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SHAPE OPTIMIZATION IN CONTACT PROBLEMS BASED ON PENALIZATION OF THE STATE INEQUALITY

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Summary. The paper deals with the approximation of optimal shape of elastic bodies, unilaterally supported by a rigid, frictionless foundation. Original state inequality, describing the behaviour of such a body is replaced by a family of penalized state problems. The relation between optimal shapes for the original state inequality and those for penalized state equations is established.

1. INTRODUCTION

This paper is concerned with optimal shape design of a two dimensional elastic body on a rigid frictionless foundation. The problem is to redesign the boundary part of the body where the unilateral boundary conditions are assumed, in such a way that the total energy of the system in the equilibrium state is minimized. The numerical results obtained show that as a by-product we can find such a shape for the contact part of the body that the contact stress is constant. This result is of great practical importance.

In the paper [10] Haslinger and Neittaanmäki give the proof of existence of a solution when the state problem is formulated in terms of the variational inequality on a variable domain. In this paper a different approach is used. The variational inequality is replaced by a family of penalized problems, each of which is given as a classical elliptic (nonlinear) boundary value problem. We show that the corresponding optimal designs (associated with the penalized problems) are close (in an appropriate sense) to an optimal design of the original problem (see Chapter 4).

In Chapter 5 we present finite element discretization of our design problem and discuss the convergence of the approximations. Some numerical results are presented in Chapter 6.

Mathematical theory of shape optimization problems, including their approximation by finite elements, can be found in [1, 5, 12, 13, 14, 20] (state problem governed by equations) and in [8, 9, 10, 11, 15, 17] (state problem governed by variational inequalities). For sensitivity analysis we refer to [2, 4, 13, 19, 21, 22, 23]. The existing literature on contact problems in connection with the shape optimization seems to be minor. We mention [2, 3, 9, 10, 13].

2. THE OPTIMAL SHAPE DESIGN PROBLEM

Let us consider a two-dimensional elastic body $\Omega = \Omega(\alpha) \subset R^2$ having the following geometrical structure

$$\Omega(\alpha) = \{(x_1, x_2) \in R^2 \mid a < x_1 < b, 0 \leqq \alpha(x_1) < x_2 < \gamma\},$$

$a, b, \gamma > 0$ given constants and $\alpha \in C^{0,1}(\langle a, b\rangle)$ a function; $\partial\Omega(\alpha) = \bar{\Gamma}_D \cup \bar{\Gamma}_F \cup \bar{\Gamma}_C(\alpha)$, $\bar{\Gamma}_D \neq \emptyset$ (a possible partition of $\partial\Omega(\alpha)$ is given by Fig. 2.1). The shape of the contact surface $\Gamma_C(\alpha)$ is defined by a control parameter α from the set $\mathscr{U}_{ad}$ of admissible controls,

$$(2.1)\quad \mathscr{U}_{ad} = \left\{\alpha \in C^{0,1}(\langle a, b\rangle) \mid 0 \leqq \alpha(x_1) \leqq C_0 < \gamma, \quad \alpha(a) = A, \alpha(b) = B, \right.$$
$$\left. \left|\frac{d}{dx_1}\alpha\right| \leqq C_1, \quad \text{meas}\,(\Omega(\alpha)) = C_2\right\},$$

A, B, C_0, C_1 and C_2 are given positive constants.

Suppose that the body $\Omega(\alpha)$ is unilaterally supported by a rigid frictionless foundation (here by the set $\{(x_1, x_2) \in R^2 \mid x_2 \leqq 0\}$) and subjected to a body force $\boldsymbol{F} = (F_1, F_2)$ and to a surface traction $\boldsymbol{P} = (P_1, P_2)$ on Γ_F.

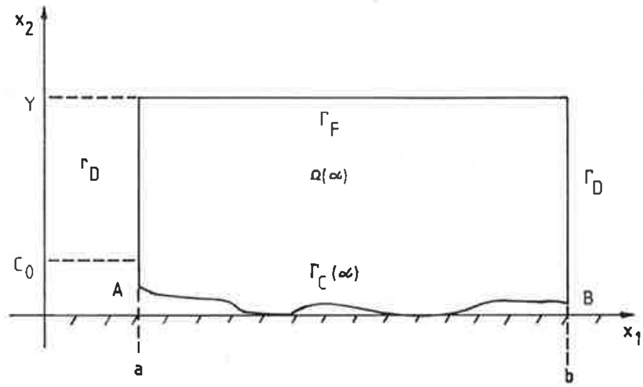

Fig. 2.1.

In the classical formulation of contact problems one looks for a displacement field $\boldsymbol{u} = \boldsymbol{u}(\alpha) = (u_1(\alpha), u_2(\alpha))$ satisfying the equilibrium equations (the dependence of $\boldsymbol{u}$ on α is emphasized by writting $\boldsymbol{u} = \boldsymbol{u}(\alpha)$)

$$(2.2)\qquad \frac{\partial}{\partial x_j}\tau_{ij}(\boldsymbol{u}) + F_i = 0 \quad \text{in} \quad \Omega(\alpha), \quad i = 1, 2$$

where the stress tensor $\tau(\boldsymbol{u}) = \{\tau_{ij}(\boldsymbol{u})\}_{i,j=1}^2$ is related to the strain tensor $\varepsilon(\boldsymbol{u}) = \{\varepsilon_{ij}(\boldsymbol{u})\}_{i,j=1}^2$ by means of the linear Hooke's law:

$$\tau_{ij}(\boldsymbol{u}) = C_{ijkl}\,\varepsilon_{kl}(\boldsymbol{u}), \quad \varepsilon_{kl}(\boldsymbol{u}) = \frac{1}{2}\left\{\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right\}$$

with elasticity coefficients C_{ijkl} satisfying the usual symmetry and ellipticity conditions. The following boundary conditions will be assumed (cf. Fig. 2.1):

(2.3) $$u_i = 0 \quad \text{on} \quad \Gamma_D, \quad i = 1, 2\,;$$

(2.4) $$\tau_{ij}(\boldsymbol{u})\, n_j = P_i \quad \text{on} \quad \Gamma_F, \quad i = 1, 2\,;$$

(2.5) $$u_2(x_1, \alpha(x_1)) \geqq -\alpha(x_1)\ \forall x_1 \in [a, b]\,;$$

(2.6) $$T_1(\boldsymbol{u}) = 0 \quad \text{on} \quad \Gamma_C(\alpha) \quad \text{(frictionless case)}\,;$$

(2.7) $$T_2(\boldsymbol{u}) \geqq 0\,, \quad (u_2 + \alpha)\, T_2(\boldsymbol{u}) = 0 \quad \text{on} \quad \Gamma_C(\alpha)\,,$$

where the standard notations and conventions of elasticity are used ([16]).

In order to give the variational inequality formulation of (2.2)–(2.7) we introduce a Hilbert space $V(\alpha)$ of virtual displacements

(2.8) $$V(\alpha) \equiv V(\Omega(\alpha)) = \{\boldsymbol{v} \in (H^1(\Omega(\alpha)))^2 \mid v_i = 0 \quad \text{on} \quad \Gamma_D,\ i = 1, 2\}$$

and its closed convex subset $K(\alpha)$ of admissible displacements

(2.9) $$K(\alpha) \equiv K(\Omega(\alpha)) = \{\boldsymbol{v} \in V(\alpha) \mid v_2(x_1, \alpha(x_1)) \geqq -\alpha(x_1)\ \forall x_1 \in [a, b]\}\,.$$

The variational form of (2.2)–(2.7) reads as follows ([16]): find $\boldsymbol{u} = \boldsymbol{u}(\alpha) \in K(\alpha)$ such that

($\mathscr{P}(\alpha)$) $$(\tau(\boldsymbol{u}), \varepsilon(\boldsymbol{v} - \boldsymbol{u}))_{0,\Omega(\alpha)} \geqq \langle L, \boldsymbol{v} - \boldsymbol{u}\rangle_\alpha\ \forall \boldsymbol{v} \in K(\alpha)\,,$$

where L is the given distribution of external forces,

(2.10) $$\langle L, \xi\rangle_\alpha = \int_{\Omega(\alpha)} F_i \xi_i \,\mathrm{d}x + \int_{\Gamma_F} P_i \xi_i \,\mathrm{d}s$$

with $\boldsymbol{F} \in (L^2(\Omega_\gamma))^2$, $\boldsymbol{P} \in (L^2(\Gamma_F))^2$, $\Omega_\gamma = (a, b) \times (0, \gamma)$ and

$$(\tau(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{0,\Omega(\alpha)} = \int_{\Omega(\alpha)} \tau_{ij}(\boldsymbol{u})\, \varepsilon_{ij}(\boldsymbol{v})\,\mathrm{d}x\,.$$

We state the shape optimization problem.

Problem $\mathbb{P}$. *Find* $\alpha^* \in \mathscr{U}_{\mathrm{ad}}$ *such that*

(2.11) $$\mathscr{E}(\boldsymbol{u}^*(\alpha^*), \alpha^*) = \min \mathscr{E}(\boldsymbol{u}(\alpha), \alpha)\,,$$

where $\mathscr{E}\colon K(\alpha) \times \mathscr{U}_{\mathrm{ad}} \to \mathbb{R}$ *is the total potential energy functional,*

$$\mathscr{E}(\boldsymbol{u}(\alpha), \alpha) = \tfrac{1}{2}(\tau(\boldsymbol{u}(\alpha)), \varepsilon(\boldsymbol{u}(\alpha)))_{0,\Omega(\alpha)} - \langle L, \boldsymbol{u}(\alpha)\rangle_\alpha\,,$$

in which $\boldsymbol{u}(\alpha)$ *is the solution of the state inequality* $(\mathscr{P}(\alpha))$.

According to [10] we have

Theorem 1.1. *Let* $\mathscr{U}_{\mathrm{ad}}$ *be given by* (2.1). *Then there exists at least one solution* α^* *of Problem* $(\mathbb{P})$.

To avoid the difficulties with possible non-differentiability of the mapping $\alpha \to \boldsymbol{u}(\alpha)$, the following approach will be used. Instead of the state variational inequality $(\mathscr{P}(\alpha))$ a family of penalized problems will be introduced: find $\boldsymbol{u}_\varepsilon \equiv \boldsymbol{u}_\varepsilon(\alpha) \in V(\alpha)$ such that

$$(\mathscr{P}_\varepsilon(\alpha)) \qquad (\tau(\boldsymbol{u}_\varepsilon)\, \varepsilon(\boldsymbol{v}))_{0,\Omega(\alpha)} + \frac{1}{\varepsilon} B_\alpha(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) = \langle L, \boldsymbol{v} \rangle_\alpha\,, \quad \forall \boldsymbol{v} \in V(\alpha)\,,$$

where B_α is the *penalty operator*,

$$(2.12) \qquad B_\alpha(\boldsymbol{u}, \boldsymbol{v}) = \int_a^b [(u_2(x_1, \alpha(x_1)) + \alpha(x_1))^-]^2\, v_2(x_1, \alpha(x_1))\, \mathrm{d}x_1 \equiv$$

$$\equiv \int_a^b ((u_2(\alpha) + \alpha)^-)^2\, v_2(\alpha)\, \mathrm{d}x_1\,,$$

$\varepsilon \to 0+$ is a *penalty parameter* and a^- denotes the negative part of a ($a^- := (|a| - a)/2$).

