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FINITE ELEMENT APPROXIMATION FOR A DIV-ROT SYSTEM WITH MIXED BOUNDARY CONDITIONS IN NON-SMOOTH PLANE DOMAINS

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1. INTRODUCTION

In this paper we are concerned with the mixed boundary value problem for the following div-rot system:

$$\begin{cases} \operatorname{div} \boldsymbol{u} = f & \text{in} \quad \Omega\,, \\ \operatorname{rot} \boldsymbol{u} = g & \text{in} \quad \Omega\,, \end{cases} \tag{1.1}$$

$$\begin{cases} \boldsymbol{n} \,.\, \boldsymbol{u} = 0 & \text{on} \quad \Gamma_1\,, \\ \boldsymbol{n} \wedge \boldsymbol{u} = 0 & \text{on} \quad \Gamma_2\,. \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, Γ_0 is a finite set of points where one type of boundary conditions changes into another and Γ_1, Γ_2 are disjoint and open in $\partial\Omega$. Functions f and g are given; for a differentiable vector field $\boldsymbol{u} = (u_1, u_2)$, $\operatorname{div} \boldsymbol{u} = \partial_1 u_1 + \partial_2 u_2$, $\operatorname{rot} \boldsymbol{u} = \partial_1 u_2 - \partial_2 u_1$; $\boldsymbol{n} = (n_1, n_2)$ is the outward unit normal to $\partial\Omega$, which exists almost everywhere, $\boldsymbol{n} \,.\, \boldsymbol{u} = n_1 u_1 + n_2 u_2$, $\boldsymbol{n} \wedge \boldsymbol{u} = n_1 u_2 - n_2 u_1$. If $\Gamma_1 = \emptyset$ of $\Gamma_2 = \emptyset$ the usual compatibility condition is assumed.

Many physically interesting phenomena can be described by a system like (1.1) to (1.2) (e.g. the steady state Maxwell equations, the ideal fluid flow and mechanics problems, see [2, 3, 5, 8, 11, 13, 14, 15, 16, 19, 21]). Such a problem is also obtained when the gradient of a second order elliptic problem with mixed Dirichlet and Neumann boundary conditions is looked for. For an extensive collection of examples we refer to [5, 15, 19, 21] and references therein.

A finite element approximation of the system (1.1)–(1.2) in smooth domains for $\Gamma_1 = \emptyset$ (or $\Gamma_2 = \emptyset$) is investigated in [3, 14, 18]. The aim of this paper is to generalize these results to non-smooth domains and also to cover combined boundary conditions. Because of non-smoothness, a technique quite different from that in [14, 18] is used to prove the V-ellipticity or uniform V_h – ellipticity (see Theorems 3.1 and 4.3). We utilize the concept of a stream function ([6]). The paper is organized as follows:

In Chapter 2 we introduce some special function spaces. A variational formulation of the problem (1.1)–(1.2) is given in Chapter 3 and its solvability is proved for Γ_1 and Γ_2 connected. Chapter 4 contains a finite element approximation of the corresponding variational continuous problem. Finally, in Chapter 5 some numerical examples are presented.

2. PRELIMINARIES

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary. The notation $H^k(\Omega)$, $k > 0$, is used for the Sobolev space, see [12]; especially $L^2(\Omega) = H^0(\Omega)$ with the scalar product $(\cdot, \cdot)_0$. The usual norm in $H^k(\Omega)$ or in $(H^k(\Omega))^2$ will be denoted by $\|\cdot\|_{k,\Omega}$ and the subscript Ω will often be omitted. We shall also denote by $\|\cdot\|_{0,\Gamma}$ the norm in $L^2(\Gamma)$ for a measurable part Γ of $\partial\Omega$. The notation $H^{1/2}(\Gamma)$ is used for the space of traces $\varphi|_\Gamma$ for $\varphi \in H^1(\Omega)$.

Let $C^k(\bar{\Omega})$ be the space of functions, the (classical) derivatives of which up to order k are continuous in $\bar{\Omega}$. We write $\partial_i = \partial/\partial x_i$ and put $C(\bar{\Omega}) = C^0(\bar{\Omega})$.

We note (see [6], p. 16) that the functional $\mathbf{v} \mapsto \mathbf{n} \,.\, \mathbf{v}|_{\partial\Omega}$ defined on $(C^\infty(\bar{\Omega}))^2$ can be extended by continuity to a linear continuous mapping from the space

$$H(\mathrm{div}; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^2 \mid \mathrm{div}\, \mathbf{v} \in L^2(\Omega)\}$$

into $H^{-1/2}(\partial\Omega)$, which is the dual space to the space $H^{1/2}(\partial\Omega)$. In this case, the Green formula is of the form

$$(2.1) \qquad (\mathrm{div}\, \mathbf{v}, \varphi)_0 + (\mathbf{v}, \mathrm{grad}\, \varphi)_0 = \langle \mathbf{n} \,.\, \mathbf{v}, \varphi \rangle_{\partial\Omega} \quad \forall \mathbf{v} \in H(\mathrm{div}; \Omega) \quad \forall \varphi \in H^1(\Omega)\,.$$

Here $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$, and $\mathbf{n} \,.\, \mathbf{v}$ is called the normal component of $\mathbf{v}$. In particular, if $\mathbf{n} \,.\, \mathbf{v}|_{\partial\Omega} \in L^2(\partial\Omega)$ then

$$(2.2) \qquad \langle \mathbf{n} \,.\, \mathbf{v}, \varphi \rangle_{\partial\Omega} = \int_{\partial\Omega} (\mathbf{n} \,.\, \mathbf{v})\, \varphi \,\mathrm{d}s \quad \forall \varphi \in H^1(\Omega)\,.$$

