ON SUPERCONVERGENCE TECHNIQUES

M. Křížek and P. Neittaanmäki

Preprint 34 December 1984

AMS Subject Classification (1980): 65D, 65J

University of Jyväskylä, Jyväskylä 2017

978-951-39-7029-1 PDF

ISSN 0780-8569

ON SUPERCONVERGENCE TECHNIQUES

Michal Křížek[(1)] and Pekka Neittaanmäki[(2),(3)]

Abstract. A brief survey with a bibliography of superconvergence phenomena in finding a numerical solution of differential and integral equations is presented. A particular emphasis is laid on superconvergent schemes for elliptic problems in plane employing the finite element method.

Keywords: superconvergence, finite elements, differential and integral equations

Subject classifications: AMS (MOS) 65D, 65J, 65L, 65M, 65N, 65P, 65R

1. Introduction

The purpose of this paper is to give a survey of the existing literature on superconvergence techniques for differential and integral equations. Especially, we shall concentrate (Section 2) on efficient superconvergent schemes for a class of important problems - the second order elliptic boundary value problems solved by the finite element method.

During the development of this method it has been found out (see e.g. [16, 46, 70, 73, 98]) that the rate of convergence of FE-approximations at some exceptional points of a domain exceeds the possible global rate. This phenomenon has come to be known as "superconvergence". Such points of exceptional accuracy of the derivatives of FE-approximations (the so-called stress points) have been observed e.g. in [17, 47, 99, 166, 179, 192].

---

(1) Mathematical Institute, Czechoslovak Academy of Sciences, Žitná 25, CS-11567 Praha 1, Czechoslovakia.

(2) Department of Mathematics, University of Jyväskylä, Seminaarinkatu 15, SF-40100 Jyväskylä 10, Finland.

(3) Department of Physics and Mathematics, Lappeenranta University of Technology, Box 20, SF-53851, Lappeenranta 85, Finland.

A systematic study of superconvergence phenomena seems to have its beginning in the seventies. There has been a great effort focused on the superconvergence at nodal points and also at the Gauss-Legendre, Jacobi, and Lobatto points etc., see [73, 112, 33, 16]. However, at the present time the term superconvergence is used in much broader sense than before. One can recover the Galerkin solution or its derivatives by means of various post-processing techniques to produce an acceleration of convergence, and this is also called by many authors a superconvergence if the post-processing is easily computable. After such a post-processing one can often get an increase of accuracy not only at some isolated points, but also in a subdomain (local superconvergence) or even in the whole domain (global superconvergence).

In Section 2 we introduce several superconvergent FE-schemes for plane elliptic problems which possibly will characterize what was done in this field. Further we only mention superconvergence phenomena for two-point boundary value problems and other related problems.

Section 3 is intended to be a brief survey of extensive literature on superconvergence for parabolic and hyperbolic problems, integral equations, and other problems if the Galerkin, collocation or least squares method are employed.

Recently the number of papers concerning superconvergence has considerably grown and most of them have been written during the last five years. Thus this paper should facilitate the orientation in the literature. In order to help the reader, the bibliography is equipped with the reference numbers to Mathematical Reviews.

2. Superconvergence schemes for elliptic problems

2.1. The aim. The main aim of this section is to present several superconvergent FE-schemes for a 2^{nd} order elliptic boundary value problem in plane. For the sake of clarity, we demonstrate these schemes in their simplest setting and only for the Poisson equation

$$(2.1)\qquad \begin{aligned} -\Delta u &= f \quad \text{in } \Omega , \\ u &= 0 \quad \text{on } \partial\Omega , \end{aligned}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with a Lipschitz boundary $\partial\Omega$ and $f \in L^2(\Omega)$. Furthermore, we note superconvergence results for two-point boundary value problems, for higher order elliptic problems in plane and for the computation of the boundary flux. We close this section with a remark on other techniques for the acceleration of convergence.

2.2. Preliminaries. Let us introduce several notations and definitions. The Euclidean norm is denoted by $|\cdot|$. We write $\Omega_0 \subset\subset \Omega$ if Ω_0 is a domain such that $\bar{\Omega}_0 \subset \Omega$. By $P_k(\Omega)$ we mean the space of polynomials of degree at most k. The usual norm and seminorm in the Sobolev space $(W_p^k(\Omega))^d$ ($k \in \{0,1,\ldots\}$, $d \in \{1, 2\}$, $p \in [1, \infty]$) are denoted by $\|\cdot\|_{k,p,\Omega}$ and $|\cdot|_{k,p,\Omega}$, respectively. In particular, we write $H^k(\Omega) = W_p^k(\Omega)$ and $\|\cdot\|_k = \|\cdot\|_{k,\Omega} = \|\cdot\|_{k,p,\Omega}$ for $p = 2$. The space $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the $\|\cdot\|_1$-norm.

We denote by T_h a partition (triangulation) of the domain Ω in the usual sense [53, 166]. Up to the end of subsection 2.3.7, the families $\{T_h\}$ of partitions of $\bar{\Omega}$ are supposed to be strongly regular, i.e. any $K \in T_h$ contains a circle of diameter Ch and is contained in a circle of diameter h, $0 < C < 1$ independent of h and K.(4)

(4) Here and in what follows the letter C stands for a generic positive constant which may vary with context.

We denote by $V_h \subset H_0^1(\Omega)$ a finite element space corresponding to $\mathcal{T}_h$. Throughout the subsections 2.3, 2.6 and 2.7, the notation u is used for the weak solution of (2.1) and u is supposed to be sufficiently smooth. By $u_h \in V_h$ we mean the Galerkin approximation of u, i.e.

$$(\nabla u_h, \nabla v_h)_0 = (f, v_h)_0 \qquad \forall v_h \in V_h ,$$

where $(\cdot,\cdot)_0$ is the inner product in $(L^2(\Omega))^d$, $d = 1,2$.

2.3. The superconvergence results for the problem (2.1). For a "reasonable" choice of k and p, the best possible global estimate for the Galerkin method is usually of the form

$$\| Au - Au_h \|_{k,p,\Omega} \leq C(u)\,\eta(h) , \qquad h \to 0 ,$$

where $A = I$ (the identity operator) or $A = \nabla$, $C(u)$ depends on some norm of u, and $\eta(h)$ which is independent of u, has mostly the form

$$\eta(h) = h^\alpha |\ln h|^\beta , \qquad \alpha > 0 , \ \beta \geq 0 \ \text{ real}.$$

We do not give an exact general definition of the superconvergence since there is a lot of very different approaches to it. Nevertheless, we outline roughly what we mean by superconvergence in this section. Consider a linear continuous post-processing operator

$$(2.2) \qquad \sim : Au_h \mapsto \widetilde{Au}_h .$$

The post-processing (2.2) is said to cause the superconvergence if

$$\| Au - \widetilde{Au}_h \| = o(\eta(h)) \quad \text{as} \quad h \to 0 .$$

Here, the norm $\|\cdot\|$ has to be close to the $\|\cdot\|_{k,p,\Omega}$-norm in some sense. For instance, $\|\cdot\|$ may be a discrete analog of the $\|\cdot\|_{k,p,\Omega}$-norm, or $\|\cdot\| = \|\cdot\|_{k,p,\Omega_0}$ for $\Omega_0 \subset\subset \Omega$, or $\|\cdot\| = \|\cdot\|_{k,p,\Omega}$, etc.

Early papers established superconvergence phenomena mainly when (2.2) was a restriction operator. Recently, however, there is a growing literature on superconvergence, where the post-processing (2.2) is an averaging, convolution, or smoothing operator. We will illustrate various types of (2.2) in the following examples.

2.3.1. In the first place we introduce the superconvergence phenomenon at nodal points, which was analysed by Douglas, Dupont and Wheeler [73]. Here $A = I$ and the post-processing (2.2) is a restriction.

For a partition $0 = n_0 < n_1 \dots < n_m = 1$, let $I_i = [n_{i-1}, n_i]$. Fix an integer $k \geq 3$ and define

(2.3) $$S_h = \{s \in H_0^1((0,1)) \mid s|_{I_i} \in P_k(I_i),\ i = 1,\dots,m\}$$

and the finite element space on $\Omega = (0,1) \times (0,1)$ via the tensor product

$$V_h = S_h \otimes S_h .$$

Then

$$\max_{x \in N_h} |u(x) - \tilde{u}_h(x)| \leq C h^{k+2} \|u\|_{k+3},$$ (5)

where $N_h = \{(n_i, n_j)\}$ is the set of nodal points and $\tilde{u}_h(x) = u_h(x)$ for $x \in N_h$. This is a superconvergence result in the sense that the rate of convergence at nodes is greater than globally possible, that is

$$\|u - u_h\|_{0,\infty,\Omega} \leq C h^{k+1} (\|u\|_{k+2,2,\Omega} + \|u\|_{k+1,\infty,\Omega}).$$

2.3.2. Next, we present the result of Zlámal [192], where $\sim$ is again a restriction operator but $A = \nabla$.

Let the domain $\bar{\Omega}$ be a finite union of rectangles with sides parallel to the coordinate axes. Partitions T_h are formed by rec-

(5) In what follows all the statements hold only for a sufficiently small discretization parameter h.

tangles and $V_h \subset H_0^1(\Omega)$ consists of continuous functions which are incomplete polynomials of the third degree on every $K \in T_h$ (the two terms x_1^3 and x_2^3 are missing in the cubic polynomials). These polynomials of the so-called Serendipity family are uniquely determined by the values at the corners and at the midpoints of the sides, and it is well-known that

$$\| \nabla u - \nabla u_h \|_0 \leq C h^2 \| u \|_3$$

is the best possible rate. Let us denote by G_h the set of all maps of the four Gaussian points $(\pm\sqrt{3}/3 , \pm\sqrt{3}/3)$ of the square $\hat{K} = [-1 , 1] \times [-1 , 1]$ through one-to-one linear continuous mappings $F_K : \hat{K} \to K$, $K \in T_h$. Then the arithmetic mean μ of the values $|\nabla u(x) - \tilde{\nabla} u_h(x)|$, $x \in G_h$, (where $\tilde{\nabla} u_h(x) = \nabla u_h(x)$, $x \in G_h$) is bounded by

$$\mu \leq C h^3 (|u|_3 + |u|_4)$$

or, equivalently,

$$(2.4) \qquad h^2 \sum_{x \in G_h} |\nabla u(x) - \tilde{\nabla} u_h(x)| \leq C h^3 (|u|_3 + |u|_4).$$

This estimate is valid even for a more general second order elliptic equation with variable coefficients and also for the homogeneous Newton boundary condition [192]. In fact, Zlámal [193] proved more than (2.4), namely that

$$(2.5) \qquad h\Big(\sum_{x \in G_h} |\nabla u(x) - \tilde{\nabla} u_h(x)|^2 \Big)^{1/2} = O(h^3) .$$

This is the discrete L^2-norm estimate. The bound (2.4) can be generalized to the one- and three-dimensional cases and for other elements of the Serendipity family, e.g. for rectangular bilinear elements, the sampling at centroids leads to the $O(h^2)$ superconvergence. The generalization for curved isoparametric elements (including the reduced integration) can be found in Zlámal [193] and Lesaint and Zlámal [112]. Related papers include Zlámal [194, 195]. Some extensions of (2.4) under a weaker condition on the ellipticity can be found in Leyk [115], see also Chen [48, 50].