The penalized optimal shape design problem $(\mathbb{P}_\varepsilon)$ now reads as follows:

Problem $(\mathbb{P}_\varepsilon)$. *Find* $\alpha_\varepsilon^* \in \mathscr{U}_{\mathrm{ad}}$ *such that*

$$(2.13) \qquad \mathscr{E}(\boldsymbol{u}_\varepsilon^*(\alpha_\varepsilon^*); \alpha_\varepsilon^*) = \min_{\alpha \in \mathscr{U}_{\mathrm{ad}}} \mathscr{E}(\boldsymbol{u}_\varepsilon(\alpha); \alpha)\,,$$

where $\boldsymbol{u}_\varepsilon(\alpha) \in V(\alpha)$ *is the solution of* $(\mathscr{P}_\varepsilon(\alpha))$.

3. SOLVABILITY OF PROBLEM $(\mathbb{P}_\varepsilon)$

In order to prove the solvability of Problem $(\mathbb{P}_\varepsilon)$ we need some preliminary results.

Lemma 3.1. *Let* $\alpha_n \to \alpha$ *in* $C^0(\langle a, b \rangle)$ *and let* $\varphi \in K(\alpha)$ *be given. Then there exists* $\{\varphi_j\}_{j=1}^\infty$, $\varphi_j \in (C^\infty(\bar{\Omega}_\gamma))^2$ *such that*

$$(3.1) \qquad \varphi_j|_{\Omega(\alpha_{n(j)})} \in K(\alpha_{n(j)})\,,$$

$$(3.2) \qquad \varphi_j \to \tilde{\varphi} \quad \text{in} \quad (H^1(\Omega_\gamma))^2\,,$$

where $\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2)$ *denotes the Calderon extension of* $\varphi = (\varphi_1, \varphi_2)$ *from* $\Omega(\alpha)$ *on* Ω_γ *and* $\{\alpha_{n(j)}\}$ is a *subsequence of* $\{\alpha_n\}$.

Proof. The existence of $\{\varphi_j\}$, $\varphi_j \in (H^1(\Omega_\gamma))^2$, satisfying (3.1), (3.2) has been proved in [10], Lemma 2.4. It remains to prove that φ_j can be chosen more regular, namely

$\varphi_j \in (C^\infty(\bar{\Omega}_\gamma))^2$. From the proof of the above mentioned result it follows that φ_j can be written as

$$\varphi_j = \psi + \Phi_j\,, \quad \text{where} \quad \Phi_j = (\Phi_{1j}, \Phi_{2j}) \in (C^\infty(\bar{\Omega}_\gamma))^2$$

and

$$\psi = (\psi_1, \psi_2) \in (H^1(\Omega_\gamma))^2 \quad \text{satisfies} \quad \psi|_{\Omega(\alpha)} \in K(\alpha)\,, \quad \psi_2 \geqq -x_2 \quad \text{on} \quad \Omega_\gamma\,.$$

Using the density results (see [7], pp. 35–38) one can approximate ψ_2 by a sequence $\{\psi_{2j}\}$, $\psi_{2j} \in C^\infty(\bar{\Omega}_\gamma)$ such that

$$\psi_{2j} \geqq -x_2 \quad \text{in} \quad \Omega_\gamma$$

and vanishing in the neighbourhood of Γ_D. As the approximation of ψ_1 is standard we finish the proof. □

Theorem 3.2. *Let* $\mathscr{U}_{\text{ad}}$ *be given by* (2.1). *For every* $\varepsilon > 0$ *there exists a solution* $\alpha^* \in \mathscr{U}_{\text{ad}}$ *of Problem* $(\mathbb{P}_\varepsilon)$.

Proof. To simplify notation we set the parameter $\varepsilon = 1$ and suppress the subscript ε. Let $\alpha_n \in \mathscr{U}_{\text{ad}}$ be a minimizing sequence of $\mathscr{E}$, i.e.

$$q := \inf_{\alpha \in \mathscr{U}_{\text{ad}}} \mathscr{E}(\alpha) = \lim_{n \to \infty} \mathscr{E}(\alpha_n)\,. \text{ [1]}$$

We set $\Omega_n := \Omega(\alpha_n)$ and denote by $\boldsymbol{u}_n$ the solution of the state equation $(\mathscr{P}_1(\alpha_n))$ in Ω_n. From Lemma A (see Appendix), the definition of B_α and the state equation $(\mathscr{P}_1(\alpha))$ we obtain

$$(3.3) \qquad C\|\boldsymbol{u}_n\|^2_{1,\Omega_n} \leqq (\tau(\boldsymbol{u}_n), \varepsilon(\boldsymbol{u}_n))_{0,\Omega_n} \leqq (\tau(\boldsymbol{u}_n), \varepsilon(\boldsymbol{u}_n))_{0,\Omega_n} + B_n(\boldsymbol{u}_n, \boldsymbol{u}_n) =$$
$$= \langle L, \boldsymbol{u}_n \rangle_{\alpha_n} \leqq (\|\boldsymbol{F}\|_{0,\Omega_\gamma} + \|\boldsymbol{P}\|_{0,\Gamma_F})\, \|\boldsymbol{u}_n\|_{1,\Omega_n}$$

where $B_n \equiv B_{\alpha_n}$. Thus

$$(3.4) \qquad \|\boldsymbol{u}_n\|_{1,\Omega_n} \leqq C \quad \text{for} \quad n = 1, 2, 3 \ldots \text{ [2]}$$

As $\mathscr{U}_{\text{ad}}$ is compact in the $C^0([a, b])$-topology, there exists a subsequence of $\{\alpha_n\}$, also denoted by $\{\alpha_n\}$, which converges to $\alpha^* \in \mathscr{U}_{\text{ad}}$ uniformly on $[a, b]$. Hence for every $m \in \mathbb{N}$ there exists $n_0 = n_0(m)$ such that for any $n \geqq n_0(m)$

$$\Omega_n \supset \bar{G}_m(\alpha^*)\,,$$

where

$$(3.5) \qquad G_m \equiv G_m(\alpha^*) := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (a, b), \alpha^*(x_1) + 1/m < x_2 < \gamma\}\,.$$

As

$$\|\boldsymbol{u}_n\|_{1,G_m} \leqq \|\boldsymbol{u}_n\|_{1,\Omega_n} \leqq C \quad \forall n \geqq n_0(m)$$

[1]) We shall write simply $\mathscr{E}(\alpha)$ instead of $\mathscr{E}(\boldsymbol{u}(\alpha); \alpha)$.

[2]) In what follows, C will denote a generic strictly positive constant with different values in different places.

one may extract a subsequence $\{u_{n_1}\} \subset \{u_n\}$ such that

$$u_{n_1} \rightharpoonup u^{(m)} \quad \text{(weakly) in} \quad (H^1(G_m))^2 ,$$

where $u^{(m)} \in V(G_m) := \{u \in (H^1(G_m))^2 \mid u_i = 0 \quad \text{on} \quad \Gamma_D \cap \partial G_m\}$.

As there exists $n_0(m + 1)$ such that for any $n \geqq n_0(m + 1)$, $\Omega_n \supset \bar{G}_{m+1} \supset \bar{G}_m$ we can extract a subsequence $\{u_{n_2}\} \subset \{u_{n_1}\}$ that converges weakly in $(H^1(G_{m+1}))^2$ to $u^{(m+1)} \in V(G_{m+1})$.

Trivially, $u^{(m)} = u^{(m+1)}$ in G_m. Similarly, for each $k > 0$ there exists $n_0(m + k)$ such that for $n > n_0(m + k)$, $\Omega_n \supset \bar{G}_{m+k} \supset \bar{G}_{m+k-1}$ and consequently we can extract a subsequence $\{u_{n_k}\} \subset \{u_{n_{k-1}}\}$ that converges weakly in $(H^1(G_{m+k}))^2$ to $u^{(m+k)} \in V(G_{m+k})$.

Denoting by $\{u_n^D\}$ the diagonal sequence determined by $\{u_{n_k}\}$ we have

$$u_n^D \rightharpoonup u|_{G_m} \quad \text{in} \quad (H^1(G_m))^2 \quad \text{for every } m , \tag{3.6}$$

where $u|_{G_m} = u^{(m)}$ on G_m. Clearly $u \in V(\alpha^*)$. To simplify notation we shall write again $\{u_n\}$ instead of $\{u_n^D\}$.

The next step is to show that u is the solution of the state problem $(\mathscr{P}_\varepsilon(\alpha^*))$. For every n we have

$$(\tau(u_n), \varepsilon(w))_{0,\Omega_n} + B_n(u_n, w) = (F, w)_{0,\Omega_n} + (P, w)_{0,\Gamma_F} \quad \forall w \in \hat{V} , \tag{3.7}$$

where $\hat{V} = \{u \in (H^1(\Omega_\gamma))^2 \mid u_i = 0 \text{ on } \Gamma_D\}$. As $\alpha_n \to \alpha^*$ in $C^0([a, b])$ topology, we obtain from the right hand side of (3.7) that

$$\lim_{n \to \infty} (F, w)_{0,\Omega_n} + (P, w)_{0,\Gamma_F} = (F, w)_{0,\Omega(\alpha^*)} + (P, w)_{0,\Gamma_F} . \tag{3.8}$$

Next we shall prove that

$$B_n(u_n, w) \to B_{\alpha^*}(u, w) \quad \forall w \in (C_0^\infty(\bar{\Omega}_\gamma))^2 . \tag{3.9}$$

We have

$$\left| \int_a^b ((u_{2n}(\alpha_n) + \alpha_n)^-)^2 \, w_2(\alpha_n) \, dx_1 - \int_a^b ((u_2(\alpha^*) + \alpha^*)^-)^2 \, w_2(\alpha^*) \, dx_1 \right| \leqq$$

$$\leqq \int_a^b |((u_{2n}(\alpha_n) + \alpha_n)^-)^2 - ((u_2(\alpha^*) + \alpha^*)^-)^2| \, |w_2(\alpha_n)| \, dx_1 +$$

$$+ \int_a^b ((u_2(\alpha^*) + \alpha^*)^-)^2 \, |w_2(\alpha_n) - w_2(\alpha^*)| \, dx_1 = I_1 + I_2 .$$

As $\alpha_n \to \alpha^*$ uniformly in $[a, b]$, $I_2 \to 0$ as $n \to \infty$. Further, as $|w_2(\alpha_n)|$ remains bounded we can estimate using some elementary facts like $|a^2 - b^2| = |a - b|$ $|a + b|$ and $|a^- - b^-| \leqq |a - b|$ and the Hölder inequality:

$$I_1 \leqq C \int_a^b |((u_{2n}(\alpha_n) + \alpha_n)^-)^2 - ((u_2(\alpha^*) + \alpha^*)^-)^2| \, dx_1 \leqq$$

$$\leqq C\left(\int_a^b (u_{2n}(\alpha_n) - u_2(\alpha^*) + \alpha_n - \alpha^*)^2 \,\mathrm{d}x_1\right)^{1/2} \cdot$$

$$\cdot\left(\int_a^b ((u_{2n}(\alpha_n) + \alpha_n)^- + (u_2(\alpha^*) + \alpha^*)^-)^2 \,\mathrm{d}x_1\right)^{1/2} .$$

