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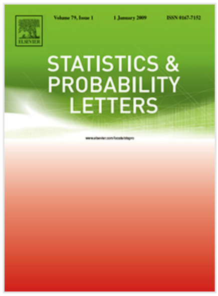

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GENERICITY OF DIMENSION DROP ON SELF-AFFINE SETS

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ABSTRACT. We prove that generically, for a self-affine set in $\mathbb{R}^d$, removing one of the affine maps which defines the set results in a strict reduction of the Hausdorff dimension. This gives a partial positive answer to a folklore open question.

1. Introduction and statement of main results

Let $f_i\colon \mathbb{R}^d \to \mathbb{R}^d$ be a contractive invertible mapping for all $i \in \{1,\dots,N\}$. Throughout of the paper, let N be an integer such that $N \geq 2$. By Hutchinson [6], there exists a unique nonempty compact set $E \subset \mathbb{R}^d$ satisfying

$$E = \bigcup_{i=1}^{N} f_i(E).$$

If the mappings f_i are affine, then the set E is called *self-affine*. For every $\mathsf{v} = (v_1,\dots,v_N) \in (\mathbb{R}^d)^N$ and $\mathsf{A} = (A_1,\dots,A_N) \in GL_d(\mathbb{R})^N$ with $\|A_i\| < 1$ for all $i \in \{1,\dots,N\}$ we denote the self-affine set corresponding to A and v by

$$E_{\mathsf{A},\mathsf{v}} = \bigcup_{i=1}^{N} A_i(E_{\mathsf{A},\mathsf{v}}) + v_i.$$

Then, by setting $\mathsf{A}' = (A_1,\dots,A_{N-1}) \in GL_d(\mathbb{R})^{N-1}$ and $\mathsf{v}' = (v_1,\dots,v_{N-1}) \in (\mathbb{R}^d)^{N-1}$, the self-affine set

$$E_{\mathsf{A}',\mathsf{v}'} = \bigcup_{i=1}^{N-1} A_i(E_{\mathsf{A}',\mathsf{v}'}) + v_i \subset E_{\mathsf{A},\mathsf{v}}$$

is obtained from $E_{\mathsf{A},\mathsf{v}}$ by removing one of the affine maps which defines the set.

A folklore conjecture suggests that $\dim_{\mathrm{H}}(E_{\mathsf{A}',\mathsf{v}'}) < \dim_{\mathrm{H}}(E_{\mathsf{A},\mathsf{v}})$. There exist simple counter-examples showing that this cannot be the case for all self-affine sets; for example, see [9, Example 9.3]. Therefore the conjecture is about generic behavior. During recent years, the question has been propagated by Schmeling. It follows from Feng and Käenmäki [4, §3] and Falconer [1, Theorem 5.3] that the conjecture holds in $\mathbb{R}^2$ for Lebesgue almost every choice of translation vectors v. Very recently, Käenmäki and Morris [9, Theorem B] solved the problem in dimension three. They showed that the conjecture holds in $\mathbb{R}^3$ again for Lebesgue almost every choice of translation vectors v. We remark that, by Falconer and Miao [2, Theorem 2.5] and Falconer [1, Theorem 5.3], the conjecture holds in arbitrary dimension whenever the matrix tuple is upper triangular.

In this note, we present a simple proof to show that the conjecture holds in arbitrary dimension for a generic choice of the matrix tuple. This is a partial solution to the problem since one expects the conjecture to be true for all matrix tuples. At least, it is the case for rational dimensions:

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Theorem A. *Let $0 \le s \le d$ be rational and $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$ be such that $P_{\mathsf{A}}(\varphi^s) = 0$ and $\|A_i\| < 1/2$ for all $i \in \{1, \ldots, N\}$. Then*

$$\dim_{\mathrm{H}}(E_{\mathsf{A}',\mathsf{v}'}) < \dim_{\mathrm{H}}(E_{\mathsf{A},\mathsf{v}}) = s$$

for $\mathcal{L}^{dN}$-almost all $\mathsf{v} \in \mathbb{R}^{dN}$.

The proof of Theorem A relies on understanding the structure and properties of equilibrium states obtained from the associated sub-additive dynamical system. Equilibrium state is a probability measure maximizing the dimension and it will be defined in §2.5.

For irrational dimensions, we verify the conjecture in an open and dense set of matrix tuples. The proof of this result is based on a property of matrix tuples called s-irreducibility. It is a sufficient condition for the uniqueness of the equilibrium state; see Theorem 2.5.

Theorem B. *For $\mathcal{L}^{d^2N}$-almost every $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$ with $P_{\mathsf{A}}(\varphi^d) \le 0$ and $\|A_i\| < 1/2$ for all $i \in \{1, \ldots, N\}$ it holds that*

$$\dim_{\mathrm{H}}(E_{\mathsf{A}',\mathsf{v}'}) < \dim_{\mathrm{H}}(E_{\mathsf{A},\mathsf{v}})$$

for $\mathcal{L}^{dN}$-almost all $\mathsf{v} \in \mathbb{R}^{dN}$. In particular, the exceptional set of tuples $\mathsf{A} \in GL_d(\mathbb{R})^N$ for which the conclusion does not hold is contained in a finite union of (d^2N-1)-dimensional algebraic varieties and thus, has Hausdorff dimension at most d^2N-1.

In the above theorems, P_{A} is the singular value pressure of A defined in (2.7).