2.3.3. We introduce a simple averaging post-processing for the gradient ($A = \nabla$) suggested, among others, by Lin, Lü and Shen [121].

Let $\mathcal{T}_h$ consist of triangles and let

$$(2.6) \qquad V_h = \{v_h \in H_0^1(\Omega) \mid v_h|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\}.$$

Moreover, we assume that each $\mathcal{T}_h$ is uniform, i.e. any two adjacent triangles of $\mathcal{T}_h$ form a parallelogram.

Denote by M_h the set of the midpoints of all sides of the triangulation $\mathcal{T}_h$. As the gradient of $u_h \in V_h$ is constant on every $K \in \mathcal{T}_h$, one may define

$$(2.7) \qquad \tilde{\nabla} u_h(x) = \frac{1}{2}(\nabla u_h|_K + \nabla u_h|_{K'}), \quad x \in M_h \cap \Omega,$$

where $K, K' \in \mathcal{T}_h$ are those adjacent triangles for which $x \in K \cap K'$. Now if $u \in C^3(\bar{\Omega}) \cap H_0^1(\Omega)$ then

$$(2.8) \qquad \max_{x \in M_h \cap \Omega} |\nabla u(x) - \tilde{\nabla} u_h(x)| = \mathcal{O}(h^2 |\ln h|).$$

Note that

$$\|\nabla u - \nabla u_h\|_{0,\infty,\Omega} = \mathcal{O}(h |\ln h|),$$

or even $\mathcal{O}(h)$ for convex polygons [148].

The bound (2.8) was derived (see [122]) also for quasi-uniform triangulations in which any two adjacent triangles of $\mathcal{T}_h$ form only an approximate parallelogram, i.e. it holds that

$$|\ell(S) - \ell(S')| \leq C h^2,$$

where $\ell(S)$ and $\ell(S')$ are the lengths of the opposite sides of the parallelogram. For another type of triangulations we further refer to Lin and Lü [119]. The $O(h^2)$ superconvergence in the discrete L^2-norm (even for variable coefficients) is given by Andreev [2] and Levine [113, 114].

The averaging technique (2.7) is based on the fact that only the tangential component of ∇u_h is a superconvergent approximation to the tangential component of ∇u at the midpoints of sides [2, 114]. Also sampling at the two Gaussian points of each side of triangular quadratic elements leads to the superconvergence of the tangential component of the gradient, see Andreev [3].

2.3.4. A superconvergent recovery of the gradient at centroids is proposed by Levine [114].

The space V_h is as defined in (2.6) over a quasi-uniform triangulation. Denote by C_h the set of centroids of all $K \in T_h$. Then for $x \in C_h \cap \Omega_0$ we define

$$(2.9) \qquad \widetilde{\nabla u}_h(x) = \frac{1}{6}\left(3\,\nabla u_h|_K + \sum_{i=1}^{3} \nabla u_h|_{K_i}\right),$$

where K_i are triangles adjacent to K, $x \in K$, and $\Omega_0 \subset\subset \Omega$ is fixed. Now we have the estimate

$$h\Big(\sum_{x\in C_h\cap\Omega_0} |\nabla u(x) - \widetilde{\nabla u}_h(x)|^2\Big)^{1/2} \leq C h^2 \|u\|_3 ,$$

whereas

$$(2.10) \qquad \|\nabla u - \nabla u_h\|_0 \leq C h \|u\|_2 .$$

The $O(h^2)$ superconvergence is also proved when an appropriate numerical quadrature (e.g. the centroid rule) is used.

The relation (2.9) can be interpreted as follows. We first recover the gradient at the midpoint of each side of a triangle (cf. (2.7)) and then average these three gradients to obtain an approximation to the gradient at the centroid. The $O(h^2)$ superconvergence of the gradient at centroids, when the linear elements are used for solving a denenerated elliptic problem, is proved in El Hatri [89]. Obviously, we may uniquely determine a piecewise linear discontinuous field $\tilde{\nabla}u_h$ on $\Omega_0 \subset\subset \Omega$, which fulfils (2.7). Then we can easily derive from (2.8), see Neittaanmäki and Křížek [136], that $\tilde{\nabla}u_h$ recovers the gradient at any point of Ω_0, i.e.

(2.11) $$\| \nabla u - \tilde{\nabla}u_h \|_{0,\infty,\Omega_0} = O(h^2 \, |\ln h|).$$

Analogously, for the $\|\cdot\|_{0,\Omega_0}$ -norm, the $O(h^2)$ superconvergence can be obtained. Some extensions to a global superconvergence are discussed in the forthcoming paper of Lin and Xu [123].

2.3.5. Another simple averaging technique has been analysed by Křížek and Neittaanmäki [104].

Again we assume that V_h is of the form (2.6). Thus, when one needs an approximation of ∇u at some nodal point $x \in N_h$, then it is quite natural to calculate an average of all the constant vectors $\nabla u_h|_K$, where $K \in T_h$ are incident with x. This technique is, in fact, often used in practice, see e.g. [90, 138, 177, 191]. Putting

(2.12) $$\tilde{\nabla}u_h(x) = \frac{1}{6} \sum_{K \cap \{x\} \neq \emptyset} \nabla u_h|_K \quad \forall\, x \in N_h \cap \Omega,$$

we can uniquely define the continuous piecewise linear field $\tilde{\nabla}u_h$ over a fixed $\Omega_0 \subset\subset \Omega$. Suppose now that Ω is a parallelogram and that T_h are uniform. Then (see [104]) the following local superconvergence estimate holds (cf. (2.10))

$$\| \nabla u - \tilde{\nabla}u_h \|_{0,\Omega_0} \leq C h^2 \|u\|_{3,\Omega}.$$

When $\partial\Omega$ is curved, the local $O(h^{3/2})$ superconvergence can be achieved also for the Newton boundary condition and smooth coefficients [104]. If Ω is a polygonal domain covered by uniform triangulations we can also get (see Křížek and Neittaanmäki [106]) the global superconvergence of $\tilde{\nabla} u_h \in W_h \times W_h$:

$$(2.13) \qquad \| \nabla u - \tilde{\nabla} u_h \|_{0,\Omega} \leq C h^2 \| u \|_{3,\Omega} ,$$

where

$$(2.14) \qquad W_h = \{ w_h \in H^1(\Omega) \mid w_h|_K \in P_1(K) \quad \forall K \in T_h \} .$$

However, this requires to define the averaged gradient at boundary nodes, for instance as

$$(2.15) \qquad \begin{cases} \tilde{\nabla} u_h(x) = 0 & \text{for all vertices } x \text{ of } \partial\Omega , \text{ and} \\ \tilde{\nabla} u_h(x) = \frac{1}{2} (\sum_{i=1}^{3} \nabla u_h|_{K_i} - \nabla u_h|_{K_0}) & \end{cases}$$

for the other boundary nodes $x \in N_h \cap \partial\Omega$. Here K_i and K_3 form a parallelogram for every $i = 0, 1, 2$, and $K_1 \cap K_2 \cap K_3 = \{x\}$.

2.3.6. Next, we introduce the smoothing technique suggested by Oganesijan and Ruhovec [144], p. 94 and p. 189.

Let the boundary $\partial\Omega$ be from the class C^3 and let V_h be again as in (2.6). Suppose that triangulations consist of right-angled isosceles triangles in every $\Omega_0 \subset\subset \Omega$ for a sufficiently small h. For details about the triangulation near the boundary $\partial\Omega$, we refer to [143, 144]. We set, see [144] (or [142], p. 148)

$$\tilde{u}_h(x) = 4 h^{-2} \int_{D_h} u_h(x + y) \, dy ,$$

where $D_h = (-h, h) \times (-h, h)$. Then for $\Omega_0 \subset\subset \Omega$, we have

$$\|u - \tilde{u}_h\|_{1,\Omega_0} \leq C\, h^{3/2} \|u\|_{3,\Omega} .$$

By comparison with (2.10) we find that the above error bound is a superconvergent estimate for the gradient. For the global $\mathcal{O}(h^{3/2})$ superconvergence, it is necessary to extend u_h outside of Ω in an appropriate way (see [144], p. 21). Note that the averaged gradients in (2.11) and (2.13) are not potential fields in contradistinction to the case introduced here.

2.3.7. The Galerkin solution can be post-processed also by a convolution with the kernel proposed by Bramble and Schatz [24].

For simplicity, assume that $\bar{\Omega}$ can be decomposed into a finite number of identical squares and let the nodal points of rectangular partitions T_h be of the form (ih, jh), where i, j are integers. For a fixed $k \geq 2$, define the space of the two-dimensional B-splines

$$V_h = \{v_h \in H_0^1(\Omega) \mid v_h(x_1, x_2) =$$

$$= \sum_{(ih,jh)\in\Omega} a_{ij}\, g_k(x_1/h - i)\, g_k(x_2/h - j)\} ,$$

where $a_{ij} \in \mathbb{R}^1$ and g_k is the one-dimensional B-spline (of order $r = k+1$) given recurrently by the convolution

$$g_p = g_0 * g_{p-1} , \quad p = 1, 2, \ldots, k ,$$

g_0 being the characteristic function of the interval $[-\frac{1}{2}, \frac{1}{2}]$. Let us define $\tilde{u}_h$ via the convolution

$$\text{(2.16)} \qquad \tilde{u}_h(x) = h^{-2} \int_{\mathbb{R}^2} \Big(\prod_{i=1}^{2} \sum_{j=1-k}^{k-1} k'_j\, g_{k-1}((x_i - y_i)/h - j) \Big)\, u_h(y)\, dy ,$$

where $k'_{-j} = k'_j = \frac{1}{2} k_j$ for $j = 1, \ldots, k-1$, $k'_0 = k_0$, and where k_j, $j = 0, 1, \ldots, k-1$, are determined as the unique solution of the linear algebraic system

$$\sum_{j=0}^{k-1} k_j \int_{\mathbb{R}^1} g_{k-1}(y)\,(y+j)^{2m}\,dy = \delta_{0m}, \quad m = 0, 1, \ldots, k-1 .$$

Hence, k_j depend only on the choice of k (a table of the constants k_j' for $2 \le k \le 5$ can be found in [24], p. 110). In (2.16), $\tilde{u}_h(x)$ can be calculated analytically at any point $x \in \Omega_0 \subset\subset \Omega$, especially at nodes it is simple.

