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GENERALIZED LEBESGUE POINTS FOR SOBOLEV FUNCTIONS

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Abstract. In many recent articles, medians have been used as a replacement of integral averages when the function fails to be locally integrable. A point x in a metric measure space (X, d, μ) is called a generalized Lebesgue point of a measurable function f if the medians of f over the balls $B(x, r)$ converge to $f(x)$ when r converges to 0. We know that almost every point of a measurable, almost everywhere finite function is a generalized Lebesgue point and the same is true for every point of a continuous function. We show that a function $f \in M^{s,p}(X)$, $0 < s \leqslant 1$, $0 < p < 1$, where X is a doubling metric measure space, has generalized Lebesgue points outside a set of $\mathcal{H}^h$-Hausdorff measure zero for a suitable gauge function h.

Keywords: Sobolev space; metric measure space; median; generalized Lebesgue point

MSC 2010: 46E35, 28A78

1. Introduction

By the Lebesgue differentiation theorem, almost every point in $\mathbb{R}^n$ is a Lebesgue point of a locally integrable function, that is

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\,\mathrm{d}y = u(x)$$

for almost every $x \in \mathbb{R}^n$ and for a locally integrable function u. It is a well-known fact that a function $f \in W^{1,p}(\mathbb{R}^n)$, $1 \leqslant p \leqslant n$, has Lebesgue points outside a set of p-capacity zero, e.g. [4], [27], [13]. Recently, there has been some interests in studying Lebesgue points for Sobolev functions on metric measure spaces, especially for functions in Hajłasz-Sobolev space $M^{1,p}(X)$ and in Newtonian space (or Sobolev

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space) $N^{1,p}(X)$ defined by Hajłasz in [8] and Shanmugalingam in [24], respectively. The usual argument for obtaining the existence of Lebesgue points outside a small set for a Sobolev function goes as follows. First, Lebesgue points exist outside a set of capacity zero, see [17], [16] for Sobolev functions on metric measure spaces. Second, each set of positive Hausdorff h-measure for a suitable h is of positive capacity, see [20], [1], [22] for sets in $\mathbb{R}^n$ and [2], [14], [16] for sets in metric measure spaces. Combining these results, one gets the existence of Lebesgue points outside a set of Hausdorff h-measure zero for a suitable h, see [15] for more details on this.

In this paper, we study the existence of Lebesgue points of a function in Hajłasz-Sobolev space $M^{s,p}(X)$ for $0 < s \leqslant 1$, $0 < p < 1$, outside a small set in terms of Hausdorff h-measure. Recall that a measurable function $f\colon X \to \mathbb{R}$, where $X = (X, d, \mu)$ is a metric measure space, belongs to the Hajłasz-Sobolev space $M^{s,p}(X)$, $0 < s \leqslant 1$, $p > 0$ if and only if $f \in L^p(X)$ and there exists a nonnegative function $g \in L^p(X)$ such that the inequality

$$|f(x) - f(y)| \leqslant d(x,y)^s(g(x) + g(y)) \tag{1.1}$$

holds for all $x, y \in X \setminus E$, where $\mu(E) = 0$. This definition is due to Hajłasz for $s = 1$, see [8], and to Yang for fractional scales, see [26].

Recently, Heikkinen, Koskela and Tuominen have studied the existence of generalized Lebesgue points for functions in $M^{s,p}(X)$, $0 < s \leqslant 1$, $0 < p < \infty$, outside a set of capacity zero, see [12]. They have also studied the same question for functions in Hajłasz-Besov spaces $N^s_{p,q}$ and Hajłasz-Triebel-Lizorkin spaces $M^s_{p,q}$. Notice that $M^s_{p,\infty}(X) = M^{s,p}(X)$, see [19]. The existence of Lebesgue points outside a small set in terms of capacity for Besov and Triebel-Lizorkin functions has been studied in [1], [11], [21] and the relation between Besov capacity and Hausdorff measures has been studied in [3]. In this paper we only consider functions in Hajłasz-Sobolev spaces and avoid the use of capacity. The idea of avoiding capacity and proving the result directly for Hausdorff measures has appeared in many papers, see for example [9] and [15]. Here we use medians to define generalized Lebesgue points, as our functions may fail to be locally integrable. Medians allow us to study the oscillation of measurable functions. See Section 2 for the definitions of medians and generalized Lebesgue points. Medians have been studied for example in [23], [7], [6], [25]. Our result is the following.