Remark 1.1. Since $P_{\mathsf{A}}(\varphi^s)$ is continuous as a function of s and also, by Feng and Shmerkin [5, Theorem 1.2], as a function of A, it is tempting to try to solve the full conjecture just by taking a limit. Unfortunately, this approach does not seem to work – at least not without any further modifications.

2. Preliminaries

2.1. **Set of infinite words.** Fix $N \in \mathbb{N}$ such that $N \ge 2$ and equip the set of all infinite words $\Sigma = \{1, \ldots, N\}^{\mathbb{N}}$ with the usual ultrametric: the distance between two different words is defined to be 2^{-n}, where n is the first place at which the words differ. It is straightforward to see that Σ is compact. The *left shift* is a continuous map $\sigma \colon \Sigma \to \Sigma$ defined by setting $\sigma(\mathtt{i}) = i_2 i_3 \cdots$ for all $\mathtt{i} = i_1 i_2 \cdots \in \Sigma$.

Let Σ_* be the free monoid on $\{1, \ldots, N\}$. The concatenation of two words $\mathtt{i} \in \Sigma_*$ and $\mathtt{j} \in \Sigma_* \cup \Sigma$ is denoted by $\mathtt{ij} \in \Sigma_* \cup \Sigma$. The set Σ_* is the set of all finite words $\{\varnothing\} \cup \bigcup_{n\in\mathbb{N}} \Sigma_n$, where $\Sigma_n = \{1, \ldots, N\}^n$ for all $n \in \mathbb{N}$ and $\varnothing$ satisfies $\varnothing\mathtt{i} = \mathtt{i}\varnothing = \mathtt{i}$ for all $\mathtt{i} \in \Sigma_*$. For notational convenience, we set $\Sigma_0 = \{\varnothing\}$. The word $i_2 \cdots i_n \in \Sigma_{n-1}$ is denoted by $\sigma(\mathtt{i})$ for all $n \in \mathbb{N}$ and $\mathtt{i} = i_1 \cdots i_n \in \Sigma_n$.

The length of $\mathtt{i} \in \Sigma_* \cup \Sigma$ is denoted by $|\mathtt{i}|$. If $\mathtt{i} \in \Sigma_*$, then we set $[\mathtt{i}] = \{\mathtt{ij} \in \Sigma : \mathtt{j} \in \Sigma\}$ and call it a *cylinder set*. If $\mathtt{j} \in \Sigma_* \cup \Sigma$ and $1 \le n < |\mathtt{j}|$, we define $\mathtt{j}|_n$ to be the unique word $\mathtt{i} \in \Sigma_n$ for which $\mathtt{j} \in [\mathtt{i}]$. If $\mathtt{j} \in \Sigma_*$ and $n \ge |\mathtt{j}|$, then $\mathtt{j}|_n = \mathtt{j}$.

2.2. **Multilinear algebra.** We recall some basic facts about the exterior algebra and tensor products. Let $\{e_1, \ldots, e_d\}$ be the standard orthonormal basis of $\mathbb{R}^d$ and define

$$\wedge^k\mathbb{R}^d = \mathrm{span}\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \le i_1 < \cdots < i_k \le d\}$$

for all $k \in \{1, \ldots, d\}$ with the convention that $\wedge^0\mathbb{R}^d = \mathbb{R}$. Recall that the wedge product $\wedge \colon \wedge^k \mathbb{R}^d \times \wedge^j\mathbb{R}^d \to \wedge^{k+j}\mathbb{R}^d$ is an associative and bilinear operator, anticommutative on the elements of $\mathbb{R}^d$. For each $v \in \wedge^k\mathbb{R}^d$ and $1 \le i_1 < \cdots < i_k \le d$ there exists a real number $v_{i_1\cdots i_k}$ so that

$$v = \sum_{1\le i_1<\cdots<i_k\le d} v_{i_1\cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} \tag{2.1}$$

Moreover, if $v_j = (v_j^1, \dots, v_j^d) \in \mathbb{R}^d$ for all $j \in \{1, \dots, k\}$, then

$$v_1 \wedge \cdots \wedge v_k = \sum_{1 \le i_1 < \cdots < i_k \le d} \det \begin{pmatrix} v_1^{i_1} & \cdots & v_1^{i_k} \\ \vdots & \ddots & \vdots \\ v_k^{i_1} & \cdots & v_k^{i_k} \end{pmatrix} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

If $v \in \wedge^k \mathbb{R}^d$ can be expressed as a wedge product of k vectors of $\mathbb{R}^d$, then v is said to be *decomposable*. Observe that e.g. $e_1 \wedge e_2 + e_3 \wedge e_4 \in \wedge^2 \mathbb{R}^4$ is not decomposable.

The group of $d \times d$ invertible matrices of real numbers is denoted by $GL_d(\mathbb{R})$. This space has a topology induced from $\mathbb{R}^{d^2}$. If $A \in GL_d(\mathbb{R})$, we define an invertible linear map $A^{\wedge k} \colon \wedge^k \mathbb{R}^d \to \wedge^k \mathbb{R}^d$ by setting

$$(A^{\wedge k})(e_{i_1} \wedge \cdots \wedge e_{i_k}) = Ae_{i_1} \wedge \cdots \wedge Ae_{i_k}$$

and extending by linearity. Observe that $A^{\wedge k}$ can be represented by a $\binom{d}{k} \times \binom{d}{k}$ matrix whose entries are the $k \times k$ minors of A. Using this and standard properties of determinants, it may be shown that

$$(AB)^{\wedge k} = (A^{\wedge k})(B^{\wedge k}), \tag{2.2}$$

i.e. $A \mapsto A^{\wedge k}$ is a morphism between the corresponding multiplicative linear groups. Furthermore, if $\alpha_1(A) \ge \cdots \ge \alpha_d(A) > 0$ are the singular values of A, that is, the square roots of the eigenvalues of the positive definite matrix $A^T A$, where A^T is the transpose of A, then the products $\alpha_{i_1}(A) \cdots \alpha_{i_k}(A)$ are the singular values of $A^{\wedge k}$, for each $1 \le i_1 < \cdots < i_k \le d$.

