbb{R}^2$ with Gaussian curvature K is

$$\lim_{r\to\infty} \inf_{|x|\geq r} K(x) \leq 0. \tag{2.1}$$

This provides a wide class of Riemannian metrics satisfying the assumptions of Theorem 1.1 in dimension two. However, this does not directly give an example of a manifold satisfying the assumptions of Theorem 1.2 since the condition (2.1) is given with respect to the Euclidean metric of $\mathbb{R}^2$.

Examples of manifolds satisfying the assumptions of Theorem 1.2 can be constructed using warped products. Let (r, θ) be the polar coordinates in $\mathbb{R}^2$ and consider a warped product

$$ds^2 = dr^2 + f^2(r)d\theta^2, \tag{2.2}$$

where f is a smooth function that is positive for $r > 0$ and satisfies $f(0) = 0$ and $f'(0) = 1$. This is a Riemannian metric on $\mathbb{R}^2$ having Gaussian curvature

$$K(x) = -\frac{f''(|x|)}{f(|x|)}, \tag{2.3}$$

which depends only on the Euclidean distance $|x| := r(x)$ to the origin. We remark that distances to the origin in the Euclidean metric and in the warped metric coincide. It is shown in [GW79, Proposition 4.2] that for every $k \in C^\infty([0,\infty))$ with $k \leq 0$ there exists a unique warped metric of the form (2.2) such that $k(|x|) = K(x)$. Hence warped products provide many examples of two-dimensional manifolds for which $\mathcal{K}(x) \leq C(1 + |x|)^{-\kappa}$ with $\kappa > 0$, i.e. $\mathcal{K} \in P_\kappa(M)$.

2.2. **Higher dimensions.** Warped products can also be used to construct examples of higher dimensional Cartan-Hadamard manifolds satisfying the assumptions of Theorem 1.1, see e.g. [BO69].

In the case of Theorem 1.2 it turns out that the decay condition for curvature is very restrictive in higher dimensions: the only possible geometry is the Euclidean one. This follows directly from a theorem by Greene and Wu in [GW82]. If M is a Cartan-Hadamard manifold with $n = \dim(M) \geq 3$, $k(s) = \sup\{\, \mathcal{K}(x)\,;\, x \in M, d(x,o) = s \,\}$, where o is a fixed point, and one of the following holds:

(1) n is odd and $\liminf_{s\to\infty} s^2 k(s) \to 0$ or
(2) n is even and $\int_0^\infty sk(s)\,\mathrm{d}s$ is finite,

then M is isometric to $\mathbb{R}^n$.

3. Geometric facts

Throughout this work we will assume (M, g) to be an n-dimensional Cartan-Hadamard manifold with $n \geq 2$ unless otherwise stated. We also assume unit speed parametrization for geodesics.

In this section we collect some preliminary facts on geodesics on Cartan-Hadamard manifolds, derivatives on the unit tangent bundle and related Jacobi fields, and tensor fields. These facts will be used in the subsequent sections.

3.1. **Behaviour of geodesics.** By the Cartan-Hadamard theorem the exponential map $\exp_x$ is defined on all of T_xM and is a diffeomorphism for every $x \in M$. Hence every pair of points can be joined by a unique geodesic. Let $SM = \{(x, v) \in TM\,;\, |v| = 1\}$ be the unit sphere bundle, and if $(x, v) \in SM$ denote by $\gamma_{x,v}$ the unique geodesic with $\gamma(0) = x$ and $\dot\gamma(0) = v$. The triangle inequality implies that

$$d_g(\gamma_{x,v}(t), o) \geq |t| - d_g(x, o) \tag{3.1}$$

for all $t \in \mathbb{R}, o \in M$.

We say that a geodesic γ is escaping with respect to the point o if the function $t \mapsto d_g(\gamma(t), o)$ is strictly increasing on the interval $[0, \infty)$. The set of all such geodesics is denoted by $\mathcal{E}_o$. For $\gamma_{x,v} \in \mathcal{E}_o$ the triangle inequality gives

$$d_g(\gamma_{x,v}(t), o) \geq \begin{cases} d_g(x, o), & \text{if } 0 \leq t \leq 2d_g(x, o), \\ t - d_g(x, o), & \text{if } 2d_g(x, o) < t. \end{cases} \tag{3.2}$$

However, since (M, g) is a Cartan-Hadamard manifold, Jacobi field estimates give a stronger bound. For $\gamma_{x,v} \in \mathcal{E}_o$ one has (see [Jos08, Corollary 4.8.5] or [Pet06, Section 6.3])

$$d_g(\gamma_{x,v}(t), o) \geq \sqrt{d_g(x, o)^2 + t^2}, \qquad t \geq 0. \tag{3.3}$$

The following lemma is proved in [Leh16] in two dimensions. The proof in higher dimensions is identical, but we include a short argument for completeness.

Lemma 3.1. *Suppose $o \in M$. At least one of the geodesics $\gamma_{x,v}$ and $\gamma_{x,-v}$ is in $\mathcal{E}_o$.*

Proof. Since (M, g) is a Cartan-Hadamard manifold, the function $h(t) = d_g(\gamma_{x,v}(t), o)^2$ is strictly convex, $h'' > 0$, on $\mathbb{R}$. If $h'(0) \geq 0$ then $\gamma_{x,v}$ is escaping, and if $h'(0) \leq 0$ then $\gamma_{x,-v}$ is escaping. □

3.2. **On the geometry of the unit tangent bundle.** We first briefly explain the splitting of the tangent bundle into horizontal and vertical bundles. Then we give a short discussion on geodesics of SM. Finally, we include a proof that SM is complete when M is.

3.2.1. *The structure of the tangent bundle.* The following discussion is based on [Pat99], [PSU15], where these topics are considered in more detail. We denote by $\pi\colon TM \to M$ the usual base point map $\pi(x,v) = x$. The connection map $K_\nabla\colon T(TM) \to TM$ of the Levi-Civita connection ∇ of M is defined as follows. Let $\xi \in T_{x,v}TM$ and $c\colon (-\varepsilon,\varepsilon) \to TM$ be a curve such that $\dot{c}(0) = \xi$. Write $c(t) = (\gamma(t), Z(t))$, where $Z(t)$ is a vector field along the curve γ, and define

$$K_\nabla(\xi) \coloneqq D_t Z(0) \in T_x M.$$

The maps K_∇ and $d\pi$ yield a splitting

$$T_{x,v}TM = \tilde{\mathcal{H}}(x,v) \oplus \tilde{\mathcal{V}}(x,v) \tag{3.4}$$

where $\tilde{\mathcal{H}}(x,v) = \ker K_\nabla$ is the horizontal bundle and $\tilde{\mathcal{V}}(x,v) = \ker d_{x,v}\pi$ is the vertical bundle. Both are n-dimensional subspaces of $T_{x,v}TM$.

On TM we define the Sasaki metric g_s by

$$\langle v, w\rangle_{g_s} = \langle K_\nabla(v), K_\nabla(w)\rangle_g + \langle d\pi(v), d\pi(w)\rangle_g,$$

which makes (TM, g_s) a Riemannian manifold of dimension $2n$. The maps $K_\nabla\colon \tilde{\mathcal{V}}(x,v) \to T_xM$ and $d\pi\colon \tilde{\mathcal{H}}(x,v) \to T_xM$ are linear isomorphisms. Furthermore, the splitting (3.4) is orthogonal with respect to g_s. Using the maps K_∇ and $d\pi$, we will identify vectors in the horizontal and vertical bundles with corresponding vectors on T_xM.

The unit sphere bundle SM was defined as

$$SM \coloneqq \bigcup_{x\in M} S_xM, \qquad S_xM \coloneqq \{(x,v) \in T_xM\,;\, |v|_g = 1\}.$$

We will equip SM with the metric induced by the Sasaki metric on TM. The geodesic flow $\phi_t(x,v)\colon \mathbb{R}\times SM \to SM$ is defined as

$$\phi_t(x,v) \coloneqq (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)).$$

The associated vector field is called the geodesic vector field and denoted by X.

For SM we obtain an orthogonal splitting

$$T_{x,v}SM = \mathbb{R}X(x,v) \oplus \mathcal{H}(x,v) \oplus \mathcal{V}(x,v) \tag{3.5}$$

where $\mathbb{R}X \oplus \mathcal{H}(x,v) = \tilde{\mathcal{H}}(x,v)$ and $\mathcal{V}(x,v) = \ker d_{x,v}(\pi|_{SM})$. Both $\mathcal{H}(x,v)$ and $\mathcal{V}(x,v)$ have dimension $n-1$ and can be canonically identified with elements in the codimension one subspace $\{v\}^\perp \subset T_xM$ via $d\pi$ and K_∇, respectively. We will freely use this identification.

Following [PSU15], if $u \in C^1(SM)$, then the gradient $\nabla_{SM}u$ has the decomposition

$$\nabla_{SM}u = (Xu)X + \overset{h}{\nabla}u + \overset{v}{\nabla}u,$$

according to (3.5). The quantities $\overset{h}{\nabla}u$ and $\overset{v}{\nabla}u$ are called the horizontal and the vertical gradients, respectively. It holds that $\langle\overset{v}{\nabla}u(x,v), v\rangle = 0$ and $\langle\overset{h}{\nabla}u(x,v), v\rangle = 0$ for all $(x,v) \in SM$.

As discussed in [PSU15], on two-dimensional manifolds the horizontal and vertical gradients reduce to the horizontal and vertical vector fields $X_\perp$ and V via

$$\overset{h}{\nabla}u(x,v) = -(X_\perp u(x,v))v^\perp \quad \text{and} \quad \overset{v}{\nabla}u(x,v) = (Vu(x,v))v^\perp$$

where $v^\perp$ is such that $\{v, v^\perp\}$ is a positive orthonormal basis of T_xM. In [Leh16] the flows associated with $X_\perp$ and V were used to derive estimates for $X_\perp u$ and Vu. We will proceed in a similar manner in the higher dimensional case.