The second term of this product remains bounded because of the compactness of the imbedding from $H^1(\Omega_n)$ to $L^p(\Gamma(\alpha_n))$ (in our geometry even uniformly with respect to n) for $1 \leqq p < \infty$. Thus it remains to prove that the first term tends to zero. Now, using the notations

$$\Gamma_m = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = \alpha^*(x_1) + 1/m, x_1 \in (a, b)\} ,$$
$$u_2(\Gamma_m) \equiv u_2(x_1, \alpha^*(x_1) + 1/m), x_1 \in (a, b)$$

we obtain from the triangle inequality

$$\int_a^b (u_{2n}(\alpha_n) - u_2(\alpha^*) + \alpha_n - \alpha^*)^2 \,\mathrm{d}x_1 \leqq$$

$$\leqq C\left\{\int_a^b (\alpha_n - \alpha^*)^2 \,\mathrm{d}x_1 + \int_a^b (u_2(\alpha^*) - u_2(\Gamma_m))^2 \,\mathrm{d}x_1 + \int_a^b (u_2(\Gamma_m) - u_{2n}(\Gamma_m))^2 \,\mathrm{d}x_1 + \right.$$
$$\left. + \int_a^b (u_{2n}(\Gamma_m) - u_{2n}(\alpha_n))^2 \,\mathrm{d}x_1\right\} = C(I_{11} + I_{12} + I_{13} + I_{14}) .$$

Evidently $I_{11} \to 0$ as $n \to \infty$ and analogously to [8] we can estimate

$$I_{12} = \int_a^b (u_2(\alpha^*) - u_2(\Gamma_m))^2 \,\mathrm{d}x_1 \leqq \int_a^b \left(\int_{\alpha^*}^{\Gamma_m} \frac{\partial}{\partial x_2} u_2(x_1, x_2) \,\mathrm{d}x_2\right)^2 \mathrm{d}x_1 \leqq m^{-1} \|u_2\|_{1,\Omega(\alpha^*)}^2 ,$$

$$I_{13} = \int_a^b |u_2(\Gamma_m) - u_{2n}(\Gamma_m)|^2 \,\mathrm{d}x_1 \leqq C\|u_2 - u_{2n}\|_{0,\Gamma_m}^2 \leqq C\|u_2 - u_{2n}\|_{1,G_m(\alpha^*)}^2 ,$$

$$I_{14} = \int_a^b |u_{2n}(\Gamma_m) - u_{2n}(\alpha_n)|^2 \,\mathrm{d}x_1 \leqq \int_a^b \left(\int_{\Gamma_m}^{\alpha_n} \frac{\partial}{\partial x_2} u_2(x_1, x_2) \,\mathrm{d}x_2\right)^2 \mathrm{d}x_1 \leqq$$
$$\leqq C \max_{x_1 \in [a,b]} |\alpha^*(x_1) + 1/m - \alpha_n(x_1)| \, \|u_{2n}\|_{1,\Omega(\alpha_n)}^2 .$$

For every $\mu > 0$ there exists m_0 such that $I_{12} < \mu$. Using (3.4) we see that there exists $n_0 = n_0(m_0)$ such that for any $n > n_0$, $I_{13} < \mu$ and $I_{14} < \mu$. Thus we have proved (3.9)

Now fix m and let n be such that $\Omega_n \supset \bar{G}_m$. Then

$$(\tau(\boldsymbol{u}_n), \varepsilon(\boldsymbol{w}))_{0,\Omega_n} = (\tau(\boldsymbol{u}_n), \varepsilon(\boldsymbol{w}))_{0,G_m} + (\tau(\boldsymbol{u}_n), \varepsilon(\boldsymbol{w}))_{0,\Omega_n \setminus \Omega(\alpha^*)} +$$
$$+ (\tau(\boldsymbol{u}_n), \varepsilon(\boldsymbol{w}))_{0,(\Omega(\alpha^*) \setminus G_m) \cap \Omega_n} .$$

As $\boldsymbol{u}_n \to \boldsymbol{u}$ in $(H^1(G_m))^2$ and $(\tau(\cdot), \varepsilon(\boldsymbol{w}))_{0,G_m}$ is weakly continuous, we obtain

$$(\tau(u_n), \varepsilon(w))_{0,G_m} \to (\tau(u), \varepsilon(w))_{0,G_m} \quad \text{as} \quad n \to \infty ,$$

$$|(\tau(u_n), \varepsilon(w))_{0,\Omega_n \setminus \Omega(\alpha^*)}| \leqq \|u_n\|_{1,\Omega_n \setminus \Omega(\alpha^*)} \|w\|_{1,\Omega_n \setminus \Omega(\alpha^*)} ,$$

$$|(\tau(u_n), \varepsilon(w))_{0,(\Omega(\alpha^*)\setminus G_m)\cap\Omega_n}| \leqq \|u_n\|_{1,\Omega_n} \|w\|_{1,\Omega(\alpha^*)\setminus G_m} .$$

Passing to the limit with n and using (3.4) we have

$$\overline{\lim_{n\to\infty}} (\tau(u_n), \varepsilon(w))_{0,\Omega_n} \leqq (\tau(u), \varepsilon(w))_{0,G_m} + C\|w\|_{1,\Omega(\alpha^*)\setminus G_m(\alpha^*)} ,$$

$$\underline{\lim}_{n\to\infty} (\tau(u_n), \varepsilon(w))_{0,\Omega_n} \geqq (\tau(u), \varepsilon(w))_{0,G_m} - C\|w\|_{1,\Omega(\alpha^*)\setminus G_m(\alpha^*)} .$$

Combining this with (3.8) and (3.9) and letting m tend to infinity we finally obtain

$$(\tau(u), \varepsilon(w))_{0,\Omega(\alpha^*)} + B_{\alpha^*}(u, w) = (F, w)_{0,\Omega(\alpha^*)} + (P, w)_{0,\Gamma_F} \quad \forall w \in \hat{V} .$$

As any w in $V(\alpha^*)$ can be continuously prolongated to a function w in $\hat{V}$ we have that $u = u(\alpha^*)$, i.e. u is the solution of $(\mathscr{P}_\varepsilon(\alpha^*))$.

It remains to prove that $\mathscr{E}(\alpha^*) = \inf_{\alpha\in\mathscr{U}_{ad}} \mathscr{E}(\alpha)$. Indeed,

$$\mathscr{E}(\alpha_n) = \mathscr{E}_{G_m(\alpha^*)}(\alpha_n) + \mathscr{E}_{\Omega_n\setminus G_m(\alpha^*)}(\alpha_n) , \tag{3.10}$$

where

$$\mathscr{E}_{G_m(\alpha^*)}(\alpha_n) = \tfrac{1}{2}(\tau(u_n), \varepsilon(u_n))_{0,G_m(\alpha^*)} - (F, u_n)_{0,G_m(\alpha^*)} - (P, u_n)_{0,\Gamma_F\cap\overline{G_m(\alpha^*)}} \tag{3.11}$$

and

$$\mathscr{E}_{\Omega_n\setminus G_m(\alpha^*)} = \tfrac{1}{2}(\tau(u_n), \varepsilon(u_n))_{0,\Omega_n\setminus G_m(\alpha^*)} - (F, u_n)_{0,\Omega_n\setminus G_m(\alpha^*)} - (P, u_n)_{0,M_m} \geqq \tag{3.12}$$
$$\geqq -(F, u_n)_{0,\Omega_n\setminus G_m(\alpha^*)} - (P, u_n)_{0,M_m}$$

with

$$M_m = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = a \quad \text{and} \quad x_2 \in (\alpha(a), \alpha(a) + 1/m) , x_1 = b$$
$$\text{and } x_2 \in (\alpha(b), \alpha(b) + 1/m)\}$$

(this consideration can be omitted if dist $(\bar{\Gamma}_F, \bar{\Gamma}_C(\alpha^*)) > 0$). Now

$$\underline{\lim}_{n\to\infty} \mathscr{E}(\alpha_n) \geqq \underline{\lim}_{n\to\infty} \mathscr{E}_{G_m(\alpha^*)}(\alpha_n) + \underline{\lim}_{n\to\infty} \mathscr{E}_{\Omega_n\setminus G_m(\alpha^*)}(\alpha_n) \geqq \tag{3.13}$$
$$\geqq \mathscr{E}_{G_m(\alpha^*)}(\alpha^*) + \underline{\lim}_{n\to\infty} (-(F, u_n)_{0,\Omega_n\setminus G_m(\alpha^*)} - (P, u_n)_{0,M_m}) \geqq$$
$$\geqq \mathscr{E}_{G_m(\alpha^*)}(\alpha^*) - C(\|F\|_{0,\Omega(\alpha^*)\setminus G_m(\alpha^*)} + \|P\|_{0,M_m}) .$$

Here we hawe used the lower semicontinuity of $\mathscr{E}$. Letting $m \to \infty$ we hawe that

$$q = \lim_{n\to\infty} \mathscr{E}(\alpha_n) \geqq \mathscr{E}(\alpha^*) .$$

Thus α^* solves Problem $(\mathbb{P}_\varepsilon)$. □

Let us choose a sequence $\{\varepsilon_k\}$ of positive numbers such that $\varepsilon_k \to 0$ as $k \to \infty$. Denote by α_k^* the solution of the optimal design problem ($\mathbb{P}_{\varepsilon_k}$) and by $\boldsymbol{u}_k^*$ the solution of the associated state problem. Concerning the behaviour of the problem ($\mathbb{P}_{\varepsilon_k}$) as $\varepsilon_k \to 0$ we have the following theorem.

Theorem 4.1. *There exists a subsequence* $\{\alpha_{k_j}^*, \boldsymbol{u}_{k_j}^*(\alpha_{k_j}^*)\}$ *of* $\{\alpha_k^*, \boldsymbol{u}_k^*(\alpha_k^*)\}$ *and elements* $\alpha^* \in \mathscr{U}_{\mathrm{ad}}$, $\boldsymbol{u}^*(\alpha^*) \in K(\alpha^*)$ *such that*

$$\alpha_{k_j}^* \to \alpha^* \quad in \quad C^0([a, b]), \quad j \to \infty \tag{4.1}$$

and

$$\boldsymbol{u}_{k_j}^*(\alpha_{k_j}^*) \rightharpoonup \boldsymbol{u}^*(\alpha^*) \quad in \quad (H^1(G_m(\alpha^*)))^2 \quad for\ any \quad m \tag{4.2}$$

where α^*, $\boldsymbol{u}^*(\alpha^*)$ *are the solutions of* ($\mathbb{P}$) *and* ($\mathscr{P}(\alpha^*)$), *respectively, and* $G_m(\alpha^*)$ *is defined by* (3.5).