The inner product on $\wedge^k \mathbb{R}^d$ is defined by setting

$$\langle v, w \rangle_k = \sum_{1 \le i_1 < \cdots < i_k \le d} v_{i_1 \cdots i_k} w_{i_1 \cdots i_k}$$

for all $v, w \in \wedge^k \mathbb{R}^d$, where $v_{i_1 \cdots i_k}$ and $w_{i_1 \cdots i_k}$ are the coefficients of the corresponding linear combinations; see (2.1). The norm is defined by setting $|v|_k = \langle v, v \rangle_k^{1/2}$ for all $v \in \wedge^k \mathbb{R}^d$. It follows that $|v_1 \wedge \cdots \wedge v_k|_k$ is the k-dimensional volume of the parallelepiped with the vectors $v_1, \dots, v_k$ as sides. The operator norm of the induced linear mapping $A^{\wedge k}$ is

$$\|A^{\wedge k}\|_k = \max\{|A^{\wedge k} v|_k : |v|_k = 1\} = \alpha_1(A) \cdots \alpha_k(A). \tag{2.3}$$

The *tensor product* of two inner product spaces V and W over $\mathbb{R}$ is the inner product space $V \otimes W$. Its elements are equivalence classes of formal sums of vectors in $V \times W$ with coefficients in $\mathbb{R}$ under a natural equivalence relation. If $v \in V$ and $w \in W$, then the equivalence class of (v, w) is denoted by $v \otimes w$, which is called the *tensor product* of v with w. An element of $V \otimes W$ is called *decomposable* if it can be expressed as a tensor product of two vectors in V and W. Observe that if $v_1, v_2 \in V$ and $w_1, w_2 \in W$ are both linearly independent, then $v_1 \otimes w_1 + v_2 \otimes w_2$ is not decomposable. If $\{e_i\}_i$ is a basis for V and $\{e'_j\}_j$ is a basis for W, then $\{e_i \otimes e'_j\}_{i,j}$ is a basis for $V \otimes W$. The dimension of the tensor product space therefore is the product of dimensions of the original spaces.

The inner product on $V \otimes W$ is defined by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle$$

for decomposable elements of $V \otimes W$ and by bilinear extension for general elements of $V \otimes W$. If $T \colon V \to V$ and $U \colon W \to W$ are linear maps, then, by setting

$$T \otimes U(v \otimes w) = T(v) \otimes U(w)$$

for decomposable elements and extending by linearity, defines a linear map $T \otimes U \colon V \otimes W \to V \otimes W$ which is called the *tensor product* of T and U. If the linear maps T and U are considered to be matrices, then the matrix describing the tensor product $T \otimes U$ is the usual *Kronecker product* of the two matrices. Since the norm is defined by $|v| = \langle v, v \rangle^{1/2}$ for all $v \in V \otimes W$ the operator norm of the tensor product of T and U is the product of the operator norms of T and U, i.e.

$$\|T \otimes U\| = \|T\| \|U\|. \tag{2.4}$$

2.3. **Irreducibility.** Let $\mathcal{A}$ be a set of matrices in $GL_d(\mathbb{R})$. We say that $\mathcal{A}$ is *irreducible* if there is no proper nontrivial linear subspace V of $\mathbb{R}^d$ such that $A(V) \subset V$ for all $A \in \mathcal{A}$; otherwise $\mathcal{A}$ is called *reducible.* A tuple $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$ is *irreducible* if the corresponding set $\{A_1, \ldots, A_N\}$ is irreducible. If $\mathsf{A}^{\wedge k} = (A_1^{\wedge k}, \ldots, A_N^{\wedge k})$ is irreducible for some $k \in \{0, \ldots, d\}$, then we say that A is *k-irreducible.* For each $n \in \mathbb{N}$ and $\mathtt{i} = i_1 \cdots i_n \in \Sigma_n$ we write $A_\mathtt{i} = A_{i_1} \cdots A_{i_n} \in GL_d(\mathbb{R})$.

Lemma 2.1. *If $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$, then the following conditions are equivalent:*

1. *The tuple A is irreducible.*
2. *For every $0 \ne v, w \in \mathbb{R}^d$ there is $\mathtt{i} \in \Sigma_*$ such that $\langle v, A_\mathtt{i} w\rangle \ne 0$.*
3. *For every $0 \ne w \in \mathbb{R}^d$ it holds that $\operatorname{span}(\{A_\mathtt{i} w : \mathtt{i} \in \Sigma_*\}) = \mathbb{R}^d$.*
4. *The set $\{A_\mathtt{i} : \mathtt{i} \in \Sigma_*\}$ is irreducible.*

Proof. Although the proof is similar to that of [3, Lemma 2.6], we give it here for the convenience of the reader. To show that (1) $\Rightarrow$ (2), suppose that the condition (2) is not satisfied. Then there exist $0 \ne v, w \in \mathbb{R}^d$ such that $\langle v, A_\mathtt{i} w\rangle = 0$ for all $\mathtt{i} \in \Sigma_*$. This means that v is orthogonal to the non-trivial proper linear subspace $V = \operatorname{span}\{A_\mathtt{i} w : \mathtt{i} \in \Sigma_*\}$ of $\mathbb{R}^d$. Since trivially $A_i(V) \subset V$ for all $i \in \{1, \ldots, N\}$ we have shown that (1) does not hold, and thus finished the proof of the implication.