If $k \ge 2$ and $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ then we have the superconvergence estimates

$$\|u - \tilde{u}_h\|_{0,\Omega_0} \le C\, h^{2k} (\|u\|_{2k,\Omega_1} + \|u\|_{k+1,\Omega}),$$

$$\|u - \tilde{u}_h\|_{0,\infty,\Omega_0} \le C\, h^{2k} (\|u\|_{2k+2,\Omega_1} + \|u\|_{k+1,\Omega}),$$

whereas

$$\|u - u_h\|_0 = O(h^{k+1}), \quad \|u - u_h\|_{0,\infty,\Omega} = O(h^{k+1}).$$

The above technique of [24] is presented for more general classes of splines in $\mathbb{R}^d$ and also for negative norms. Moreover, the authors show how to obtain the superconvergence up to the boundary when Ω is the unit square.

The generalization of the above superconvergent interior approximations also to derivatives of u is studied by Thomée [170]. For the post-processing by the convolution with the Bramble-Schatz kernel, see also Douglas [66, 67].

2.3.8. We are mostly far from superconvergence on general meshes, which seems to be open problem for many schemes. However, the next averaging (smoothing) technique proposed by Louis [128] is applicable even for irregular partitions of a convex polygon $\bar{\Omega} \subset R^2$. We only assume that the finite dimensional spaces $V_h \subset H_0^1(\Omega)$ have the following standard approximation property:

For any $v \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$, $k \ge 0$, there exists a $v_h \in V_h$

with

$$\| v - v_h \|_m \leq C\, h^{k-m+1} \| v \|_{k+1} , \quad m \in \{0 , 1\}.$$

Let $x \in \Omega$ and $r > 0$ be fixed with

$$U(x , r) = \{y \in \mathbb{R}^2 \mid |x - y| \leq r\} \subset \Omega .$$

Define the average of u_h by

$$(2.17) \qquad \tilde{u}_h(x) = \int_{U(x,r)} (f(y)(\gamma(y-x) + \psi(y-x)) + u_h(y)\, \Delta\psi(y-x))\, dy ,$$

where

$$\gamma(z) = -\frac{1}{2\pi} \ln |z| , \quad z \in \mathbb{R}^2 ,$$

$$\psi(z) = -\frac{1}{8\pi r^4} (|z|^4 - 4 r^2 |z|^2 + (3 - 4 \ln r) r^4), \quad z \in \mathbb{R}^2 .$$

Then for $u \in H^{k+1}(\Omega)$, $k \geq 2$, it is

$$(2.18) \qquad |u(x) - \tilde{u}_h(x)| = O(h^{2k}) .$$

Louis' method requires more effort than that of Bramble and Schatz (cf. (2.16)), since in (2.17) the domain of integration is independent of h. Moreover, (2.17) cannot be calculated analytically, in general. So it is reasonable to use this procedure either only at points of special interest or at those points where other methods do not work (e.g. on irregular meshes). Louis [128] gives also a formula for averages of ∇u_h with the same accuracy $O(h^{2k})$ like in (2.18).

2.3.9. Also a least squares smoothing proposed by Hinton and Campbell [90] can be performed to achieve a higher accuracy of ∇u_h. The field ∇u_h is discontinuous in general and thus a continuous field $\widetilde{\nabla u}_h$ may be computed from ∇u_h through the local or global L^2-least squares method. For instance, when V_h is given by (2.6), ∇u_h is piecewise constant and we can thus choose $\widetilde{\nabla u_h}$ in the class of continuous piecewise linear functions. Although the least squares method requires more arithmetic operations than (2.12) or (2.15) do, the numerical results are better, see [90]. Related works include Hinton and Owen [91], Hinton, Scott and Ricketts [92].

2.3.10. A number of papers are devoted to the interior estimates of higher accuracy, where (2.2) is the restriction $\widetilde{Au_h} = Au_h|_{\Omega_0}$, $\Omega_0 \subset\subset \Omega$. Let us mention, for instance, Bramble and Thomée [25], Descloux [61], Haslinger [86], Nakao [135], Nitsche and Schatz [141], Schatz and Wahlbin [156].

2.3.11. For other superconvergence results for second order problems in $\mathbb{R}^n$ $(n \geq 2)$, the reader is referred to Babuška, Izadpanah and Szabo [13], Bramble and Schatz [23], Dautov [56], Dautov and Lapin [58], Dautov, Lapin and Lyashko [60], Douglas and Milner [76], El Hatri [88,89], Korneev [103], Lin and Xu [123], Nakao [135], Zhu [189, 190]. Related references further include e.g. Arnold and Brezzi [4], Babuška and Miller [14], Carey and Oden [34], Mansfield [129], etc. Numerical tests can be found in [13, 14, 104, 106, 114, 128, 136, 137, 193].

2.4. Two-point boundary value problems. For the sake of completeness, let us further recall the superconvergence results for the two-point problem

$$(2.19) \qquad -u'' + a(x)u' + b(x)u = f(x), \quad x \in (0,1),$$

with the Dirichlet conditions $u(0) = u(1) = 0$. The functions a, b and f are supposed to be sufficiently smooth and let $f = 0$ imply $u = 0$. The Galerkin method chooses $u_h \in S_h$ (S_h given by (2.3)) such that for uniform partition

$$(2.20) \qquad (u_h', s_h')_0 + (a u_h' + b u_h, s_h)_0 = (f, s_h)_0 \quad \forall s_h \in S_h.$$

This method exhibits the $\mathcal{O}(h^{2k})$ superconvergence at nodal points, see Douglas and Dupont [69] (or [70])

$$\max_{0 \leq i \leq m} |u(n_i) - u_h(n_i)| \leq C h^{2k} \|u\|_{k+1} .$$

Moreover, on any segment I_i there are $k-1$ interior points (the Lobatto points), where $u - u_h$ is $\mathcal{O}(h^{k+2})$, i.e. one order better than globally, see Bakker [16]. At the k Gauss-Legendre points of each I_i, the derivative u_h' has $\mathcal{O}(h^{k+1})$ convergence instead of $\mathcal{O}(h^k)$, see Lesaint and Zlámal [112]. A local $\mathcal{O}(h^{2k})$ accuracy for the first derivative can be obtained from a system which requires very little more computing than (2.20), see Douglas and Dupont [70]. In Dupont [77], a simple post-processing is performed to produce a superconvergence for both the values and the derivatives at any point of the interval. For a post-processing applied to the C^1-Galerkin approximation of (2.19), see Douglas and Dupont [68]. Other superconvergence results for the Galerkin method are described by Babuska and Miller [14], Chen [46], Douglas, Dupont and Wahlbin [71], de Groen [85], Huang and Wu [97], Marshall [132], Wheller [182].

In the one-dimensional case similar to (2.1) it holds that $u(n_i) = u_h(n_i)$ when linear elements are used (see [166], p. 107), i.e. there is no discretization error at nodes for an arbitrary partition. Analogous results for higher order one-dimensional problems can be found e.g. in [105, 185].

Finally, let us mention important papers on superconvergence for two-point problems using other than the Galerkin method. For the least squares method, see Ascher [11], Locker and Prenter [126, 145]; for the method of moments, see Mock [133], Rachford and Wheeler [147]; for the collocation or collocation-Galerkin method, see Ascher and Weiss [12], de Boor and Swartz [20, 22], Carey and Wheeler [35], Christiansen and Russell [51, 52], Diaz [62], de Hoog and Weiss [93], Houstis [95], Pereyra and Sewell [146], Wheeler [184]. Several papers given above contain also a superconvergence analysis for higher order problems. For more facts about superconvergence results in $\mathbb{R}^1$, we refer to the survey paper on two-point boundary value problems written by Reddien [151], (or Reddien [150], Nitsche [140]).

2.5. Higher order problems in plane. For superconvergence phenomena in higher order (fourth order) elliptic problems in $\mathbb{R}^2$ which are similar to those of second order problems, we quote e.g. Dautov [57], Dautov and Lapin [59], Korneev [103], Zlámal [192].

2.6. Boundary flux. We describe a method for finding a superconvergent approximation to the boundary flux $q = \nabla u|_{\partial\Omega} \cdot \nu$, where u is the weak solution of (2.1) and ν is the outward unit normal to $\partial\Omega$. The method was first proposed and analysed by Douglas, Dupont and Wheeler [72] for rectangular elements. For simplicity, we restrict ourselves only to the usual linear elements, see e.g. Glowinski [80], p. 398.

Let V_h and W_h be given by (2.6) and (2.14), respectively. We seek a function $\tilde{q}_h = w_h|_{\partial\Omega}$ (for some $w_h \in W_h$) which is a better approximation to q than $q_h = \nabla u_h|_{\partial\Omega} \cdot \nu$. By analogy to the Green formula, we may uniquely determine $\tilde{q}_h$ by

$$\int_{\partial\Omega} \tilde{q}_h v_h \, ds = (\nabla u_h, \nabla v_h)_0 - (f, v_h)_0 \quad \forall v_h \in W_h .$$

Taking v_h as the usual Courant basis functions at boundary nodes, we obtain a system of algebraic equations with a sparse matrix. Solving this system requires only $O(m)$ operations, where m is the number of boundary nodes.

As suggested in [106], the post-processing technique (2.15) can also be used for approximating q. Namely, define $\tilde{q}_h = \tilde{\nabla} u_h|_{\partial\Omega} \cdot \nu$, where $\tilde{\nabla} u_h$ is defined by (2.15).

Further superconvergence techniques for the boundary flux for second order problems in $\mathbb{R}^d$ $(d \geq 1)$ are characterized in Carey, Humphrey and Wheeler [33], King and Serbin [102], Louis [128], Wheller [183].