The tangential component $\mathbf{n} \wedge \mathbf{v} \in H^{-1/2}(\partial\Omega)$ can be defined (see also [6], p. 20) for $\mathbf{v}$ from the space

$$H(\mathrm{rot}; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^2 \mid \mathrm{rot}\, \mathbf{v} \in L^2(\Omega)\}\,.$$

The Green formula now reads

$$(2.3) \qquad (\mathrm{rot}\, \mathbf{v}, \varphi)_0 - (\mathbf{v}, \mathrm{curl}\, \varphi)_0 = \langle \mathbf{n} \wedge \mathbf{v}, \varphi \rangle_{\partial\Omega} \quad \forall \mathbf{v} \in H(\mathrm{rot}; \Omega) \quad \forall \varphi \in H^1(\Omega)\,,$$

where $\mathrm{curl}\, \varphi = (\partial_2 \varphi, -\partial_1 \varphi)$. More details about the spaces $H(\mathrm{div}; \Omega)$ and $H(\mathrm{rot}; \Omega)$ can be found in [6, 10, 11, 20]. Further, we define some subspaces of these spaces in the following way:

$$H(\mathrm{div}; \Omega, \Gamma_1) = \{\mathbf{v} \in (L^2(\Omega))^2 \mid \exists \bar{f} \in L^2(\Omega) : (\mathbf{v}, \mathrm{grad}\, \psi)_0 = (-\bar{f}, \psi)_0 \quad \forall \psi \in H^1(\Omega, \Gamma_2)\},$$

$$H(\mathrm{rot}; \Omega, \Gamma_2) = \{\mathbf{v} \in (L^2(\Omega))^2 \mid \exists \bar{g} \in L^2(\Omega) : (\mathbf{v}, \mathrm{curl}\, \varphi)_0 = = (\bar{g}, \varphi)_0 \quad \forall \varphi \in H^1(\Omega, \Gamma_1)\},$$

where

$$H^1(\Omega, \Gamma_i) = \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } \Gamma_i\}, \quad i = 1, 2.$$

The functions $\bar{f}$ and $\bar{g}$ are the divergence and rotation of $\mathbf{v}$, respectively. Note that if $\mathbf{v} \in H(\mathrm{div}; \Omega, \Gamma_1) \cap (H^1(\Omega))^2$, then $\mathbf{n} \cdot \mathbf{v} = 0$ on Γ_1, and analogously $\mathbf{n} \wedge \mathbf{v} = 0$ on Γ_2 for $\mathbf{v} \in H(\mathrm{rot}; \Omega, \Gamma_2) \cap (H^1(\Omega))^2$.

The symbols C, C_1, C_2, ... are reserved for the so-called generic constants which may vary with context. Let us still emphasize that all statements will always hold only for a sufficiently small triangulation parameter h.

3. ON THE CONTINUOUS PROBLEM

We shall now give a variational formulation of the problem (1.1)–(1.2). We equip the space

$$\mathscr{V} = H(\mathrm{div}; \Omega) \cap H(\mathrm{rot}; \Omega)$$

with the norm

$$|||\cdot||| = (\|\cdot\|_0^2 + \|\mathrm{div}\cdot\|_0^2 + \|\mathrm{rot}\cdot\|_0^2)^{1/2}.$$

For f, $g \in L^2(\Omega)$ we define the linear form

$$b(\mathbf{v}) = (f, \mathrm{div}\, \mathbf{v})_0 + (g, \mathrm{rot}\, \mathbf{v})_0, \quad \mathbf{v} \in \mathscr{V}, \tag{3.1}$$

and the bilinear form

$$a(\mathbf{v}, \mathbf{v}') = (\mathrm{div}\, \mathbf{v}, \mathrm{div}\, \mathbf{v}')_0 + (\mathrm{rot}\, \mathbf{v}, \mathrm{rot}\, \mathbf{v}')_0, \quad \mathbf{v}, \mathbf{v}' \in \mathscr{V}. \tag{3.2}$$

Further, let us introduce the space of trial functions

$$V = H(\mathrm{div}; \Omega, \Gamma_1) \cap H(\mathrm{rot}; \Omega, \Gamma_2)$$

with the norm $|||\cdot|||$.

By a (*weak*) *variational* formulation of the problem (1.1)–(1.2) we understand the problem of finding $\mathbf{u} \in V$ which satisfies

$$a(\mathbf{u}, \mathbf{v}) = b(\mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in V. \tag{3.3}$$

We shall call $\mathbf{u}$ the weak solution of the problem (1.1)–(1.2), since evidently any sufficiently smooth $\mathbf{u}$ satisfies also (3.3). Conversely, any sufficiently smooth solution $\mathbf{u} \in V$ of (3.3) satisfies (1.1)–(1.2), too (see the proof of Theorem 4.6). Before we consider the unique solvability of (3.3) we have to prove two theorems.