To show that (2) $\Rightarrow$ (3), suppose to the contrary that there exists $0 \ne w \in \mathbb{R}^d$ such that $V = \operatorname{span}(\{A_\mathtt{i} w : \mathtt{i} \in \Sigma_*\})$ is non-trivial proper subspace of $\mathbb{R}^d$. Thus there exists $0 \ne v \in V^\perp$. Since now $\langle v, A_\mathtt{i} w\rangle = 0$ for all $\mathtt{i} \in \Sigma_*$ we have shown that (2) does not hold, and thus finished the proof of the implication.

To show that (3) $\Rightarrow$ (4), assume contrarily that there exists a non-trivial proper linear subspace V of $\mathbb{R}^d$ such that $A_\mathtt{i}(V) \subset V$ for all $\mathtt{i} \in \Sigma_*$. Let $0 \ne w \in V$. Since $A_\mathtt{i} w \subset V$ for all $\mathtt{i} \in \Sigma_*$ we have shown that $\operatorname{span}(\{A_\mathtt{i} w : \mathtt{i} \in \Sigma_*\})$ is a non-trivial proper linear subspace of $\mathbb{R}^d$. Thus (3) does not hold and we have finished the proof of this implication.

Since the implication (4) $\Rightarrow$ (1) is trivial we have finished the whole proof. □

Let $k \in \{0, \ldots, d-1\}$ and $k < s < k+1$. We say that A is *s-irreducible* if for every $v_1, w_1 \in \wedge^k\mathbb{R}^d$ and $v_2, w_2 \in \wedge^{k+1}\mathbb{R}^d$ there is $\mathtt{i} \in \Sigma_*$ such that

$$\langle v_1, A_\mathtt{i}^{\wedge k} w_1\rangle_k \ne 0 \quad \text{and} \quad \langle v_2, A_\mathtt{i}^{\wedge(k+1)} w_2\rangle_{k+1} \ne 0.$$

Observe that, by Lemma 2.1, if A is s-irreducible, then it is k-irreducible and $(k+1)$-irreducible. It follows from [9, Example 9.2 and Lemma 3.3] and Theorem 2.5 that the converse is not true. We emphasize that if A is s-irreducible for some $k < s < k+1$, then it is s-irreducible for all $k < s < k+1$. Note that s-irreducibility for $0 < s < 1$ is just 1-irreducibility. The following lemma gives a connection between the s-irreducibility and irreducibility.

Lemma 2.2. *Let $k \in \{0, \ldots, d-1\}$ and $k < s < k+1$. If $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$ is not s-irreducible, then $\mathsf{A}' = (A_1^{\wedge k} \otimes A_1^{\wedge(k+1)}, \ldots, A_N^{\wedge k} \otimes A_N^{\wedge(k+1)})$ is not irreducible.*

Proof. Write $A_i' = A_i^{\wedge k} \otimes A_i^{\wedge(k+1)}$ for all i and observe that A_i' is an invertible linear map acting on $\wedge^k\mathbb{R}^d \otimes \wedge^{k+1}\mathbb{R}^d$. Assume to the contrary that A' is irreducible. Then, by Lemma 2.1, for every $v, w \in \wedge^k\mathbb{R}^d \otimes \wedge^{k+1}\mathbb{R}^d$ there is $\mathtt{i} \in \Sigma_*$ such that $\langle v, A_\mathtt{i}' w\rangle \ne 0$. Since, in particular, this holds for decomposable elements, we see that for every $(v_1, v_2), (w_1, w_2) \in \wedge^k\mathbb{R}^d \times \wedge^{k+1}\mathbb{R}^d$ there is $\mathtt{i} \in \Sigma_*$ such that

$$\langle v_1, A_\mathtt{i}^{\wedge k} w_1\rangle_k \langle v_2, A_\mathtt{i}^{\wedge(k+1)} w_2\rangle_{k+1} = \langle v_1 \otimes v_2, A_\mathtt{i}'(w_1 \otimes w_2)\rangle \ne 0.$$

Therefore, A is s-irreducible. □

Remark 2.3. We remark that the above lemma is far from being a characterization. For example, the set $\{A \otimes A^{\wedge 2} : A \in GL_3(\mathbb{R})\}$ is not irreducible. To see this, observe that the 1-dimensional subspace spanned by $v = e_1 \otimes (e_2 \wedge e_3) + e_2 \otimes (e_3 \wedge e_1) + e_3 \otimes (e_1 \wedge e_2)$ is invariant for all $A \otimes A^{\wedge 2}$.

Indeed, it is straightforward to see that $(A \otimes A^{\wedge 2})v = \det(A)v$ for all upper and lower triangular $A \in GL_3(\mathbb{R})$ and hence, by LU decomposition, for all $A \in GL_3(\mathbb{R})$.