2.7. Other techniques for the acceleration of convergence. We close this section remainding some post-processing techniques which increase the accuracy but differ from (2.2).

A widely used technique is the Richardson extrapolation. We outline this method in the case of the problem (2.1). Let V_h be as given in (2.6) over a quasi-uniform triangulation T_h with the nodes N_h. Define

$$(2.21) \qquad \tilde{u}_h = \frac{1}{3}(4u_{h/2} - u_h),$$

where $u_{h/2}$ is the Galerkin approximation over the refinement of T_h by midlines. Then for the most favourable case,

$$\max_{x \in N_h} |u(x) - \tilde{u}_h(x)| = O(h^4),$$

whereas

$$\| u - u_h \|_{0,\infty,\Omega} = O(h^2).$$

Hence, the accuracy of the extrapolation $(4u_{h/2} - u_h)/3$ is much better than $u_{h/2}$. This very old and well known technique is proposed for finite element method e.g. in Lin, Lü and Shen [122], Marchuk and Shaidurov [130, 131], Urvancev and Shaidurov [176] (and Chen [49] for the one-dimensional case). The simultaneous use of (2.12) and (2.21) was studied by Lin and Zhu [124].

Other techniques for the acceleration of convergence, especially for the finite difference method, were treated by Lin and Lü [117] (splitting extrapolation), Lin and Lü [118, 120] (correction techniques). See also Marchuk and Shaidurov [131], Neittaanmäki and Lin [137], etc.

3. Superconvergence results for other problems

In this section we mention some superconvergence phenomena which were discovered for other than elliptic problems.

3.1. Parabolic problems. Superconvergence results for parabolic problems have been reported in the text books of Fairweather [78], Fletcher [79] and Thomée [174]. Here, superconvergence phenomena analogous to that for elliptic problems can be expected. For convenience, consider the problem

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } \Omega \times (0, T), \quad \Omega \subset \mathbb{R}^2, \quad T > 0,$$

$$u(x, 0) = u^0(x) \quad \text{in } \Omega, \tag{3.1}$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T).$$

The superconvergence analysis of (3.1) with the usual Crank-Nicolson scheme was done by El Hatri [87], among others. For instance, biquadratic elements on uniform rectangular partitions T_h over $\bar{\Omega}$ give the accuracy $O(h^3 + \tau^2)$ (where τ is the time step) for the space gradient at the Gauss-Legendre points in the discrete L^2-norm. Thus one sees that this is similar to (2.5). The method of [87] has been considered also for variable coefficients and higher order elements (including numerical integration).

The superconvergence technique (2.7) for the problem (3.1) was justified by Andreev [2]. He used a semidiscrete Galerkin scheme (that is, the discretization in space only). Arnold and Douglas [5] also applied a semidiscrete scheme (with $\Omega \subset \mathbb{R}^1$) and obtained superconvergence results for a quasiparabolic problem at nodal points. See also Douglas [65] for a collocation method. With the help of the Laplace transform, an alternative proof of superconvergence is given in Bakker [15] and Adeboye [1].

In fact, the literature on superconvergence results for parabolic problems is very rich. Especially, let us recall the series of Thomée's works [167, 169, 171-174]. Furthermore, we cite the papers by Douglas [66, 67], Douglas, Dupont and Wheller [72, 74], Douglas, Ewing and Wheller [75], Kendall and Wheller [101], Lazarov, Andreev and El Hatri [110], Nakao [134], Wheeler [184].

3.2. Hyperbolic problems. Relatively few papers are devoted to superconvergence phenomena when solving hyperbolic problems. For the initial-boundary value problem for the first order equations (particularly $\partial u/\partial t + \partial u/\partial x = 0$), see Cullen [55], Dougalis [63], Houstis [94], Thomée [168], Thomée and Wendroff [175], Winther [187]. Second order problems are treated in Dougalis [63], Dougalis and Serbin [64]. Let us mention, for example, the result of [64], where the problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right) + b(x)u = f(x,t) \quad \text{in} \quad (0,1)\times(0,T), \quad T \geq 0,$$

$$(3.2) \qquad u(x,0) = u^0(x), \quad 0 \leq x \leq 1,$$

$$\frac{\partial u}{\partial t}(x,0) = u_t^0(x), \quad 0 \leq x \leq 1,$$

with 1-periodicity of u^0, u_t^0, a, b and f in the x range, is considered. The functions a, b are supposed to be infinitely differentiable and $a(x) \geq \alpha > 0$, $b(x) \geq 0$. The Galerkin subspaces V_h consist of periodic B-splines of order $r = k+1$ on uniform partitions of $[0,1]$ ($k \geq 1$ is the degree of the polynomials). A semidiscrete approximation of (3.2) with a suitable choice of the initial conditions for the Galerkin equation leads to an increased accuracy of the approximate solution which is $O(h^{2k})$-accurate at nodes whereas $O(h^{k+1})$ is the optimum L^2-error.

Analogous superconvergence results have been obtained also for a multi-step fully discrete Galerkin approximation. The effect of numerical integration is analyzed in [64] as well.

3.3. Some special equations. There are also superconvergence results for partial differential equations of various special types. For the Boussinesq equation, see Winther [188], for the Korteweg-de Vries equation, see Arnold and Winther [10], for the Sobolev equation, see Arnold, Douglas and Thomée [6]. An averaging technique for the Stokes problem can be found in Johnson and Pitkäranta [100]. A superconvergence result for the neutron transport equation has been established by Lesaint and Raviart [111]. We also mention the paper [139] of Neta and Victory, where superconvergence phenomena have been presented for cell-edge and cell-average fluxes.

3.4. Integral and integro-differential equations. The investigation of superconvergence phenomena for integral equations had its origin about 5 years later than that for differential equations. Nevertheless, the existing literature is quite extensive.

Fredholm integral equations of the first and second kind that arise from elliptic and some more general problems have been analysed by Hsiao and Wendland [96], Arnold and Wendland [8], Wendland et al. [54, 107, 165, 180, 181], see also Chandler [39] and Sloan and Thomée [163].

Superconvergence results for the integral equation of the first kind were obtained by Locker and Prenter [127] when they employed the least squares method. The Fredholm integral equation of the second kind is the most studied case. Let us confine ourselves to a model example

$$(3.3) \qquad u(x) - \int_0^1 k(x, s)\, u(s)\, ds = f(x), \qquad x \in \Omega = (0, 1),$$

where u is sought for given f and k. The Galerkin approximation of (3.3) consists in finding $u_h \in V_h$ such that

$$(u_h - Ku_h, v_h)_0 = (f, v_h)_0 \quad \forall v_h \in V_h,$$

where K is the integral operator occurring in (3.3) and V_h is a piecewise-polynomial space (see e.g. (2.3)) based on polynomials of degree at most k.

The study of the post-processing

$$(3.4) \qquad \tilde{u}_h = Ku_h + f$$

(which is proposed to be used also recurrently) is of particular interest here. In the most convenient case we have

$$\| u - \tilde{u}_h \|_{1,\infty,\Omega} = \mathcal{O}(h^{2k+2}),$$

whereas

$$\| u - u_h \|_{0,\infty,\Omega} = \mathcal{O}(h^{k+1}),$$

see Sloan and Thomée [163]. For other related works concerning the iterated Galerkin scheme (3.4), we quote Chandler [36-38], Chatelin [40 - 42], Chatelin and Lebbar [44, 45], Graham [83], Hsiao and Wendland [96], Sloan [161, 162], Spence and Thomas [164]. Superconvergence properties of the Galerkin approximation were also studied by Richter [153].

For collocation methods, where the superconvergence at collocation points is obtained, we quote Vainikko and Uba [178]. For superapproximation results on collocation methods, see Arnold and Wendland [8], Saranen and Wendland [154]. A comparison of Galerkin and collocation schemes and their iterated variants has been presented by Arnold and Wendland [9], Graham, Joe and Sloan [84].

Next, let us mention the works of Brunner [26-31], Brunner and Nørsett [32] which are concerned with superconvergence when solving Volterra integral equations of the first and second kind by collocation methods. Some special equations have been studied by Goldberg, Lea and Miel [82] (airfoil equation), Larsen and Nelson [109] (discrete-ordinate equations in slab geometry). More details about the convergence acceleration (superconvergence) for integral equations are included in the survey papers by Chandler [38], Goldberg [81], Lin and Liu [116], Sloan [160].

For integro-differential equations, see e.g. Brunner [30], von Seggern [157, 158].

3.5. Remarks. We close with a few comments about superconvergence in other fields. Some results for eigenvalue problems were obtained by de Boor and Swartz [21], Chatelin [40, 42, 43], Chatelin and Lebbar [45], Regińska [152], Schäfer [155], Sloan [159]. For the theory of spline approximation, see Beatson [18], Behforooz and Papamichael [19], for superapproximation, see Arnold and Saranen [7], Costabel, Stephan and Wendland [54], Stephan and Wendland [165], for variational inequalities, see Lapin [108], for the theory of optimal control, see Reddien [149], and, finally, for a general operator theory, see Lindberg [125], Winther [186].

REFERENCES

[1] K.R, Adeboye, Superconvergence results for Galerkin method for parabolic initial value problems via Laplace transform. Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.) 26 (1982), pp. 115-127. MR 84h:65095.

[2] A.B. Andreev, Superconvergence of the gradient for linear triangle elements for elliptic and parabolic equations. C. R. Acad. Bulgare Sci. 37 (1984), pp. 293-296.

[3] A.B. Andreev, Error estimate of type superconvergence of the gradient for quadratic triangular elements. C. R. Acad. Bulgare Sci. 37 (1984), pp. 1179-1182.

[4] D.N. Arnold and F. Brezzi, Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates. RAIRO Anal. Numér. (to appear).

[5] D.N. Arnold and J.J. Douglas, Superconvergence of the Galerkin approximation of a quasilinear parabolic equation in a single space variable. Calcolo 16 (1979), pp. 345-369. MR 83b:65118.

[6] D.N. Arnold, J.J. Douglas and V. Thomée, Superconvergence of a finite element approximation to the solution of a Sobolev equation in a single space variable. Math. Comp. 36 (1981), pp. 53-63. MR 82f:65108.

[7] D.N. Arnold and J. Saranen, On the asymptotic convergence of spline collocation methods for partial differential equations. SIAM J. Numer. Anal. 21 (1984), pp. 459-472.

[8] D.N. Arnold and W.L. Wendland: On the asymptotic convergence of collocation methods. Math. Comp. 41 (1983), pp. 349-381.