Let $(x,v)\in SM$ and $w\in S_xM$, $w\perp v$. We define $\phi^h_{w,t}\colon \mathbb{R}\to SM$ by $\phi^h_{w,t}(x,v)=(\gamma_{x,w}(t),V(t))$, where $V(t)$ is the parallel transport of v along $\gamma_{x,w}$. It holds that

$$K_\nabla\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi^h_{w,t}(x,v)\Big|_{t=0}\right)=0 \quad\text{and}\quad d\pi\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi^h_{w,t}(x,v)\Big|_{t=0}\right)=w. \tag{3.6}$$

We define $\phi^v_{w,t}\colon \mathbb{R}\to SM$ by $\phi^v_{w,t}(x,v)=(x,(\cos t)v+(\sin t)w)$. It holds that

$$K_\nabla\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi^v_{w,t}(x,v)\Big|_{t=0}\right)=w \quad\text{and}\quad d\pi\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi^v_{w,t}(x,v)\Big|_{t=0}\right)=0. \tag{3.7}$$

The following lemma states the relation between $\phi^h_{w,t}$ and $\phi^v_{w,t}$ and the horizontal and the vertical gradients of a function.

Lemma 3.2. *Suppose u is differentiable at $(x,v)\in SM$. Fix $w\in S_xM, w\perp v$. Then it holds that*

$$\langle \overset{h}{\nabla} u(x,v),w\rangle=\frac{d}{dt}u(\phi^h_{w,t}(x,v))\Big|_{t=0}$$

and

$$\langle \overset{v}{\nabla} u(x,v),w\rangle=\frac{d}{dt}u(\phi^v_{w,t}(x,v))\Big|_{t=0}.$$

Proof. Using the chain rule and the equations (3.6) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}u(\phi^h_{w,t}(x,v))\Big|_{t=0}=\langle\nabla_{SM}u(\phi^h_{w,t}(x,v)),\frac{\mathrm{d}}{\mathrm{d}t}\phi^h_{w,t}(x,v)\rangle_{g_s}\Big|_{t=0}=\langle \overset{h}{\nabla} u(x,v),w\rangle.$$

For $\overset{v}{\nabla}$ we use the equations (3.7) in a similar fashion. □

The maps $\phi^h_{w,t}$ and $\phi^v_{w,t}$ are related to normal Jacobi fields along geodesics. We can define

$$J^h_w(t):=\frac{\mathrm{d}}{\mathrm{d}s}\pi\left(\phi_t(\phi^h_{w,s}(x,v))\right)\Big|_{s=0}=d_{\phi_t(x,v)}\pi\left(\frac{\mathrm{d}}{\mathrm{d}s}\phi_t(\phi^h_{w,s}(x,v))\Big|_{s=0}\right).$$

Since $\Gamma(s,t)=\pi\left(\phi_t(\phi^h_{w,s}(x,v))\right)$ is a variation of $\gamma_{x,v}$ along geodesics, $J^h_w(t)$ is a Jacobi field along $\gamma_{x,v}$. It has the initial conditions $J^h_w(0)=w$ and $D_tJ^h_w(0)=0$ by the symmetry lemma (see e.g. [Lee97]).

Replacing $\phi^h_{w,s}$ with $\phi^v_{w,s}$ gives a Jacobi field $J^v_w(t)$ with the initial conditions $J^v_w(t)(0)=0$ and $D_tJ^v_w(t)(0)=w$. In the both cases the Jacobi field is normal because $\langle v,w\rangle_g=0$.

By the symmetry lemma

$$K_\nabla\left(\frac{\mathrm{d}}{\mathrm{d}s}\phi_t(\phi^h_{w,s}(x,v))\Big|_{s=0}\right)=D_s\partial_t\gamma_{\phi^h_{w,s}(x,v)}(t)\Big|_{s=0}=D_t\partial_s\gamma_{\phi^h_{w,s}(x,v)}(t)\Big|_{s=0}=D_tJ^h_w(t).$$

From the definition of the Sasaki metric we then see that

$$\langle\nabla_{SM}u(x,v),\frac{\mathrm{d}}{\mathrm{d}s}\phi_t(\phi^h_{w,s}(x,v))\Big|_{s=0}\rangle_{g_s}=\langle \overset{h}{\nabla} u(x,v),J^h_w(t)\rangle_g+\langle \overset{v}{\nabla} u(x,v),D_tJ^h_w(t)\rangle_g.$$

and

$$\langle\nabla_{SM}u(x,v),\frac{\mathrm{d}}{\mathrm{d}s}\phi_t(\phi^v_{w,s}(x,v))\Big|_{s=0}\rangle_{g_s}=\langle \overset{h}{\nabla} u(x,v),J^v_w(t)\rangle_g+\langle \overset{v}{\nabla} u(x,v),D_tJ^v_w(t)\rangle_g.$$

Remark 1. The constructions in this subsection remain valid at a.e. $(x,v)\in SM$ if one assumes that u is in the space $W^{1,\infty}_{\mathrm{loc}}(SM)$. Functions in $W^{1,\infty}_{\mathrm{loc}}(SM)$ are characterized as locally Lipschitz functions, and further by Rademacher's theorem, differentiable almost everywhere and weak gradients equal to gradients almost everywhere (see e.g. [Eva98, Chapters 5.8.2–5.8.3]).

3.2.2. *Geodesics on the unit tangent bundle.* Next we describe some facts related to geodesics on SM (see e.g. [BBNV03] and references therein). Let $R(U,V)$ denote the Riemannian curvature tensor. A curve $\Gamma(t) = (x(t), V(t))$ on SM is a geodesic if and only if

$$\begin{cases} \nabla_{\dot{x}}\dot{x} = -R(V, \nabla_{\dot{x}}V)\dot{x} \\ \nabla_{\dot{x}}\nabla_{\dot{x}}V = -\left|\nabla_{\dot{x}}V\right|_g^2 V, \quad \left|\nabla_{\dot{x}}V\right|_g^2 \text{ is a constant along } x(t) \end{cases} \tag{3.8}$$

holds for every t in the domain of Γ (see [Sas62, Equations 5.2]). Given $(x,v) \in SM$, the horizontal lift of $w \in T_xM$ is denoted by w^h, i.e. the unique vector $w^h \in T_{x,v}(SM)$ such that $d(\pi|_{SM})(w^h) = w$ and $K_\nabla(w^h) = 0$, and the vertical lift w^v is defined similarly. Initial conditions for $x, \dot{x}, V$ and $\nabla_{\dot{x}}V$ at $t=0$ with $g(V(0), \nabla_{\dot{x}(0)}V(0)) = 0$ and $\left|V(0)\right|_g = 1$ determine a unique geodesic $\Gamma = (x,V)$, by (3.8), which satisfies the initial conditions $\Gamma(0) = (x(0), V(0))$ and $\dot{\Gamma}(0) = \dot{x}(0)^h + (\nabla_{\dot{x}(0)}V(0))^v$ where the lifts are done with respect to $(x(0), V(0)) \in SM$. The geodesics of SM are of the following three types:

(1) If $\nabla_{\dot{x}(0)}V(0) = 0$, then Γ is a parallel transport of $V(0)$ along the geodesic x on M (horizontal geodesics).
(2) If $\dot{x}(0) = 0$, then Γ is a great circle on the fibre $\pi^{-1}(x(0))$ and $x(t) = x(0)$ (vertical geodesics, in this case one interprets the system (3.8) via $\nabla_{\dot{x}} = D_t$).
(3) All the rest, i.e. solutions of (3.8) with initial conditions $\dot{x}(0) \neq 0$ and $\nabla_{\dot{x}(0)}V(0) \neq 0$ (oblique geodesics).

We state the following lemma for the sake of clarity.

Lemma 3.3. *Fix $(x,v) \in SM$ and $w \in S_xM$, $w \perp v$. Then $\phi_t(x,v)$ and $\phi^h_{w,t}(x,v)$ are horizontal unit speed geodesics and $\phi^v_{w,t}(x,v)$ is a vertical unit speed geodesic with respect to t.*

Proof. The fact that $\phi_t(x,v)$ and $\phi^h_{w,t}(x,v)$ are horizontal geodesics and $\phi^v_{w,t}(x,v)$ is a vertical geodesic follows immediately from their definitions and the above discussion based on the system of differential equations (3.8). The fact that $\phi_t(x,v)$, $\phi^h_{w,t}(x,v)$ and $\phi^v_{w,t}(x,v)$ are unit speed follows from the equations (3.6) and (3.7) and the definition of the Sasaki metric. □

Lemma 3.3 allows us to derive the following formulas which are used in the proof of Lemma 4.7.

Corollary 3.4. *Let $(x,v) \in SM$. Assume that $Y \in T_{x,v}(SM)$ has the decomposition*

$$Y = aX|_{x,v} + H + V, \quad H \in \mathcal{H}(x,v), V \in \mathcal{V}(x,v), a \in \mathbb{R}.$$

Then

$$\begin{aligned} (D\phi_t)_{x,v}(aX|_{x,v}) &= aX|_{\phi_t(x,v)}, \\ (D\phi_t)_{x,v}(H) &= |H|_{g_s}\left[(J^h_{w_h}(t))^h + (D_tJ^h_{w_h}(t))^v\right], \\ (D\phi_t)_{x,v}(V) &= |V|_{g_s}\left[(J^v_{w_v}(t))^h + (D_tJ^v_{w_v}(t))^v\right], \end{aligned}$$

where $D\phi_t$ is the differential of ϕ_t, $w_h = d\pi(H)/\left|d\pi(H)\right|$ and $w_v = K_\nabla(V)/\left|K_\nabla(V)\right|$. Moreover, $(D\phi_t)_{x,v}(X|_{x,v})$ is orthogonal to $(D\phi_t)_{x,v}(H)$ and $(D\phi_t)_{x,v}(V)$.

