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A PARALLEL FE-SPLITTING-UP METHOD TO PARABOLIC PROBLEMS

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Abstract. A new efficient method for solving parabolic systems is presented. The proposed method is based on the splitting up principle in which the problem is reduced to a series of independent 1-D problems. This enables the use of parallel processors. We can solve multidimensional problems only by applying the 1-D FE-method and consequently avoid the difficulties in constructing a FE-space for multidimensional problems. The method is suitable for general domains as well as rectangle domains. Every 1-D subproblem is solved by applying cubic B-splines. Several numerical examples are presented.

1. Introduction

We consider the parabolic boundary value problem

$$(1.1) \quad \begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(a_1 \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(a_2 \frac{\partial u}{\partial y} \right) + a_3 \frac{\partial u}{\partial x} + a_4 \frac{\partial u}{\partial y} + a_5 u + f & \text{in } \Omega \times [0, T] \\ u(x, y, 0) = u_0(x, y) & , \quad (x, y) \in \Omega \\ u = 0 & \text{on } \partial\Omega \times [0, T]. \end{cases}$$

We shall suppose for simplicity that $\Omega \subset \mathbf{R}^2$ is convex. Here $a_5 \leq 0$.

We shall propose the following method to solve problem (1.1). First, we divide the time interval $[0, T]$ into N subintervals $[t_i, t_{i+1}]$, $i = 0, 1, 2, \dots, N-1$, $t_{i+1} - t_i = \tau$. Then in each interval $[t_i, t_{i+1}]$ we first solve for fixed $y \in (y_{\min}, y_{\max})$ the subproblem

$$(1.2) \quad \begin{cases} \frac{\partial u_1(x, y, t)}{\partial t} = \frac{\partial}{\partial x} \left(a_1(x, y) \frac{\partial u_1(x, y, t)}{\partial x} \right) \\ \quad + a_3(x, y) \frac{\partial u_1(x, y, t)}{\partial x} & , \quad x \in (d_1(y), d_2(y)), \quad t \in [t_i, t_{i+1}] \\ u_1(x, y, t_i) = u_2(x, y, t_i) \\ u_1(d_1(y), y, t) = u_1(d_2(y), y, t) = 0 & , \quad t \in [t_i, t_{i+1}]. \end{cases}$$

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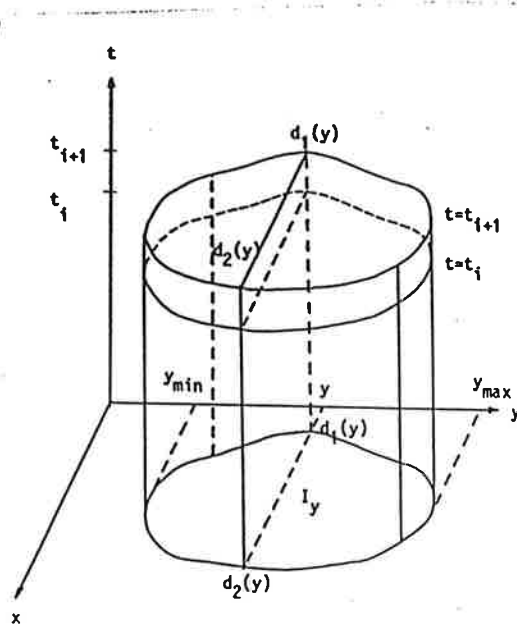


Figure 1.1

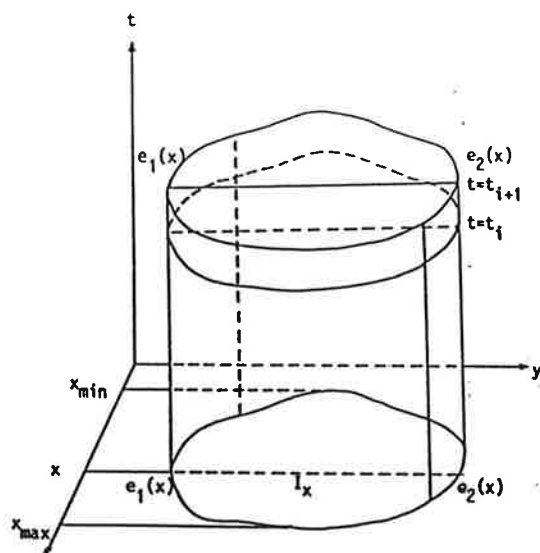


Figure 1.2

For $t_0 = 0$, we take $u_1(x, y, t_0) = u_0(x, y)$.

For the used notations we refer to Figure 1.1. With every fixed $y \in (y_{\min}, y_{\max})$, (1.2) is a 1-D partial differential equation on $I_y \equiv (d_1(y), d_2(y))$ and we can use an efficient 1-D FE-method to solve it. In fact, we use a cubic B-spline function as the FE-basis to solve this 1-D equation with $O(h^4)$ accuracy. Another very important point of this method is that we can use parallel processors to solve the 1-D equation (1.2) for various y . In numerical realization we solve (1.2) for finite number of lines in the interval $[y_{\min}, y_{\max}]$.