Theorem 1.1. *Let (X, d, μ) be a doubling metric measure space. Let $f \in M^{s,p}(X)$, where $0 < s \leqslant 1$, $0 < p < 1$. Then $\lim\limits_{r\to 0} m_f^\gamma(B(z,r))$ exists outside a set E_ε with $\mathcal{H}^h(E_\varepsilon) = 0$ whenever*

$$h(B(x,\varrho)) = \frac{\mu(B(x,\varrho))}{\varrho^{sp}} \log^{-p-\varepsilon}\left(\frac{1}{\varrho}\right) \quad \textit{and} \quad \varepsilon > 0.$$

We refer to Section 2 for the definition of generalized Hausdorff h-measure and also for the existence of the above limit outside a small set for a measurable and almost everywhere finite function.

Our result seems to be new even in $\mathbb{R}^n$. For $f \in M^{1,p}(\mathbb{R}^n)$, where $n/(n+1) < p < 1$, we can use integral averages instead of medians by the recent result of Koskela and Saksman in [18]. They have proved that each function $f \in M^{1,p}(\mathbb{R}^n)$ for $p > n/(n+1)$ is locally integrable. More details are given in Remark 3.2.

2. Notation and preliminaries

We assume throughout that $X = (X, d, \mu)$ is a metric measure space equipped with a metric d and a Borel regular outer measure μ. We call such a μ a measure. The Borel regularity of the measure μ means that all Borel sets are μ-measurable and that for every set $A \subset X$ there is a Borel set D such that $A \subset D$ and $\mu(A) = \mu(D)$. We denote the open ball in X with center $x \in X$ and radius $0 < r < \infty$ by $B(x,r) = \{y \in X\colon\ d(y,x) < r\}$. A Borel regular measure μ on a metric space (X, d) is called a *doubling measure* if every ball in X has a positive and finite measure and there exists a constant $C_\mu \geqslant 1$ such that $\mu(B(x,2r)) \leqslant C_\mu\, \mu(B(x,r))$ holds for each $x \in X$ and $r > 0$. We call the triple (X, d, μ) a *doubling metric measure space* if μ is a doubling measure on X.

Definition 2.1. Let $0 < \gamma \leqslant 1/2$. The γ-median $m_f^\gamma(A)$ of a measurable, almost everywhere finite function f over a set $A \subset X$ of finite measure is

$$m_f^\gamma(A) = \sup\{M \in \mathbb{R}\colon\ \mu(\{x \in A\colon\ f(x) < M\}) \leqslant \gamma\mu(A)\}.$$

Definition 2.2. Let $0 < \gamma \leqslant 1/2$ and let f be a measurable, almost everywhere finite function. A point $x \in X$ is a *generalized Lebesgue point* of f if

$$\lim_{r\to 0} m_f^\gamma(B(x,r)) = f(x).$$

We mention here the basic property of medians, for more properties and details see [23], [12]. These two references guarantee that almost every point is a generalized Lebesgue point.

Theorem 2.3. *There exists a set E with $\mu(E) = 0$ such that*

$$\lim_{r\to 0} m_f^\gamma(B(x,r)) = f(x)$$

for every $0 < \gamma \leqslant 1/2$ and $x \in X \setminus E$.

We recall that the *generalized Hausdorff h-measure* is defined by

$$\mathcal{H}^h(E) = \limsup_{\delta\to 0} H^h_\delta(E),$$

where

$$H^h_\delta(E) = \inf\Big\{\sum h(B(x_i,r_i))\colon\ E\subset\bigcup B(x_i,r_i),\ r_i\leqslant\delta\Big\},$$

where the dimension gauge function h is required to be continuous and increasing with $h(0)=0$, see [16].

For the readers's convenience we state here a fundamental covering lemma (for the proof see [5], 2.8.4–6, or [27], Theorem 1.3.1).

Lemma 2.4 (5B-covering lemma). *Every family $\mathcal{F}$ of balls of uniformly bounded diameters in a metric space X contains a pairwise disjoint subfamily $\mathcal{G}$ such that for every $B\in\mathcal{F}$ there exists $B'\in\mathcal{G}$ such as $B\cap B'\neq\emptyset$ and* $\operatorname{diam}(B)<2\operatorname{diam}(B')$. *In particular, we have that*

$$\bigcup_{B\in\mathcal{F}} B\subset\bigcup_{B\in\mathcal{G}} 5B.$$

3. Proof of Theorem 1.1

Fix $\varepsilon>0$. Let h be as in the statement of Theorem 1.1. Let us write $B_j = B(z,2^{-j})$ for $z\in X$ and $j\in\mathbb{N}$. Our first aim is to show that the sequence $(m^\gamma_f(B_j))_j$ is a Cauchy sequence outside a set of $\mathcal{H}^h$-measure zero. Let us also write $B^l_j = \{x\in B_j\colon\ f(x)\leqslant m^\gamma_f(B_j)\}$ and $B^u_j=\{y\in B_j\colon\ f(y)\geqslant m^\gamma_f(B_j)\}$ for all $j\in\mathbb{N}$. Then from the definition of the median it easily follows that for all $j\in\mathbb{N}$

$$\mu(B^l_j)\geqslant\gamma\mu(B_j)\quad\text{and}\quad\mu(B^u_j)\geqslant(1-\gamma)\mu(B_j). \tag{3.1}$$