Proof. Lemma 3.3 gives that $\phi_s(x,v)$, $\phi^h_{w_h,s}(x,v)$ and $\phi^v_{w_v,s}(x,v)$ are unit speed geodesics on SM. If $\Gamma(s) = \phi_s(x,v)$, then $\Gamma(s)$ is a unit speed geodesic on SM, $\dot{\Gamma}(0) = X|_{x,v}$, and

$$(D\phi_t)_{x,v}(X|_{x,v}) = D\phi_t(\dot{\Gamma}(0)) = (\phi_t \circ \Gamma)'(0) = X|_{\phi_t(x,v)}.$$

Moreover, using the unit speed geodesic $\Gamma(s) = \phi^h_{w_h,s}(x,v)$ on SM, and using the formulas after Lemma 3.2, gives

$$\begin{aligned}(D\phi_t)_{x,v}(H) &= D\phi_t(|H|_{g_s}\dot\Gamma(0)) = |H|_{g_s}(\phi_t\circ\Gamma)'(0)\\ &= |H|_{g_s}\left[(J^h_{w_h}(t))^h + (D_tJ^h_{w_h}(t))^v\right]\end{aligned}$$

which is orthogonal to $X|_{\phi_t(x,v)}$. Finally, the unit speed geodesic $\Gamma(s) = \phi^v_{w_v,s}(x,v)$ on SM gives

$$\begin{aligned}(D\phi_t)_{x,v}(V) &= D\phi_t(|V|_{g_s}\dot\Gamma(0)) = |V|_{g_s}(\phi_t\circ\Gamma)'(0)\\ &= |V|_{g_s}\left[(J^v_{w_v}(t))^h + (D_tJ^v_{w_v}(t))^v\right]\end{aligned}$$

which is also orthogonal to $X|_{\phi_t(x,v)}$. □

3.2.3. *Completeness of the unit tangent bundle.* We will need the fact that SM is complete when M is complete. This need arises from theory of Sobolev spaces on manifolds (see Section 5). We could not find a reference so a proof is included.

Lemma 3.5. *Let M be a complete Riemannian manifold with or without boundary. Then SM is complete.*

Proof. Let $(y^{(j)})$ be a Cauchy sequence in (SM, d_{g_s}). We show that it converges in the topology induced by g_s. The definition of the Sasaki metric implies that

$$L_{g_s}(\Gamma) \geq \int_0^\tau \left|d\pi_{\Gamma(t)}(\dot\Gamma(t))\right|_g \,\mathrm{d}t = L_g(\pi\circ\Gamma) \geq d_g(\pi(\Gamma(0)), \pi(\Gamma(\tau)))$$

where $\Gamma : [0,\tau] \to SM$ is any piecewise C^1-smooth curve. Hence

$$d_{g_s}(a,b) \geq d_g(\pi(a),\pi(b)) \qquad (3.9)$$

for all $a, b \in SM$. The above inequality implies that $(\pi(y^{(j)}))$ is a Cauchy sequence in (M, g) and converges, say to $p \in M$, by completeness of M.

Consider a coordinate neighborhood U of p in M, so that $\pi^{-1}(U)$ is diffeomorphic to $U \times S^{n-1}$. Choose an open set V and a compact set K so that $p \in V \subset K \subset U$. Now $\pi^{-1}(K)$ is homeomorphic to $K \times S^{n-1}$ which is compact as a product of two compact sets. Since $\pi(y^{(j)}) \to p$, there exists N such that $\pi(y^{(j)}) \in V$ for all $j \geq N$, and this implies $y^{(j)} \in \pi^{-1}(K)$ for all $j \geq N$. Hence $(y^{(j)})$ has a limit in $(\pi^{-1}(K), d_{g_s}|_{\pi^{-1}(K)})$ since it is a Cauchy sequence, and thus $(y^{(j)})$ converges also in (SM, d_{g_s}). □

3.3. **Symmetric covariant tensors fields.** We denote by $S^m(M)$ the set of C^1-smooth symmetric covariant m-tensor fields and by $S^m_x(M)$ the symmetric covariant m-tensors at point x. Following [DS10] (where more details are also given), we define the map $\lambda_x\colon S^m_x(M) \to C^\infty(S_xM)$,

$$\lambda_x(f)(v) = f_x(v,\dots,v)$$

which is given in local coordinates by

$$\lambda_x(f_{i_1\dots i_m}dx^{i_1}\otimes\cdots\otimes dx^{i_m})(v) = f_{i_1\dots i_m}(x)v^{i_1}\dots v^{i_m}.$$

If $S^m_x(M)$ and $C^\infty(S_xM)$ are endowed with their usual L^2-inner products, then λ_x is an isomorphism and even isometry up to a factor. It smoothly depends on x and hence we get an embedding $\lambda\colon S^m(M) \to C^1(SM)$. The mapping λ identifies symmetric covariant m-tensor fields with homogeneous polynomials (with respect to v) of degree m on SM. We will use this identification and do not always write λ explicitly.

The symmetrization of a tensor is defined by

$$\sigma(\omega_1 \otimes \cdots \otimes \omega_m) = \frac{1}{m!} \sum_{\pi \in \Pi_m} \omega_{\pi(1)} \otimes \cdots \otimes \omega_{\pi(m)},$$

where Π_m is the permutation group of $\{1, \ldots, m\}$. From the above expression we see that if a covariant m-tensor field f is in $E^1_\eta(M)$ or $P^1_\eta(M)$ for some $\eta > 0$, then so is σf too. Furthermore, for $f \in S^m(M)$ one has

$$\lambda(\sigma \nabla f) = X\lambda(f). \tag{3.10}$$

It follows from the last identity and the fundamental theorem of calculus that if $f \in P^1_\eta(M)$ for some $\eta > 0$, then $I_m(\sigma \nabla f) = 0$. This shows that I_m always has a nontrivial kernel for $m \geq 1$, as described in the introduction.

The next lemma states how the decay properties of a tensor field carry over to functions on SM.

Lemma 3.6. *Suppose $f \in S^m(SM)$ and $\eta > 0$.*

(a) *If $f \in E^1_\eta(M)$, then*

$$\sup_{v \in S_xM} |Xf(x,v)| \in E_\eta(M), \qquad \sup_{v \in S_xM} |\overset{h}{\nabla} f(x,v)| \in E_\eta(M) \quad \text{and} \quad \sup_{v \in S_xM} |\overset{v}{\nabla} f(x,v)| \in E_\eta(M).$$

(b) *If $f \in P^1_\eta(M)$, then*

$$\sup_{v \in S_xM} |Xf(x,v)| \in P_{\eta+1}(M), \qquad \sup_{v \in S_xM} |\overset{h}{\nabla} f(x,v)| \in P_{\eta+1}(M) \quad \text{and} \quad \sup_{v \in S_xM} |\overset{v}{\nabla} f(x,v)| \in P_\eta(M).$$

Proof. (a) The result for Xf follows from (3.10). To prove the other statements we take $x \in M$ and use local normal coordinates $(x^1, \ldots, x^n)$ centered at x and the associated coordinates $(v^1, \ldots, v^n)$ for T_xM. In these coordinates $f(x) = f_{i_1 \ldots i_m}(x)\, dx^{i_1} \otimes \cdots \otimes dx^{i_m}$ and $\nabla f(x) = \partial_{x_j} f_{i_1 \ldots i_m}(x)\, dx^j \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_m}$. We see that

$$|f(x)|_g = \left(\sum_{i_1, \ldots, i_m} |f_{i_1 \ldots i_m}(x)|^2 \right)^{1/2} \quad \text{and} \quad |\nabla f(x)|_g = \left(\sum_{j, i_1, \ldots, i_m} \left|\partial_{x_j} f_{i_1 \ldots i_m}(x)\right|^2 \right)^{1/2}.$$

For $Xf, \overset{h}{\nabla} f$ and $\overset{v}{\nabla} f$ at x we have coordinate representations (see [PSU15, Appendix A])

$$\begin{aligned}
Xf(x,v) &= v^j \partial_{x_j} f, \\
\overset{h}{\nabla} f(x,v) &= \left(\partial^{x_j} f - (v^k \partial_{x_k} f) v^j \right) \partial_{x_j}, \\
\overset{v}{\nabla} f(x,v) &= \partial^{v_j} f \partial_{x_j}.
\end{aligned}$$

We get that

$$Xf(x,v)X(x,v) + \overset{h}{\nabla} f(x,v) = \partial^{x_j} f \partial_{x_j} = \partial^{x_j} f_{i_1 \ldots i_m}(x) v^{i_1} \ldots v^{i_m} \partial_{x_j}$$

and, using the orthogonality of $Xf(x,v)X(x,v)$ and $\overset{h}{\nabla} f(x,v)$ and the Cauchy-Schwarz inequality,

$$\sup_{v \in S_xM} |\overset{h}{\nabla} f(x,v)| \leq \left(\sum_{j, i_1, \ldots, i_m} \left|\partial_{x_j} f_{i_1 \ldots i_m}(x)\right|^2 \right)^{1/2} = |\nabla f(x)|_g.$$

This implies that $\sup_{v \in S_xM} |\overset{h}{\nabla} f(x,v)| \in E_\eta(M)$.