Theorem 3.1. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with a Lipschitz boundary. Then*

$$\|\mathbf{v}\|_0 \leqq C(\|\operatorname{div} \mathbf{v}\|_0 + \|\operatorname{rot} \mathbf{v}\|_0) \quad \text{for all} \quad \mathbf{v} \in V, \tag{3.4}$$

if and only if Γ_1 *and* Γ_2 *are connected.*

Proof. "$\Rightarrow$": 1° Suppose that Γ_1 is not connected and let Γ_1' be one of its components. Let

$$\mathbf{v} = \operatorname{curl} z\,, \tag{3.5}$$

where $z \in H^1(\Omega)$ is a weak solution of the following problem:

$$\begin{aligned} \Delta z &= 0 \quad \text{in} \quad \Omega\,, \\ z &= 1 \quad \text{on} \quad \Gamma_1'\,, \\ z &= 0 \quad \text{on} \quad \Gamma_1 - \Gamma_1'\,, \\ \partial_n z &= 0 \quad \text{on} \quad \Gamma_2\,, \end{aligned} \tag{3.6}$$

$(\partial_n = \partial/\partial n)$. We show that in this case $\mathbf{v}$ does not satisfy (3.4) by verifying first that $\mathbf{v} \in V$.

For the unit tangent $\mathbf{t} = (n_2, -n_1)$ to $\partial\Omega$ we write $\partial_{\mathbf{t}} = \partial/\partial\mathbf{t}$. Let $\psi \in H^1(\Omega, \Gamma_2) \cap C^\infty(\bar{\Omega})$ be arbitrary. Since $\psi = 0$ on Γ_2, we have by (3.5), (3.6) and by the Green formula (2.3) that

$$\begin{aligned} (\mathbf{v}, \operatorname{grad} \psi)_0 &= (\operatorname{curl} z, \operatorname{grad} \psi)_0 = -\int_{\partial\Omega} z(\mathbf{n} \wedge \operatorname{grad} \psi)\,\mathrm{d}s = \\ &= \int_{\partial\Omega} z\,\partial_{\mathbf{t}}\psi\,\mathrm{d}s = \int_{\Gamma_1'} \partial_{\mathbf{t}}\psi\,\mathrm{d}s\,. \end{aligned} \tag{3.7}$$

But the last integral is zero, since either Γ_1' is a closed curve or $\psi = 0$ at the end points of $\bar{\Gamma}_1'$. According to [4], p. 618, $H^1(\Omega, \Gamma_2) \cap C^\infty(\bar{\Omega})$ is dense in $H^1(\Omega, \Gamma_2)$ with respect to the $\|\cdot\|_1$ – norm. Consequently, the relation (3.7) yields

$$(\mathbf{v}, \operatorname{grad} \psi)_0 = 0 \quad \forall \psi \in H^1(\Omega, \Gamma_2)\,, \tag{3.8}$$

i.e. $\mathbf{v} \in H(\operatorname{div}; \Omega, \Gamma_1)$.

Further, using (3.5) and the fact that z is a weak solution of (3.6), we get

$$(\mathbf{v}, \operatorname{curl} \varphi)_0 = (\operatorname{curl} z, \operatorname{curl} \varphi)_0 = (\operatorname{grad} z, \operatorname{grad} \varphi)_0 = 0 \tag{3.9}$$

for all $\varphi \in H^1(\Omega, \Gamma_1)$, i.e. $\mathbf{v} \in H(\operatorname{rot}; \Omega, \Gamma_2)$.

Now, from (3.8), (3.9) and (3.6) we find that $\operatorname{div} \mathbf{v} = \operatorname{rot} \mathbf{v} = 0$ in Ω, but $\|\mathbf{v}\|_0 \neq 0$, i.e. (3.4) does not hold.

2° Secondly, we suppose that Γ_2 is not connected and that Γ_2' is one of its components. Then by an analogous argument as in past 1°, it can be shown that (3.4) does not hold for $\mathbf{v} = \operatorname{grad} z$, where $z \in H^1(\Omega)$ is the weak solution of the problem

$$\Delta z = 0 \quad \text{in} \quad \Omega\,,$$

$$\partial_{\mathbf{n}} z = 0 \quad \text{on} \quad \Gamma_1 ,$$
$$z = 1 \quad \text{on} \quad \Gamma_2' ,$$
$$z = 0 \quad \text{on} \quad \Gamma_2 - \Gamma_2' .$$

"$\Leftarrow$": Conversely, let Γ_1 and Γ_2 be connected. Both the cases $\Gamma_1 = \emptyset$ and $\Gamma_2 = \emptyset$ are proved in [10]. So let $\Gamma_1 \neq \emptyset, \Gamma_2 \neq \emptyset$, and let $\mathbf{v} \in V$ be arbitrary. Let $p \in H^1(\Omega, \Gamma_2)$ be a weak solution of the problem

$$\begin{aligned} \Delta p &= \operatorname{div} \mathbf{v} \quad \text{in} \quad \Omega , \\ \partial_{\mathbf{n}} p &= 0 \quad \text{on} \quad \Gamma_1 , \\ p &= 0 \quad \text{on} \quad \Gamma_2 . \end{aligned} \tag{3.10}$$

Hence

$$(\operatorname{grad} p, \operatorname{grad} \psi)_0 = (-\operatorname{div} \mathbf{v}, \psi)_0 \quad \forall \psi \in H^1(\Omega, \Gamma_2) \tag{3.11}$$

and

$$\|p\|_1 \leqq C_1 \|\operatorname{div} \mathbf{v}\|_0 . \tag{3.12}$$

Utilizing (3.11) and the definition of $H(\operatorname{div}; \Omega, \Gamma_1)$, we get for $\mathbf{w} = \mathbf{v} - \operatorname{grad} p$ that

$$(\mathbf{w}, \operatorname{grad} \psi)_0 = 0 \quad \forall \psi \in H^1(\Omega, \Gamma_2) , \tag{3.13}$$

i.e. the vector function $\mathbf{w}$ is divergence-free.