[9] D.N. Arnold and W.L. Wendland, Collocation versus Galerkin procedures for boundary integral methods. Preprint No. 671, TH Darmstadt, 1982, pp. 1-16.

[10] D.N. Arnold and R. Winther, A superconvergent finite element method for the Korteweg-de Vries equation. Math. Comp. 38 (1982), pp. 23-36. MR 82m:65087.

[11] U. Ascher, Discrete least squares approximations for ordinary differential equations. SIAM J. Numer. Anal. 15 (1978), pp. 478-496. MR 81e:65043.

[12] U. Ascher and R. Weiss, Collocation for singular perturbation. First order systems with constant coefficients. SIAM J. Numer. Anal. 20 (1983), pp. 537-557.

[13] I. Babuška, K. Izadpanah and B. Szabo, The post-processing technique in the finite element method. The theory and experience. Technical Note BN-1016, Univ. of Maryland, 1984.

[14] I. Babuška and A. Miller, The post-processing in the finite element method, Parts I-III. Internat. J. Numer. Methods Engrg. 20 (1984), pp. 1085-1109, 1111-1129.

[15] M. Bakker, On the numerical solution of parabolic equations in a single space variable by the continuous time Galerkin method. SIAM J. Numer. Anal. 17 (1980), pp. 162-177. MR 81m:65167.

[16] M. Bakker, A note on C^0 Galerkin methods for two-point boundary problems. Numer. Math. 38 (1981/82), pp. 447-453. MR 83e:65140.

[17] J. Barlow, Optimal stress location in finite-element models. Internat. J. Numer. Methods Engrg. 10 (1976), pp. 243-251.

[18] R. Beatson, Joint approximation of a function and its derivatives. Approximation Theory III (Proc. Conf., Univ. Texas, Austin, Tex., 1980), Academic Press, New York, 1980, pp. 199-206. MR 82c:41019.

[19] G.H. Behforooz and N. Papamichael, Improved orders of approximation derived from interpolatory cubic splines. BIT 19 (1979), pp. 19-26. MR 80e:41004.

[20] C. de Boor and B. Swartz, Collocation at Gaussian points. SIAM J. Numer. Anal. 10 (1973), pp. 582-606. MR 51#9528.

[21] C. de Boor and B. Swartz, Collocation approximation to eigenvalues of an ordinary differential equation: The principle of the thing. Math. Comp. 35 (1980), pp. 679-694. MR 81k:65097.

[22] C. de Boor and B. Swartz, Local piecewise polynomial projection methods for an O.D.E. which give high-order convergence at knots. Math. Comp. 36 (1981), pp. 21-33. MR 82f:65091.

[23] J.H. Bramble and A.H. Schatz, Estimates for spline projection. RAIRO Anal. Numér. 10 (1976), pp.5-37. MR 55#9563.

[24] J.H. Bramble and A.H. Schatz, Higher order local accuracy by averaging in the finite element method. Math. Comp. 31 (1977), pp. 94-111. MR 55#4739.

[25] J.H. Bramble and V. Thomée, Interior maximum norm estimates for some simple finite element methods. RAIRO Anal. Numér. 8 (1974), pp. 5-18. MR 50#11808.

[26] H. Brunner, On superconvergence in collocation methods for Abel integral equations. Numerical Mathematics and Computing (Proc. Eighth Manitoba Conf., Winnipeg, 1978). Utilitas Mathematica Publishing, Inc., Winnipeg, Man, 1979, pp. 117-128. MR 80m:45023.

[27] H. Brunner, Superconvergence of collocation methods for Volterra integral equations of the first kind. Computing 21 (1979), pp. 151-157. MR 83a:65125.

[28] H. Brunner, A note on collocation methods for Volterra integral equations of the first kind. Computing 23 (1979), pp. 179-187. MR 83b:65145.

[29] H. Brunner, Superconvergence in collocation and implicit Runge-Kutta methods for Volterra-type integral equations of the second kind. Numerical Treatment of Integral Equations (Workshop, Math. Res. Inst., Oberwolfach, 1979). Internat. Ser. Numer. Math., 53, Birkhäuser, Basel 1980, pp. 54-72. MR 81m:65194.

[30] H. Brunner, The application of the variation of constants formulas in the numerical analysis of integral and integro-differential equations. Utilitas Math. 19 (1981), pp. 255-290. MR 83b:65146.

[31] H. Brunner, Iterated collocation methods and their discretizations for Volterra integral equations, 1983, (to appear).

[32] H. Brunner and S.P. Nørsett, Superconvergence of collocation methods for Volterra and Abel integral equations of the second kind. Numer. Math. 36 (1981), pp. 347-358. MR 83e:65202.

[33] G.F. Carey, D. Humphrey and M.F. Wheeler, Galerkin and collocation-Galerkin methods with superconvergence and optimal fluxes. Internat. J. Numer. Methods Engrg. 17 (1981), pp. 939-950. MR 82g:80008.

[34] G.F. Carey and J.T. Oden, Finite Elements. Vol. II: A second course. The Texas Finite Element Series, Vol. 2. Englewood Cliffs, New Jersey, Prentice-Hall, 1983.

[35] G.F. Carey and M.F. Wheeler, C^0-collocation-Galerkin methods. Codes for Boundary-Value Problems in Ordinary Differential Equations (Proc. Conf., Houston, 1978). LN in Comp. Sci., Vol.76, Springer, Berlin, 1979, pp. 250-256. MR 82b:65074.

[36] G.A. Chandler, Global superconvergence of iterated Galerkin solutions for second kind integral equations. Technical Report, Australian Nat. Univ., Canberra, 1978.

[37] G.A. Chandler, Superconvergence of numerical solutions to second kind integral equations. Ph. D. thesis, Australian Nat. Univ., Canberra, 1979.

[38] G.A. Chandler, Superconvergence for second kind integral equations. The Application and Numerical Solution of Integral Equations (Proc. Conf., Canberra, 1978). Sijthoff & Noordhoff, Alphen aan den Rijn, 1980, pp. 103-117. MR 81h:45027.

[39] G.A. Chandler, Galerkin's method for boundary integral equations on polygonal domains. Numer. Anal. Report No 83/1, Univ. of Queensland, 1983, pp. 1-19.

[40] F. Chatelin, Sur les bornes d'erreur a posteriori pour les éléments propres d'opérateurs linéaires. Numer. Math. 32 (1979), pp. 233-246. MR 81d:65027.

[41] F. Chatelin, Des resultats de superconvergence. Seminaire d'analyse numérique, No. 331. Univ. Scientifique et Medicale de Grenoble, Laboratoire IMAG, 1980.

[42] F. Chatelin, The spectral approximation of linear operators with applications to the computation of eigenelements of differential and integral operators. SIAM Rev. 23 (1981), pp. 495-522. MR 83b:47023.

[43] F. Chatelin, Spectral Approximation of Linear Operators. Academic Press, New York, 1983.

[44] F. Chatelin and R. Lebbar, The iterated projection solution for the Fredholm integral equation of second kind. J. Austral. Math. Soc. Ser. B 22 (1981), pp. 439-451. MR 82h:65096.

[45] F. Chatelin and R. Lebbar, Superconvergence results for the iterated projection method applied to a second kind Fredholm integral equation and eigenvalue problem. Preprint, 1981.

[46] C.M. Chen, The good points of the approximation solution for Galerkin method for two-point boundary problem. Numer. Math. J. Chinese Univ. 1 (1979), pp. 73-79. MR 82e:65085.

[47] C.M. Chen, Optimal points of the stresses for triangular linear element. Numer. Math. J. Chinese Univ. 2 (1980), pp. 12-20. MR 83d:65279.

[48] C.M. Chen, Superconvergence of finite element solutions and their derivatives. Numer. Math. J. Chinese Univ. 3 (1981), pp. 118-125. MR 82m:65100.

[49] C.M.Chen, Extrapolation of Galerkin solutions. Nat. Sci. J. Xiangtan Univ. 2 (1981), pp. 1-6.

[50] C.M. Chen, Superconvergence of finite elements for nonlinear problems. Numer. Math. J. Chinese Univ. 4 (1982), pp. 222-228. MR 83m:65083.

[51] J. Christiansen and R.D. Russell, Adaptive mesh selection strategies for solving boundary value problems. SIAM J. Numer. Anal. 15 (1978), pp. 59-80. MR 57#11071.

[52] J. Christiansen and R.D. Russell, Error analysis for spline collocation methods with application to knot selection. Math. Comp. 32 (1978), pp. 415-419. MR 58#13736.

[53] P.G. Ciarlet, The Finite Element Method for Elliptic Problems. North-Holland Publishing Co., Amsterdam, New York, Oxford, 1978. MR 58#25001.

[54] M. Costabel, E. Stephan and W.L. Wendland, On the boundary integral equation of the first kind for the bi-Laplacian in a polygonal plane domain. Ann. Scuola Norm. Sup. Pisa Cl. Sci., Ser. IV, 10 (1983), pp. 197-242.

[55] M.J.P. Cullen, The use of quadratic finite element methods and irregular grids in the solution of hyperbolic problems. J. Comput. Phys. 45 (1982), pp. 221-245. MR 83f:65152.

[56] R.Z. Dautov, Superconvergence of difference schemes for the third boundary value problem for quasilinear elliptic equations. Chisl. Metody Mekh. Sploshn. Sredy 11 (1980), pp. 62-80. MR 84b:65102.

[57] R.Z. Dautov, Superconvergence of finite-element schemes with numerical integration for quasilinear fourth-order elliptic equations. Differencial'nye Uravnenija 18 (1982), pp. 1172-1181, Differential Equations 18 (1982), pp. 818-824. MR 83k:65080.

[58] R.Z. Dautov and A.V. Lapin, Difference schemes of an arbitrary order of accuracy for quasilinear elliptic equations. Izv. Vysš. Učebn. Zaved. Matematika 209 (1979), pp. 24-37. MR 81g:65128.

[59] R.Z. Dautov and A.V. Lapin, Investigation of the convergence, in mesh norms, of finite-element-method schemes with numerical integration for fourth-order elliptic equations. Differential Equations 17 (1981), pp. 807-817. MR 83h:65122.

[60] R.Z. Dautov, A.V. Lapin and A.D. Lyashko, Some mesh schemes for quasilinear elliptic equations. Ž. Vyčisl. Mat. i Mat. Fiz. 20 (1980), pp. 334-339, U.S.S.R. Computational Math. and Math. Phys. 20 (1980), pp. 62-78. MR 81j:65121.