For $\overset{v}{\nabla} f$, the identity $\partial_{v_j} v^k = \delta_j^k - v_j v^k$ (see [PSU15]) implies that

$$\begin{aligned}\overset{v}{\nabla} f(x,v) &= \sum_{j=1}^{n} (f_{j i_2 \dots i_m} v^{i_2} \dots v^{i_m} - f(x,v) v_j) \partial_{x_j} + \dots + \sum_{j=1}^{n} (f_{i_1 \dots i_{m-1} j} v^{i_1} \dots v^{i_{m-1}} - f(x,v) v_j) \partial_{x_j} \\ &= m \sum_{j=1}^{n} (f_{j i_2 \dots i_m} v^{i_2} \dots v^{i_m} - f(x,v) v_j) \partial_{x_j}\end{aligned}$$

Thus orthogonality and expanding the squares gives

$$\left| \overset{v}{\nabla} f(x,v) \right|^2 = m^2 \sum_{j=1}^{n} \left| f_{j i_2 \dots i_m}(x) v^{i_2} \dots v^{i_m} \right|^2 \leq m^2 \sum_{i_1, \dots, i_m} |f_{i_1 \dots i_m}(x)|^2 = m^2 \, |f(x)|_g^2$$

which in turn implies that $\sup_{v \in S_x M} |\overset{v}{\nabla} f(x,v)| \in E_\eta(M)$. The proof for (b) is the same. □

4. Growth estimates

Throughout this section we assume that f is a symmetric covariant m-tensor field in $P_\eta(M)$ for some $\eta > 1$. The main results in this section are Lemmas 4.3 and 4.7. They state that if f is such a tensor field, possibly with some additional decay at infinity, then the corresponding solution u^f of the transport equation will have decay at infinity.

We begin by observing that the geodesic X-ray transform is well defined for such f.

Lemma 4.1. *Let $f \in P_\eta(M)$ for some $\eta > 1$. For any $(x,v) \in SM$ one has*

$$\int_{-\infty}^{\infty} |f_{\gamma_{x,v}(t)}(\dot{\gamma}_{x,v}(t), \dots, \dot{\gamma}_{x,v}(t))| \, dt < \infty.$$

Proof. The assumption implies that $|f_{\gamma_{x,v}(t)}(\dot{\gamma}_{x,v}(t), \dots, \dot{\gamma}_{x,v}(t))| \leq C(1 + d(\gamma_{x,v}(t), o))^{-\eta}$. One can then change variables so that $t = 0$ corresponds to the point on the geodesic that is closest to o, split the integral over $t \geq 0$ and $t \leq 0$, and use the fact that the integrands are $\leq C(1 + |t|)^{-\eta}$ by the estimate (3.3). □

If $f \in P_\eta(M)$ for some $\eta > 1$, we may now define

$$u^f(x,v) := \int_0^\infty f_{\gamma_{x,v}(t)}(\dot{\gamma}_{x,v}(t), \dots, \dot{\gamma}_{x,v}(t)) \, \mathrm{d}t.$$

It easy to see that

$$u^f(x,v) + (-1)^m u^f(x,-v) = If(x,v)$$

for all $(x,v) \in SM$.

We have the usual reduction to the transport equation.

Lemma 4.2. *Let $f \in P_\eta(M)$ for some $\eta > 1$. Then $Xu^f = -f$.*

Proof. By definition

$$Xu^f(x,v) = \lim_{s \to 0} -\frac{1}{s} \int_0^s f_{\gamma_{x,v}(t)}(\dot{\gamma}_{x,v}(t), \dots, \dot{\gamma}_{x,v}(t)) \, \mathrm{d}t = -f_x(v, \dots, v). \qquad \square$$

Next we derive decay estimates for u^f under the assumption that $If = 0$.

Lemma 4.3. *Suppose that $If = 0$.*

(a) *If* $f \in E_\eta(M)$ *for* $\eta > 0$*, then*

$$\left|u^f(x,v)\right| \le C(1+d_g(x,o))e^{-\eta d_g(x.o)}$$

for all $(x,v) \in SM$*.*

(b) *If* $f \in P_\eta(M)$ *for* $\eta > 1$*, then*

$$\left|u^f(x,v)\right| \le \frac{C}{(1+d_g(x,o))^{\eta-1}}$$

for all $(x,v) \in SM$*.*

Proof. Since $If = 0$, one has $\left|u^f(x,v)\right| = \left|u^f(x,-v)\right|$. By Lemma 3.1, possibly after replacing (x,v) by $(x,-v)$, we may assume that $\gamma_{x,v}$ is escaping. We have

$$\left|u^f(x,v)\right| = \left|\int_0^\infty f(\gamma(t))(\dot\gamma(t),\dots,\dot\gamma(t))\,\mathrm{d}t\right| \le \int_0^\infty |f(\gamma(t))|_g\,\mathrm{d}t.$$

The rest of the proof is as in [Leh16, Lemma 3.2]. □

Lemma 4.4. *Let* $f \in P_\eta(M)$ *for some* $\eta > 1$*. If* $If = 0$ *and* u^f *is differentiable at* $(x,v) \in SM$*, then*

$$\overset{h}{\nabla}u^f(x,-v) = (-1)^{m-1}\overset{h}{\nabla}u^f(x,v) \quad \textit{and} \quad \overset{v}{\nabla}u^f(x,-v) = (-1)^m\overset{v}{\nabla}u^f(x,v).$$

Proof. From $If = 0$ it follows that

$$u^f(x,v) + (-1)^m u^f(x,-v) = 0.$$

Fix $w \in S_xM$, $w \perp v$. We note that

$$u^f(\phi^h_{w,s}(x,-v)) + (-1)^m u^f(\phi^h_{-w,-s}(x,v)) = 0$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}s}u^f(\phi^h_{w,s}(x,-v))\Big|_{s=0} = -(-1)^m\frac{\mathrm{d}}{\mathrm{d}s}(u^f(\phi^h_{-w,-s}(x,v)))\Big|_{s=0} = (-1)^m\frac{\mathrm{d}}{\mathrm{d}s}(u^f(\phi^h_{-w,s}(x,v)))\Big|_{s=0}.$$

By Lemma 3.2

$$\langle\overset{h}{\nabla}u^f(x,-v),w\rangle = (-1)^m\langle\overset{h}{\nabla}u^f(x,v),-w\rangle = -(-1)^m\langle\overset{h}{\nabla}u^f(x,v),w\rangle.$$

For $\overset{v}{\nabla}u^f$ we use that

$$u^f(\phi^v_{w,s}(x,-v)) + (-1)^m u^f(\phi^v_{-w,s}(x,v)) = 0$$

and by Lemma 3.2 we get that

$$\langle\overset{v}{\nabla}u^f(x,-v),w\rangle = (-1)^{m-1}\langle\overset{v}{\nabla}u^f(x,v),-w\rangle = (-1)^m\langle\overset{v}{\nabla}u^f(x,v),w\rangle.$$ □

We move on to prove growth estimates for Jacobi fields. These estimates will be used to derive estimates for $\overset{h}{\nabla}u^f$ and $\overset{v}{\nabla}u^f$.

Lemma 4.5. *Suppose* $J(t)$ *is a normal Jacobi field along a geodesic* γ*.*

(a) *If all sectional curvatures along* $\gamma([0,\tau])$ *are* $\ge -K_0$ *for some constant* $K_0 > 0$*, and if* $J(0) = 0$ *or* $D_tJ(0) = 0$*, then*

$$|J(t)| \le |J(0)|\cosh(\sqrt{K_0}t) + |D_tJ(0)|\,\frac{\sinh(\sqrt{K_0}t)}{\sqrt{K_0}}$$

for $t \in [0,\tau]$*.*

(b) *If $t_0 \in (0, \tau)$, then*

$$|D_t J(t)| + \left|\frac{J(t)}{t} - D_t J(t)\right| \leq \left[|D_t J(t_0)| + \left|\frac{J(t_0)}{t_0} - D_t J(t_0)\right|\right] e^{2\int_{t_0}^t s\mathcal{K}(\gamma(s))\,ds}$$

for $t \in [t_0, \tau]$.

Proof. (a) follows from the Rauch comparison theorem [Jos08, Theorem 4.5.2]. For (b), we follow the argument in [Leh16]. Consider an orthonormal frame $\{\dot\gamma(t), E_1(t), \ldots, E_{n-1}(t)\}$ obtained by parallel transporting an orthonormal basis of $T_{\gamma(0)}M$ along γ. Write $J(t) = u^j(t)E_j(t)$, so that the Jacobi equation becomes

$$\ddot u(t) + R(t)u(t) = 0 \tag{4.1}$$

where $u(t) = (u^1(t), \ldots, u^{n-1}(t))$ and $R_{jk} = R(E_j, \dot\gamma, \dot\gamma, E_k)$. We wish to estimate $v(t) = \frac{u(t)}{t}$, and we do this by writing $v(t) = A(t) + \frac{B(t)}{t}$ where

$$A(t) = \dot u(t), \qquad B(t) = u(t) - t\dot u(t).$$

By using the equation, we see that

$$A(t) - A(t_0) = -\int_{t_0}^t sR(s)v(s)\,\mathrm{d}s,$$

$$B(t) - B(t_0) = \int_{t_0}^t s^2 R(s)v(s)\,\mathrm{d}s.$$

Write $g(t) = |A(t)| + \left|\frac{B(t)}{t}\right|$. If $t \geq t_0$ one has

$$g(t) = \left|A(t_0) - \int_{t_0}^t sR(s)v(s)\,\mathrm{d}s\right| + \frac{1}{t}\left|B(t_0) + \int_{t_0}^t s^2R(s)v(s)\,\mathrm{d}s\right| \leq g(t_0) + 2\int_{t_0}^t s\|R(s)\|g(s)\,\mathrm{d}s.$$

The Gronwall inequality implies that

$$g(t) \leq g(t_0)\mathrm{e}^{2\int_{t_0}^t s\|R(s)\|\,\mathrm{d}s}.$$

The result follows from this, since $\|R(s)\| = \sup_{|\xi|=1} R(s)\xi\cdot\xi = \sup_{\dot\gamma(s)\in\Pi} K(\Pi) \leq \mathcal{K}(\gamma(s))$. ☐

Corollary 4.6. *Suppose that (M, g) is a Cartan-Hadamard manifold. Let γ be a geodesic and J a normal Jacobi field along it, satisfying either $J(0) = 0$ and $|D_tJ(0)| \leq 1$ or $|J(0)| \leq 1$ and $D_tJ(0) = 0$.*

(a) *If $-K_0 \leq K \leq 0$ and $K_0 > 0$, then*

$$|J(t)| \leq Ce^{\sqrt{K_0}t} \quad \text{and} \quad |D_tJ(t)| \leq Ce^{\sqrt{K_0}t}$$

for $t \geq 0$ where the constants do not depend on the geodesic γ.