Further, let $\varphi \in H^1(\Omega, \Gamma_1) \cap C^\infty(\bar{\Omega})$ be arbitrary. Then the Green formula (2.1) and (2.2) yields

$$(\operatorname{grad} p, \operatorname{curl} \varphi)_0 = \int_{\partial\Omega} p(\mathbf{n} \,.\, \operatorname{curl} \varphi) \, \mathrm{d}s = - \int_{\partial\Omega} p \, \partial_{\mathbf{t}} \varphi \, \mathrm{d}s = 0 ,$$

as $p = 0$ on Γ_2 and $\varphi = 0$ on Γ_1. Due to the density of $H^1(\Omega, \Gamma_1) \cap C^\infty(\bar{\Omega})$ in $H^1(\Omega, \Gamma_1)$ we get

$$(\operatorname{grad} p, \operatorname{curl} \varphi)_0 = 0 \quad \forall \varphi \in H^1(\Omega, \Gamma_1) . \tag{3.14}$$

Next, from the connectivity of Γ_1 and Γ_2 we see that Ω is either simply connected or doubly connected. When Ω is doubly connected, then Γ_1 is just one of the two components of $\partial\Omega$ and we have

$$\mathbf{n} \,.\, \mathbf{w} = 0 \quad \text{in} \quad H^{-1/2}(\Gamma_1) \tag{3.15}$$

($H^{-1/2}(\Gamma_1)$ is the dual space to $H^{1/2}(\Gamma_1)$, see [6, 9]). As $\mathbf{w}$ is divergence-free we find by the Green formula (2.1) that $\langle \mathbf{n} \,.\, \mathbf{w}, 1 \rangle_{\partial\Omega} = 0$. Consequently, by (3.15) it also holds that $\langle \mathbf{n} \,.\, \mathbf{w}, 1 \rangle_{\Gamma_2} = 0$, where $\langle \cdot, \cdot \rangle_{\Gamma_2}$ denotes the duality pairling between $H^{-1/2}(\Gamma_2)$ and $H^{1/2}(\Gamma_2)$.

By [6], p. 22, there exists a stream function $q \in H^1(\Omega)$ (unique apart from an additive constant, which will be chosen later) such that

$$\text{curl } q = \mathbf{w}(= \mathbf{v} - \text{grad } p)\,. \tag{3.16}$$

The relations (3.13), (3.16) and (2.3) imply that for any $\psi \in H^1(\Omega, \Gamma_2) \cap C^\infty(\bar{\Omega})$,

$$0 = (\text{curl } q, \text{grad } \psi)_0 = -(\text{curl } \psi, \text{grad } q)_0 = \int_{\Gamma_1} \psi\, \partial_t q \, \mathrm{d}s\,,$$

as $\psi = 0$ on Γ_2. Thus $q \in H^1(\Omega, \Gamma_1)$ is constant on (connected) Γ_1 and we can choose q to be zero on Γ_1, i.e. $q \in H^1(\Omega, \Gamma_1)$.

By (3.16), (3.14) and by the definition of $H(\text{rot}; \Omega, \Gamma_2)$ we obtain

$$\begin{aligned}(\text{grad } q, \text{grad } \varphi)_0 &= (\text{curl } q, \text{curl } \varphi) = (\mathbf{w}, \text{curl } \varphi)_0 = \\ &= (\mathbf{v} - \text{grad } p, \text{curl } \varphi)_0 = (\mathbf{v}, \text{curl } \varphi)_0 = (\text{rot } \mathbf{v}, \varphi)_0\end{aligned}$$

for all $\varphi \in H(\Omega, \Gamma_1)$, i.e. $q \in H^1(\Omega, \Gamma_1)$ is a weak solution of the problem

$$\begin{aligned} -\Delta q &= \text{rot } \mathbf{v} \quad \text{in} \quad \Omega\,, \\ q &= 0 \quad \text{on} \quad \Gamma_1\,, \\ \partial_n q &= 0 \quad \text{on} \quad \Gamma_2\,, \end{aligned} \tag{3.17}$$

and

$$\|q\|_1 \leqq C_2 \|\text{rot } \mathbf{v}\|_0\,. \tag{3.18}$$

Finally, (3.16), (3.12) and (3.18) imply

$$\|\mathbf{v}\|_0 \leqq \|\text{grad } p\|_0 + \|\text{curl } q\|_0 \leqq C(\|\text{div } \mathbf{v}\|_0 + \|\text{rot } \mathbf{v}\|_0)\,. \qquad \square$$

Convention. 3.2. *For simplicity we suppose from now on that Γ_1 and Γ_2 are connected.*

Remark. 3.3. By Theorem 3.2 the bilinear form (3.2) is a scalar product in V and $\sqrt{a(\mathbf{v}, \mathbf{v})}$ is equivalent to the norm $|||\mathbf{v}|||$, i.e.

$$C|||\mathbf{v}|||^2 \leqq a(\mathbf{v}, \mathbf{v}) \leqq |||\mathbf{v}|||^2, \ \mathbf{v} \in V\,. \tag{3.19}$$

Theorem 3.4. *The space V equipped with the scalar product $a(\cdot, \cdot)$ is a Hilbert space.*