Proof. Using Lemma A (see Appendix), we obtain

$$\|\boldsymbol{u}_k^*\|_{1,\Omega(\alpha_k^*)} \leqq C \quad \forall k = 1, 2, \ldots \tag{4.3}$$

As $\mathscr{U}_{\mathrm{ad}}$ is compact, there exists a subsequence of $\{\alpha_k^*\}$ and an element $\alpha^* \in \mathscr{U}_{\mathrm{ad}}$ such that

$$\alpha_k^* \to \alpha^* \quad \text{uniformly in } [a, b] .$$

The construction of a weakly convergent subsequence $\{\boldsymbol{u}_{k_j}^*\}$ satisfying (4.2) is analogous to the method applied in the proof of Theorem 3.2. We shall denote the diagonal sequence by $\{\boldsymbol{u}_k^*\}$ and the weak limit by $\boldsymbol{u}^* \in (H^1(\Omega(\alpha^*)))^2$. As in the proof of Theorem 3.2 we obtain that

$$B_{\alpha_k^*}(\boldsymbol{u}_k^*, \boldsymbol{w}) \to B_{\alpha^*}(\boldsymbol{u}) \quad \forall \boldsymbol{w} \in \hat{V} \quad \text{as} \quad k \to \infty . \tag{4.4}$$

On the other hand, from the state equation we have using (4.3)

$$|B_{\alpha_k^*}(\boldsymbol{u}_k^*, \boldsymbol{w})| \leqq C\varepsilon_k \|\boldsymbol{w}\|_{1,\hat{\Omega}} \to 0 \quad \text{as} \quad k \to \infty . \tag{4.5}$$

Combining (4.4) and (4.5) we see that $\boldsymbol{u}^* \in \operatorname{Ker} B_{\alpha^*} = K(\alpha^*)$.

As the next step we prove that $\boldsymbol{u}^*$ is the solution of the variational inequality ($\mathscr{P}(\alpha^*)$) in $\Omega(\alpha^*)$. In other words, we have to prove that

$$(\tau(\boldsymbol{u}^*), \varepsilon(\boldsymbol{w} - \boldsymbol{u}^*))_{0,\Omega(\alpha^*)} \geqq (\boldsymbol{F}, \boldsymbol{w} - \boldsymbol{u}^*)_{0,\Omega(\alpha^*)} + (\boldsymbol{P}, \boldsymbol{w} - \boldsymbol{u}^*)_{0,\Gamma_F}$$
$$\forall \boldsymbol{w} \in K(\alpha^*) .$$

According to Lemma 3.1, for each $\boldsymbol{w} \in K(\alpha^*)$ we can find a sequence $\{\boldsymbol{w}_k\}$, $\boldsymbol{w}_k \in (H^1(\Omega_\gamma))^2$, $\boldsymbol{w}_{k|\Omega_k} \in K(\alpha_k^*)$ and such that $\boldsymbol{w}_k \to \tilde{\boldsymbol{w}}$ in $(H^1(\Omega_\gamma))^2$. Now from the definition of $B_k \equiv B_{\alpha_k^*}$ we obtain

$$(4.6) \qquad (\tau(u_k), \varepsilon(w_k - u_k))_{0,\Omega_k} \geqq (F, w_k - u_k)_{0,\Omega_k} + (P, w_k - u_k)_{0,\Gamma_F},$$

where $\Omega_k = \Omega(\alpha_k^*)$.

Concerning the left hand side of (4.6) we can estimate

$$(\tau(u_k^*), \varepsilon(w_k - u_k^*))_{0,\Omega_k} = (\tau(u_k^*), \varepsilon(w_k - u_k^*))_{0,G_m} + (\tau(u_k^*), \varepsilon(w_k - u_k^*))_{0,\Omega_k\setminus\Omega(\alpha^*)} +$$
$$+ (\tau(u_k^*), \varepsilon(w_k - u_k^*))_{0,(\Omega(\alpha^*)\setminus G_m)\cap\Omega_k} \leqq (\tau(u_k^*), \varepsilon(w_k - u_k^*))_{0,G_m} +$$
$$+ (\tau(u_k^*), \varepsilon(w_k))_{0,\Omega_k\setminus\Omega(\alpha^*)} + (\tau(u_k^*), \varepsilon(w_k))_{0,(\Omega(\alpha^*)\setminus G_m)\cap\Omega_k}.$$

Thus we easily get

$$(4.7) \qquad \overline{\lim_{k\to\infty}}\,(\tau(u_k^*), \varepsilon(w_k - u_k^*))_{0,\Omega_k} \leqq (\tau(u^*), \varepsilon(w - u^*))_{0,G_m} + C\|w\|_{1,\Omega(\alpha^*)\setminus G_m},$$

where $C > 0$ does not depend on m.

Similarly,

$$(F, w_k - u_k^*)_{0,\Omega_k} = (F, w_k - u_k^*)_{0,G_m} + (F, w_k - u_k^*)_{0,\Omega_k\setminus\Omega(\alpha^*)} +$$
$$+ (F, w_k - u_k^*)_{0,(\Omega(\alpha^*)\setminus G_m)\cap\Omega_k}$$

and consequently,

$$(4.8) \qquad \underline{\lim_{k\to\infty}}\,(F, w_k - u_k^*)_{0,\Omega_k} \geqq (F, w - u_k^*)_{0,G_m} - C(\|F\|_{0,\Omega(\alpha^*)\setminus G_m} + \|w\|_{0,\Omega(\alpha^*)\setminus G_m}).$$

Finally, if we define M_m as in the proof of Theorem 3.2 we obtain

$$(P, w_k - u_k^*)_{0,\Gamma_F} = (P, w_k - u_k^*)_{0,\Gamma_F\setminus M_m} + (P, w_k - u_k^*)_{0,M_m}$$

and passing to the limit:

$$(4.9) \qquad \underline{\lim_{k\to\infty}}\,(P, w_k - u_k^*)_{0,\Gamma_F} \geqq (P, w - u^*)_{0,\Gamma_F\setminus M_m} - C(\|P\|_{0,M_m} + \|w\|_{0,M_m}).$$

Combining (4.7), (4.8) and (4.9) and passing to the limit with m we find that u^* solves Problem $(\mathscr{P}(\alpha^*))$.

It remains to prove that $\mathscr{E}(\alpha^*) \leqq \mathscr{E}(\alpha)\ \forall\alpha \in \mathscr{U}_{ad}$. Let $\hat{\alpha} \in \mathscr{U}_{ad}$ be fixed and let $u(\hat{\alpha}) \in K(\hat{\alpha})$ be the solution of $(\mathscr{P}(\hat{\alpha}))$. Using the classical results we known that there exist $u_k(\hat{\alpha}) \in V(\Omega(\hat{\alpha}))$ (solutions of $(\mathscr{P}_{\varepsilon_k}(\hat{\alpha}))$ such that

$$u_k(\hat{\alpha}) \to u(\hat{\alpha}) \quad \text{in} \quad (H^1(\Omega(\hat{\alpha})))^2.$$

Consequently,

$$(4.10) \qquad \mathscr{E}(u_k(\hat{\alpha}), \hat{\alpha}) \to \mathscr{E}(u(\hat{\alpha}), \hat{\alpha}) \equiv \mathscr{E}(\hat{\alpha}).$$

On the other hand,

$$(4.11) \qquad \mathscr{E}(u_k^*(\alpha_k^*), \alpha_k^*) \leqq \mathscr{E}(u_k(\hat{\alpha}), \hat{\alpha})$$

as follows from the definition of (P_{ε_k}). From (4.10), (4.11) and using the same approach as at the end of the proof of Theorem 3.3 we are led to

$$\mathscr{E}(\alpha^*) \leqq \varliminf_{k\to\infty} \mathscr{E}(\alpha_k^*) \leqq \mathscr{E}(\hat{\alpha}) \quad \forall \hat{\alpha} \in \mathscr{U}_{\text{ad}} .$$

This completes the proof of Theorem 4.1. □

5. THE FINITE ELEMENT APPROXIMATION OF $(\mathbb{P}_\varepsilon)$

Approximation of Problem $(\mathbb{P})$ will be defined starting from $(\mathbb{P}_\varepsilon)$ by means of finite elements. We suppose that $\mathscr{U}_{\text{ad}}$ is replaced by the set

$$\mathscr{U}_{\text{ad}}^h = \{\alpha \in C([a, b]) \mid \alpha \mid_{[a_{i-1}, a_i]} \in P_1([a_{i-1}, a_i])\} \cap \mathscr{U}_{\text{ad}}$$

where $a = a_0 < a_1 < \ldots < a_N = b$ is a partition of $[a, b]$, P_1 denotes the set of *linear functions*. For any $\alpha_h \in \mathscr{U}_{\text{ad}}^h$ we define

$$\Omega(\alpha_h) \equiv \{(x_1, x_2) \in \mathbb{R}^2 \mid a < x_1 < b, \alpha_h(x_1) < x_2 < \gamma\} ,$$

i.e., the variable part of the boundary $\Gamma_C(\alpha)$ is now approximated by a piecewise linear arc $\Gamma_C(\alpha_h)$.

By $\mathscr{T}_h(\alpha_h)$, $\alpha_h \in \mathscr{U}_{\text{ad}}^h$, we denote a triangulation of $\Omega(\alpha_h)$ such that the whole segment $I_i = \{(x_1, x_2) \mid x_1 \in [a_{i-1}, a_i],\ x_2 = \alpha_h(x_2)\}$ is the whole side of a triangle $T_i \in \mathscr{T}_h(\alpha_h)$, and satisfying the usual requirements concerning the mutual position of two triangles belonging to $\mathscr{T}_h(\alpha_h)$. Moreover, we shall consider only such families of $\{\mathscr{T}_h(\alpha_h)\}$, which are *regular uniformly* for $h \to 0+$ with respect to $\alpha_h \in \mathscr{U}_{\text{ad}}^h$, i.e. there exists $\delta_0 > 0$ independently of $h > 0$ and $\alpha_h \in \mathscr{U}_{\text{ad}}^h$, such that all interior angles of all triangles belonging to $\mathscr{T}_h(\alpha_h)$ are greater or equal to δ_0 (for practical applications some other technical restrictions will be added, see Chapter 7). Finally suppose that $\mathscr{T}_h(\alpha_h)$ as a function α_h is continuous for $\forall h$. Next, the symbol $\Omega_h(\alpha_h)$ will denote the set $\Omega(\alpha_h)$ with a given triangulation $\mathscr{T}_h(\alpha_h)$; we also use the abbreviation Ω_h for $\Omega_h(\alpha_h)$.