[61] J. Descloux, Interior regularity and local convergence of Galerkin finite element approximations for elliptic equations. Topics in Numerical Analysis, II (Proc. Royal Irish Acad. Conf., Dublin, 1974), Academic Press, New York, 1975, pp. 24-41. MR 54#1658.

[62] J.C. Diaz, A collocation-Galerkin method for two-point boundary value problem using continuous piecewise polynomial spaces. SIAM J. Numer. Anal. 14 (1977), pp. 844-858. MR 58#3481.

[63] V.A. Dougalis, Multistep-Galerkin methods for hyperbolic equations. Math. Comp. 33 (1979), pp. 563-584. MR 81b:65081.

[64] V.A. Dougalis and S.M. Serbin, On the superconvergence of Galerkin approximations to second-order hyperbolic equations. SIAM J. Numer. Anal. 17 (1980), pp. 431-446. MR 81j:65106.

[65] J.J. Douglas, A superconvergence result for the approximate solution of the heat equation by a collocation method. The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations (Proc. Sympos., Univ. Maryland, Baltimore, 1972). Academic Press, New York, 1972, pp. 475-490. MR 53#7063.

[66] J.J. Douglas, Improved accuracy through superconvergence in the pressure in the simulation of miscible displacement. Computing Methods in Applied Sciences and Engineering (Proc. Conf., Versailles, 1983), LN in Comp. Sci., Springer Verlag, 1984.

[67] J.J. Douglas, Superconvergence in the pressure in the simulation of miscible displacement (to appear).

[68] J.J. Douglas and T. Dupont, Superconvergence for Galerkin methods for the two-point boundary problem via local projections. Numer. Math. 21 (1973), pp. 270-278. MR 48#10130.

[69] J.J. Douglas and T. Dupont, Some superconvergence results for Galerkin methods for the approximate solution of two-point boundary value problems. Topics in Numerical Analysis, II (Proc. Royal Irish Acad. Conf., Dublin, 1972), Academic Press, London, 1973, pp. 89-92. MR 51#2295.

[70] J.J. Douglas and T. Dupont, Galerkin approximations for the two-point boundary problem using continuous, piecewise polynomial spaces. Numer. Math. 22 (1974), pp. 99-109. MR 50#15360.

[71] J.J. Douglas, T. Dupont and L. Wahlbin, Optimal L_∞ error estimates for Galerkin approximations to solutions of two-point boundary value problems. Math. Comp. 29 (1975), pp. 475-483. MR 51#7298.

[72] J.J. Douglas, T. Dupont and M.F. Wheeler, A Galerkin procedure for approximating the flux on the boundary for elliptic and parabolic boundary value problems. RAIRO Anal. Numér. 8 (1974), pp. 47-59. MR 50#11811.

[73] J.J. Douglas, T. Dupont and M.F. Wheeler, An L^∞ estimate and a superconvergence result for a Galerkin method for elliptic equations based on tensor products of piecewise polynomials. RAIRO Anal. Numér. 8 (1974), pp. 61-66. MR 50#11812.

[74] J.J. Douglas, T. Dupont and M.F. Wheeler, Some superconvergence results for an H^1-Galerkin procedure for the heat equation. Comp. Methods in Appl. Sci. and Engrg. Part 1 (Proc. Internat. Sympos., Versailles, 1973), Lecture Notes in Comp. Sci., Vol. 10, Springer Verlag, Berlin, New York, 1974, pp. 288-311. MR 56#10056.

[75] J.J. Douglas, R.E. Ewing and M.F. Wheeler, The approximation of the pressure by a mixed method in the simulation of misicble displacement. RAIRO Anal. Numér. 17 (1983), pp. 17-33.

[76] J.J. Douglas and F.A. Milner, Interior and superconvergence estimates for mixed methods for second order elliptic equations (to appear).

[77] T. Dupont, A unified theory of superconvergence for Galerkin methods for two-point boundary problems. SIAM J. Numer. Anal. 13 (1976), pp. 362-368. MR 53#12021.

[78] G. Fairweather, Finite Element Galerkin Methods for Differential Equations. Lecture Notes in Pure and Applied Mathematics, Vol. 34. Marcel Dekker, Inc., New York, Basel, 1978, MR 58#13781.

[79] C.A.J. Fletcher, Computational Galerkin Methods. Springer Series in Computational Physics, Springer Verlag, New York, 1984.

[80] R. Glowinski, Numerical Methods for Nonlinear Variational Problems. Springer Series in Computational Physics. Springer Verlag, New York, 1984.

[81] M.A. Goldberg, A survey of numerical methods for integral equations. Solution Methods for Integral Equations. Math. Concepts and Methods in Sci. and Engrg., 18. Plenum Press, New York, London, 1979, pp. 1-58. MR 80m:65090.

[82] M.A. Goldberg, M. Lea and G. Miel, A superconvergence result for the generalized airfoil equation with application to the flap problem. J. Integral Equations 5 (1983), pp. 175-186. MR 84e:65127.

[83] I.G. Graham, Galerkin methods for second kind integral equations with singularities. Math. Comp. 39 (1982), pp. 519-533. MR 84d:65090.

[84] I.G. Graham, S. Joe and I.H. Sloan, Galerkin versus collocation for integral equations of the second kind. IMA J. Numer. Anal. (to appear).

[85] P.P.N. de Groen, A finite element method with a large mesh-width for a stiff two-point boundary value problem. J. Comput. Appl. Math. 7 (1981), pp. 3-15. MR 82d:65059.

[86] J. Haslinger, Elements finis et l'estimation de l'erreur interieur de la convergence. Comment. Math. Univ. Carolin. 15 (1974), pp. 85-102. MR 49#1799.

[87] M. El Hatri, Superconvergence of the lumped mass approximation of parabolic equations. C. R. Acad. Bulgare Sci. 36 (1983), pp. 575-578.

[88] M. El Hatri, Superconvergence of axisymmetrical boundary-value problem. C. R. Acad. Bulgare Sci. 36 (1983), pp. 1499-1502.

[89] M. El Hatri, Superconvergence in the finite element method for a degenerated boundary value problem. C. R. Acad. Bulgare Sci. (to appear).

[90] E. Hinton and J.S. Campbell, Local and global smoothing of discontinuous finite element functions using a least squares method. Internat. J. Numer. Methods Engrg. 8 (1974), pp. 461-480. MR 53#14859.

[91] E. Hinton and D.R.J. Owen, Finite Element Programming. Academic Press, London, New York, San Francisco, 1977.

[92] E. Hinton, F.C. Scott and R.E. Ricketts, Local least squares stress smoothing for parabolic isoparametric elements. Internat. J. Numer. Methods Engrg. 9 (1975), pp. 235-238.

[93] F.R. de Hoog and R. Weiss, Collocation methods for singular boundary value problems. SIAM J. Numer. Anal. 15 (1978), pp. 198-217. MR 57#8041.

[94] E.N. Houstis, Application of method of collocation on lines for solving nonlinear hyperbolic problems. Math. Comp. 31 (1977), pp. 443-456. MR 56#1749.

[95] E.N. Houstis, A collocation method for systems of nonlinear ordinary differential equations. J. Math. Anal. Appl. 62 (1978), pp. 24-37. MR 58#8295.

[96] G.C. Hsiao and W.L. Wendland, The Aubin-Nitsche lemma for integral equations. J. Integral Equations 3 (1981), pp. 299-315. MR 83j:45019.

[97] D. Huang and D. Wu, The superconvergence of the spline finite element solution and its second order derivate for the two-point boundary problem of a fourth order differential equation. J. Zhejiang Univ. 3 (1982), pp. 92-99.

[98] B.L. Hulme, One-step piecewise polynomial Galerkin methods for initial value problems. Math. Comp. 26 (1972), pp. 415-426. MR 47#9834.

[99] B.M. Irons and A. Razzaque, Experience with the patch test for convergence of finite elements. Mathematical Foundations of the Finite Element Method with Applications to P.D.E., Academic Press, New York, 1972, pp.557-587. MR 54#11813.

[100] C. Johnson and J. Pitkäranta, Analysis of some mixed finite element methods related to reduced integration. Math. Comp. 33 (1982), pp. 375-400. MR 83d:65287.

[101] R.P. Kendall and M.F. Wheeler, A Crank-Nicolson H^{-1}-Galerkin procedure for parabolic problems in a single space variable. SIAM J. Numer. Anal. 13 (1976), pp. 861-876. MR 58#13785.

[102] J.T. King and S.M. Serbin, Boundary flux estimates for elliptic problems by the perturbed variational method. Computing 16 (1976), pp. 339-347. MR 54#6524.

[103] V.G. Korneev, Superconvergence of solutions of the finite element method in mesh norms. Zh. Vychisl. Mat. i Mat. Fiz. 22 (1982), pp. 1133-1148. MR 84b:65114.

[104] M. Křížek and P. Neittaanmäki, Superconvergence phenomenon in the finite element method arising from averaging gradients. Numer. Math. 45 (1984), pp. 105-116.

[105] M. Křížek and P. Neittaanmäki, A phenomenon arising from the use of harmonic and biharmonic elements. Research Report 5/1984, Lappeenranta Univ. of Technology, 1984, pp. 1-11.

[106] M. Křížek and P. Neittaanmäki, On a global superconvergent recovery technique for the gradient from piecewise linear FE-approximations. Preprint No. 33, Univ. of Jyväskylä, 1984, pp. 1-17.

[107] U. Lamp, T. Schleicher, E. Stephan and W.L. Wendland, Theoretical and experimental asymptotic convergence of the boundary integral method for plane mixed boundary value problem. Boundary Element Methods in Engineering (Proc. Fourth Int. Semin., Southampton, Engl., 1982). Southampton, 1982, pp. 3-17.

[108] A.V. Lapin, Schemes of the finite element methods for some classes of variational inequalities, error estimates, algorithms (to appear).

[109] E.W. Larsen and P. Nelson, Finite-difference approximations and superconvergence for the discrete-ordinate equations in slab geometry. SIAM J. Numer. Anal. 19 (1982), pp. 334-348. MR 83# 82050.

[110] R.D. Lazarov, A.B. Andreev and M. El Hatri, Superconvergence of the gradient in the finite-element method for some elliptic and parabolic problems. Variational-Difference Methods in Mathematical Physics. Part 2 (Proc. Fifth Int. Conf., Moscow, 1983), Viniti, Moscow, 1984, pp. 13-25.