After we solve (1.2) for every y in $[t_i, t_{i+1}]$, we take $u_1(x, y, t_{i+1})$ as an initial value for $u_2(x, y, t)$ at $t = t_i$ in Ω . Then we proceed to solve for every fixed $x \in (x_{\min}, x_{\max})$ the

following problem:

$$(1.3) \quad \begin{cases} \frac{\partial u_2(x, y, t)}{\partial t} = \frac{\partial}{\partial y} \left(a_2(x, y) \frac{\partial u_2(x, y, t)}{\partial y} \right) + a_4(x, y) \frac{\partial u_2(x, y, t)}{\partial y} \\ + a_5(x, y) u_2(x, y, t) + f(x, y, t), \quad t \in [t_i, t_{i+1}], \quad y \in (e_1(x), e_2(x)) \\ u_2(x, y, t_i) = u_1(x, y, t_{i+1}) \\ u_2(x, e_1(x), t) = u_2(x, e_2(x), t) = 0, \quad t \in [t_i, t_{i+1}]. \end{cases}$$

For the notations used see Figure 1.2. We note that (1.3) is a 1-D problem on $I_x = (e_1(x), e_2(x))$. We also use a 1-D cubic B-spline function as a FE-basis to solve (1.3). When we get the value of u_2 at $t = t_{i+1}$ on Ω we take $u_2(x, y, t_{i+1})$ as a solution $u(x, y, t_{i+1})$ and then proceed to the next interval $[t_{i+1}, t_{i+2}]$ by solving (1.2) first and then (1.3) etc. Also, in practical computing we only solve (1.3) on finite number of lines.

The splitting-up method is already a well studied method, especially for linear problems [5,9,15]. But in the literature, a lot of work involves the finite difference method. As in [9] and cited papers therein, it seems that authors always first discretize the problem and then split the matrix. This technique is suitable for the finite difference method. In [4] the finite element method was used only for rectangular domains with the splitting-up method. For a general domain, the authors in paper [10] again first discretize the problem by using a multidimensional finite element space and then split the matrix into four parts. In a recent paper [2] the one dimensional finite element method was used in combination with the splitting-up method and some other methods, but the study is restricted only to rectangle domains. In this paper we use a 1-D finite element to solve multidimensional problems in general domains. We give a new settlement which gives a clear view on how to use the 1-D finite element techniques for the splitting-up method. The computing of this method can also be don by parallel processors.

We shall restrict our consideration to 2-D problems only because of the notational simplicity. We consider the homogenous Dirichlet boundary, for the sake of simplicity, in the proof. The method also works for nonhomogenous boundary conditions.

In §2, we shall give the detailed analysis of the method and prove the convergence rate of the splitting-up method. Because, by the splitting-up method, the multidimensional problems are solved only by a 1-D FE-method, in §3 we describe the numerical implementation with cubic splines for the 1-D FE-method. With the method from §2 and §3, in §4 we shall give the numerical experiments for 2-D and 3-D problems. Because only a 1-D FE-method is used, this gives crucial advantages in the distributed parameter identification problems [14].

2. The splitting-up method

For simplicity we suppose that $a_3 = 0$, $a_4 = 0$. Let the matrix $a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ satisfy the ellipticity conditions. In our paper, we assume functions u , f , u_0 , a and $\partial\Omega$ are smooth enough. For extensions see Remark 5.1 in §5.

In our paper, we use standard notations for Sobolev spaces $H^k(\Omega)$ and $H_0^k(\Omega)$, $k \geq 0$ is an integer and spaces $C^k(\Omega)$, $C_0^\infty(\Omega)$ with continuous functions and test functions. Especially we write $L^2(\Omega)$ for $H^0(\Omega)$ with scalar product $(\cdot, \cdot)_{0, \Omega}$ and norm $\|\cdot\|_{L^2(\Omega)}$.

We define the bilinear forms in $H_0^1(\Omega) \times H_0^1(\Omega)$

$$(2.1) \quad \begin{aligned} b(u, v) &= - \int_{\Omega} a \nabla u \cdot \nabla v \, dx dy + \int_{\Omega} a_5 uv \, dx dy & u, v \in H_0^1(\Omega) \\ b_1(u, v) &= - \int_{\Omega} a_1 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx dy & u, v \in H_0^1(\Omega) \\ b_2(u, v) &= - \int_{\Omega} a_2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \, dx dy + \int_{\Omega} a_5 uv \, dx dy & u, v \in H_0^1(\Omega) \end{aligned}$$

and three operators A, A_1, A_2

$$\begin{aligned} (Au, v)_{0, \Omega} &= b(u, v) & u \in \text{Dom}(A) & \forall v \in H_0^1(\Omega) \\ (A_1 u, v)_{0, \Omega} &= b_1(u, v) & u \in \text{Dom}(A_1) & \forall v \in H_0^1(\Omega) \\ (A_2 u, v)_{0, \Omega} &= b_2(u, v) & u \in \text{Dom}(A_2) & \forall v \in H_0^1(\Omega). \end{aligned}$$

To explain the meaning of operator A above, let $u \in H_0^1(\Omega)$ be given. If $v \rightarrow b(u, v)$ is continuous in the $L^2(\Omega)$ -topology, then we say $u \in \text{Dom}(A)$. Because $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, we can extend it to $L^2(\Omega)$, then by the Riesz representation theorem, we know there exists a unique element $Au \in L^2(\Omega)$, which satisfies

$$(Au, v)_{0, \Omega} = b(u, v) \quad \forall v \in H_0^1(\Omega).$$

We define this relation from u to Au as the operator A . A_1, A_2 are defined in the same way.