With any $\mathscr{T}_h$ a finite dimensional space $V_h(\alpha_h) \subset V(\Omega(\alpha_h))$ will be associated:

$$V_h(\alpha_h) = \{\boldsymbol{v}_h \in (C(\bar{\Omega}(\alpha_h)))^2 \mid \boldsymbol{v}_{h|T_i} \in (P_1(T_i))^2 \text{ for any } T_i \in \mathscr{T}_h(\alpha_h),\ \boldsymbol{v}_h = 0 \text{ on } \Gamma_D\} .$$

The approximation of $(\mathbb{P}_\varepsilon)$ is now defined as follows:

Problem $(\mathbb{P}_{\varepsilon(h)})_h$. *Find* $\alpha_{\varepsilon h}^* \in \mathscr{U}_{\text{ad}}^h$ *such that*

$$\mathscr{E}_h(\boldsymbol{u}_{\varepsilon h}^*(\alpha_{\varepsilon h}^*); \alpha_{\varepsilon h}^*) = \min_{\alpha_h \in \mathscr{U}_{\text{ad}}^h} \mathscr{E}_h(\boldsymbol{u}_{\varepsilon h}(\alpha_h), \alpha_h) ,$$

where

$$\mathscr{E}_h(\boldsymbol{u}_{\varepsilon h}(\alpha_h); \alpha_h) \equiv \tfrac{1}{2}(\tau(\boldsymbol{u}_{\varepsilon h}(\alpha_h)), \varepsilon(\boldsymbol{u}_{\varepsilon h}(\alpha_h)))_{0,\Omega(\alpha_h)} - \langle L, \boldsymbol{u}_{\varepsilon h}(\alpha_h)\rangle_{\alpha_h} \tag{5.1}$$

in which $\boldsymbol{u}_{\varepsilon h}(\alpha_h) \in V_h(\alpha_h)$ *is the solution of the discrete state equation*

$$(\mathscr{P}_{\varepsilon(h)}(\alpha_h))_h \qquad (\tau(\boldsymbol{u}_h), \varepsilon(\boldsymbol{v}_h))_{0,\Omega(\alpha_h)} + \frac{1}{\varepsilon(h)} B_{\alpha_h}(\boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h) = \langle L, \boldsymbol{v}_h \rangle_{\alpha_h} \quad \forall \boldsymbol{v}_h \in V_h(\Omega(\alpha_h))$$

with $\varepsilon = \varepsilon(h) \to 0+$ *iff* $h \to 0+$.

Here B_{α_h} denotes the penalty operator:

$$B_{\alpha_h}(\boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h) \equiv \int_a^b ([u_{2\varepsilon h} + \alpha_h]^-)^2 \, v_{2h} \, \mathrm{d}x_1 \, .$$

The solvability of Problem $(\mathbb{P}_{\varepsilon(h)})_h$ can be proved with compactness arguments. Assuming the properties of $\{\mathscr{T}_h\}$ we shall prove

Theorem 5.1. *Let* $\alpha_h^* \in \mathscr{U}_{\mathrm{ad}}^h$ *be a solution of Problem* $(\mathbb{P}_{\varepsilon(h)})_h$ *and let* $\boldsymbol{u}_{\varepsilon h}^*$ *be the corresponding solution of the state equation* $(\mathscr{P}_{\varepsilon_h}(\alpha_h))_h$. *Then there exists a subsequence* $\{\alpha_{h_j}^*\} \subset \{\alpha_h^*\}$, *an element* $\alpha^* \in \mathscr{U}_{\mathrm{ad}}$ *and* $\boldsymbol{u}^*(\alpha^*) \in K(\alpha^*)$ *such that*

$$\alpha_{h_j}^* \to \alpha^* \quad in \quad C^0([a, b]) \quad for \quad h_j \to 0+ \, , \tag{5.2}$$

$$\boldsymbol{u}_{\varepsilon h_j}^*(\alpha_{h_j}^*) \to \boldsymbol{u}^*(\alpha^*) \quad in \quad (H^1(G_m(\alpha^*)))^2 \quad for \quad h_j \to 0+ \, , \tag{5.3}$$

for any $m \geqq m_0$, *where* α^* *is the solution of Problem* $(\mathbb{P})$, $\boldsymbol{u}^*(\alpha^*)$ *the corresponding state and* $G_m(\alpha^*)$ *is defined in* (3.5).

To prove this theorem we shall use the following result:

Lemma 5.2. *Let* $\alpha_h \in \mathscr{U}_{\mathrm{ad}}^h$, $\alpha \in \mathscr{U}_{\mathrm{ad}}$ *be such that* $\alpha_h \to \alpha$ *uniformly in* $[a, b]$ *as* $h \to 0+$. *Let* $\boldsymbol{u}_h = \boldsymbol{u}_h(\alpha_h)$ *be the solution of* $(\mathscr{P}_{\varepsilon(h)}(\alpha_h))_h$. *Then there exists a subsequence* $\{\boldsymbol{u}_{h_j}\} \subset \{\boldsymbol{u}_h\}$ *such that*

$$\boldsymbol{u}_{h_j}(\alpha_{h_j}) \to \boldsymbol{u}(\alpha) \quad in \quad (H^1(G_m(\alpha)))^2 \quad as \quad j \to \infty \, , \quad for\ any \quad m > 0 \, , \tag{5.4}$$

where $\boldsymbol{u}(\alpha) \in K(\alpha)$ *is the solution of* $(\mathscr{P}(\alpha))$.

Proof. Since $\boldsymbol{u}_h \in V(\alpha_h)$ we obtain from the definition of $(\mathscr{P}_{\varepsilon(h)}(\alpha_h))_h$ using Lemma A (see Appendix) that $\|\boldsymbol{u}_h\|_{1,\Omega_h} \leqq C$ independently of h. Now let m be fixed. Then there exists $h_0 = h_0(m)$ such that $\bar{G}_m(\alpha) \subset \Omega_h$ for all $h \leqq h_0$. Consequently,

$$\|\boldsymbol{u}_h\|_{1,G_m(\alpha)} \leqq \|\boldsymbol{u}_h\|_{1,\Omega_h} \leqq C \quad \forall h \leqq h_0 \, . \tag{5.5}$$

The weakly convergent subsequence $\{\boldsymbol{u}_{h_j}\}$ and an element $\boldsymbol{u} \in V(\alpha)$ satisfying (5.4) can now be constructed exactly as in the previous proofs.

The next step is to prove that $\boldsymbol{u}(\alpha) \in K(\alpha)$. As in [9] we shall show that $B_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{w}) = 0 \ \forall \boldsymbol{w} \in (C_0^\infty(\bar{\Omega}_\gamma))^2$ which implies that $\boldsymbol{u}(\alpha) \in K(\alpha)$. Let h_j be a filter of indices for which (5.4) holds. Let $\boldsymbol{w}$ be an arbitrary function in $(C_0^\infty(\bar{\Omega}_\gamma))^2$. For each h_j with $\boldsymbol{w}$ we associate its linear interplate $\boldsymbol{w}_{h_j} := \pi_{h_j} \boldsymbol{w}|_{\Omega_{h_j}} \in V_{h_j}(\Omega_{h_j})$. Using $\boldsymbol{w}_{h_j}$ as a test function in $(\mathscr{P}_{\varepsilon(h_j)}(\alpha_{h_j}))_{h_j}$ we get

$$(\tau(u_{h_j}), \varepsilon(w_{h_j}))_{0,\Omega_{h_j}} + \frac{1}{\varepsilon(h_j)} B_{\alpha_{h_j}}(u_{h_j}, w_{h_j}) = (F, w_{h_j})_{0,\Omega_{h_j}} + (P, w_{h_j})_{0,\Gamma_F} .$$

Thus from (5.5)

$$(5.6) \qquad 0 \leqq |B_{\alpha_{h_j}}(u_{h_j}, w_{h_j})| \leqq \varepsilon(h_j)\, C\|w_h\|_{1,\Omega_{h_j}} .$$

As the linear interpolates have the approximation property

$$\|w_{h_j} - w\|_{1,\Omega_{h_j}} \leqq Ch_j\|w\|_{2,\Omega_{h_j}} \leqq Ch_j\|w\|_{2,\Omega_\gamma}\ ^{3)}$$

we have that $\|w_h\|_{1,\Omega_{h_j}} \leqq C$. Thus

$$(5.7) \qquad \lim_{j\to\infty} B_{\alpha_{h_j}}(u_{h_j}, w_{h_j}) = 0 .$$

On the other hand,

$$|B_{\alpha_{h_j}}(u_{h_j}, w_{h_j}) - B_\alpha(u, w)| \leqq |B_\alpha(u, w) - B_{\alpha_{h_j}}(u_{h_j}, w)| + |B_{\alpha_{h_j}}(u_{h_j}, w - w_{h_j})| .$$

The first term on the right tends to zero as $h_j \to 0$ as follows from (3.9) and

$$|B_{\alpha_{h_j}}(u_{h_j}, w - w_{h_j})| \leqq C\|u_{2h_j} + \alpha_{h_j}\|^2_{0,\Gamma_C(\alpha_{h_j})} \|w - w_{h_j}\|_{L^\infty(\Gamma_C(\alpha_{h_j}))} \leqq$$
$$\leqq C(\|u_{h_j}\|_{0,\Gamma_C(\alpha_{h_j})} + \|\alpha_{h_j}\|_{0,\Gamma_C(\alpha_{h_j})})^2 \|w - w_{h_j}\|_{L^\infty(\Gamma_C(\alpha_{h_j}))} \leqq$$
$$\leqq C(\|u_{h_j}\|^2_{1,\Omega_{h_j}} + C_1) \|w - w_{h_j}\|_{L^\infty(\Omega_{h_j})} \to 0 \quad \text{as} \quad h_j \to 0 .$$

Thus $B_\alpha(u(\alpha), w) = 0$ $\forall w \in (C_0^\infty(\bar{\Omega}_\gamma))^2$, i.e. $u(\alpha) \in K(\alpha)$.

It remains to show that $u = u(\alpha)$ solves Problem $(\mathscr{P}(\alpha))$, i.e. we have to verify that

$$(5.8) \quad (\tau(u), \varepsilon(w - u))_{0,\Omega(\alpha)} \geqq (F, w - u)_{0,\Omega(\alpha)} + (P, w - u)_{0,\Gamma_F} \quad \forall w \in K(\alpha) .$$

Let $w \in K(\alpha)$ be fixed and let $\tilde{w}$ be its Calderon extension. According to Lemma 3.1 one can find a sequence $\{w_i\}$. $w_i \in (H^2(\Omega_\gamma))^2$ (even more regular) such that $w_i \to \tilde{w}$ in $(H^1(\Omega_\gamma))^2$ for $i \to \infty$. Let i be fixed. Then $w_i \in K(\alpha_h)$ provided h is sufficiently small, i.e. for $h \leqq h_0(i)$. Thus, as $\alpha_{h_j} \to \alpha$, uniformly in $[a, b]$, we notice that

$$w_{ih_j}|_{\Omega_{h_j}} := \pi_{h_j} w_i|_{\Omega_{h_j}} \in K(\alpha_{h_j})$$

for h_j sufficiently small.