A slightly modified version of the condition $C(s)$ introduced and used by Falconer and Sloan [3] implies s-irreducibility. We say that the set $\mathcal{A}$ of matrices in $GL_d(\mathbb{R})$ satisfies the *condition* $C(s)$ if for every $v_1, w_1 \in \wedge^k\mathbb{R}^d$ and $v_2, w_2 \in \wedge^{k+1}\mathbb{R}^d$ there is $A \in \mathcal{A}$ such that

$$\langle v_1, A^{\wedge k} w_1\rangle_k \neq 0 \quad \text{and} \quad \langle v_2, A^{\wedge(k+1)} w_2\rangle_{k+1} \neq 0. \tag{2.5}$$

A tuple $\mathsf{A} = (A_1 \ldots, A_N) \in GL_d(\mathbb{R})^N$ satisfies the condition $C(s)$ if the corresponding set $\{A_1 \ldots, A_N\}$ satisfies the condition $C(s)$. The condition $C(k)$ holds if the left-hand side equation in (2.5) is satisfied. We remark that the condition $C(s)$ in [3] is slightly weaker: instead of arbitrary $v_2, w_2 \in \wedge^{k+1}\mathbb{R}^d$ they require (2.5) only for vectors of the form $v_2 = v_1 \wedge v$ and $w_2 = w_1 \wedge w$. The following lemma follows immediately from the definitions.

Lemma 2.4. *Let $k \in \{0, \ldots, d-1\}$ and $k \le s < k+1$. Then $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$ is s-irreducible if and only if $\{A_{\mathtt{i}} : \mathtt{i} \in \Sigma_*\}$ satisfies the condition $C(s)$. In particular, if A satisfies the condition $C(s)$, then A is s-irreducible.*

2.4. **Singular value function.** Let $k \in \{0, \ldots, d-1\}$ and $k \le s < k+1$. We define the *singular value function* to be

$$\varphi^s(A) = \|A^{\wedge k}\|_k^{k+1-s}\, \|A^{\wedge(k+1)}\|_{k+1}^{s-k} = \alpha_1(A) \cdots \alpha_k(A)\alpha_{k+1}(A)^{s-k}$$

for all $A \in GL_d(\mathbb{R})$ with the convention that $\|A^{\wedge 0}\|_0 = 1$. Observe that (2.2) and the submultiplicativity of the operator norm imply

$$\begin{aligned}\varphi^s(AB) &= \|(AB)^{\wedge k}\|_k^{k+1-s}\, \|(AB)^{\wedge(k+1)}\|_{k+1}^{s-k} \\ &\le \|A^{\wedge k}\|_k^{k+1-s}\, \|B^{\wedge k}\|_k^{k+1-s}\, \|A^{\wedge(k+1)}\|_{k+1}^{s-k}\, \|B^{\wedge(k+1)}\|_{k+1}^{s-k} = \varphi^s(A)\varphi^s(B)\end{aligned} \tag{2.6}$$

for all $A, B \in GL_d(\mathbb{R})$. When $s \ge d$, we set $\varphi^s(A) = |\det(A)|^{s/d}$ for completeness.

For a given $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$ we define

$$P_{\mathsf{A}}(\varphi^s) = \lim_{n\to\infty} \tfrac{1}{n} \log \sum_{\mathtt{i}\in\Sigma_n} \varphi^s(A_{\mathtt{i}}) \tag{2.7}$$

and call it the *singular value pressure* of A. The limit above exists by the standard theory of subadditive sequences. It is easy to see that, as a function of s, the singular value pressure is continuous, strictly decreasing, and convex between any two consecutive integers. Furthermore, since $P_{\mathsf{A}}(\varphi^0) = \log N > 0$ and $\lim_{s\to\infty} P_{\mathsf{A}}(\varphi^s) = -\infty$ there exists unique $s \ge 0$ for which $P_{\mathsf{A}}(\varphi^s) = 0$. The minimum of d and this s is called the *affinity dimension* of A and is denoted by $\dim_{\mathrm{aff}}(\mathsf{A})$.

2.5. **Equilibrium states.** We denote the collection of all Borel probability measures on Σ by $\mathcal{M}(\Sigma)$, and endow it with the weak* topology. We say that $\mu \in \mathcal{M}(\Sigma)$ is *fully supported* if $\mu([\mathtt{i}]) > 0$ for all $\mathtt{i} \in \Sigma_*$. Let

$$\mathcal{M}_\sigma(\Sigma) = \{\mu \in \mathcal{M}(\Sigma) : \mu \text{ is } \sigma\text{-invariant}\},$$

where *σ-invariance* of μ means that $\mu([\mathtt{i}]) = \mu(\sigma^{-1}([\mathtt{i}])) = \sum_{i=1}^N \mu([i\mathtt{i}])$ for all $\mathtt{i} \in \Sigma_*$. Observe that if $\mu \in \mathcal{M}_\sigma(\Sigma)$, then $\mu(A) = \mu(\sigma^{-1}(A))$ for all Borel sets $A \subset \Sigma$. We say that μ is *ergodic* if $\mu(A) = 0$ or $\mu(A) = 1$ for every Borel set $A \subset \Sigma$ with $A = \sigma^{-1}(A)$. Recall that the set $\mathcal{M}_\sigma(\Sigma)$ is compact and convex with ergodic measures as its extreme points.