From the definition, it is obvious that $\text{Dom}(A_1) \cap \text{Dom}(A_2) = \text{Dom}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. For $u \in \text{Dom}(A)$

$$Au = A_1 u + A_2 u.$$

By operator A , we can write (1.1) as an abstract evolution equation in $L^2(\Omega)$.

$$(2.2) \quad \begin{cases} \frac{du}{dt} = Au + f, & u \in \text{Dom}(A) \\ u(0) = u_0. \end{cases}$$

The concepts about semigroups used here can be found in [11].

Concerning operators A, A_1 and A_2 , we can prove the following theorem:

THEOREM 1. *Operator A, A_1 and A_2 generate a C_0 -semigroup of contractions $T_0(t), T_1(t)$ and $T_2(t)$, respectively.*

REMARK 2.1. In the proof, we not only prove the result of the theorem, but we also show the 2-D variational equations are equivalent to a series of 1-D variational equations.

PROOF: Operator A obviously generates a C_0 -semigroup of contractions. So we only need to prove the same is true for A_1, A_2 . Because

$$(A_1 u, u) = - \int_{\Omega} a_1 \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} \, dx \leq 0.$$

and from the Lumer–Philips theorem [11, p. 14], we only need to prove that for $\lambda > 0$

$$R(\lambda I - A_1) = L^2(\Omega).$$

This means for every $\lambda > 0$, $f \in L^2(\Omega)$, there exists $u_1 \in \text{Dom}(A_1)$ which satisfies

$$(\lambda I - A_1)u_1 = f.$$

This is equivalent to showing that $u_1 \in H_0^1(\Omega)$ is the solution of the variational equation

$$(2.3) \quad \int_{\Omega} \left(a_1 \frac{\partial u_1}{\partial x} \frac{\partial v}{\partial x} + \lambda u_1 v \right) dx dy = \int_{\Omega} f v dx dy \quad \forall v \in H_0^1(\Omega).$$

Next we shall show that the 2-D problem (2.3) can be reduced to 1-D problems.

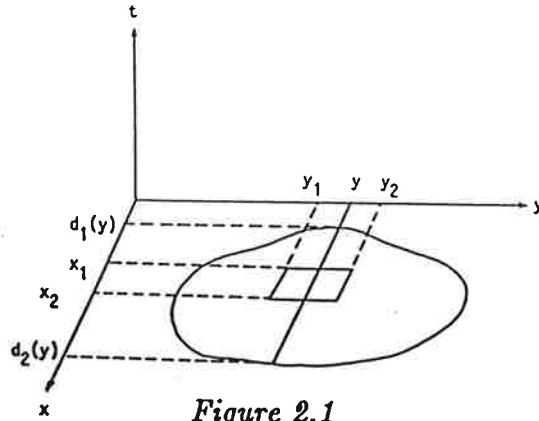


Figure 2.1

For every fixed y , we consider the following 1-D problem (see Fig.2.1):

$$\begin{cases} -\frac{\partial}{\partial x} \left(a_1 \frac{\partial u_1}{\partial x} \right) + \lambda u_1 = f(x, y) \\ u_1 \in H_0^1((d_1(y), d_2(y))). \end{cases}$$

The Lax–Milgram theorem implies that there exists a unique $g^y(x) \in H_0^1((d_1(y), d_2(y)))$ such that

$$(2.4) \quad \int_{d_1(y)}^{d_2(y)} \left(a_1 \frac{\partial g^y}{\partial x} \frac{\partial v_1}{\partial x} + \lambda g^y v_1 \right) dx = \int_{d_1(y)}^{d_2(y)} f(\cdot, y) v_1 dx \quad \forall v_1 \in H_0^1((d_1(y), d_2(y))).$$