[111] P. Lesaint and P.A. Raviart, On a finite element method for solving the neutron transport equation. Mathematical Aspects of Finite Elements in P.D.E. (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, 1974). Univ. of Wisconsin, Madison, Academic Press, New York, 1974, pp. 89-123. MR 58#31918.

[112] P. Lesaint and M. Zlámal, Superconvergence of the gradient of finite element solutions. RAIRO Anal. Numér. 13 (1979), pp. 139-166. MR 80g:65112.

[113] N. Levine, Stress sampling points for linear triangles in the finite element method. Numerical analysis report 10/82, Univ. of Reading, 1982.

[114] N. Levine, Superconvergent recovery of the gradient from piecewise linear finite element approximations. IMA J. Numer. Anal. (to appear).

[115] Z. Leyk, Superconvergence in the finite element method. Matematyka Stosowana 20 (1982), pp. 93-107.

[116] Q. Lin and J.Q. Liu, Extrapolation method for Fredholm integral equations with nonsmooth kernels. Numer. Math. 35 (1980), pp. 459-464. MR 81m:65198.

[117] Q. Lin and T. Lü, The splitting extrapolation method for multidimensional problems. J. Comput. Math. 1 (1983), pp. 45-51.

[118] Q. Lin and T. Lü, Correction and splitting procedures for solving partial differential equations. Research report IMS-9, Inst. Math. Sci. Chengdu Branch of Acad. Sinica, 1983, pp. 1-28.

[119] Q. Lin and T. Lü, Asymtotic expansions for finite element approximation of elliptic problem on polygonal domains. Computing Methods in Applied Sciences and Engineering (Proc. Sixth Int. Conf. Versailles, 1983), LN in Comp. Sci., Springer Verlag, 1984.

[120] Q. Lin and T. Lü, The combination of approximate solutions for accelerating the convergence. RAIRO Anal. Numér. 18 (1984), pp. 153-160.

[121] Q. Lin, T. Lü and S. Shen, Asymptotic expansion for finite element approximations. Research Report IMS-11, Inst. Math. Sci. Chengdu Branch of Acad. Sinica, 1983, pp. 1-6.

[122] Q. Lin, T. Lü and S. Shen, Maximum norm estimate, extrapolation and optimal point of stresses for the finite element methods on the strongly regular triangulation. J. Comput. Math. 1 (1983), pp. 376-383.

[123] Q. Lin and J.Ch. Xu, Linear finite elements with high accuracy, J. Comput. Math., 3 (1985), No 1.

[124] Q. Lin and Q.D. Zhu, Asymptotic expansion for the derivative of finite elements. J. Comput. Math. 2 (1984), pp. 361-363.

[125] B. Lindberg, Error estimation and iterative improvement for discretization algorithms. BIT 20 (1980), pp. 486-500. MR 82j:65036.

[126] J. Locker and P.M. Prenter, On least squares methods for linear two-point boundary value problems. Functional Analysis Methods in Numerical Analysis (Proc. Spec. Sess., AMS, St. Louis, 1977). LN in Math., Vol. 701, Springer, Berlin, 1979, pp. 149-168. MR 80j:65029.

[127] J. Locker and P.M. Prenter, Regularization with differential operators. II: Weak least squares finite element solutions to first kind integral equations. SIAM J. Numer. Anal. 17 (1980), pp. 247-267. MR 83j:65062b.

[128] A. Louis, Acceleration of convergence for finite element solutions of the Poisson equation. Numer. Math. 33 (1979), pp. 43-53. MR 81i:65091.

[129] L. Mansfield, On mixed finite element methods for elliptic equations. Comput. Math. Appl. 7 (1981), pp. 59-66. MR 82b:65133.

[130] G.I. Marchuk and V.V. Shaidurov, Increase of accuracy of projective-difference schemes. Comp. Methods in Appl. Sci. and Engrg. Part 2 (Proc. Internat. Sympos., Versailles, 1973), LN in Comp. Sci., Springer Verlag, Berlin, 1974, pp. 240-260. MR 49#4198.

[131] G.I. Marchuk and V.V. Shaidurov, Difference Methods and Their Extrapolations, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[132] R.S. Marshall, An alternative interpretation of superconvergence. Internat. J. Numer. Methods Engrg. 14 (1979), pp. 1707-1709. MR 80m:65079.

[133] M.S. Mock, Projection methods with different trial and test spaces. Math. Comp. 30 (1976), pp. 400-416. MR 54#11814.

[134] M.Nakao, Some superconvergence estimates for a collocation H^1-Galerkin method for parabolic problems. Mem. Fac. Sci. Kyushu Univ. Ser. A 35 (1981), pp. 291-306. MR 83a:65111.

[135] M. Nakao, Interior estimates and superconvergence for H^{-1}-Galerkin method to elliptic equations. Bull. Kyushu Inst. Tech. Math. Natur. Sci. 30 (1983), pp. 19-30.

[136] P. Neittaanmäki and M. Křížek, Superconvergence of the finite element schemes arising from the use of averaged gradients. Accuracy Estimates and Adaptive Refinements in Finite Element Computations (Proc. Int. Conf., Lisbon, 1984), pp. 169-178.

[137] P. Neittaanmäki and Q. Lin, Acceleration of the convergence in finite difference methods. Lappeenranta Univ. of Technology, Res. Rep. 91 (1984), pp. 1-17.

[138] P. Neittaanmäki and J. Saranen, A modified least squares FE-method for ideal fluid flow problems. J. Comput. Appl. Math. 8 (1982), pp. 165-169.

[139] B. Neta and H.D. Victory, A new fourth-order finite-difference method for solving discrete-ordinates slab transport equations. SIAM J. Numer. Anal. 20 (1983), pp. 94-105.

[140] J.A. Nitsche, L_∞-error analysis for finite elements. Mathematics of Finite Elements and Applications, III (Proc. Third MAFELAP Conf., Brunel Univ., Uxbridge, 1978), Academic Press, London, 1979, pp. 173-186. MR 81f:65084.

[141] J.A. Nitsche and A.H. Schatz, Interior estimates for Ritz-Galerkin methods. Math. Comp. 28 (1974), pp. 937-958. MR 51#9525.

[142] L.A. Oganesjan, V.J. Rivkind and L.A. Ruhovec, Variation-difference methods for the solution of elliptic equations. Part I (Proc. Sem., Issue 5, Vilnius, 1973), Inst. of Phys. and Math., Vilnius, 1973, pp. 3-389, MR 57#18146.

[143] L.A. Oganesjan and L.A. Ruhovec, An investigation of the rate of convergence of variational-difference schemes for second order elliptic equations in a two-dimensional region with smooth boundary. Ž. Vyčisl. Mat. i Mat. Fiz. 9 (1969), pp. 1102-1120. MR 45#4665.

[144] L.A. Oganesjan and L.A. Ruhovec, Variational-Difference Methods for the Solution of Elliptic Equations. Izd. Akad. Nauk Armjanskoi SSR, Jerevan, 1979. MR 82m:65105.

[145] P.M. Prenter and J. Locker, Optimal L^2 and L^∞ error estimates for continuous and discrete least squares methods for boundary value problems. SIAM J. Numer. Anal. 15 (1978), pp. 1151-1160. MR 80d:65097.

[146] W. Pereyra and E.G. Sewell, Mesh selection for discrete solution of boundary value problems in ordinary differential equations. Numer. Math. 23 (1975), pp. 261-268. MR 57#4527.

[147] H.H. Rachford and M.F. Wheeler, An H^{-1}-Galerkin procedure for the two-point boundary value problem. Mathematical Aspects of Finite Elements in Partial Diff. Equations, Academic Press, New York, 1974, pp. 353-382. MR 50#1525.

[148] R. Rannacher and R. Scott, Some optimal error estimates for piecewise linear finite element approximations. Math. Comp. 38 (1982), pp. 437-446.

[149] G.W. Reddien, Collocation at Gauss points as a discretization in optimal control. SIAM J. Control Optim. 17 (1979), pp. 298-306. MR 80b:93171.

[150] G.W. Reddien, Codes for boundary-value problems in ordinary differential equations. (Proc. Conf. Houston, 1978.) LN in Comp. Sci., Vol. 76, Springer, Berlin, 1979, pp. 206-227.

[151] G.W. Reddien, Projection methods for two-point boundary value problems. SIAM Rev. 22 (1980), pp. 156-171. MR 81e:65049.

[152] T. Regińska, Superconvergence of eigenvalue external approximation for ordinary differential operators (to appear).

[153] G.R. Richter, Superconvergence of piecewise polynomial Galerkin approximations, for Fredholm integral equations of the second kind. Numer. Math. 31 (1978/79), pp. 63-70. MR 80a:65273.

[154] J. Saranen and W.L. Wendland, On the asymptotic convergence of collocation methods with spline function of even degree. Math. Comp. (to appear).

[155] E. Schäfer, Fehlerabschätzungen für Eigenwertnäherungen nach der Ersatzkernmethode bei Integralgleichungen. Numer. Math. 32 (1979), pp. 281-290. MR 80e:65129.

[156] A.H. Schatz and L.B. Wahlbin, Interior maximum norm estimates for finite element methods. Math. Comp. 31 (1977), pp. 414-442. MR 55#4748.

[157] R. von Seggern, Die Methode der finiten Elemente be der numerischen Behandlung linear Integrodifferentialgleichungen. Kernforschungs-anlage Julich, Zentralinstitut für Angewandte Mathematik, Julich, 1979. MR 83e:65215.

[158] R. von Seggern, Ein Superkonvergenzresultat bei Anwendung der Methode der finiten Elemente auf lineare Integrodifferential-gleichungen. Z. Angew. Math. Mech. 61 (1981), pp. 320-321.

[159] I.H. Sloan, Iterated Galerkin method for eigenvalue problems. SIAM J. Numer. Anal. 13 (1976), pp. 753-760. MR 55#1727.

[160] I.H. Sloan, A review of numerical methods for Fredholm integral equations of the second kind. Application and Numerical Solution of Integral Equations. (Proc. Sem., Australian Nat. Univ., Canberra, 1978), Sijthoff & Noordhoff, Alphen aan der Rijn, 1980, pp. 51-74. MR 81h:65133.

[161] I.H. Sloan, Superconvergence and the Galerkin method for integral equations of the second kind. Treatment of Integral Equations by Numerical Methods (Proc. Symp., Durham, 1982). Academic Press, New York, London, 1982, pp. 197-207.