Proof. Let $\{\mathbf{v}_k\} \subset V$ be a Cauchy sequence. One sees by Theorem 3.1 that $\mathbf{v}_k$ converges to a function $\mathbf{v} \in (L^2(\Omega))^2$ in the $\|\cdot\|_0$-norm. Evidently also div $\mathbf{v}_k$ converges to some $\bar{f} \in L^2(\Omega)$. Hence,

$$\begin{aligned}(\text{div } \mathbf{v}_k, \psi)_0 &\to (\bar{f}, \psi)_0 \quad \forall \psi \in H^1(\Omega, \Gamma_2)\,, \\ (\mathbf{v}_k, \text{grad } \psi)_0 &\to (\mathbf{v}, \text{grad } \psi)_0 \quad \forall \psi \in H^1(\Omega, \Gamma_2)\,.\end{aligned}$$

By the definition of $H(\text{div}; \Omega, \Gamma_1)$ we see that $\mathbf{v} \in H(\text{div}; \Omega, \Gamma_1)$ with div $\mathbf{v} = \bar{f}$.

Analogously, it can be verified that $\mathbf{v} \in H(\mathrm{rot}; \Omega, \Gamma_2)$. $\square$

Due to the Riesz theorem, the foregoing theorem and (3.19), we come to the following assertion.

Corollary 3.4. *The problem (3.3) has a unique solution.*

Remark. 3.5. Assuming the H^2-regularity for the problems (3.10) and (3.17) we can derive from (3.16) the so-called Friedrichs inequality:

$$\|\mathbf{v}\|_1 \leqq C(\|\mathrm{div}\, \mathbf{v}\|_0 + \|\mathrm{rot}\, \mathbf{v}\|_0) \quad \text{for all} \quad \mathbf{v} \in V. \tag{3.20}$$

Sufficient conditions for the H^2-regularity of the mixed problem (3.10) in polygonal domains can be found e.g. in [7], p. 210. In [10] necessary and sufficient conditions for the validity of (3.20) are given in the case $\Gamma_1 = \emptyset$ or $\Gamma_2 = \emptyset$.

4. ON THE DISCRETE PROBLEM

We suppose that $\partial\Omega$ is piecewise twice differentiable and has a finite number of corners. By $\{\mathscr{T}_h\}$ we denote a strongly regular family or triangulations of $\bar{\Omega} : \bar{\Omega} = \bigcup_{K \in \mathscr{T}_h} K$, i.e. $\{\mathscr{T}_h\}$ satisfies the inverse assumption (see [2], p. 140). Here h is the usual discretization parameter. Further, we assume that the common sides of neighbouring triangles of $\mathscr{T}_h$ are always straight segments, while the other sides are in general curved (i.e. they coincide with the corresponding parts of the boundary $\partial\Omega$). Moreover, let the interior of any side of $K \in \mathscr{T}_h$ be disjoint with $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ (the so-called consistence condition).

Let Z_h^i be the set of all nodal points of $\mathscr{T}_h$ lying on $\bar{\Gamma}_i$ $(i = 1, 2)$. Let $\hat{Z}$ be the union of Γ_0 and all corner points $\partial\Omega$ and let $Z = \hat{Z} \setminus \{\mathbf{x} \in \Gamma_0 \mid \text{the tangents to } \bar{\Gamma}_1 \text{ and to } \bar{\Gamma}_2 \text{ are perpendicular at } \mathbf{x}\}$.

We define finite element subspaces $\mathscr{V}_h$, $V_h \subset (H^1(\Omega))^2$, by

$$\begin{aligned} \mathscr{V}_h &= \{\mathbf{v} \in (C(\bar{\Omega}))^2 \; \mathbf{v}|_K \in (P_1(K))^2 \;\; \forall K \in \mathscr{T}_h\}\,, \\ V_h &= \{\mathbf{v} \in \mathscr{V}_h \mid v(\mathbf{x}) = 0 \;\; \forall \mathbf{x} \in Z, (\mathbf{n}\,.\,\mathbf{v})(\mathbf{x}) = 0 \\ &\quad \forall \mathbf{x} \in Z_h^1 \setminus Z\,, \quad (\mathbf{n} \wedge \mathbf{v})(\mathbf{x}) = 0 \;\; \forall \mathbf{x} \in Z_h^2 \setminus Z\}\,. \end{aligned} \tag{4.1}$$

Here $P_1(K)$ denotes the space of polynomials on K of degree at most one. Thus, V_h consists of piecewise linear continuous vector fields satisfying only pointwise the boundary conditions. Consequently, we get the inclusion $V_h \subset V$ if and only if Ω is polygonal, i.e. the following finite element approximation of the problem (3.3) will be conforming just for polygonal domains.

Find $\mathbf{u}_h \in V_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = b(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h\,, \tag{4.2}$$

where a and b are defined by (3.2) and (3.1), respectively.

A basis of V_h can be constructed as follows. Let $\mathbf{P}_1, \ldots, \mathbf{P}_k$ be the interior nodal points of $\mathscr{T}_h$, let $\mathbf{P}_{k+1}, \ldots, \mathbf{P}_{k+l}$ be the nodal points of $\Gamma_1 \setminus Z$, and let $\mathbf{P}_{k+l+1}, \ldots$ $\ldots, \mathbf{P}_{k+l+m}$ be the nodal points of $\bar{\Gamma}_2 \setminus Z$. Further, let φ_i be the usual Courant tetrafunctions such that

$$\varphi_i(\mathbf{P}_j) = \delta_{ij}, \quad i, j = 1, \ldots, k + l + m .$$

Then obviously

$$\{(\varphi_i, 0)\}_{i=1}^{k} \cup \{(0, \varphi_i)\}_{i=1}^{k} \cup \{\mathbf{t}(\mathbf{P}_i)\, \varphi_i\}_{i=k+1}^{k+l} \cup \{\mathbf{n}(\mathbf{P}_i)\, \varphi_i\}_{i=k+l+1}^{k+l+m}$$

form a basis in V_h and this is just the basis used in Chap. 5.