Suppose first that $m^\gamma_f(B_j)\geqslant m^\gamma_f(B_{j+1})$. Using inequality (1.1) and the Fubini theorem, we obtain

$$\begin{aligned}
&\mu(B^u_j)\,\mu(B^l_{j+1})|m^\gamma_f(B_j)-m^\gamma_f(B_{j+1})|^p\\
&\qquad\leqslant\int_{B^u_j}\int_{B^l_{j+1}}|f(x)-f(y)|^p\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\
&\qquad\leqslant\int_{B^u_j}\int_{B^l_{j+1}}d(x,y)^{sp}(g(x)+g(y))^p\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\
&\qquad\leqslant 2^p\int_{B^u_j}\int_{B^l_{j+1}}d(x,y)^{sp}(g^p(x)+g^p(y))\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)
\end{aligned}$$

$$
\begin{aligned}
&= 2^{2p}2^{-spj}\mu(B_j^u)\int_{B_{j+1}^l} g^p(x)\,\mathrm{d}\mu(x)\\
&\quad + 2^{2p}2^{-spj}\mu(B_{j+1}^l)\int_{B_j^u} g^p(x)\,\mathrm{d}\mu(x).
\end{aligned}
$$

Using the doubling property and inequalities in (3.1), we get

$$
\begin{aligned}
(3.2)\quad |m_f^\gamma(B_j) - m_f^\gamma(B_{j+1})|^p &\leqslant 2^{2p}2^{-spj}\left[\frac{\mu(B_j)}{\mu(B_{j+1}^l)} + \frac{\mu(B_j)}{\mu(B_j^u)}\right]\fint_{B_j} g^p(x)\,\mathrm{d}\mu(x)\\
&\leqslant 2^{2p}2^{-spj}\left[\frac{C_\mu}{\gamma} + \frac{1}{1-\gamma}\right]\fint_{B_j} g^p(x)\,\mathrm{d}\mu(x)\\
&= C_1 2^{-spj}\fint_{B_j} g^p(x)\,\mathrm{d}\mu(x),
\end{aligned}
$$

where $C_1 = 2^{2p}[C_\mu/\gamma + 1/(1-\gamma)]$.

Next, suppose that $m_f^\gamma(B_j) \leqslant m_f^\gamma(B_{j+1})$. Replacing B_j^u by B_j^l and B_{j+1}^l by B_{j+1}^u we repeat the above argument to obtain inequality (3.2) with the constant $C_2 = 2^{2p}[C_\mu/(1-\gamma) + 1/\gamma]$.

By choosing $C = \max\{C_1, C_2\}$ we conclude that

$$
|m_f^\gamma(B_j) - m_f^\gamma(B_{j+1})|^p \leqslant C2^{-spj}\fint_{B_j} g^p(x)\,\mathrm{d}\mu(x)
$$

holds for all possible values of $m_f^\gamma(B_j)$, $m_f^\gamma(B_{j+1})$ and all j. For $m, l \in \mathbb{N}$, $m < l$ this gives the estimate

$$
\begin{aligned}
(3.3)\qquad |m_f^\gamma(B_l) - m_f^\gamma(B_m)| &\leqslant \sum_{j=m}^{l-1} |m_f^\gamma(B_j) - m_f^\gamma(B_{j+1})|\\
&\leqslant C\sum_{j=m}^{l-1} 2^{-sj}\Big(\fint_{B_j} g^p(x)\,\mathrm{d}\mu(x)\Big)^{1/p}.
\end{aligned}
$$