For different y , the corresponding solution $g^y(x)$ is also different. Let $\tilde{u}_1(x, y) := g^y(x)$. For a square $(x_1, x_2) \times (y_1, y_2) \subset \Omega$, we choose $v_1(x) \in H_0^1((x_1, x_2))$, and let it be zero elsewhere. So $v_1(x) \in H_0^1((d_1(y), d_2(y)))$ for every $y \in (y_1, y_2)$. We also choose $w_2(y) \in H_0^1((y_1, y_2))$ and let it be zero elsewhere. Consequently, $v_1(x)w_2(y) \in H_0^1(\Omega)$ and we get

by (2.4)

$$\begin{aligned}
 & \int_{\Omega} \left(a_1 \frac{\partial \tilde{u}_1}{\partial x} \frac{\partial v_1}{\partial x} + \lambda \tilde{u}_1 v_1 \right) w_2 \, dx dy \\
 (2.5) \quad & = \int_{y_1}^{y_2} w_2(y) \int_{x_1}^{x_2} \left(a_1 \frac{\partial \tilde{u}_1}{\partial x} \frac{\partial v_1}{\partial x} + \lambda \tilde{u}_1 v_1 \right) \, dx dy \\
 & = \int_{y_1}^{y_2} w_2(y) \int_{d_1(y)}^{d_2(y)} \left(a_1 \frac{\partial g^y}{\partial x} \frac{\partial v_1}{\partial x} + \lambda g^y v_1 \right) \, dx dy \\
 & = \int_{\Omega} f v_1 w_2 \, dx dy.
 \end{aligned}$$

In (2.4) g^y depends continuously on a_1 and f . From this we can see that \tilde{u} is measurable on Ω . So, the conditions for the Fubini theorem are satisfied.

The product functions, $v_1(x)w_2(y)$, in rectangles inside Ω are dense in $C_0^\infty(\Omega)$. For example, we can divide Ω into rectangles as in Figure 4.1 with uniform mesh and get Ω_h and take S^h to be the linear product FE-space on Ω_h [3, p. 56, Th. 2]. Then if $\partial\Omega$ is regular enough, for every $w \in C_0^\infty(\Omega)$ there exists $v_h w_h \in S^h$ such that

$$\begin{aligned}
 & \|w - v_h w_h\|_{H^1(\Omega)} \\
 & \leq \|w - v_h w_h\|_{H^1(\Omega \setminus \Omega_h)} + \|w - v_h w_h\|_{H^1(\Omega_h)} \\
 & \leq C \text{meas}(\Omega \setminus \Omega_h) + Ch.
 \end{aligned}$$

By choosing $v_1 = v_h$ and $w_2 = w_h$ in (2.5) and letting $h \rightarrow 0$ we obtain

$$(2.6) \quad \int_{\Omega} \left(a_1 \frac{\partial \tilde{u}_1}{\partial x} \frac{\partial w}{\partial x} + \lambda \tilde{u}_1 w \right) \, dx dy = \int_{\Omega} f w \, dx dy \quad \forall w(x, y) \in C_0^\infty(\Omega).$$

Because $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, (2.6) is also valid for every $w \in H_0^1(\Omega)$. This proves $\tilde{u}_1(x, y)$ is a solution of (2.3). Therefore

$$R(\lambda I - A_1) = L^2(\Omega).$$

So A_1 generates a C_0 -semigroup of contractions. The same is for A_2 . □

THEOREM 2. For $t > 0$ we have

$$(2.7) \quad \|T_0(t)u_0 - T_2(t)T_1(t)u_0\|_{L^2(\Omega)} = o(t) \quad \text{if } u_0 \in H^2(\Omega) \cap H_0^1(\Omega),$$

and

$$(2.8) \quad \|T_0(t)u_0 - T_2(t)T_1(t)u_0\|_{L^2(\Omega)} \leq Ct^2 \quad \text{if } u_0 \in H^4(\Omega) \cap H_0^1(\Omega),$$

where C is independent of t .

PROOF: From [11, Th. 2.4] we obtain

$$\begin{aligned}
 & T_0(t)u_0 - u_0 - tAu_0 \\
 (2.9) \quad & = \int_0^t (T_0(\tau) - I) Au_0 \, d\tau \\
 & = \int_0^t \int_0^\tau AT_0(s)Au_0 \, ds d\tau = \int_0^t (t - \tau)AT_0(\tau)Au_0 \, d\tau.
 \end{aligned}$$

As the operators are closed and also $u_0 \in D(A)$, we get

$$\frac{d}{dt} T_2(t)T_1(t)u_0 = T_2(t)(A_1 + A_2)T_1(t)u_0$$

$$\frac{d}{dt} T_2(t)(A_1 + A_2)T_1(t)u_0 = T_2(t)A_1T_1(t)A_1u_0 + 2A_2T_2(t)T_1(t)A_1u_0 + A_2T_2(t)A_2T_1(t)u_0.$$

Consequently,

$$\begin{aligned} & T_2(t)T_1(t)u_0 - u_0 - t(A_1 + A_2)u_0 \\ &= \int_0^t T_2(\tau)(A_1 + A_2)T_1(\tau)u_0 d\tau - t(A_1 + A_2)u_0 \\ (2.10) \quad &= \int_0^t \int_0^\tau (T_2(s)A_1T_1(s)A_1u_0 + 2A_2T_2(s)T_1(s)A_1u_0 + A_2T_2(s)A_2T_1(s)u_0) dsd\tau \\ &= \int_0^t (t - \tau)(T_2(\tau)A_1T_1(\tau)A_1u_0 + 2A_2T_2(\tau)T_1(\tau)A_1u_0 + A_2T_2(\tau)A_2T_1(\tau)u_0) d\tau. \end{aligned}$$