Concerning the unique solvability of the problem (4.2), we first prove an auxiliary lemma.

Lemma 4.1. *Let* $\mathbf{v}_h \in \mathscr{V}_h$ *and let* $\operatorname{div} \mathbf{v}_h = \operatorname{rot} \mathbf{v}_h = 0$. *Then* $\mathbf{v}_h$ *is from* $(P_1(\bar{\Omega}))^2$ *and has the form*

$$\mathbf{v}_h(\mathbf{x}) = (\alpha x_1 + \beta x_2 + \gamma, \beta x_1 - \alpha x_2 + \delta), \mathbf{x} = (x_1. x_2) \in \bar{\Omega} , \tag{4.3}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}^1$.

Proof. Let the assumptions of the lemma be satisfied and let $K, K' \in \mathscr{T}_h$ be two neighbouring triangles. Then evidently $\mathbf{v}_h$ is on K and K' of the form

$$\begin{aligned} \mathbf{v}_h|_K(\mathbf{x}) &= (\alpha x_1 + \beta x_2 + \gamma, \beta x_1 - \alpha x_2 + \delta) , \\ \mathbf{v}_h|_{K'}(\mathbf{x}) &= (\alpha' x_1 + \beta' x_2 + \gamma', \beta' x_1 - \alpha' x_2 + \delta') . \end{aligned} \tag{4.4}$$

We distinguish the following two cases:

1) Let the common side S of the triangles K and K' coincide with the line $x_2 = kx_1 + q$. From (4.4) and from the continuity of $\mathbf{v}_h$ on S, one gets

$$\alpha x_1 + \beta k x_1 + \beta q + \gamma = \alpha' x_1 + \beta' k x_1 + \beta' q + \gamma' ,$$
$$\beta x_1 - \alpha k x_1 - \alpha q + \delta = \beta' x_1 - \alpha' k x_1 + \alpha' q + \delta' ,$$

which implies

$$\alpha - \alpha' = k(\beta' - \beta) \quad \text{and} \quad \beta - \beta' = k(\alpha - \alpha') ,$$

i.e. $\alpha = \alpha'$ and $\beta = \beta'$.

2) Let $S = K \cap K'$ coincide with the line $x_1 = \text{const.}$, then by (4.4) we see that $\alpha = \alpha'$ and $\beta = \beta'$.

Further, from (4.4) and the continuity of $\mathbf{v}_h$, we get also $\gamma = \gamma'$ and $\delta = \delta'$. ☐

Theorem 4.2. *If* card $(Z) > 1$, *then the discrete problem* (4.2) *always has a unique solution.*

Proof. We prove that $a(\cdot, \cdot)$ is a scalar product on V_h. Then the unique solvability of the problem (4.2) will follow from the Riesz theorem.

So let $a(\mathbf{v}_h, \mathbf{v}_h) = 0$ for some $\mathbf{v}_h \in V_h$. Thus, (4.3) and (4.1) yield

$$\mathbf{v}_h(\mathbf{x}) = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = 0 \quad \forall \mathbf{x} = (x_1, x_2) \in Z .$$

Since card $(Z) > 1$, the matrix of this system is singular and, therefore, $\alpha = \beta = 0$ and consequently also $\gamma = \delta = 0$. Hence, $\mathbf{v}_h = 0$. ☐

In the case card $(Z) \leqq 1$, it is easy to give an example when (4.2) has more solutions. Fortunately, it results from the following theorem that (4.2) has a unique solution at least for sufficiently small h.

Theorem 4.3. *For sufficiently small h, the bilinear form $a(\cdot, \cdot)$ is uniformly V_h-elliptic (with respect to h), i.e. there exist constants $C > 0$ and $h_0 > 0$ such that for any $\mathscr{T}_h$ with $h \in (0, h_0)$,*

$$a(\mathbf{v}_h, \mathbf{v}_h) \geqq C |||\mathbf{v}_h|||^2 \quad \forall \mathbf{v}_h \in V_h . \tag{4.5}$$

The proof is based on the following two lemmas.

Lemma 4.4. *There exists a constant $C > 0$ such that*

$$C|||\mathbf{v}|||^2 \leqq a(\mathbf{v}, \mathbf{v}) + \|\mathbf{n} \,.\, \mathbf{v}\|^2_{0,\Gamma_1} + \|\mathbf{n} \wedge \mathbf{v}\|^2_{0,\Gamma_2} \tag{4.6}$$

for all $\mathbf{v} \in (H^1(\Omega))^2$.