(b) *If $\mathcal{K} \in P_\kappa(M)$ for some $\kappa > 2$, then*

$$|J(t)| \leq C(t+1) \quad \text{and} \quad |D_tJ(t)| \leq C$$

for $t \geq 0$. If in addition $\gamma \in \mathcal{E}_o$, then the constants do not depend on the geodesic γ.

Proof. (a) The estimate for $|J(t)|$ follows directly from Lemma 4.5. Using the same notations as in the proof of that Lemma we have $|D_tJ(t)| = |\dot u(t)|$ and by integrating (4.1) from 0 to t we get

$$\begin{aligned}|\dot u(t)| &\leq |\dot u(0)| + \int_0^t \|R(s)\||u(s)|\,\mathrm{d}s\\ &\leq |D_tJ(0)| + \int_0^t K_0|J(s)|\,\mathrm{d}s\\ &\leq C\mathrm{e}^{\sqrt{K_0}t}.\end{aligned}$$

(b) For a fixed geodesic, the estimates follow from Lemma 4.5. If $\mathcal{K} \in P_\kappa(M)$ for $\kappa > 2$, then

$$A := \sup_{\gamma \in \mathcal{E}_o} \int_0^\infty s\mathcal{K}(\gamma(s))\,\mathrm{d}s \le C \sup_{\gamma \in \mathcal{E}_o} \int_0^\infty s(1 + d_g(\gamma(s), o))^{-\kappa}\,\mathrm{d}s < \infty$$

by using (3.3). Let us fix $t_0 = 1$ and suppose that J is a Jacobi field along a geodesic in $\mathcal{E}_o$ whose initial values satisfy the given assumptions. From Lemma 4.5 and (a) we then get that

$$\begin{aligned}|J(t)| &\le \mathrm{e}^{2A}\left(2\,|D_t J(1)| + |J(1)|\right) t \\ &\le \mathrm{e}^{2A} C \mathrm{e}^{\sqrt{K_0}} t\end{aligned}$$

for $t \ge 1$, where $K_0 = \sup_{x \in M} |\mathcal{K}(x)|$.

For $t \in [0, 1]$ we can estimate $|J(t)| \le C\mathrm{e}^{\sqrt{K_0}}$. By combining these two estimates we get

$$|J(t)| \le C(1 + \mathrm{e}^{2A})t$$

for $t \ge 0$, and the constants do not depend on $\gamma \in \mathcal{E}_o$.

For $|D_t J(t)|$, Lemma 4.5 gives the estimate

$$|D_t J(t)| \le \mathrm{e}^{2A}\left(2\,|D_t J(1)| + |J(1)|\right)$$

for $t \ge 1$, and for $t \in [0, 1]$ we get a bound from (a). Neither of these bounds depends on $\gamma \in \mathcal{E}_o$. □

Lemma 4.7. *Suppose that $If = 0$.*

(a) *If $-K_0 \le K \le 0$, $K_0 > 0$ and $f \in E^1_\eta(M)$ for some $\eta > \sqrt{K_0}$, then u^f is differentiable along every geodesic on SM, $u^f \in W^{1,\infty}(SM)$ and*

$$\left|\overset{h}{\nabla} u^f(x, v)\right| \le Ce^{-(\eta - \sqrt{K_0})d_g(x,o)}$$

for a.e. $(x, v) \in SM$.

(b) *If $\mathcal{K} \in P_\kappa(M)$ for some $\kappa > 2$ and $f \in P^1_\eta(M)$ for some $\eta > 1$, then u^f is differentiable along every geodesic on SM, $u^f \in W^{1,\infty}(SM)$ and*

$$\left|\overset{h}{\nabla} u^f(x, v)\right| \le \frac{C}{(1 + d_g(x, o))^{\eta - 1}}$$

for a.e. $(x, v) \in SM$.

The same estimates hold for $\overset{v}{\nabla} u^f$ with the same assumptions.

Proof of $u^f \in W^{1,\infty}_{loc}(SM)$. We show that u^f is locally Lipschitz continuous. Fix $(x_0, v_0) \in SM$, and suppose that $\Gamma(s)$ is a unit speed geodesic on SM through (x_0, v_0). We have

$$\begin{aligned}\frac{u^f(\Gamma(r)) - u^f(\Gamma(0))}{r} &= \int_0^\infty \frac{f(\phi_t(\Gamma(r))) - f(\phi_t(\Gamma(0)))}{r}\,\mathrm{d}t \\ &= \int_0^\infty \frac{1}{r}\int_0^r \frac{\partial}{\partial s}\left[f(\phi_t(\Gamma(s)))\right]\mathrm{d}s\,\mathrm{d}t \qquad (4.2)\\ &= \int_0^\infty \frac{1}{r}\int_0^r \langle \nabla_{SM} f(\phi_t(\Gamma(s))), D\phi_t(\Gamma(s))\dot{\Gamma}(s)\rangle\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

When we apply Corollary 3.4 to the right hand side of (4.2) (and omit the identifications), we find that

$$
\begin{aligned}
\frac{u^f(\Gamma(r)) - u^f(\Gamma(0))}{r} &= \int_0^\infty \frac{1}{r} \int_0^r \Bigg[Xf(\phi_t(\Gamma(s)))\langle \dot\Gamma(s), X\rangle \\
&+ \langle \overset{h}{\nabla} f(\phi_t(\Gamma(s))), |\dot\Gamma(s)^h| J^h_{w_h(s)}(t) + |\dot\Gamma(s)^v| J^v_{w_v(s)}(t)\rangle \\
&+ \langle \overset{v}{\nabla} f(\phi_t(\Gamma(s))), |\dot\Gamma(s)^h| D_t J^h_{w_h(s)}(t) + |\dot\Gamma(s)^v| D_t J^v_{w_v(s)}(t)\rangle \Bigg] \,\mathrm{d}s\,\mathrm{d}t
\end{aligned} \tag{4.3}
$$

where $w_h(s) = \dot\Gamma(s)^h/|\dot\Gamma(s)^h|$ and $w_v(s) = \dot\Gamma(s)^v/|\dot\Gamma(s)^v|$. Here the Jacobi fields are along the geodesic $\gamma_{\Gamma(s)}(t) := \pi(\phi_t(\Gamma(s)))$. By definition their initial values fulfill the assumptions of Corollary 4.6.

From this point on we will work under assumptions of (b). The proof under assumptions of (a) is similar but simpler. We fix a small $\varepsilon > 0$. We show that the integral (4.3) has a uniform upper bound for every $r \in (0, 1]$ and every geodesic Γ through a point in $B_{(x_0,v_0)}(\varepsilon) \subset SM$. For $(x, v) \in SM$ we denote by $\mathcal{G}(x, v)$ the set of unit speed geodesics on SM through (x, v), and define

$$J(x_0, v_0, \varepsilon) := \{\Gamma \in \mathcal{G}(x, v)\,;\, (x, v) \in B_{(x_0,v_0)}(\varepsilon)\}.$$

For all $\Gamma \in J(x_0, v_0, \varepsilon), \Gamma(0) = (x, v)$, and $s \in (0, r]$ the estimate (3.9) gives that $d_g(x, x_0) \le \varepsilon$ and

$$d_g(\gamma_{\Gamma(s)}(0), x) = d_g(\pi(\Gamma(s)), x) \le d_{g_s}(\Gamma(s), (x, v)) \le s.$$

The estimate (3.1) implies that

$$
\begin{aligned}
d_g(\pi(\phi_t(\Gamma(s))), o) &= d_g(\gamma_{\Gamma(s)}(t), o) \ge t - d_g(\gamma_{\Gamma(s)}(0), x_0) \\
&\ge t - \sup_{s\in(0,r]} d_g(\gamma_{\Gamma(s)}(0), o) \ge t - d_g(x, o) - r \\
&\ge t - d_g(x_0, o) - \varepsilon - r
\end{aligned} \tag{4.4}
$$

for all $t \ge t_0$ where $t_0 := d_g(x_0, o) + r + \varepsilon$. We can use a trivial estimate $d_g(\pi(\phi_t(\Gamma(s))), o) \ge 0$ on the interval $[0, t_0]$. Further, the estimate (4.4) gives

$$\mathcal{K}(\gamma_{\Gamma(s)}(t)) \le \frac{C}{(1 + d_g(\gamma_{\Gamma(s)}(t), o))^\eta} \le \frac{C}{(1 + t - d_g(x_0, o) - \varepsilon - r)^\eta} \tag{4.5}$$

for all $t \ge t_0$ where the constant C does not depend on $s \in (0, r]$ or the geodesic $\Gamma \in J(x_0, v_0, \varepsilon)$, and hence

$$\sup_{\substack{\Gamma\in J(x_0,v_0,\varepsilon),\\ s\in(0,r]}} \int_0^\infty t\mathcal{K}(\gamma_{\Gamma(s)}(t))\,\mathrm{d}t < \infty. \tag{4.6}$$

Using the proof of Corollary 4.6 together with (4.6), we can find a constant C which does not depend on $s \in (0, r]$ so that one has

$$|J^h_{w_h(s)}(t)| \le Ct, \quad |D_t J^h_{w_h(s)}(t)| \le C$$

for all $t \ge 0$ and $\Gamma \in J(x_0, v_0, \varepsilon)$. Similar estimates hold also uniformly for $J^v_{w_v(s)}(t)$ and $D_t J^v_{w_v(s)}(t)$.