If $\mu \in \mathcal{M}_\sigma(\Sigma)$, then we define the *entropy* h of μ by setting

$$h(\mu) = \lim_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in\Sigma_n} -\mu([\mathtt{i}])\log\mu([\mathtt{i}]).$$

In addition, if $0 \le s \le d$ and $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$, then we define

$$\lambda_{\mathsf{A}}(\varphi^s, \mu) = \lim_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in\Sigma_n} \mu([\mathtt{i}])\log\varphi^s(A_{\mathtt{i}}).$$

Recalling (2.6) and the fact that μ is invariant, the limits above exist and equal the infimums by the standard theory of subadditive sequences.

An application of Jensen's inequality yields $P_{\mathsf{A}}(\varphi^s) \ge h(\mu) + \lambda_{\mathsf{A}}(\varphi^s, \mu)$ for all $\mu \in \mathcal{M}_\sigma(\Sigma)$ and $s \ge 0$. A measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ is called an *φ^s-equilibrium state* of A if it satisfies the following variational principle:

$$P_{\mathsf{A}}(\varphi^s) = h(\mu) + \lambda_{\mathsf{A}}(\varphi^s, \mu).$$

Käenmäki [8, Theorems 2.6 and 4.1] proved that for each $\mathsf{A} \in GL_d(\mathbb{R})^N$ and $s \ge 0$ there exists an ergodic φ^s-equilibrium state of A; see also [11, Theorem 3.3]. The example of Käenmäki and Vilppolainen [11, Example 6.2] shows that such an equilibrium state is not necessarily unique.

The following theorem shows that s-irreducibility is a sufficient condition for the uniqueness of the φ^s-equilibrium state. Its proof follows by applying [9, Lemma 2.5] in [10, Theorem A].

Theorem 2.5. *Let $k \in \{0, \ldots, d-1\}$ and $k < t < k+1$. If $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$ is t-irreducible, then for every $k < s < k+1$ there exists a unique φ^s-equilibrium state μ of A and it satisfies the following condition: there exists $C \ge 1$ depending only on A and s such that*

$$C^{-1} e^{-nP_{\mathsf{A}}(\varphi^s)} \varphi^s(A_{\mathtt{i}}) \le \mu([\mathtt{i}]) \le C e^{-nP_{\mathsf{A}}(\varphi^s)} \varphi^s(A_{\mathtt{i}})$$

for all $\mathtt{i} \in \Sigma_*$.

Similarly as in (2.7), given $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$ and $s \ge 0$, we define

$$P_{\mathsf{A}}(\|\cdot\|^s) = \lim_{n\to\infty} \tfrac{1}{n} \log \sum_{\mathtt{i}\in\Sigma_n} \|A_{\mathtt{i}}\|^s$$

and call it the *norm pressure* of A. Note that $P_{\mathsf{A}}(\|\cdot\|^s) = P_{\mathsf{A}}(\varphi^s)$ for all $0 \le s \le 1$. If $\mu \in \mathcal{M}_\sigma(\Sigma)$, then we also set

$$\lambda_{\mathsf{A}}(\|\cdot\|^s, \mu) = \lim_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in\Sigma_n} \mu([\mathtt{i}]) \log \|A_{\mathtt{i}}\|^s.$$

It follows that $P_{\mathsf{A}}(\|\cdot\|^s) \ge h(\mu) + \lambda_{\mathsf{A}}(\|\cdot\|^s, \mu)$ for all $\mu \in \mathcal{M}_\sigma(\Sigma)$ and $s \ge 0$. A measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ is called a *$\|\cdot\|^s$-equilibrium state* of A if

$$P_{\mathsf{A}}(\|\cdot\|^s) = h(\mu) + \lambda_{\mathsf{A}}(\|\cdot\|^s, \mu).$$

The following theorem is proved by Feng and Käenmäki [4, Theorem 1.7].

Theorem 2.6. *If $0 \le s \le d$ and $\mathsf{A} \in GL_d(\mathbb{R})^N$, then there exist at most d distinct ergodic $\|\cdot\|^s$-equilibrium states of A and they are all fully supported. Furthermore, if A is irreducible, then the equilibrium state is unique.*

As became apparent in [9], this result is useful also in the study of φ^s-equilibrium states.

3. Proofs of the main results

To prove Theorems A and B, we rely on the following proposition. It is proved in [9, Proposition 8.1] and its proof is a simple consequence of the variational principle.

Proposition 3.1. *Let $0 \le s \le d$ and $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$. If all the φ^s-equilibrium states of A are fully supported and $\mathsf{A}' = (A_1, \ldots, A_{N-1}) \in GL_d(\mathbb{R})^{N-1}$, then $P_{\mathsf{A}'}(\varphi^s) < P_{\mathsf{A}}(\varphi^s)$. Moreover, if $P_{\mathsf{A}}(\varphi^d) \le 0$, then $\dim_{\mathrm{aff}}(\mathsf{A}') < \dim_{\mathrm{aff}}(\mathsf{A})$.*

Let us first focus on Theorem A.