As $w_{ih_j} \in K(\alpha_{h_j})$ we obtain using the definition of $B_{\alpha_{h_j}}$ that

$$(\tau(u_{h_j}), \varepsilon(u_{h_j} - w_{ih_j}))_{0,\Omega_{h_j}} \leqq (F, u_{h_j} - w_{ih_j})_{0,\Omega_{h_j}} + (P, u_{h_j} - w_{ih_j})_{0,\Gamma_F} .$$

Since $u_{h_j} \rightharpoonup u$ in $(H^1(G_m(\alpha)))^2$ and $w_{ih_j} \to w_i$ in $(H^1(\Omega_\gamma))^2$ we easily obtain that

$$\varliminf_{h_j\to 0} (\tau(u_{h_j}), \varepsilon(w_{ih_j} - u_{h_j}))_{0,\Omega_{h_j}} \leqq (\tau(u), \varepsilon(w_i - u))_{0,G_m(\alpha)} + C\|w_i\|_{1,\Omega(\alpha)\setminus G_m} ,$$

---

[3]) Here we use the fact that the family $\{\mathscr{T}_h(\alpha_h)\}$ is uniformly regular with respect to $\alpha_h \in \mathscr{U}^h_{ad}$.

$$\overline{\lim_{h_j \to 0}} \{(F, w_{ih_j} - u_{h_j})_{0,\Omega_{h_j}} + (P, w_{ih_j} - u_{h_j})_{0,\Gamma_F}\} \geqq (F, w_i - u)_{G_m(\alpha)} +$$

$$+ (P, w_i - u)_{0,\Gamma_F \cap \bar{G}_m} - C(i) \left(\|F\|_{0,\Omega(\alpha)\setminus G_m(\alpha)} + \|P\|_{0,\Gamma_F \setminus \bar{G}_m)}\right)$$

and passing to the limit with $m \to \infty$ and then with $i \to \infty$ we have assertion (5.8). This completes the proof Lemma 4.2. □

Proof *of Theorem* 5.1. Let $\{\alpha_h^*\}$, $h \to 0+$, $\alpha_h^* \in \mathcal{U}_{ad}^h$ be solutions of Problems $(\mathbb{P}_{\varepsilon(h)})_h$. As $\mathcal{U}_{ad}^h \subset \mathcal{U}_{ad}$ for all h and $\mathcal{U}_{ad}$ is compact in the $C^0([a, b])$-topology, there exists a subsequence $\{\alpha_{h_j}^*\} \subset \{\alpha_h^*\}$ and $\alpha^* \in \mathcal{U}_{ad}$ such that

$$\alpha_{h_j}^* \to \alpha^* \quad \text{uniformly in} \quad [a, b] .$$

From Lemma 5.2 we obtain the existence of a subsequence $\{u_{h_j}(\alpha_{h_j}^*)\}$ of solutions of state problems $(\mathcal{P}_{\varepsilon(h_j)}(\alpha_{h_j}^*))_{h_j}$ such that

$$u_{h_j}(\alpha_{h_j}^*) \rightharpoonup u^*(\alpha^*) \quad \text{in} \quad (H^1(G_m(\alpha^*)))^2 \quad \forall m \in \mathbb{N} ,$$

where $u(\alpha^*) \in K(\alpha^*)$ and solves $(\mathcal{P}(\alpha^*))$.

To complete the proof of Theorem 5.1 it remains to show that α^* is a minimizer of $\mathscr{E}$ in $\mathcal{U}_{ad}$. Now with the same notations (cf. (3.10)–(3.11)) and techniques as in the proof of Theorem 3.2 (cf. (3.13))[4])

(5.9) $$\underline{\lim_{h_j \to 0}} \mathscr{E}_{h_j}(\alpha_{h_j}^*) \geqq \underline{\lim_{h_j \to 0}} \mathscr{E}_{h_j, G_m(\alpha^*)}(\alpha_{h_j}^*) + \underline{\lim_{h_j \to 0}} \mathscr{E}_{h_j, \Omega_{h_j} \setminus G_m(\alpha^*)}(\alpha_{h_j}^*) \geqq$$

$$\geqq \mathscr{E}_{G_m(\alpha^*)}(\alpha^*) + \underline{\lim_{h_j \to 0}} \left(-(F, u_{h_j})_{0,\Omega_{h_j} \setminus G_m(\alpha^*)} - (P, u_{h_j})_{0,M_m}\right) .$$

Letting $m \to \infty$ in (5.9) we see that

(5.10) $$\underline{\lim_{h_j \to 0}} \mathscr{E}_{h_j}(\alpha_{h_j}^*) \geqq \mathscr{E}(\alpha^*) .$$

Let $\hat{\alpha} \in \mathcal{U}_{ad}$ be arbitrary and let $\hat{\alpha}_h \in \mathcal{U}_{ad}^h$ be a sequence such that

(5.11) $$\hat{\alpha}_h \to \hat{\alpha} \quad \text{uniformly in } [a, b] .$$

The existence of such a sequence has been proved in [1]. The solution $u(\hat{\alpha})$ of $(\mathcal{P}(\hat{\alpha}))$ can be approximated by a sequence $\{v_{h_j}\}$, $v_{h_j} \in (H^1(\Omega_\gamma))^2$ such that $v_{h_j | \Omega(\hat{\alpha}_{h_j})} \in K(\hat{\alpha}_{h_j})$ and

(5.12) $$v_{h_j} \to \tilde{u}(\hat{\alpha}) \quad \text{in} \quad (H^1(\Omega_\gamma))^2 ,$$

where $\tilde{u}(\hat{\alpha})$ denotes the Calderon extension of $u(\hat{\alpha})$ (see the proof of Lemma 5.2, especially the construction of functions w_{ih_j}). Then by (5.10), (5.11), (5.12) and definition of $\mathscr{E}_h$ and $\mathscr{E}$,

$$\mathscr{E}(\alpha^*) \leqq \underline{\lim_{h_j \to 0}} \mathscr{E}_{h_j}(\alpha_{h_j}^*) \leqq \underline{\lim_{h_j \to 0}} \mathscr{E}_{h_j}(\hat{\alpha}_{h_j}) \leqq \underline{\lim_{h_j \to 0}} J_{\hat{\alpha}_{h_j}}(v_{h_j}) = J_{\hat{\alpha}}(u(\hat{\alpha})) = \mathscr{E}(\hat{\alpha}) \quad \forall \hat{\alpha} \in \mathcal{U}_{ad} ,$$

[4]) We write simply $\mathscr{E}_h(\alpha_h)$ for $\mathscr{E}_h(u_h(\alpha_h); \alpha_h)$

where

$$J_\alpha(\phi) := \tfrac{1}{2}(\tau(\phi), \varepsilon(\phi))_{0,\Omega(\alpha)} - \langle L, \phi \rangle_\alpha .$$

This completes the proof of Theorem 5.1. □

6. NUMERICAL REALIZATION

Taking into account the geometry of $\Omega_h(\alpha_h)$ and the piecewise linearity of $\Gamma_C(\alpha_h)$, for finding an optimal $\Gamma_C(\alpha_h)$ it is sufficient to find the x_2-coordinates of the *design nodes*

(6.1) $$A_i = (a_i, \alpha_h(a_i)) \quad i = 0, \ldots, N(h), \quad a_i = ih ,$$

such that $\mathscr{E}_h(\boldsymbol{u}_{\varepsilon h}(\alpha_h); \alpha_h)$ is minimum.

As for fixed ε and h, the shape of $\Omega(\alpha_h)$ and the value of $\boldsymbol{u}_{\varepsilon h}$ depend on the *design variables* (x_2-coordinates of A_i)

(6.2) $$\boldsymbol{X} \equiv (x_0, \ldots, x_{N(h)}) \equiv (\alpha_h(a_0), \ldots, \alpha_h(a_{N(h)})) ,$$

we shall write $\Omega(\boldsymbol{X}) \equiv \Omega_h(\alpha_h)$, $\boldsymbol{u}_\varepsilon(\boldsymbol{X}) \equiv \boldsymbol{u}_{\varepsilon h}(\alpha_h)$.

For fixed $\Omega(\boldsymbol{X})$, $\boldsymbol{u}_\varepsilon(\boldsymbol{X})$ can be obtained by solving the algebraic form of $(\mathscr{P}_{\varepsilon(h)}(\alpha_h))_h$ for nodal displacements $\boldsymbol{Q}_\varepsilon(\boldsymbol{X})$,

$$\boldsymbol{Q}_\varepsilon(\boldsymbol{X}) = (q_{\varepsilon 1}, \ldots, q_{\varepsilon n(h)}) , \quad n(h) = \dim V_h(\alpha_h):$$

(6.3) $$A(\boldsymbol{X})\, \boldsymbol{Q}_\varepsilon(\boldsymbol{X}) + \frac{1}{\varepsilon} B(\boldsymbol{Q}_\varepsilon(\boldsymbol{X}), \boldsymbol{X}) = \boldsymbol{F}(\boldsymbol{X}) + \boldsymbol{P} .$$

Here $A(\boldsymbol{X})$ is the stiffness matrix, $\boldsymbol{F}(\boldsymbol{X})$ is the force vector arising from the body force, $\boldsymbol{P}$ is the contribution of the surface traction (independent of $\boldsymbol{X}$). The operator $B(\cdot, \boldsymbol{X})$: $\mathbb{R}^{n(h)} \to \mathbb{R}^{n(h)}$ is a nonlinear mapping corresponding to the discretization of the penalty operator B_{α_h}.

The discrete cost functional, written in terms of $\boldsymbol{X}$ and $\boldsymbol{Q}_\varepsilon(\boldsymbol{X})$, reads now

(6.4) $$\mathscr{E}(\boldsymbol{Q}_\varepsilon(\boldsymbol{X}); \boldsymbol{X}) = \tfrac{1}{2}(\boldsymbol{Q}_\varepsilon(\boldsymbol{X}), A(\boldsymbol{X})\, \boldsymbol{Q}_\varepsilon(\boldsymbol{X})) - (\boldsymbol{F}(\boldsymbol{X}) + \boldsymbol{P}, \boldsymbol{Q}_\varepsilon(\boldsymbol{X})) .$$

Problem $(\mathbb{P}_{\varepsilon(h)})_h$ is equivalent to Problem $(\mathbb{P}(\boldsymbol{X}))$, defined as follows:

Problem $(\mathbb{P}(\boldsymbol{X}))$. *Find* $\boldsymbol{X}^* \in \mathscr{D}$ *such that*

(6.5) $$\mathscr{E}(\boldsymbol{Q}^*_\varepsilon(\boldsymbol{X}^*); \boldsymbol{X}^*) \leqq \mathscr{E}(\boldsymbol{Q}_\varepsilon(\boldsymbol{X}); \boldsymbol{X}) \quad \textit{for all} \quad \boldsymbol{X} \in \mathscr{D} ,$$

where $\boldsymbol{Q}_\varepsilon(\boldsymbol{X})$ *is the solution of* (6.3) *and where* $\mathscr{D}$ *is the set of admissible design points* (*cf. definition* (2.1) *of* $\mathscr{U}_{\text{ad}}$):

$$\mathscr{D} = \Big\{ \boldsymbol{X} \in \mathbb{R}^{N(h)+1} \mid 0 \leqq x_i \leqq C_0 , \quad x_0 = A , \quad x_{N(h)} = B ,$$

$$- \frac{C_1}{N(h)} \leqq x_i - x_{i-1} \leqq \frac{C_1}{N(h)}, \quad i = 1, \dots, N(h), \quad \sum_{i=1}^{N(h)} (x_i + x_{i-1}) = \text{const.} \Big\} .$$

Hence $(\mathbb{P}(X))$ is a nonlinear programming problem with box constraints, linear inequality constraints and a linear equality constraint.