Let $h_1(B(x,\varrho)) = \mu(B(x,\varrho))/\varrho^{sp}\log^{-p-\varepsilon/2}(1/\varrho)$. If $\int_{B(z,r)} g^p\,\mathrm{d}x \leqslant Ch_1(B(z,r))$ for all sufficiently small $0 < r < 1/5$, then $(m_f^\gamma(B_j))_j$ is a Cauchy sequence, by (3.3). On the other hand, let us consider the set

$$
\begin{aligned}
E_\varepsilon = \Big\{&z \in X\colon \text{ there exists arbitrarily small } 0 < r_z < \frac{1}{5}\\
&\text{such that } \int_{B(z,r_z)} g^p\,\mathrm{d}\mu(x) \geqslant Ch_1(B(z,r_z))\Big\}.
\end{aligned}
$$

Let $0 < \delta < 1/5$. By using the 5B-covering lemma, we get a pairwise disjoint family $\mathcal{G}$ consisting of balls as above such that

$$E_\varepsilon \subset \bigcup_{B\in\mathcal{G}} 5B,$$

where $\operatorname{diam}(B) < 2\delta$ for $B \in \mathcal{G}$. Then

$$\begin{aligned}\mathcal{H}^{h_1}_{10\delta}(E_\varepsilon) &\leqslant C \sum_{B\in\mathcal{G}} h_1(B(z, 5r_z)) \\ &\leqslant C \sum_{B\in\mathcal{G}} \int_B g^p \,\mathrm{d}\mu(x) \\ &\leqslant C \int_{\bigcup\limits_{B\in\mathcal{G}} B} g^p \,\mathrm{d}\mu(x) < \infty.\end{aligned}$$

It follows that $\mathcal{H}^{h_1}(E_\varepsilon) < \infty$ and hence we have that $\mathcal{H}^h(E_\varepsilon) = 0$, which yields the existence of $\lim\limits_{j\to\infty} m_f^\gamma(B(z, 2^{-j}))$ for $\mathcal{H}^h$-a.e. $z \in X$.

For given $r > 0$, we can always find $j \in \mathbb{N}$ such that $2^{-(j+1)} < r < 2^{-j}$. Using the same method as above, we conclude that

$$|m_f^\gamma(B_j) - m_f^\gamma(B(z, r))| \leqslant C2^{-spj} \fint_{B_j} g^p(x) \,\mathrm{d}\mu(x)$$

and that $\lim\limits_{r\to 0} m_f^\gamma(B(z, r))$ exists outside E_ε. □

Remark 3.1. It is known that $f \in M^{1,1}(X)$ has Lebesgue points outside a set E with $\mathcal{H}^h(E) = 0$ with $h(B(x, \varrho)) = \mu(B(x, \varrho))/\varrho$ provided X supports a 1-Poincaré inequality, [16]. We do not know if one can obtain a better result than Theorem 1.1 for $f \in M^{1,p}(X)$ by showing that the exceptional set has $\mathcal{H}^h$-Hausdorff measure zero with $h(B(x, \varrho)) = \mu(B(x, \varrho))/\varrho^p$. In $\mathbb{R}^n$, one possible approach is to use the Riesz potential after inequality (3.3), as shown below.

It is easy to see from (3.3) that

$$\begin{aligned}|m_f^\gamma(B_l) - m_f^\gamma(B_m)| &\leqslant C\biggl(\sum_{j=m}^{l-1} 2^{-jp} \fint_{B_j} g^p(x) \,\mathrm{d}x\biggr)^{1/p} \\ &\leqslant C\biggl(\int_{B_m} \frac{g^p(x)}{|z - x|^{n-p}} \,\mathrm{d}x\biggr)^{1/p} \\ &= CI_p^{B_m} g^p(z),\end{aligned}$$

where $I_p^{B_m} g^p(z)$ is the Riesz potential (local version) of g^p. Then we use Theorem 3.1.4 (a) of [1] to conclude that $\lim\limits_{r\to 0} m_f^\gamma(B(z, r))$ exists outside E with $\mathcal{L}^n(E) = 0$.

It would be interesting to know if there is an estimate similar to that in Theorem 3.1.4 (a) of [1] for the $\mathcal{H}^{n-\alpha}$-Hausdorff measure of the set $\{z\colon\ I_\alpha u(z) > \lambda\}$, for $u \in L^1(\mathbb{R}^n)$, $0 < \alpha < n$ and for all $\lambda > 0$. This would improve our result in this case.

Remark 3.2. In $\mathbb{R}^n$, for the case when $n/(n+1) < p < 1$, we use telescoping arguments between the centred balls and also inequality (1.1) to get an estimate similar to that in (3.3) for the integral averages instead of medians. Similar technique can be found in [10]. Then it is easy to see that $\lim\limits_{r\to 0} f_{B(z,r)}$ exists outside a set of $\mathcal{H}^h$-measure zero with the same h as in Theorem 1.1.

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