Theorem 3.2. *If $0 \le s \le d$ is rational and $\mathsf{A} \in GL_d(\mathbb{R})^N$, then there exist at most finitely many distinct ergodic φ^s-equilibrium states of A and they are all fully supported.*

Proof. Following [12, §5], we will express the singular value function as a norm of a tensor product. The assumption that s is rational is essential here. Note that if s is an integer, then the claim follows immediately from [4, §3]. Let $p, q \in \mathbb{N}$ be such that $s = p/q$ and let $k \in \{0, \ldots, d-1\}$ be such that $k < p/q < k+1$. If $A \in GL_d(\mathbb{R})$, then

$$A' = (\otimes_{i=1}^{(k+1)q-p} A^{\wedge k}) \otimes (\otimes_{j=1}^{p-kq} A^{\wedge(k+1)}) \tag{3.1}$$

is an invertible linear map acting on the $\binom{d}{k}^{(k+1)q-p}\binom{d}{k+1}^{p-kq}$-dimensional inner product space $(\otimes_{i=1}^{(k+1)q-p} \wedge^k \mathbb{R}^d) \otimes (\otimes_{j=1}^{p-kq} \wedge^{k+1} \mathbb{R}^d)$. Observe that (2.4) and (2.3) give

$$\begin{aligned}\|A'\| &= \|A^{\wedge k}\|_k^{(k+1)q-p} \|A^{\wedge(k+1)}\|_{k+1}^{p-kq} = (\alpha_1(A)\cdots\alpha_k(A))^{(k+1)q-p}(\alpha_1(A)\cdots\alpha_{k+1}(A))^{p-kq} \\ &= (\alpha_1(A)\cdots\alpha_k(A))^q \alpha_{k+1}(A)^{p-\ell q} = \varphi^s(A)^q\end{aligned}$$

and hence

$$\varphi^s(A) = \|A'\|^{1/q}.$$

The set of φ^s-equilibrium states of A is therefore precisely the set of $\|\cdot\|^{1/q}$-equilibrium states of A', where $\mathsf{A}' = (A_1', \ldots, A_N') \in GL_{d'}(\mathbb{R})^N$, $d' = \binom{d}{k}^{(k+1)q-p}\binom{d}{k+1}^{p-kq}$, and each A_i' is of the form (3.1). Thus, by Theorem 2.6, there exist at most d' distinct ergodic φ^s-equilibrium states of A and they are all fully supported. □

Remark 3.3. (1) Unfortunately the proof of the previous theorem does not show that the number of φ^s-equilibrium states is bounded over s.

(2) A small modification of Lemma 2.2 shows that if $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$ is not s-irreducible, then $\mathsf{A}' = (A_1', \ldots, A_N')$, where each A_i' is of the form (3.1), is not irreducible.

Proof of Theorem A. If $\mathsf{A}' = (A_1, \ldots, A_{N-1}) \in GL_d(\mathbb{R})^{N-1}$, then Theorem 3.2 and Proposition 3.1 imply that $\dim_{\mathrm{aff}}(\mathsf{A}') < \dim_{\mathrm{aff}}(\mathsf{A})$. Therefore, the result of Falconer [1, Theorem 5.3] finishes the proof. □

Let us then turn to Theorem B. We will first give a sufficient and checkable condition for s-irreducibility. We say that $(A_1, A_2) \in GL_d(\mathbb{R})^2$ satisfies the *eigenvalue condition* if both matrices have d distinct eigenvalues (real or complex) and the following two conditions are satisfied:

(E1) The eigenvalues of A_1 and A_2, respectively denoted by $\lambda_1, \ldots, \lambda_d$ and $\lambda_1', \ldots, \lambda_d'$, satisfy

$$\lambda_{i_1}\cdots\lambda_{i_k} \neq \lambda_{j_1}\cdots\lambda_{j_k} \quad \text{and} \quad \lambda_{i_1}'\cdots\lambda_{i_k}' \neq \lambda_{j_1}'\cdots\lambda_{j_k}'$$

for all pairs $(i_1, \ldots, i_k) \neq (j_1, \ldots, j_k)$ and for all $k \in \{1, \ldots, d\}$.

(E2) If the eigenvectors of A_1 and A_2 corresponding to the eigenvalues are $e_1, \ldots, e_d$ and $e_1', \ldots, e_d'$, respectively, and $X \in GL_d(\mathbb{R})$ is the change of basis matrix for which $Xe_i' = e_i$ for all $i \in \{1, \ldots, d\}$, then all the minors of X are non-zero.

In (E2), if an eigenvalue λ is complex, then its complex conjugate $\overline{\lambda}$ is also an eigenvalue. In this case, we choose eigenvectors to be any two linearly independent vectors spanning the invariant plane corresponding to λ and $\overline{\lambda}$.

Furthermore, we say that $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$ satisfies the eigenvalue condition if there exist $i, j \in \{1, \ldots, N\}$ such that $i \neq j$ and (A_i, A_j) satisfies the eigenvalue condition. The following proposition shows that the eigenvalue condition is generic.

Proposition 3.4. *The set $\{\mathsf{A} \in GL_d(\mathbb{R})^N : \mathsf{A}$ satisfies the eigenvalue condition$\}$ is open, dense, and of full Lebesgue measure in $GL_d(\mathbb{R})^N$. In fact, the complement of the set is a finite union of (d^2N-1)-dimensional algebraic varieties.*