Recall that $|\dot\Gamma(s)^h|, |\dot\Gamma(s)^v| \le |\dot\Gamma(s)| = 1$, and that $w_h(s), w_v(s)$ depend on Γ. By combining the above estimates for Jacobi fields with estimate (4.4) and Lemma 3.6 we get for the integrand in (4.3) that

$$
\begin{aligned}
&\big|Xf(\phi_t(\Gamma(s)))\langle\dot\Gamma(s),X\rangle\\
&\quad+\langle\overset{h}{\nabla}f(\phi_t(\Gamma(s))),|\dot\Gamma(s)^h|J^h_{w_h(s)}(t)+|\dot\Gamma(s)^v|J^v_{w_v(s)}(t)\rangle\\
&\quad+\langle\overset{v}{\nabla}f(\phi_t(\Gamma(s))),|\dot\Gamma(s)^h|D_tJ^h_{w_h(s)}(t)+|\dot\Gamma(s)^v|D_tJ^v_{w_v(s)}(t)\rangle\big|\\
&\le\big|Xf(\gamma_{\Gamma(s)}(t))\big|+\left|\overset{h}{\nabla}f(\gamma_{\Gamma(s)}(t))\right|\left||\dot\Gamma(s)^h|J^h_{w_h(s)}(t)+|\dot\Gamma(s)^v|J^v_{w_v(s)}(t)\right|\\
&\quad+\left|\overset{v}{\nabla}f(\gamma_{\Gamma(s)}(t))\right|\left||\dot\Gamma(s)^h|D_tJ^h_{w_h(s)}(t)+|\dot\Gamma(s)^v|D_tJ^v_{w_v(s)}(t)\right|\\
&\le\big|Xf(\gamma_{\Gamma(s)}(t))\big|+\left|\overset{h}{\nabla}f(\gamma_{\Gamma(s)}(t))\right|\Big(|J^h_{w_h(s)}(t)|+|J^v_{w_v(s)}(t)|\Big)\\
&\quad+\left|\overset{v}{\nabla}f(\gamma_{\Gamma(s)}(t))\right|\Big(|D_tJ^h_{w_h(s)}(t)|+|D_tJ^v_{w_v(s)}(t)|\Big)\\
&\le\frac{Ct}{(1+t-d_g(x_0,o)-\varepsilon-r)^{\eta+1}}+\frac{C}{(1+t-d_g(x_0,o)-\varepsilon-r)^{\eta}}
\end{aligned}
\tag{4.7}
$$

for all $t\in[t_0,\infty)$, $s\in(0,r]$ and $\Gamma\in J(x_0,v_0,\varepsilon)$. On the interval $[0,t_0]$ we also get a uniform upper bound since f, its covariant derivative and sectional curvatures are all bounded.

We can conclude that integral on the right hand side of (4.3) converges absolutely with some uniform bound $C<\infty$ over $r\in(0,1]$ and the set $J(x_0,v_0,\varepsilon)$. This shows that u^f is locally Lipschitz, i.e. $u^f\in W^{1,\infty}_{\mathrm{loc}}(SM)$ (cf. Remark 1). Moreover, the uniform estimate together with the dominated convergence theorem guarantees that the limit $r\to 0$ of (4.2) exists for all geodesics Γ on SM. This finishes the first part of the proof. □

Proof of the gradient estimates. By Rademacher's theorem u^f is differentiable almost everywhere, and thus we can assume that u^f is differentiable at $(x,v)\in SM$. By Lemmas 3.1 and 4.4 we can assume that (x,v) satisfies $\gamma=\gamma_{x,v}\in\mathcal{E}_o$. We may also assume that $\overset{h}{\nabla}u^f(x,v)\neq 0$. Since $\langle\overset{h}{\nabla}u^f(x,v),v\rangle=0$, we can take $w=\overset{h}{\nabla}u^f(x,v)/|\overset{h}{\nabla}u^f(x,v)|$ in Lemma 3.2 and get that

$$
\begin{aligned}
\left|\overset{h}{\nabla}u^f(x,v)\right|&=\frac{\mathrm{d}}{\mathrm{d}s}u^f(\phi^h_{w,s}(x,v))\Big|_{s=0}\\
&=\int_0^\infty\langle\overset{h}{\nabla}f(\phi_t(x,v)),J^h(t)\rangle+\langle\overset{v}{\nabla}f(\phi_t(x,v)),D_tJ^h(t)\rangle\,\mathrm{d}t
\end{aligned}
\tag{4.8}
$$

where J^h is again a Jacobi field along γ fulfilling the assumptions of Corollary 4.6. Under the conditions in part (a), the estimate (3.3) implies

$$
\left|\overset{h}{\nabla}u^f(x,v)\right|\le C\int_0^\infty \mathrm{e}^{-\eta d_g(\gamma(t),o)}\mathrm{e}^{\sqrt{K_0}\,t}\,\mathrm{d}t\le\int_0^\infty\mathrm{e}^{-\eta\sqrt{d_g(x,o)^2+t^2}}\mathrm{e}^{\sqrt{K_0}\,t}\,\mathrm{d}t.
$$

Writing $r=d_g(x,o)$ and splitting the integral over $[0,r)$ and $[r,\infty)$ gives

$$
\left|\overset{h}{\nabla}u^f(x,v)\right|\le C\left[\int_0^r\mathrm{e}^{-\eta r}\mathrm{e}^{\sqrt{K_0}\,t}\,\mathrm{d}t+\int_r^\infty\mathrm{e}^{-\eta t}\mathrm{e}^{\sqrt{K_0}\,t}\,\mathrm{d}t\right]\le C\mathrm{e}^{-(\eta-\sqrt{K_0})d_g(x,o)}.
$$

The above estimate also shows that $|\overset{h}{\nabla} u^f|$ is bounded. Similarly, under the conditions in part (b), Lemma 3.6, Corollary 4.6 and (3.3) imply

$$\begin{aligned}\left|\overset{h}{\nabla} u^f(x,v)\right| &\leq C\int_0^\infty \frac{1+t}{(1+d_g(\gamma(t),o))^{\eta+1}}\,\mathrm{d}t + C\int_0^\infty \frac{C}{(1+d_g(\gamma(t),o))^{\eta}}\,\mathrm{d}t \\ &\leq C\left[\int_0^r \frac{1+t}{(1+r)^{\eta+1}}\,\mathrm{d}t + \int_r^\infty \frac{1+t}{(1+t)^{\eta+1}}\,\mathrm{d}t\right] \leq C(1+r)^{-(\eta-1)}\end{aligned}$$

where $r = d_g(x,o)$. The same arguments apply to $\overset{v}{\nabla} u^f$. Hence $u^f \in W^{1,\infty}(SM)$ in the both cases, (a) and (b). □

Lemma 4.8. (a) *If* $-K_0 \leq K \leq 0$ *and* $K_0 > 0$, *then*

$$\operatorname{Vol} S_o(r) \leq Ce^{(n-1)\sqrt{K_0}r}$$

for all $r \geq 0$.

(b) *If* $\mathcal{K} \in P_\kappa(M)$ *for* $\kappa > 2$, *then*

$$\operatorname{Vol} S_o(r) \leq Cr^{n-1}$$

for all $r \geq 0$.

Proof. We define the mapping $f\colon S_oM \to S_o(r)$,

$$f(v) = (\pi \circ \phi_r)(o,v) = \exp_o(rv).$$

We denote by $\mathrm{d}\Sigma$ the volume form on $S_o(r)$ and have that

$$\operatorname{Vol} S_o(r) = \int_{S_o(r)} \mathrm{d}\Sigma = \int_{S_oM} f^*(\mathrm{d}\Sigma) = \int_{S_oM} \mu\,\mathrm{d}S,$$

where $\mathrm{d}S$ denotes the volume form on S_oM (induced by Sasaki metric) and $\mu\colon S_oM \to \mathbb{R}$.

Let $v \in S_oM$ and $\{w_i\}_{i=1}^{n-1}$ be an orthonormal basis for T_vS_oM with respect to Sasaki metric. By the Gauss lemma $\{d_vf(w_i)\}_{i=1}^{n-1}$ is an orthonormal basis for $T_{f(v)}S_o(r)$ and

$$f^*(\mathrm{d}\Sigma)_v(w_1,\dots,w_{n-1}) = \mathrm{d}\Sigma_{f(v)}(d_vf(w_1),\dots,d_vf(w_{n-1})).$$

It holds that $d_vf(w_i) = J_i(r)$ where J_i is a Jacobi field along the geodesic $\gamma_{o,v}$ with initial values $J(0) = d_v\pi(w_i)$ and $D_tJ_i(0) = K_\nabla(w_i)$. We get that

$$|\mu(v)| \leq \prod_{i=1}^{n-1} |d_vf(w_i)| = \prod_{i=1}^{n-1} |J_i(r)|.$$

Since the tangent vectors w_i lie in $\mathcal{V}(o,v)$ we have $|J_i(0)|_g = 0$ and $|D_tJ_i(0)|_g = |w_i|_{g_s} = 1$, and the estimates for the volume of $S_o(r)$ then follow from Corollary 4.6. □

5. Proof of the main theorems

In this section we will combine the facts above to prove Theorems 1.1 and 1.2. We begin by introducing some useful notation related to operators on the sphere bundle and spherical harmonics. One can find more details in [GK80b], [DS10] and [PSU15]. We prove the main theorems of this work in the end of this section.