Proof. We can proceed in the way analogous to that adopted in the proof of Theorem 3.1. Let $\mathbf{v} \in (H^1(\Omega))^2$ be given. Instead of (3.10) and (3.17) we have

$$\begin{cases} \Delta p = \operatorname{div} \mathbf{v} & \text{in} \quad \Omega , \\ p = 0 & \text{on} \quad \Gamma_2 , \\ \partial_n p = \mathbf{n} \,.\, \mathbf{v} & \text{on} \quad \Gamma_1 , \end{cases} \quad \text{and} \quad \begin{cases} -\Delta q = \operatorname{rot} \mathbf{v} & \text{in} \quad \Omega , \\ q = 0 & \text{on} \quad \Gamma_1 , \\ \partial_n q = \mathbf{n} \wedge \mathbf{v} & \text{on} \quad \Gamma_2 , \end{cases}$$

respectively. Hence,

$$\|p\|_{1,\Omega} \leqq C_1(\|\operatorname{div} \mathbf{v}\|_{0,\Omega} + \|\mathbf{n} \,.\, \mathbf{v}\|_{0,\Gamma_1}) ,$$
$$\|q\|_{1,\Omega} \leqq C_2(\|\operatorname{rot} \mathbf{v}\|_{0,\Omega} + \|\mathbf{n} \wedge \mathbf{v}\|_{0,\Gamma_2}) .$$

Finally, the assertion follows from the representation $\mathbf{v} = \operatorname{grad} p + \operatorname{curl} q$ as in the proof of Theorem 3.1. ☐

According to the pointwise boundary conditions of $\mathbf{v}_h \in V_h$ on piecewise smooth Γ_1 and Γ_2, the last two terms in (4.6) vanish for $h \to 0$. This can be proved in a way similar to [17].

Lemma 4.5. *There exists a constant $C > 0$ such that*

$$\|\boldsymbol{n} \cdot \mathbf{v}_h\|^2_{0,\Gamma_1} + \|\boldsymbol{n} \wedge \mathbf{v}_h\|^2_{0,\Gamma_2} \leqq Ch\|\mathbf{v}_h\|^2_{0,\Omega}$$

for all $\mathbf{v}_h \in V_h$ and for all sufficiently small $h > 0$.

Proof. Let $\mathbf{v}_h \in V_h$ be given, let $\mathbf{P}_i$ and $\mathbf{P}_{i+1}$ be two neighbouring nodes of $\bar{\Gamma}_2$ and let Γ_2^i be an arc (of the class $\mathscr{C}^{(2)}$) between them. As $(\boldsymbol{n} \wedge \mathbf{v}_h)(\mathbf{P}_i) = 0$ and $(\boldsymbol{n} \wedge \mathbf{v}_h)(\mathbf{P}_{i+1}) = 0$, we have (see [17], p. 23)

$$|(\boldsymbol{n} \wedge \mathbf{v}_h)(\mathbf{x})| \leqq C_1 h^2(|\mathbf{v}_h(\mathbf{x})| + |\nabla \mathbf{v}_h(\mathbf{x})|) \tag{4.7}$$

for all $\mathbf{x} \in \Gamma_2^i$, where the constant $C_1 > 0$ depends only on $\partial\Omega$, and $\nabla \mathbf{v}_h$ denotes the matrix of the first partial derivatives of $\mathbf{v}_h$, and $|\cdot|$ is the Euclidean norm.

Similarly, on every arc Γ_1^i between two neighbouring nodal points of $\bar{\Gamma}_1$ we have

$$|(\boldsymbol{n} \cdot \mathbf{v}_h)(\mathbf{x})| \leqq C_2 h^2(|\mathbf{v}_h(\mathbf{x})| + |\nabla \mathbf{v}_h(\mathbf{x})|) \tag{4.8}$$

for all $\mathbf{x} \in \Gamma_1^i$.

Now by (4.7), (4.8) and by the trace theorem

$$\|\boldsymbol{n} \cdot \mathbf{v}_h\|^2_{0,\Gamma_1} + \|\boldsymbol{n} \wedge \mathbf{v}_h\|^2_{0,\Gamma_2} \leqq \tag{4.9}$$

$$\leqq C_3 h^4 \int_{\partial\Omega} (|\mathbf{v}_h|^2 + |\nabla \mathbf{v}_h|^2)\, ds \leqq C_4 h^4 \|\mathbf{v}_h\|^2_{1,\Omega} + C_5 h^3 \|\nabla \mathbf{v}_h\|^2_{0,\Omega_h{}^0},$$

where $\Omega_h^0 = \bigcup\{K \in \mathscr{T}_h | K \cap \partial\Omega \neq \emptyset\}$. For estimating the term $\int_{\partial\Omega} |\nabla \mathbf{v}_h|^2\, ds$ in (4.9), the linearity of $\mathbf{v}_h|_K$, $K \in \mathscr{T}_h$, and the fact that $\text{meas}\,(K \cap \partial\Omega)/\text{meas}\, K = \mathcal{O}(h^{-1})$ was utilized.

The inverse property (see [2], p. 142) for the finite elements says

$$\|\mathbf{v}_h\|_{1,\Omega} \leqq Ch^{-1}\|\mathbf{v}_h\|_{0,\Omega} \quad \text{for all} \quad v_h \in V_h\,. \tag{4.10}$$

Finally, a combination of (4.9) and (4.10) gives the assertion of the lemma. ☐

Now, the proof of Theorem 4.3 immediately follows from Lemmas 4.4 and 4.5.

Finally, we shall consider the rate of convergence of the discrete solutions.