Proof. The claim basically follows from the first part of the proof of [7, Corollary 2.7]. However, because of Remark 3.5 and since the proof of [7, Corollary 2.7] omits some of the details, we give a full proof for the convenience of the reader. We may clearly assume that $N = 2$. Observe that the complement of $\{\mathsf{A} \in GL_d(\mathbb{R})^2 : \mathsf{A}$ satisfies the condition (E1)$\}$ is

$$\bigcup_{k=1}^{d} \bigcup_{(i_1,\ldots,i_k)\neq(j_1,\ldots,j_k)} \{(A_1, A_2) \in GL_d(\mathbb{R})^2 : \lambda_{i_1}\cdots\lambda_{i_k} = \lambda_{j_1}\cdots\lambda_{j_k} \text{ or } \lambda'_{i_1}\cdots\lambda'_{i_k} = \lambda'_{j_1}\cdots\lambda'_{j_k}\},$$

where $\lambda_1, \ldots, \lambda_d$ and $\lambda'_1, \ldots, \lambda'_d$ are the eigenvalues of the matrices A_1 and A_2, respectively. To show that this complement is a finite union of $(2d^2-1)$-dimensional algebraic varieties it clearly suffices to show that $C = \{(A_1, A_2) \in GL_d(\mathbb{R})^2 : \lambda_{i_1}\cdots\lambda_{i_k} = \lambda_{j_1}\cdots\lambda_{j_k}\}$ is a $(2d^2-1)$-dimensional algebraic variety. Denoting

$$g(\lambda) = \prod_{1\le i_1<\cdots<i_k\le d} (\lambda - \lambda_{i_1}\cdots\lambda_{i_k}),$$

we see that

$$\begin{aligned} C &= \{\mathsf{A} \in GL_d(\mathbb{R})^2 : \text{the polynomial } g \text{ has a multiple root}\} \\ &= \{\mathsf{A} \in GL_d(\mathbb{R})^2 : \text{the discriminant of } g \text{ is } 0\}. \end{aligned}$$

Note that the coefficients of g can be expressed by the entries of A_1. As the discriminant of a polynomial is a symmetric function in the roots, it can be expressed in terms of the coefficients of the polynomial. Therefore, the discriminant of g is a polynomial of the entries of A_1 and C is a $(2d^2-1)$-dimensional algebraic variety.

Let us next consider the condition (E2). Observe first that the complement of $\{X \in GL_d(\mathbb{R}) :$ all the minors of X are nonzero$\}$ is

$$\begin{aligned} \mathcal{A} &= \bigcup_{k=1}^{d} \{X \in GL_d(\mathbb{R}) : \text{there exists a zero minor of order } k\} \\ &= \bigcup_{k=1}^{d} \bigcup_{\substack{1\le i_1<\cdots<i_k\le d \\ 1\le j_1<\cdots<j_k\le d}} \{X \in GL_d(\mathbb{R}) : \det(X^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}) = 0\}, \end{aligned}$$

where $X^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$ is the submatrix of X corresponding to rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_k$. Therefore, if $\mathsf{A} = (A_1, A_2) \in GL_d(\mathbb{R})^2$ does not satisfy the condition (E2), then the matrix $X \in GL_d(\mathbb{R})^2$ changing the eigenbases of A_1 and A_2 is contained in $\mathcal{A}$. Since the elements of X are determined from A by some linear equations and the defining property of X is a polynomial equation, we see that the complement of $\{\mathsf{A} \in GL_d(\mathbb{R})^2 : \mathsf{A}$ satisfies the condition (E2)$\}$ is a finite union of $(2d^2-1)$-dimensional algebraic varieties.

We have now finished the proof since the second claim trivially implies the first claim. □

Remark 3.5. There is a small inaccuracy in the statement of [7, Corollary 2.7]. The corollary claims that a certain family of matrices satisfying the condition $C(s)$ is open. The proof, however, only verifies that this family contains an open set.

Proposition 3.6. *If $\mathsf{A} \in GL_d(\mathbb{R})^N$ satisfies the eigenvalue condition, then A is s-irreducible for all $0 \le s \le d$. In particular, for every $0 \le s \le d$ there exists a unique φ^s-equilibrium state μ of A and it satisfies the following condition: there exists $C \ge 1$ depending only on A and s such that*

$$C^{-1}e^{-nP_{\mathsf{A}}(\varphi^s)}\varphi^s(A_{\mathtt{i}}) \le \mu([\mathtt{i}]) \le Ce^{-nP_{\mathsf{A}}(\varphi^s)}\varphi^s(A_{\mathtt{i}})$$

for all $\mathtt{i} \in \Sigma_*$.

Proof. Since (A_i, A_j) satisfies the eigenvalue condition for some $i, j \in \{1, \ldots, N\}$ with $i \neq j$ it follows from [7, Theorem 2.6] (and the latter part of the proof of [7, Corollary 2.7] which deals with the case of complex eigenvalues) that the family $\{A_{\mathtt{k}} : \mathtt{k} \in \bigcup_{k=1}^{d'} \{i, j\}^k\}$, where $d' = 2(\max\{\binom{d}{m} : m \in \{0, \ldots, d\}\})^2$, satifies the condition $C(s)$ for all $0 \le s \le d$. By Lemma 2.4, we see that

(A_i, A_j), and consequently also A, is s-irreducible for all $0 \le s \le d$. The second claim follows now immediately from Theorem 2.5. □

Proof of Theorem B. It follows immediately from Propositions 3.4 and 3.6 that the set $\{\mathsf{A} \in GL_d(\mathbb{R})^N$: there exists a unique φ^s-equilibrium state of A and it is fully supported$\}$ contains a set in $GL_d(\mathbb{R})^N$ which is open, dense, and of full Lebesgue measure. Furthermore, the complement of the set is contained in a finite union of (d^2N-1)-dimensional algebraic varieties. Therefore, for any A in this set, Proposition 3.1 implies that $\dim_{\mathrm{aff}}(\mathsf{A}') < \dim_{\mathrm{aff}}(\mathsf{A})$. The result of Falconer [1, Theorem 5.3] finishes the proof. □

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