Because $Au_0 = A_1u_0 + A_2u_0$, by subtracting (2.10) from (2.9) we obtain

$$\begin{aligned} & \frac{\|T_0(t)u_0 - T_2(t)T_1(t)u_0\|_{L^2(\Omega)}}{t^2} \\ & \leq \frac{1}{t^2} \left\{ \left\| \int_0^t (t - \tau)AT_0(\tau)Au_0 d\tau \right\|_{L^2(\Omega)} \right. \\ & \quad \left. + \left\| \int_0^t (t - \tau)(T_2(\tau)A_1T_1(\tau)A_1u_0 + 2A_2T_2(\tau)T_1(\tau)A_1u_0 + A_2T_2(\tau)A_2T_1(\tau)u_0) d\tau \right\|_{L^2(\Omega)} \right\} \\ & \leq \frac{1}{2} \left\{ \max_{0 \leq \tau \leq t} \|AT_0(\tau)Au_0\|_{L^2(\Omega)} \right. \\ & \quad \left. + \max_{0 \leq \tau \leq t} \|T_2(\tau)A_1T_1(\tau)A_1u_0 + 2A_2T_2(\tau)T_1(\tau)A_1u_0 + A_2T_2(\tau)A_2T_1(\tau)u_0\|_{L^2(\Omega)} \right\} \leq C. \end{aligned}$$

From the definitions of the operators and semigroups and also from the regularity property for the parabolic equations, as in [8], we can see that this constant is independent of t but depends on $\|u_0\|_{H^4(\Omega)}$. This proves (2.8).

To prove (2.7) we note that: if $u_0 \in \text{Dom}(A_1) \cap \text{Dom}(A_2)$, then $Au_0 = A_1u_0 + A_2u_0$. So

$$\begin{aligned} & \lim_{t \rightarrow 0^+} \frac{T_0(t)u_0 - T_2(t)T_1(t)u_0}{t} \\ &= \lim_{t \rightarrow 0^+} \left(\frac{T_0(t)u_0 - u_0}{t} - \frac{T_2(t)T_1(t)u_0 - T_2(t)u_0}{t} - \frac{T_2(t)u_0 - u_0}{t} \right) \\ &= Au_0 - (A_1u_0 + A_2u_0) \\ &= 0. \end{aligned}$$

□

According to Theorem 1 and 2 we can discretize the original problem in the following way. We first divide $[0, T]$ into N subintervals

$$[t_i, t_{i+1}] \quad i = 0, 1, 2, \dots, N-1 \quad t_{i+1} - t_i = T/N = \tau.$$

Then at each subinterval $[t_i, t_{i+1}]$, $i = 0, 1, 2, \dots, N-1$, we first solve the subproblem (in the x -direction)

$$(2.11) \quad \begin{cases} \frac{du_1}{dt} = A_1 u_1, & u_1 \in \text{Dom}(A_1) \subset H_0^1(\Omega), \quad t \in [t_i, t_{i+1}] \\ u_1(t_i) = u_2(t_i) \\ \text{if } i = 0, u_1(0) = u_0. \end{cases}$$

Then by moving the information we obtained at the time level t_{i+1} back to the time level t_i we solve the problem (in the y -direction)

$$(2.12) \quad \begin{cases} \frac{du_2}{dt} = A_2 u_2 + f & u_2 \in \text{Dom}(A_2) \subset H_0^1(\Omega), \quad t \in [t_i, t_{i+1}] \\ u_2(t_i) = u_1(t_{i+1}). \end{cases}$$

Now $u_2(t)$ represents the final approximation for $u(t)$ for $t \in [t_i, t_{i+1}]$.

Note that (2.11), (2.12) mean

$$\begin{aligned} u_1(t_{i+1}) &= T_1(\tau)u_2(t_i) \\ u_2(t_{i+1}) &= T_2(\tau)u_1(t_{i+1}) + \int_{t_i}^{t_{i+1}} T_2(t_{i+1} - t)f(t) dt \\ &= T_2(\tau)T_1(\tau)u_2(t_i) + \tau f(t_{i+1}) + O(\tau^2). \end{aligned}$$

But

$$\begin{aligned} u(t_{i+1}) &= T_0(\tau)u(t_i) + \int_{t_i}^{t_{i+1}} T_0(t_{i+1} - t)f(t) dt \\ &= T_0(\tau)u(t_i) + \tau f(t_{i+1}) + O(\tau^2). \end{aligned}$$

Consequently

$$\begin{aligned} &\|u(t_{i+1}) - u_2(t_{i+1})\|_{L^2(\Omega)} \\ &\leq \|T_0(\tau)u(t_i) - T_2(\tau)T_1(\tau)u_2(t_i)\|_{L^2(\Omega)} + C\tau^2 \\ &\leq \|(T_0(\tau) - T_2(\tau)T_1(\tau))u(t_i)\|_{L^2(\Omega)} + \|T_2(\tau)T_1(\tau)(u(t_i) - u_2(t_i))\|_{L^2(\Omega)} + C\tau^2 \\ &\leq \sum_{j=1}^i [\|(T_0(\tau) - T_2(\tau)T_1(\tau))u(t_j)\|_{L^2(\Omega)} + C\tau^2] \\ &\begin{cases} \leq iC\tau^2 & \text{if } u(t) \text{ satisfy (2.8),} \\ = i\sigma(\tau) & \text{if } u(t) \text{ satisfy (2.7).} \end{cases} \end{aligned}$$