[162] I.H. Sloan, Four variants of the Galerkin method for integral equations of the second kind. IMA J. Numer. Anal. 4 (1984), pp. 9-17.

[163] I.H. Sloan and V. Thomée, Superconvergence of the Galerkin iterates for integral equations of the second kind. SIAM J. Numer. Anal. (to appear).

[164] A. Spence and K.S. Thomas, On superconvergence properties of Galerkin's method for compact operator equations. IMA J. Numer. Anal. 3 (1983), pp. 253-271.

[165] E. Stephan, and W.L. Wendland, An augmented Galerkin procedure for the boundary integral method applied to two-dimensional screen and crack problems. Applicable Anal. 18 (1984), pp. 183-220.

[166] G. Strang and G. Fix, An Analysis of the Finite Element Method. Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1973. MR 56# 1747.

[167] V. Thomée, Spline approximation and difference schemes for the heat equation. Math. Foundations of the Finite Element Method with Applications to P.D.E. (Proc. Sympos., Univ. of Maryland, 1972). Academic Press, New York, London, 1972, pp. 711-746. MR 49#11824.

[168] V. Thomée, Convergence estimates for semi-discrete Galerkin methods for initial-value problems. Numerische,insbesondere approximationstheoretische Behandlung von Funktionalgleichungen. LN in Math., Vol. 333, Springer Verlag, New York, 1973, pp. 243-262. MR 56#17147.

[169] V. Thomée, Some error estimates in Galerkin methods for parabolic equations. Mathematical Aspects of Finite Element Methods (Proc. Conf., Rome, 1975). LN in Math., Vol.606, Springer Verlag, Berlin, Heidelberg, New York, 1977, pp. 353-362. MR 58#31928.

[170] V. Thomée, High order local approximations to derivatives in the finite element method. Math. Comp. 31 (1977), pp. 652-660. MR 55#11572.

[171] V. Thomée, Galerkin-finite element method for parabolic equations. Bull. Iranian Math. Soc. 10 (1978), pp. 3-17. MR 81b:65107.

[172] V. Thomée, Galerkin-finite element methods for parapolic equations. (Proc. Internat. Congress of Mathematicians, Helsinki, 1978). Acad. Sci. Fennica, Helsinki, 1980, pp. 943-952. MR 81i:65100.

[173] V. Thomée, Negative norm estimates and superconvergence in Galerkin methods for parabolic problems. Math. Comp. 34 (1980), pp. 93-113. MR 81a:65092.

[174] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems. LN in Math., Vol. 1054, Springer Verlag, Berlin, 1984.

[175] V. Thomée and B. Wendroff, Convergence estimates for Galerkin method for variable coefficient initial value problems. SIAM J. Numer. Anal. 11 (1974), pp. 1059-1068. MR 51#7309.

[176] A.L. Urvancev and V.V. Shaidurov, Sharpening an approximate solution of the quasilinear Poisson equation. Variational-Difference Methods in Math. Phys. (Proc. Third Int. Conf., Novosibirsk, 1975). Vyčisl. Centr Sibirsk. Otdel Akad. Nauk. SSSR, Novosibirsk, 1976, pp. 134-144. MR 57#11099.

[177] J. Vacek, Dual variational principles for an elliptic partial differential equation. Apl. Mat. 21 (1976), pp. 5-27. MR 54#716.

[178] G. Vainikko and P. Uba, A piecewise polynomial approximation to the solution of an integral equation with weakly singular kernel. J. Austral. Math. Soc. Ser. B 22 (1981), pp. 431-438. MR 82h:65100.

[179] O.A. Veryard, Problems associated with the convergence of isoparametric and miscoparametric finite elements. M. Sc. thesis, Univ. of Wales, 1971.

[180] W.L. Wendland, Boundary element methods and their asymtotic convergence. Theoretical Acoustics and Numerical Techniques. CISM Courses No. 277, Springer, Wien, New York, 1983, pp. 135-216.

[181] W.L. Wendland, E. Stephan and G.C. Hsiao, On the integral equation method for the plane mixed boundary value problem of the Laplacian. Math. Methods Appl. Sci. 1 (1979), pp. 265-321. MR 82e:31003.

[182] M.F. Wheeler, An optimal L_∞ error estimate for Galerkin approximations to solutions of two-point boundary value problems. SIAM J. Numer. Anal. 10 (1973), pp. 914-917. MR 49#8399.

[183] M.F. Wheeler, Galerkin procedure for estimating the flux for two-point boundary value problems. SIAM J. Numer. Anal. 11 (1974), pp. 764-768. MR 52#4644.

[184] M.F. Wheeler, A C^0-collocation-finite element method for two-point boundary value problems and one space dimensional parabolic problems. SIAM J. Numer. Anal. 14 (1977), pp. 71-90. MR 56#13667.

[185] C. Wielgosz, Exact results given by finite element methods in mechanics. J. Méc. Appl. 1 (1982), pp. 323-329.

[186] R. Winther, Some superlinear convergent results for the conjugate gradient method. SIAM J. Numer. Anal. 17 (1980), pp. 14-17. MR 81k:65060.

[187] R. Winther, A stable finite element method for initial-boundary value problems for first-order hyperbolic systems. Math. Comp. 36 (1981), pp. 65-86. MR 81m:65181.

[188] R. Winther, A finite element method for a version of the Boussinesq equation. SIAM J. Numer. Anal. 19 (1982), pp. 561-570. MR 83f:65184.

[189] Q.D. Zhu, A superconvergence result for the finite element method. Numer. Math. J. Chinese Univ. 3 (1981), pp. 50-55. MR 82f:65121.

[190] Q.D. Zhu, "Nature" inner superconvergence for the finite element method. Proc. China-France Sympos. on the Finite Element Method, Beijing, 1982.

[191] O.C. Zienkiewicz and Y.K. Cheung, The Finite Element Method in Structural and Continuum Mechanics. McGraw Hill, London, 1967. MR 47#4518.

[192] M. Zlámal, Some superconvergence results in the finite element method. Mathematical Aspects of Finite Element Methods (Proc. Conf., Rome, 1975). LN in Math. Vol. 606, Springer Verlag, Berlin, Heidelberg, New York, 1977, pp. 353-362. MR 58#8365.

[193] M. Zlamál, Superconvergence and reduced integration in the finite element method. Math. Comp. 32 (1978), pp. 663-685. MR 58#13794.

[194] M. Zlámal, Superconvergence of gradients in the finite element method. Variational-Difference Methods in Mathematical Physics (Proc. All-Union Conf., Novosibirsk, 1977). Akad. Nauk. SSSR Sibirsk. Otdel., Vychisl. Centr, Novosibirsk, 1978, pp. 15-22. MR 82e:65116.

[195] M. Zlámal, Superconvergence of the gradient of finite element solutions. (Wiss. Z. Hochschule Architektur Bauwesen Weimar, 1979). Weimar, 1979, pp. 375-380.

11. J. SARANEN
On generalized harmonic fields in domains with anisotropic nonhomogeneous media
(Preprint 11, 1981, 15 pp.)

12. P. NEITTAANMÄKI and J. SARANEN
A mixed finite element method for the heat flow problem
(Preprint 12, 1981, 8 pp.)

13. S. GRANLUND, P. LINDQVIST and O. MARTIO
F-harmonic measure in space
(Preprint 13, 1981, 24 pp.)

14. M. SAARIMÄKI
Counterexamples to the algebraic closed graph theorem
(Preprint 14, 1981, 7 pp.)

15. J. HASLINGER and P. NEITTAANMÄKI
Penalty method in design optimization governed by a unilateral boundary value problem
(Preprint 15, 1982, 23 pp.)

16. S. GRANLUND, P. LINDQVIST and O. MARTIO
Note on the PWB-method in the non-linear case
(Preprint 16, 1982, 21 pp.)

17. M. KŘÍŽEK and P. NEITTAANMÄKI
On the validity of Friedrichs' inequalities
(Preprint 17, 1982, 16 pp.)

18. J. HASLINGER and P. NEITTAANMÄKI
On the method of penalization in design optimization of systems governed by some unilateral boundary value problems
(Preprint 18, 1983, 17 pp.)

19. T. KUUSALO
A direct proof of Smirnov's theorem on locally metrizable spaces
(Preprint 19, 1983, 4 pp.)

20. K. ASTALA and H.-O. TYLLI
On the bounded compact approximation property and measures of noncompactness
(Preprint 20, 1983, 22 pp.)

21. S. GRANLUND, P. LINDQVIST and O. MARTIO
Phragmén-Lindelöf's and Lindelöf's theorems
(Preprint 21, 1983, 42 pp.)

22. J. HASLINGER and P. NEITTAANMÄKI
On optimal shape design of systems governed by mixed Dirichlet-Signorini boundary value problems
(Preprint 22, 1983, 42 pp.)

23. J. HASLINGER and P. NEITTAANMÄKI
On the existence of optimal shape in contact problems
(Preprint 23, 1983, 21 pp.)

24. M. KŘÍŽEK and P. NEITTAANMÄKI
Superconvergence phenomenon in the finite element method arising from averaging gradients
(Preprint 24, 1983, 20 pp.)

25. O. MARTIO
Radial uniqueness of quasiregular mappings
(Preprint 25, 1983, 12 pp.)

26. M. KŘÍŽEK and P. NEITTAANMÄKI
Superconvergence of the finite element schemes arising from the use of averaged gradients
(Preprint 26, 1984, 10 pp.)

27. T. KUUSALO
Hurwitz's theorem for open mappings
(Preprint 27, 1984, 3 pp.)

28. A. LEHTONEN
On the Euler-Lagrange inequality of a convex variational integral
(Preprint 28, 1984, 15 pp.)

29. P. LINDQVIST and O. MARTIO
Two theorems of N. Wiener for solutions of quasilinear elliptic equations
(Preprint 29, 1984, 26 pp.)

30. J. HASLINGER, P. NEITTAANMÄKI and T. TIIHONEN
On optimal shape design of an elastic body on a rigid foundation
(Preprint 30, 1984, 8 pp.)

31. T. KILPELÄINEN
A simple proof for the lower-semicontinuity of homogeneous and convex variational integrals
(Preprint 31, 1984, 8 pp.)

32. L. KAHANPÄÄ
A Guy David curve that cannot be covered with finitely many chord-arc curves
(Preprint 32, 1984, 8 pp.)

33. M. KŘÍŽEK and P. NEITTAANMÄKI
On a global superconvergent recovery technique for the gradient from piecewise linear FE-approximations
(Preprint 33, 1984, 17 pp.)