The norm $\|\cdot\|$ in this section will always be the $L^2(SM)$-norm. We define the Sobolev space $H^1(SM)$ as the set of all $u \in L^2(SM)$ for which $\|u\|_{H^1(SM)} < \infty$, where

$$\begin{aligned}\|u\|_{H^1(SM)} &= \left(\|u\|^2 + \|\nabla_{SM} u\|^2\right)^{1/2} \\ &= \left(\|u\|^2 + \|Xu\|^2 + \|\overset{h}{\nabla} u\|^2 + \|\overset{v}{\nabla} u\|^2\right)^{1/2}.\end{aligned}$$

Let $C_c^\infty(SM)$ denote the smooth compactly supported functions on SM. It is well known that if N is complete Riemannian manifold, then $C_c^\infty(N)$ is dense in $H^1(N)$ (see [Eic88, Satz 2.3]). By Lemma 3.5 SM is complete when M is complete. Hence $C_c^\infty(SM)$ is dense in $H^1(SM)$.

For the following facts see [PSU15]. The vertical Laplacian $\Delta : C^\infty(SM) \to C^\infty(SM)$ is defined as the operator

$$\Delta := -\overset{v}{\operatorname{div}}\overset{v}{\nabla}.$$

The Laplacian Δ has eigenvalues $\lambda_k = k(k+n-2)$ for $k = 0, 1, 2, \ldots$, and its eigenfunctions are homogeneous polynomials in v. One has an orthogonal eigenspace decomposition

$$L^2(SM) = \bigoplus_{k \geq 0} H_k(SM),$$

where $H_k(SM) := \{f \in L^2(SM)\,;\, \Delta f = \lambda_k f\}$. We define $\Omega_k = H_k(SM) \cap H^1(SM)$. In particular, by Lemma 5.1 below any $u \in H^1(SM)$ can be written as

$$u = \sum_{k=0}^{\infty} u_k, \quad u_k \in \Omega_k,$$

where the series converges in $L^2(SM)$.

One can split the geodesic vector field in two parts, $X = X_+ + X_-$, so that (by Lemma 5.1) $X_+ : \Omega_k \to H_{k+1}(SM)$ and $X_- : \Omega_k \to H_{k-1}(SM)$. The next lemma gives an estimate for $X_\pm u$ in terms of Xu and $\overset{h}{\nabla} u$.

Lemma 5.1. *Suppose $u \in H^1(SM)$. Then $X_\pm u \in L^2(SM)$ and*

$$\|X_+ u\|^2 + \|X_- u\|^2 \leq \|Xu\|^2 + \|\overset{h}{\nabla} u\|^2.$$

Moreover, for each $k \geq 0$ one has $u_k \in H^1(SM)$, and there is a sequence $(u_k^{(j)})_{j=1}^\infty \subset C_c^\infty(SM) \cap H_k(SM)$ with $u_k^{(j)} \to u_k$ in $H^1(SM)$ as $j \to \infty$.

Proof. Let $u \in C_c^\infty(SM)$. One has the decomposition

$$\overset{h}{\nabla} u = \overset{v}{\nabla}\left[\sum_{l=1}^{\infty}\left(\frac{1}{l} X_+ u_{l-1} - \frac{1}{l+n-2} X_- u_{l+1}\right)\right] + Z(u)$$

where $Z(u)$ is such that $\overset{v}{\operatorname{div}}\, Z(u) = 0$ (see [PSU15, Lemma 4.4]). Hence

$$\begin{aligned}\|\overset{h}{\nabla} u\|^2 &= \sum_{l=1}^{\infty}\left(l(l+n-2)\|\frac{1}{l} X_+ u_{l-1} - \frac{1}{l+n-2} X_- u_{l+1}\|^2\right) + \|Z(u)\|^2 \\ &= \sum_{l=1}^{\infty}\left(\frac{l+n-2}{l}\|X_+ u_{l-1}\|^2 - 2\langle X_+ u_{l-1}, X_- u_{l+1}\rangle + \frac{l}{l+n-2}\|X_- u_{l+1}\|^2\right) + \|Z(u)\|^2.\end{aligned}$$

We also have

$$\|Xu\|^2 = \|X_-u_1\|^2 + \sum_{l=1}^{\infty} \left(\|X_+u_{l-1} + X_-u_{l+1}\|^2\right)$$
$$= \|X_-u_1\|^2 + \sum_{l=1}^{\infty} \left(\|X_+u_{l-1}\|^2 + 2\langle X_+u_{l-1}, X_-u_{l+1}\rangle + \|X_-u_{l+1}\|^2\right)$$

by the definition of X_+ and X_-. Adding up these estimates gives that

$$\|Xu\|^2 + \|\overset{h}{\nabla}u\|^2 = \|Z(u)\|^2 + \|X_-u_1\|^2 + \sum_{l=1}^{\infty} \left(A(n,l)\|X_+u_{l-1}\|^2 + B(n,l)\|X_-u_{l+1}\|^2\right)$$

where $A(n,l) = 2 + \frac{n-2}{l}$ and $B(n,l) = 1 + \frac{l}{l+n-2}$. Since $A(n,l) \geq 1$ and $B(n,l) \geq 1$ for all $l = 1, 2, \ldots$ and $n \geq 2$, the estimate for $\|X_+u\|^2 + \|X_-u\|^2$ follows when $u \in C_c^\infty(SM)$, and it extends to $H^1(SM)$ by density and completeness.

Moreover, if $u \in C_c^\infty(SM)$ and if $k \geq 0$, then the triangle inequality $\|Xu_k\| \leq \|X_+u_k\| + \|X_-u_k\|$ and orthogonality imply that

$$\|u_k\| + \|Xu_k\| + \|\overset{v}{\nabla}u_k\| \leq \|u\| + \|X_+u\| + \|X_-u\| + \|\overset{v}{\nabla}u\|.$$

We may also estimate $\overset{h}{\nabla}u_k$ by [PSU15, Proposition 3.4] and orthogonality to obtain

$$\|\overset{h}{\nabla}u_k\|^2 \leq (2k+n-1)\|X_+u_k\|^2 + (\sup_M K)\|\overset{v}{\nabla}u_k\|^2 \leq C_k(\|X_+u\|^2 + \|\overset{v}{\nabla}u\|^2).$$

It follows from the first part of this lemma that

$$\|u_k\|_{H^1(SM)} \leq C_k\|u\|_{H^1(SM)}, \qquad u \in C_c^\infty(SM).$$

This extends to $u \in H^1(SM)$ by density and completeness. Finally, if $u \in H^1(SM)$ and the sequence $(u^{(j)}) \subset C_c^\infty(SM)$ satisfies $u^{(j)} \to u$ in $H^1(SM)$, then also $u_k^{(j)} \to u_k$ in $H^1(SM)$ by the above inequality. $\square$

Corollary 5.2. *Suppose $u \in H^1(SM)$. Then*

$$\lim_{k\to\infty} \|X_+u_k\|_{L^2(SM)} = 0.$$

Proof. By Lemma 5.1 one has

$$\|X_+u\|^2 = \sum_{k=0}^{\infty} \|X_+u_k\|^2 < \infty$$

which implies the claim. $\square$

Lemma 5.3. *Let $u \in H^1(SM)$ and $k \geq 1$. Then one has that*

$$\|X_-u_k\| \leq D_n(k)\|X_+u_k\|$$

where

$$D_2(k) = \begin{cases} \sqrt{2}, & k = 1 \\ 1, & k \geq 2, \end{cases}$$

$$D_3(k) = \left[1 + \frac{1}{(k+1)^2(2k-1)}\right]^{1/2}$$

$$D_n(k) \leq 1 \quad \text{for } n \geq 4.$$

Proof. This result was shown for smooth compactly supported functions in [PSU15, Lemma 5.1]. The result follows for $u \in H^1(SM)$ by an approximation argument using Lemma 5.1. $\square$

The estimates from Section 4 allow us to prove the following result:

Lemma 5.4. *Suppose that f is a symmetric m-tensor field and either of the following holds:*

(a) $-K_0 \leq K \leq 0$, $K_0 > 0$ *and* $f \in E^1_\eta(M)$ *for* $\eta > \frac{(n+1)\sqrt{K_0}}{2}$

(b) $\mathcal{K} \in P_\kappa(M)$ *for* $\kappa > 2$ *and* $f \in P^1_\eta(M)$ *for* $\eta > \frac{n+2}{2}$.

Then $u^f \in H^1(SM)$.

Proof. We prove only (a), the proof for (b) is similar. By Lemma 4.7 we have that $u^f \in W^{1,\infty}(SM)$. Lemma 4.3 gives that

$$|u^f(x,v)| \leq C(1 + d_g(x,o))\mathrm{e}^{-\eta d_g(x,o)}$$

on SM. By using the coarea formula with Lemma 4.8 we get

$$\begin{aligned}\int_{SM} |u^f(x,v)|^2 \, \mathrm{d}V_{g_s} &\leq C \int_M (1 + d_g(x,o))^2 \mathrm{e}^{-2\eta d_g(x,o)} \, \mathrm{d}V_g \\ &= C \int_0^\infty (1+r)^2 \mathrm{e}^{-2\eta r} \left(\int_{S_o(r)} \mathrm{d}S \right) \mathrm{d}r \\ &\leq C \int_0^\infty (1+r)^2 \mathrm{e}^{-2\eta r} \mathrm{e}^{(n-1)\sqrt{K_0} r} \mathrm{d}r.\end{aligned}$$

The last integral above is finite and hence $u^f \in L^2(SM)$. Similar calculations using Lemmas 4.2 and 4.7 show that Xu^f, $\overset{h}{\nabla} u^f$ and $\overset{v}{\nabla} u^f$ all have finite L^2-norms under the assumption $\eta > \frac{(n+1)\sqrt{K_0}}{2}$, and therefore the H^1-norm of u^f is finite. $\square$

We are ready to prove our main theorems.