The evaluation of the cost functional $\mathscr{E}$ involves the solving of the nonlinear state problem. Consequently, the NLP-algorithm should use as few function evaluations as possible. Clearly, some gradient information is then necessary. In the following we shall evaluate the gradient of $\mathscr{E}$ with respect to X.

For brevity we write $\mathscr{E}(X)$ instead of $\mathscr{E}(Q(X); X)$.

Let $X \in \mathscr{D}$ and $V \in \mathbb{R}^{N(h)+1}$ be fixed. We denote by

$$\mathscr{E}'(X) \equiv \mathscr{E}'(X)\, V \equiv \lim_{t \to 0+} \frac{\mathscr{E}(X + tV) - \mathscr{E}(X)}{t},$$

$$Q_\varepsilon'(X) \equiv \lim_{t \to 0+} \frac{Q_\varepsilon(X + tV) - Q_\varepsilon(X)}{t},$$

$$A'(X) \equiv \lim_{t \to 0+} \frac{A(X + tV) - A(X)}{t},$$

$$F'(X) \equiv \lim_{t \to 0+} \frac{F(X + tV) - F(X)}{t},$$

$$B'(Q_\varepsilon, X) \equiv \lim_{t \to 0+} \frac{B(Q_\varepsilon, X + tV) - B(Q_\varepsilon, X)}{t}$$

the directional differentials of $\mathscr{E}$, Q_ε, A, F and B, respectively. From (6.4) we obtain

$$\mathscr{E}'(X)\, V = (Q_\varepsilon', AQ_\varepsilon - F - P) - (F', Q_\varepsilon) + \tfrac{1}{2}(Q_\varepsilon, A'Q_\varepsilon) . \tag{6.6}$$

In order to eliminate Q_ε' from (6.6) we introduce the adjoint system

$$\left(A + \frac{1}{\varepsilon} \frac{\partial}{\partial Q_\varepsilon} B(Q_\varepsilon, X) \right) \Lambda = AQ_\varepsilon - F - P . \tag{6.7}$$

From (6.6) and (6.7) we obtain the directional gradient for $\mathscr{E}$ with respect to X,

$$\mathscr{E}'(X)\, V = +\tfrac{1}{2}(Q_\varepsilon, A'Q_\varepsilon) - (F', Q_\varepsilon) + \left(\Lambda, F' - A'(X)\, Q_\varepsilon - \frac{1}{\varepsilon} B'(Q_\varepsilon, X)\right) . \tag{6.8}$$

We shall close this chapter with the following important remark. When instead of $(\mathscr{P}_\varepsilon(\alpha))$ the original inequality formulation $(\mathscr{P}(\alpha))$ is employed the mapping $\alpha \to \boldsymbol{u}(\alpha)$ is not differentiable ($\boldsymbol{u}(\alpha)$ solves $(\mathscr{P}(\alpha))$). However, in the case of the cost functional of the total potential energy it is possible to compute the gradient of $\mathscr{E}$ even when the state is governed by the variational inequality. Namely, as the following calculation shows we do not need the derivative of the state explicitly.

Indeed,

$$T \equiv AQ - F - P \tag{6.9}$$

corresponds to the contact stress on $\Gamma_C(X)$, where A, F and P are the same as in (6.6) and $Q = Q(X)$ denotes the nodal displacement vector in the finite element discretization of the state inequality $(\mathscr{P}(\alpha))$. If $T_j \neq 0$ for some $j \in 0, \ldots, N(h)$ (at nodes of $\Gamma_C(X)$) this means that the corresponding node is in contact, i.e. $Q_j = -x_j$. Consequently,

$$\frac{\partial Q_j}{\partial x_i} = -\frac{\partial x_j}{\partial x_i} = -\delta_{ij}\,. \tag{6.10}$$

Now we obtain by (6.9) and (6.10)

(6.11)

$$\frac{\partial}{\partial x_i}\,\mathscr{E}(Q(X); X) = -T_i + \left(\frac{\partial}{\partial x_i} F, Q\right) - \frac{1}{2}\left(Q, \left(\frac{\partial}{\partial x_i} A\right) Q\right), \quad i = 0, \ldots, N(h)\,.$$

We emphasize that (6.11) is possible because of the special property of the criteria functional.

7. NUMERICAL EXAMPLES

In numerical tests we suppose that the elastic body cosnsists of a homogeneous and isotropic material with the Poisson ratio $\nu = 0{\cdot}29$ and the Young modulus $E = 2{\cdot}15\,.\,10^{11}\,Nm^{-2}$.

Example 7.1. We have chosen the parameters in definition (2.1) of $\mathscr{U}_{ad}$ as follows: $a = 0$, $b = 8$, $C_0 = {\cdot}05$, $A = 0{\cdot}05$, $B = 0{\cdot}05$, $\gamma = 1$, $C_1 = 0{\cdot}025$ and $C_2 = 7{\cdot}8$, i.e. $\Omega(\alpha) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 8, \alpha(x_1) < x_2 < 1\}$. Furthermore, we suppose that

$$\Gamma_D = \{x \in \partial\Omega(\alpha) \mid x_1 = 0, x_1 = 8\}$$

and

$$\Gamma_F = \{x \in \partial\Omega(\alpha) \mid x_2 = 1\}$$

(cf. Figure 7.1).

The body force F is assumed to be zero and the surface traction $P = (0, P_2)$ where $P_2 = -5{\cdot}75\,.\,10^8$ if $x_1 \in (2,6)$ and 0 elsewhere.

The triangulation $\mathscr{T}_h(\alpha_h)$ of $\Omega_h(\alpha_h)$ is constructed with the aid of a uniform triangulation $\hat{\mathscr{T}}_h$ of $\hat{\Omega} = [0, 8] \times [0, 1]$ by mapping all nodal points $(\hat{x}_1, \hat{x}_2)$ of $\hat{\mathscr{T}}_h$ to nodal points $(x_1, x_2) \equiv (S_1^{\alpha_h}(\hat{x}_1, \hat{x}_2), S_2^{\alpha_h}(\hat{x}_1, \hat{x}_2))$ of $\mathscr{T}_h(\alpha_h)$, where

$$(S_1^{\alpha_h}(\hat{x}_1, \hat{x}_2), S_2^{\alpha_h}(\hat{x}_1, \hat{x}_2)) = \begin{cases} (\hat{x}_1, \hat{x}_2) & \text{if} \quad x_2 \geqq 0{\cdot}5\,, \\ (\hat{x}_1, \alpha_h(\hat{x}_1) + 2\hat{x}_2(\frac{1}{2} - \alpha_h(\hat{x}_1)) & \text{if} \quad 0 \leqq x_2 \leqq 0{\cdot}5\,. \end{cases}$$

Thus the domain $\Omega_h(\alpha_h)$ and its triangulation are completely determined by the values of the design variables $x_i = \alpha_h(ih)$ $i = 0, \ldots, N(h)$.

In Figure 7.1 we can see the triangulation of $\Omega_h(\alpha_h)$ for $h = 1/2$.

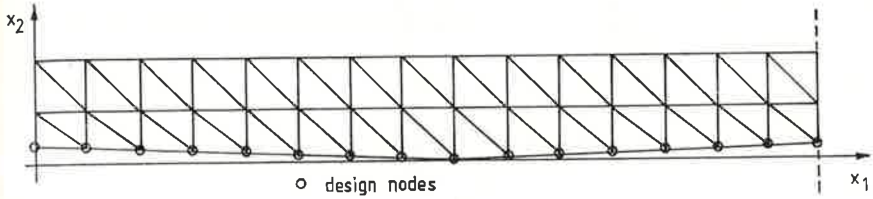

Fig. 7.1. Triangulation of $\Omega_h(\alpha_h)$ for some $\alpha_h \in \mathscr{U}^h_{\mathrm{ad}}$.

Because of the symmetry with respect to the line $x_1 = 4$ in Example 7.1 we have analyzed only one half of the domain using 128 triangles (mesh-size $h = 0{\cdot}25$). This gives us 17 design nodes on the moving boundary $\frac{1}{2}\Gamma_C(\alpha_h)$. As the value $\alpha_h(0)$ is fixed a priori this leaves us 16 degrees of freedom in minimization.

In all test examples we have used the NPSOL routine of SOL (System Optimization Laboratory, for the method see [6]). The gradient of $\mathscr{E}$ has been computed

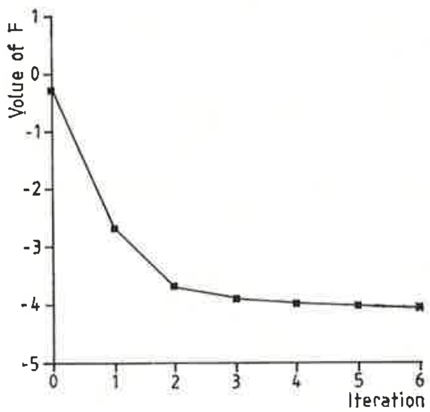

Fig. 7.2. Diminution of $\mathscr{E}$ versus iteration.

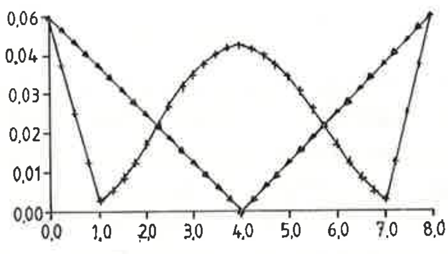

Fig. 7.3. $\Gamma_C(\alpha_h^0)$ (Δ) and $\Gamma_C(\alpha_h^6)$ $(+)$.

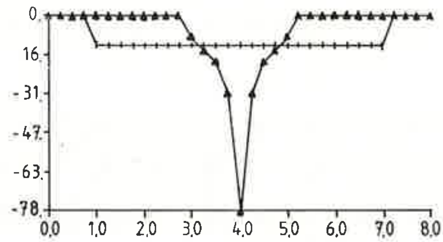

Fig. 7.4. Contact stresses for the initial guess (Δ) and for the final design (+).

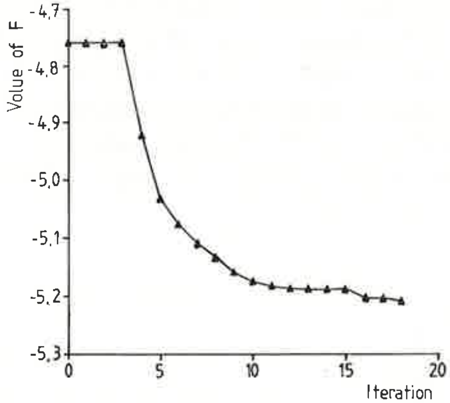

Fig. 7.5. Diminution of $\mathscr{E}$ versus iteration.

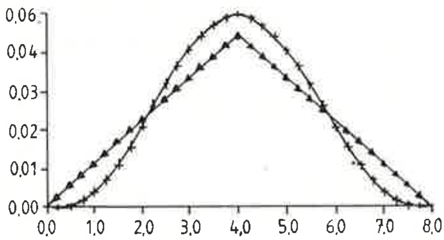

Fig. 7.6. $\Gamma_C(\alpha_h^0)$ (Δ) and $\Gamma_C(\alpha_h^{18})$ (+).

by the formula (6.11). The state problem is solved by a modified relaxation method. The computation has been carried out by VAX 11/780. with FPA in single precision. The authors are indebted to A. Kaarna for his assistance in numerical tests. In Figure 7.2 we can see the diminution of $\mathscr{E}(Q(X)^{(k)}); X^{(k)})$ versus iteration. The initial shape $(\Gamma_C(\alpha^0))$ and the final shape $(\Gamma_C(\alpha^5))$ of the contact surface Γ_C are shown in Figure 7.3.