Here we use the property that T_1, T_2 are C_0 -semigroups of contractions. Further, if we assume $u_0 \in H^4(\Omega)$ and a_1, a_2 are regular enough, then as in [8], we can prove $u(t) \in H^4(\Omega)$, $\forall t > 0$, so $u(t)$ satisfies (2.8), and also $\|u(t)\|_{H^4(\Omega)}$ is uniformly bounded with t . So the constant C here is independent of τ .

The solutions of (2.11) and (2.12) exist and are unique by Theorem 1. Because A_1, A_2 are all defined on 2-D domains, (2.11) means that u_1 is the variational solution of the problem

$$(2.13) \quad \begin{cases} \frac{d}{dt} (u_1(\cdot, \cdot, t), v(\cdot, \cdot))_{0, \Omega} + \left(a_1(\cdot, \cdot) \frac{\partial u_1}{\partial x}(\cdot, \cdot, t), \frac{\partial v}{\partial x}(\cdot, \cdot) \right)_{0, \Omega} = 0 \\ \forall v(x, y) \in H_0^1(\Omega), \quad t \in [t_i, t_{i+1}] \\ u_1(x, y, t_i) = u_2(x, y, t_i). \end{cases}$$

This is a 2-D problem, we should in practice use a 2-D FE-method to solve it. However, we have the following theorem:

THEOREM 3. *Problem (2.13) is equivalent to the following problem:*

$$(2.14) \quad \begin{cases} \frac{d}{dt} (u_1(\cdot, y, t), v)_{0, I_y} + \left(a_1(\cdot, y) \frac{\partial u_1}{\partial x}(\cdot, y, t), \frac{\partial v}{\partial x}(\cdot, y) \right)_{0, I_y} = 0 \quad \forall v \in H_0^1(I_y) \\ u_1(\cdot, y, t_i) = u_2(\cdot, y, t_i) \end{cases}$$

for every fixed $y \in (y_{\min}, y_{\max})$, where $I_y = (d_1(y), d_2(y))$. Correspondingly, one gets u_2 by solving the 1-D equivalent of form (2.12).

PROOF: As we see in the proof of Theorem 1, when we solve (2.14) for every y , the obtained solution satisfies (2.13). However, because A_1 generates a C_0 -semigroup of contractions, the solution of (2.13) is unique. So they are equivalent to each other. \square

Theorem 3 means that the solution of (2.13) satisfies (2.14) for every y , and when we solve (2.14) for every y , the solution satisfies (2.13) (see Figure 1.1). The corresponding result holds for u_2 as well. For real computings we just compute (2.14) for finite y , for example, in finite number of lines (cf. Fig. 1.1 and Fig. 1.2).

This theorem shows that we can solve (2.11) and (2.12) by a 1-D FE-method (with 1-D cubic spline functions, for example). Moreover, we can solve (2.11) and (2.12) by parallel processing because for different fixed x or y the obtained problems are independent of each other.

In Section 3 and 4 we shall describe the numerical implementation of the proposed method.

3. On solving the 1-D subproblem

For the sake of the multidimensional problems, we should first describe the numerical realization for the 1-D problem

$$(3.1) \quad \begin{cases} \frac{\partial u(x, t)}{\partial t} = \frac{\partial}{\partial x} \left(a(x) \frac{\partial u(x, t)}{\partial x} \right) + f(x, t) \quad \text{in } (x, t) \in (0, 1) \times [0, T] \\ u(x, 0) = u_0(x), \quad x \in (0, 1) \\ u(0, t) = u(1, t) = 0, \quad t \in [0, T]. \end{cases}$$

We use a FE-method to solve problem (3.1). We take a uniform subdivision of $(0, 1)$ with mesh size h to define the nodal points $x_i = ih$, $i = 0, \dots, M$, $M = \frac{1}{h}$. We choose the cubic B-spline φ_i , $i = 0, \dots, M$ as the basis function. In order to give expression for φ_i we first define the following function

$$\varphi(x) = \begin{cases} \frac{(2-x)^3}{6} - \frac{2(1-x)^3}{3} - x^3 + \frac{2(1+x)^3}{3}, & -2 \leq x \leq -1 \\ \frac{(2-x)^3}{6} - \frac{2(1-x)^3}{3} - x^3, & -1 \leq x \leq 0 \\ \frac{(2-x)^3}{6} - \frac{2(1-x)^3}{3}, & 0 \leq x \leq 1 \\ \frac{(2-x)^3}{6}, & 1 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

and then define the functions $\tilde{\varphi}_i$ as follows

$$(3.2) \quad \tilde{\varphi}_i(x) = \varphi\left(\frac{x-x_i}{h}\right).$$

Our basis functions are defined as follows:

$$\varphi_i(x) = \tilde{\varphi}_i(x), \quad i = 2, 3, \dots, M-2$$

and at the end points we define our basis functions as

$$\begin{aligned} \varphi_0 &= \tilde{\varphi}_0 - 4\tilde{\varphi}_{-1} \\ \varphi_1 &= \tilde{\varphi}_1 - \frac{1}{4}\tilde{\varphi}_0 \\ \varphi_{M-1} &= \tilde{\varphi}_{M-1} - \frac{1}{4}\tilde{\varphi}_M \\ \varphi_M &= \tilde{\varphi}_M - 4\tilde{\varphi}_{M+1}, \end{aligned}$$

which satisfy the Dirichlet boundary condition. Here $\tilde{\varphi}_{-1}$, $\tilde{\varphi}_{M+1}$, are the functions as in (3.2), but they are extended out of the interval $(0, 1)$ (see [12, p. 208, 13, p. 73] for details).

The discrete analog of (3.1) reads

$$(3.3) \quad \begin{cases} \frac{d}{dt} \int_0^1 u_h v_h dx + \int_0^1 a \frac{\partial u_h}{\partial x} \frac{\partial v_h}{\partial x} dx = \int_0^1 f v_h dx \quad \forall v_h \in S^h, \quad t \in [0, T] \\ u_h(x, 0) = u_0^h(x), \end{cases}$$

where $u_h(x, t) = \sum_{i=0}^M x_i(t) \varphi_i(x)$ and S^h is the FE-space spanned by the basis functions $\{\varphi_i\}_{i=0}^M$. Now (3.3) leads to the initial value problem:

$$(3.4) \quad \begin{cases} B\dot{x}(t) + Ax(t) = F(t), \quad t \in [0, T] \\ x(0) = x_0, \end{cases}$$

where $B = (b_{ij})_{(M+1) \times (M+1)}$, $A = (a_{ij})_{(M+1) \times (M+1)}$ and $F(t) = (f_i)_{M+1}$ with components

$$b_{ij} = \int_0^1 \varphi_i \varphi_j dx \quad a_{ij} = \int_0^1 a \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial x} dx \quad f_i(t) = \int_0^1 f(t, x) \varphi_i dx.$$

The implicit Euler scheme for solving (3.4) reads

$$(3.5) \quad \begin{cases} \left(\frac{B}{\Delta t} + A \right) x^{n+1} = F^n + \frac{B}{\Delta t} x^n, & n = 0, 1, \dots \\ x^0 = x_0. \end{cases}$$

In the above algorithm we in fact do not need to compute x^0 . Only Bx^0 is needed. As $u_h(x, 0) = \sum x_i(0) \varphi_i(x)$, multiplying both sides by φ_j ($j = 0, 1, \dots, M$) and by integrating we get

$$\sum x_i(0) (\varphi_i, \varphi_j)_{0, \Omega} = (u_h(x, 0), \varphi_j)_{0, \Omega} = (u_0(x), \varphi_j)_{0, \Omega}$$

that is

$$(3.6) \quad Bx^0 = b,$$

where $b = (b_0, \dots, b_M)$ with components $b_i = \int_0^1 u_0 \varphi_i dx$. Note that we do not need to solve the system (3.6), but only to compute b_i (analytically or by applying numerical integration).

The implicit Euler method is unconditionally stable. At the whole the accuracy in above approximation is of order $O(h^4 + \Delta t)$. In the following we give the numerical results, which illustrate the accuracy in L^∞ -norm. In fact we have proved the error estimate in L^2 -norm (which is convenient in the semigroup theory), but in numerical tests we control the pointwise error. This is necessary to control numerically as well as the error in L^2 -norm.

EXAMPLE 3.1. Let $a \equiv 1$, $u = e^{-\pi^2 t} \sin(\pi x)$, $f \equiv 0$ and $T = 1$. Table 3.1 shows the accuracy of the computed solution in L^∞ -norm (maximum norm over inner grid points).

h	τ	h^2	h^3	h^4
1/10		1.74-2	1.81-3	1.86-4
1/20		4.49-3	2.27-4	1.15-5
1/30		2.00-3	6.72-5	2.09-6

Table 3.1. $\|u - u_{h,\tau}\|_{L^\infty}$

In using this 1-D solver to multidimensional problems we encounter the situation that we know $u_0(x)$ only at the node points (see Step 3 in §4). In order to get x_0 in (3.5) we use the following technique.

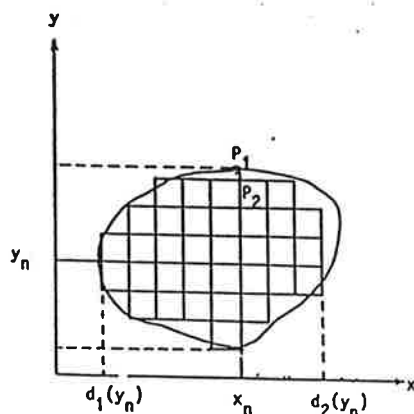


Figure 4.1

For example, when $i = 0$, we take the initial value in (2.11) as $u_1(0) = u_0$. For every y_n call the 1-D solver to solve (2.11) on $(d_1(y_n), d_2(y_n))$ to time $t = \Delta t$.