Proof of Theorems 1.1 and 1.2. Suppose that the m-tensor field f and the sectional curvature K satisfy the assumptions of Theorem 1.1 or 1.2. Recall that we identify f with a function on SM as described in Section 3.3. Then $u = u^f$ is in $H^1(SM)$ by Lemma 5.4, and Lemma 4.2 states that $Xu = -f$ on SM. Note also that $f \in H^1(SM)$, which follows as in the proof of Lemma 5.4.

Since f is of degree m it has a decomposition

$$f = \sum_{k=0}^{m} f_k, \quad f_k \in \Omega_k,$$

and u has a decomposition

$$u = \sum_{k=0}^{\infty} u_k, \quad u_k \in \Omega_k.$$

We first show that $u_k = 0$ for $k \geq m$. From $Xu = -f$ it follows that for $k \geq m$ we have

$$X_+ u_k + X_- u_{k+2} = 0.$$

This implies that

$$\|X_+ u_k\| \leq \|X_- u_{k+2}\|, \qquad k \geq m. \tag{5.1}$$

Fix $k \geq m$. We apply Lemma 5.3 and the inequality (5.1) iteratively to get

$$\begin{aligned}\|X_-u_k\| &\leq D_n(k)\|X_+u_k\| \\ &\leq D_n(k)\|X_-u_{k+2}\| \\ &\leq D_n(k)D_n(k+2)\|X_+u_{k+2}\| \\ &\leq \left[\prod_{l=0}^{N} D_n(k+2l)\right]\|X_+u_{k+2N}\|.\end{aligned}$$

By Corollary 5.2

$$\lim_{l\to\infty}\|X_+u_{k+2l}\| = 0.$$

Moreover, as stated in [PSU15, Theorem 1.1], one has

$$\prod_{l=0}^{\infty} D_n(k+2l) < \infty.$$

Thus we obtain that

$$\|X_-u_k\| = \|X_+u_k\| = 0.$$

This gives $Xu_k = 0$, which implies that $t \mapsto u_k(\varphi_t(x,v))$ is a constant function on $\mathbb{R}$ for any $(x,v) \in SM$. Since u decays to zero along any geodesic we must have $u_k = 0$, and this holds for all $k \geq m$.

It remains to verify that the equation $Xu = -f$ on SM together with the fact $u = \sum_{k=0}^{m-1} u_k$ imply the conclusions of Theorems 1.1 and 1.2. This is done as in [PSU13, end of Section 2]. Suppose that m is odd (the case where m is even is similar). The function f is a homogeneous polynomial of order m in v and hence its Fourier decomposition has only odd terms, i.e.

$$f = f_m + f_{m-2} + \cdots + f_1.$$

It follows that the decomposition of u has only even terms,

$$u = u_{m-1} + u_{m-3} + \cdots + u_0.$$

By taking tensor products with the metric g and symmetrizing it is possible to raise the degree of a symmetric tensor: if $F \in S^m(M)$, then $\alpha F := \sigma(F \otimes g) \in S^{m+2}(M)$. One has $\lambda(\alpha F) = \lambda(F)$, since $\lambda(g)$ has a constant value 1 on SM.

We define $h \in S^{m-1}(M)$ by

$$h := -\sum_{j=0}^{(m-1)/2} \alpha^j(\lambda^{-1}(u_{m-1-2j})).$$

Then $\lambda(h) = -u$, so equation (3.10) gives $\lambda(\sigma\nabla h) = \lambda(f)$, which implies $f = \sigma\nabla h$. □

References

[Arm94] D. H. Armitage. A non-constant continuous function on the plane whose integral on every line is zero. *Amer. Math. Monthly*, 101(9):892–894, 1994.

[Bal05] Guillaume Bal. Ray transforms in hyperbolic geometry. *J. Math. Pures Appl. (9)*, 84(10):1362–1392, 2005.

[BBNV03] J. Berndt, E. Boeckx, P. T. Nagy, and L. Vanhecke. Geodesics on the unit tangent bundle. *Proc. Roy. Soc. Edinburgh Sect. A*, 133(6):1209–1229, 2003.

[BCT96] Carlos A. Berenstein and Enrico Casadio Tarabusi. Integral geometry in hyperbolic spaces and electrical impedance tomography. *SIAM J. Appl. Math.*, 56(3):755–764, 1996.

[BO69] R. L. Bishop and B. O'Neill. Manifolds of negative curvature. *Trans. Amer. Math. Soc.*, 145:1–49, 1969.

[DS10] N. S. Dairbekov and V. A. Sharafutdinov. Conformal Killing symmetric tensor fields on Riemannian manifolds. *Mat. Tr.*, 13(1):85–145, 2010.

[Eic88] Jürgen Eichhorn. Sobolev-Räume, Einbettungssätze und Normungleichungen auf offenen Mannigfaltigkeiten. *Math. Nachr.*, 138:157–168, 1988.

[Eva98] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[GGG03] I. M. Gelfand, S. G. Gindikin, and M. I. Graev. *Selected topics in integral geometry*, volume 220 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2003. Translated from the 2000 Russian original by A. Shtern.

[GK80a] V. Guillemin and D. Kazhdan. Some inverse spectral results for negatively curved 2-manifolds. *Topology*, 19(3):301–312, 1980.

[GK80b] Victor Guillemin and David Kazhdan. Some inverse spectral results for negatively curved n-manifolds. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 153–180. Amer. Math. Soc., Providence, R.I., 1980.

[GPSU16] Colin Guillarmou, Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. The X-ray transform for connections in negative curvature. *Comm. Math. Phys.*, 343(1):83–127, 2016.

[GW79] R. E. Greene and H. Wu. *Function theory on manifolds which possess a pole*, volume 699 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[GW82] R. E. Greene and H. Wu. Gap theorems for noncompact Riemannian manifolds. *Duke Math. J.*, 49(3):731–756, 1982.

[Hel94] Sigurdur Helgason. *Geometric analysis on symmetric spaces*, volume 39 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1994.

[Hel99] Sigurdur Helgason. *The Radon transform*, volume 5 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 1999.

[Hel13] Sigurdur Helgason. Some personal remarks on the Radon transform. In *Geometric analysis and integral geometry*, volume 598 of *Contemp. Math.*, pages 3–19. Amer. Math. Soc., Providence, RI, 2013.

[Jen04] Sine R. Jensen. Sufficient conditions for the inversion formula for the k-plane Radon transform in $\mathbf{R}^n$. *Math. Scand.*, 94(2):207–226, 2004.

[Jos08] Jürgen Jost. *Riemannian geometry and geometric analysis*. Universitext. Springer-Verlag, Berlin, fifth edition, 2008.

[JSB00] Mark S. Joshi and Antonio Sa Barreto. Inverse scattering on asymptotically hyperbolic manifolds. *Acta Math.*, 184(1):41–86, 2000.

[Kli17] Michael V. Klibanov. A phaseless inverse scattering problem for the 3-D Helmholtz equation. *Inverse Probl. Imaging*, 11(2):263–276, 2017.

[Kni02] Gerhard Knieper. Hyperbolic dynamics and Riemannian geometry. In *Handbook of dynamical systems, Vol. 1A*, pages 453–545. North-Holland, Amsterdam, 2002.

[KS14] Carlos Kenig and Mikko Salo. Recent progress in the Calderón problem with partial data. In *Inverse problems and applications*, volume 615 of *Contemp. Math.*, pages 193–222. Amer. Math. Soc., Providence, RI, 2014.

[KW74] Jerry L. Kazdan and F. W. Warner. Curvature functions for open 2-manifolds. *Ann. of Math. (2)*, 99:203–219, 1974.

[Lee97] John M. Lee. *Riemannian manifolds*, volume 176 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. An introduction to curvature.

[Leh16] J. Lehtonen. The geodesic ray transform on two-dimensional Cartan-Hadamard manifolds. *ArXiv e-prints*, December 2016.

[LP89] Peter D. Lax and Ralph S. Phillips. *Scattering theory*, volume 26 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, second edition, 1989. With appendices by Cathleen S. Morawetz and Georg Schmidt.

[Muk77] R.G. Mukhometov. The problem of recovery of a two-dimensional Riemannian metric and integral geometry. *Sov. Math., Dokl.*, 18:27–31, 1977.

[Nat01] F. Natterer. *The mathematics of computerized tomography*, volume 32 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001. Reprint of the 1986 original.

[Pat99] Gabriel P. Paternain. *Geodesic flows*, volume 180 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1999.

[Pet06] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.

[PS88] L. N. Pestov and V. A. Sharafutdinov. Integral geometry of tensor fields on a manifold of negative curvature. *Sibirsk. Mat. Zh.*, 29(3):114–130, 221, 1988.

[PSU13] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Tensor tomography on surfaces. *Invent. Math.*, 193(1):229–247, 2013.

[PSU14] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Tensor tomography: progress and challenges. *Chin. Ann. Math. Ser. B*, 35(3):399–428, 2014.

[PSU15] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Invariant distributions, Beurling transforms and tensor tomography in higher dimensions. *Math. Ann.*, 363(1-2):305–362, 2015.

[Rad17] J. Radon. Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. Leipz. Ber. 69, 262-277 (1917)., 1917.

[Sas62] Shigeo Sasaki. On the differential geometry of tangent bundles of Riemannian manifolds. II. *Tôhoku Math. J. (2)*, 14:146–155, 1962.

[Sha94] V. A. Sharafutdinov. *Integral geometry of tensor fields*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.

[Wed14] Ricardo Weder. High-velocity estimates, inverse scattering and topological effects. In *Spectral theory and differential equations*, volume 233 of *Amer. Math. Soc. Transl. Ser. 2*, pages 225–251. Amer. Math. Soc., Providence, RI, 2014.

[Zal82] Lawrence Zalcman. Uniqueness and nonuniqueness for the Radon transform. *Bull. London Math. Soc.*, 14(3):241–245, 1982.

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