Figure 7.4 shows the corresponding contact stresses (computed by the formula (6.9)).

Example 7.2. As in Example 7.1 but suppose that $A = B = 0$ and $C_0 = 0{\cdot}1$. All calculations have been carried out in a similar way as in Example 7.1. In Figures 7.5 – 7.7 we can see the corresponding results.

Example 7.3. Let

$$\Omega(\alpha) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 4,\ 0 \leqq \alpha(x_1) < x_2 < 1\}$$

and let

$$\mathscr{U}^1_{\text{ad}} = \Big\{\alpha \in C^{0,1}([0, 1]) \mid 0 \leqq \alpha(x_1) \leqq 0{\cdot}05,\ \alpha(0) = 0{\cdot}05\,,$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}x_1}\alpha\right| \leqq 0{\cdot}05\,,\ \text{meas}\,(\Omega(\alpha)) = 3{\cdot}9\Big\}\,.$$

In Figure 7.8 we can see the partition of $\partial\Omega(\alpha)$.

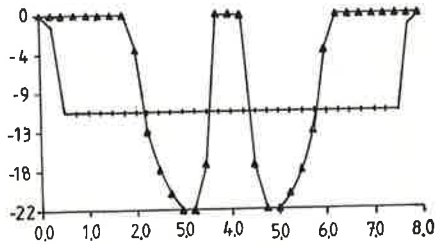

Fig. 7.7. Contact stresses for the initial guess (Δ) and for the final shape (+).

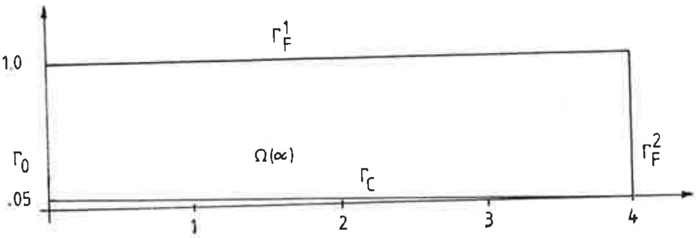

Fig. 7.8. $\Omega(\alpha)$, $\alpha \in \mathscr{U}^1_{\text{ad}}$.

We suppose that $\boldsymbol{F} \equiv 0$, $\boldsymbol{P} \equiv (P_1, P_2) = (0, 0)$ on Γ^2_F and

$$(P_1, P_2) = \begin{cases} (0, 0) & \text{if} \quad x \in \Gamma^1_F \quad \text{and} \quad 0 < x_1 < 2\,, \\ (-5{\cdot}75 \,.\, 10^8, -5{\cdot}75 \,.\, 10^8) & \text{if} \quad x \in \Gamma^1_F \quad \text{and} \quad 2 \leqq x_1 \leqq 4\,. \end{cases}$$

The solution strategy (triangulation, algorithms, gradient etc.) is the same as in Example 7.1. In Figures 7.9 – 7.11 we see the results.

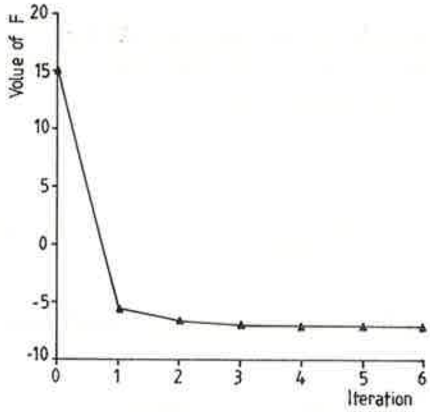

Fig. 7.9. Diminution of $\mathscr{E}$.

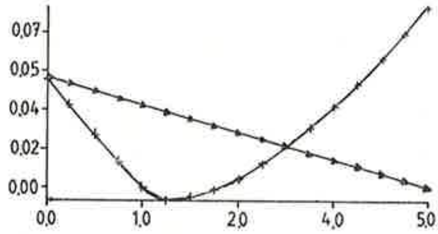

Fig. 7.10. $\Gamma_C(\alpha_h^0)$ (Δ) and $\Gamma_C(\alpha_h^6)$ (+).

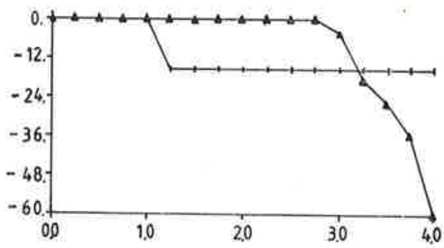

Fig. 7.11. Contact stresses for the initial guess (Δ) and for the final shape (+).

In all test examples the value of the cost functional is reduced and the part of $\Gamma_C(\alpha_h)$ where the body is in contact after deformation is enlarged. As a by-product we could find for $\Gamma_C(\alpha_h)$ such a shape that the contact stress will be evenly distributed when geometrical constraints are appropriate. This is of great practical significance for designers. From the mathematical point of view the functional $\mathscr{E}$ is easy to handle whereas the direct minimization of contact stresses is more involved.

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APPENDIX

Lemma A. *There exists a constant $c > 0$ independent of $\alpha \in \mathcal{U}_{ad}$, $v \in V(\alpha)$ and such that*

$$\|v\|^2_{1,\Omega(\alpha)} \leqq c(\varepsilon_{ij}(v), \varepsilon_{ij}(v))_{0,\Omega(\alpha)} \tag{A.1}$$

holds for any $\alpha \in \mathcal{U}_{ad}$ and any $v \in V(\alpha)$.

Proof. It is known (see [24]) that the so called second Korn's inequality

$$\|v\|^2_{1,\Omega(\alpha)} \leqq \tilde{c}\{(\varepsilon_{ij}(v), \varepsilon_{ij}(v))_{0,\Omega(\alpha)} + (v, v)_{0,\Omega(\alpha)}\} \tag{A.2}$$

holds for any $v \in V(\alpha)$ with a constant $\tilde{c} > 0$ depending on the lipschitz constant of α, only. Due to the definition of $\mathcal{U}_{ad}$, $\tilde{c}$ can be chosen independently of α. Let us prove now that (A.2) is valid even without the second term on its right hand side, in other words (A.1) is valid.

Let (A.1) be not true. Then for any n integer there exist subsequences $\{\alpha_n\}$, $\{v_n\}$, $\alpha_n \in \mathcal{U}_{ad}$, $v_n \in V(\alpha_n)$ such that

$$\|v_n\|^2_{1,\Omega_n} \geqq n(\varepsilon_{ij}(v_n), \varepsilon_{ij}(v_n))_{0,\Omega_n} . \tag{A.3}$$

Without loss of generality we can suppose that

$$\|v_n\|_{1,\Omega_n} = 1 \quad \forall n \tag{A.4}$$

and there is a function $\alpha \in \mathcal{U}_{ad}$ such that

$$\alpha_n \to \alpha \quad \text{uniformly in } [a, b] . \tag{A.5}$$

(A.3) and (A.4) imply

$$(\varepsilon_{ij}(v_n), \varepsilon_{ij}(v_n))_{0,\Omega_n} \to 0 \quad \text{if} \quad n \to \infty . \tag{A.6}$$

On the other hand for n sufficiently large

$$(v_n, v_n)_{0,\Omega_n} \geqq \tilde{c}/2 \tag{A.7}$$

as follows from (A.2), (A.4) and (A.6).

Let $G_m(\alpha)$ be defined analogously to (3.5). Using the same approach as in the proof of Theorem 3.2, we can find a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) such that

$$v_n \rightharpoonup v(\text{weakly}) \quad \text{in} \quad (H^1(G_m(\alpha))^2 \tag{A.8}$$

for any m integer. Let m be fixed. As

$$(\varepsilon_{ij}(\mathbf{v}), \varepsilon_{ij}(\mathbf{v}))_{0,G_m} \leqq \liminf (\varepsilon_{ij}(\mathbf{v}_n), \varepsilon_{ij}(\mathbf{v}_n))_{0,G_m} \to 0\,,$$

one can easily get, by applying the first Korn's inequality in $V(G_m(\alpha))$ that $\mathbf{v} \equiv 0$ in $G_m(\alpha)$ and hence in $\Omega(\alpha)$. On the other hand

$$\text{(A.9)} \qquad \|\mathbf{v}_n\|^2_{0,\Omega_n} = \|\mathbf{v}_n\|^2_{0,G_m} + \|\mathbf{v}_n\|^2_{0,\Omega_n - G_m}$$

for n sufficiently large. By a direct calculation we obtain

$$\text{(A.10)} \qquad \|\mathbf{v}_n\|^2_{0,\Omega_n - G_m} \leqq c \max_{x_1 \in [a,b]} |\alpha(x_1) + 1/m - \alpha_n(x_1)|$$

so that for n, m sufficiently large

$$\|\mathbf{v}_n\|^2_{0,\Omega_n - G_m(\alpha)} \leqq \tilde{c}/4\,.$$

From this, (A.7) and (A.9) we finally get

$$\|\mathbf{v}_n\|^2_{0,G_m} \geqq \tilde{c}/4 \quad \forall n\,,$$

which contradicts to the fact that $\mathbf{v} \equiv 0$ in $\Omega(\alpha)$.

Souhrn

OPTIMÁLNÍ NAVRHOVÁNÍ TVARU OBLASTI PRO PRUŽNÉ TĚLESO V KONTAKTU, ZALOŽENÉ NA PENALIZACI STAVOVÉ NEROVNICE

JAROSLAV HASLINGER, PEKKA NEITTAANMÄKI, TIMO TIIHONEN

Práce je věnována studiu optimálního návrhu kontaktní plochy pružného tělesa, jednostranně podpíraného dokonale tuhou oporou. Původní variační nerovnice, jež popisuje chování daného modelu, je převedena pomocí penalizační metody na systém nelineárních rovnic. Tyto penalizované úlohy nyní vystupují v úloze nových, penalizovaných relací. V práci se zkoumá vzájemný vztah mezi původní úlohou optimálního návrhu oblasti se stavovou nerovnicí a posloupností obdobných úloh se stavovými penalizovanými rovnicemi. Práce se rovněž zabývá problematikou aproximace dané úlohy pomocí metody konečných prvků. V závěru jsou uvedeny některé numerické výsledky pro modelové úlohy.

Резюме

АППРОКСИМАЦИЯ ПРОБЛЕМЫ НАХОЖДЕНИЯ ОПТИМАЛЬНОЙ ОБЛАСТИ В КОНТАКНЫХ ЗАДАЧАХ УПРУГИХ ТЕЛ

JAROSLAV HASLINGER, PEKKA NEITTAANMÄKI, TIMO TIIHONEN

В работе изучается аппроксимация задачи нахождения оптимальной геометрической формы упругово тела, находящегося в контакте с жестким препятствием.

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