Then take the value of u_1 at the time $t = \Delta t$ as the initial value for u_2 at $t = 0$ in (2.12). Note that because we solve u_1 discretely, when we solve (2.12) for $x = x_n$, we only know u_1 at the node points. Consequently we use the 1-D solver as described in the second part of §3 to solve (2.11) for every x_n from $t = 0$ to $t = \Delta t$.

Then proceed to the next time interval. Solve (2.11) first for every y_n . Then solve (2.12) for every x_n . By calling the 1-D solver we mean the method described in the second part of §3.

For parallel processors, if we solve (2.11) (see also (1.2) and Fig.1.1) on N lines in the x -direction then we use N processors to compute them and then if we solve (2.12) (see also (1.3) and Fig.1.2) on M lines in the y -direction we can use $M - N$ (we assume $M \geq N$) new processors plus the N processors to compute u_2 . The following examples contain results of the numerical tests. In the example we take $a_1 = a_2 := a, a_3 = a_4 = a_5 := 0$ in (1.1).

EXAMPLE 4.1. We take $\Omega = (0, 1) \times (0, 1)$, $a \equiv e^{xy}$, $u = e^t \sin(\pi x) \sin(\pi y)$ and $T = 1$. Moreover, f and u_0 are obtained through (1.1). The L^∞ -norm errors for various discretization parameters are in Table 4.1.

h	τ	h^2	h^3	h^4
1/5		0.545	0.136	3.208-2
1/10		1.656-1	1.823-2	2.337-3
1/20		4.423-2	2.341-3	1.198-4

Table 4.1. $\|u - u_{h,\tau}\|_{L^\infty}$

With the 1-D solver we also easily get the numerical results for 3-D problems which

show the same convergence rate (see Example 4.2). In the next example the problem

$$(4.1) \quad \begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (a \nabla u) + f(x, y, z, t) \\ u(x, y, z, 0) = u_0(x, y, z) \quad , \quad (x, y, z, t) \in \Omega \times [0, T] \\ u|_{\partial \Omega \times [0, T]} = 0 \end{cases}$$

is solved in $\Omega = (0, 1) \times (0, 1) \times (0, 1)$ with $T = 1$ and a is assumed to be a scalar parameter.

EXAMPLE 4.2. Let $a \equiv 1$, $u = e^{-3\pi^2 t} \sin(\pi x) \sin(\pi y) \sin(\pi z)$ and $f = 0$. Numerical errors at the nodal points are shown in Table 4.2.

h	τ	h^2	h^3	h^4
1/4		7.95-2	2.67-2	7.60-3
1/8		2.65-2	3.50-3	4.16-4
1/16		6.98-3	4.50-4	2.50-5

Table 4.2. $\|u - u_{h,\tau}\|_{L^\infty}$

It is remarkable that in using our method in computing the 3-D problem very small computer storage is needed and we also get very good accuracy. In the paper we require that the parameters a_i , $i = 1, 2, \dots, 5$ are time independent. But this is not essential. We have also done the computing when these parameters depend on t . They also show the same convergence property.

5. Conclusion

From the theoretical analysis and the numerical experiments we should point out that our method has the following good points:

1. We can use a parallel processor to solve the 2- and 3-dimensional problems.
2. With this splitting technique we can solve all multidimensional problems by using only a 1-D finite element method. Because only a 1-D finite space is needed, we have plenty of good finite element spaces available. For example, any order spline functions; P-version FE-spaces, H-P version FE-spaces. From this we also get another very important advantage, that is, we remove all the great difficulty in constructing these FE-spaces for multidimensional domains. In this paper only cubic spline functions are used. These have a convergence order of $(1/N^4)$, but a freedom of $N + 3$. Here only a first order scheme was used for the time variable. In order to improve the convergence order in the time variable a boundary correction is necessary as mentioned in [6].
3. In the computing we only need to solve the 1-D problems and so a small amount of computer storage is needed. This makes it possible to solve very complicated 3-D problems using micro-computers or transputers for example.
4. The input for our method are the boundary points, which is much less than in the standard FEM. Moreover, we can also use nonuniform mesh sizes in our method.

REMARK 5.1. When mixed derivatives occur in (1.1), i.e. $\frac{\partial}{\partial x}(a_{12} \frac{\partial u}{\partial y})$, $\frac{\partial}{\partial y}(a_{21} \frac{\partial u}{\partial x})$, we can use a transform to turn the equation into the form as in (1.1). Accordingly, the boundary is also changed.

REMARK 5.2. It seems quite straightforward to extend the method for the hyperbolic equation

$$u_{tt} = \nabla(a\nabla u) + f$$

and for the fourth order problem

$$u_t = \Delta(a\Delta u) + f$$

and for

$$u_{tt} = \Delta(a\Delta u) + f$$

with appropriate boundary and initial data. The proposed method seems to give excellent results in the parameter identification of these problems [14].

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