Theorem 4.6. *For $\mathbf{u} \in (H^{1+\varepsilon}(\Omega))^2$, $0 < \varepsilon \leqq 1$, the difference $\mathbf{u} - \mathbf{u}_h$ fulfils the inequality*

$$|||\mathbf{u} - \mathbf{u}_h||| \leqq Ch^{\varepsilon}\|\mathbf{u}\|_{1+\varepsilon}\,.$$

Proof. Let $\Phi \in L^2(\Omega)$ ($(\Phi, 1)_0 = 0$ in the case $\Gamma_2 = \emptyset$) be arbitrary and let $\psi \in H^1(\Omega, \Gamma_2)$ be a weak solution of the problem:

$$\begin{aligned} \text{div grad}\, \psi &= \Phi \quad \text{in} \quad \Omega\,, \\ \partial_n \psi &= 0 \quad \text{on} \quad \Gamma_1\,, \\ \psi &= 0 \quad \text{on} \quad \Gamma_2\,. \end{aligned}$$

Putting $\mathbf{v} = \operatorname{grad} \psi$, we get by (3.3) that (note that rot grad $\equiv 0$)

$$(\operatorname{div} \mathbf{u}, \Phi)_0 = (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v})_0 = a(\mathbf{u}, \mathbf{v}) = b(\mathbf{v}) = (f, \operatorname{div} \mathbf{v})_0 = (f, \Phi)_0 ,$$

i.e. $\operatorname{div} \mathbf{u} = f$ in $L^2(\Omega)$. Analogously, we find that (3.3) implies $\operatorname{rot} \mathbf{u} = g$ in $L^2(\Omega)$. Consequently, we have $a(\mathbf{u}, \mathbf{v}) = b(\mathbf{v})$ for all $\mathbf{v} \in (H^1(\Omega))^2$ and in particular,

$$a(\mathbf{u}, \mathbf{v}_h) = b(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h .$$

Applying the second Strang lemma ([2], p. 210), we obtain

$$|||\mathbf{u} - \mathbf{u}_h||| \leqq C_1 \inf_{\mathbf{v}_h \in V_h} |||\mathbf{u} - \mathbf{v}_h||| \leqq C_1 \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_1 \leqq C_2 h^{\varepsilon} \|\mathbf{u}\|_{1+\varepsilon} ,$$

where C_1, C_2 do not depend on h and where the last estimate follows from the well-known results for finite elements by the interpolation properties of Sobolev spaces (see e.g. [1], p. 10). □

Remark 4.7. If $V \cap (C^\infty(\bar{\Omega}))^2$ is dense in V with respect to the $|||\cdot|||$-norm, using a standard technique we get

$$(4.11) \qquad |||\mathbf{u} - \mathbf{u}_h||| \to 0 \quad \text{for} \quad h \to 0 .$$

In [8] it is proved that $V \cap (H^1(\Omega))^2$ contains a dense subset (in the H^1-topology) of infinitely differentiable functions for $\Gamma_1 = \emptyset$. Thus in this case, (4.11) holds for $\mathbf{u} \in (H^1(\Omega))^2$. Let us still note that under some regularity assumptions it can be proved (see [8]) for $\Gamma_1 = \emptyset$ that $\|\mathbf{u} - \mathbf{u}_h\|_0 = \mathcal{O}(h^2)$.

5. NUMERICAL TESTS

The method given above was tested in different geometries. In the following, three test examples are presented. The authors are indebted to Mr. M. Könkkölä fot his help in carrying out the computions connected with these examples.

Example 5.1. Let $\Omega = (0, 1) \times (0, 1)$ and $\Gamma_1 = \{\mathbf{x} \in \mathbb{R}^2 \mid 0 < x_1 < 1, x_2 = 0\}$, $\Gamma_2 = \partial\Omega \setminus \bar{\Gamma}_1$. For $f(\mathbf{x}) = -\frac{5}{4}\pi^2 \sin \pi x_1 \cos (\pi x_2/2)$, $g(\mathbf{x}) = 1$ the weak solution of the system (1.1)–(1.2) is $\mathbf{u}(\mathbf{x}) = \pi(\cos \pi x_1 \cos (\pi x_2/2) + 1 - x_2, -\frac{1}{2} \sin \pi x_1 \,.$ $. \sin (\pi x_2/2))$. The values of the error $\mathbf{u} - \mathbf{u}_h$ in various norms are shown in Table 5.1.

Table 5.1 Errors in Example 5.1.

h	$\|\mathbf{u} - \mathbf{u}_h\|_0$	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|$	$\|\mathbf{u} - \mathbf{u}_h\|_1$
1/2	·7982582	5·2197511	7·1782787
1/4	·1616543	2·7678407	3·7882521
1/8	·0386592	1·4152033	1·9233271
1/16	·0096729	·7070499	·9618257

From Table 5.1 it can be seen that in the $\|\cdot\|_0$-norm the convergence is quadratic whereas in the $|||\cdot|||$-norm or $\|\cdot\|_1$-norm it is linear.

Example 5.2. Let $\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1,\ x_2 > 0\}$, $\Gamma_1 = \{\mathbf{x} \in \mathbb{R}^2 \mid -1 < x_1 < 1,\ x_2 = 0\}$, $\Gamma_2 = \partial\Omega \setminus \bar{\Gamma}_1$. For $f(\mathbf{x}) = 8x_1$, $g(\mathbf{x}) = 0$, the weak solution of the system (1.1)–(1.2) is

$$\boldsymbol{u}(\mathbf{x}) = (3x_1^2 + x_2^2 - 1,\ 2x_1x_2)\,.$$

Table 5.2. Errors in Example 5.2.

h	$\|u - u_h\|_0$	$|||u - u_h|||$	$\|u - u_h\|_1$
1/2	·3116610	2·7396820	3·7690384
1/4	·0897454	1·5784481	2·0048463
1/8	·0232356	·8303512	1·0189380
1/16	·0058458	·4234565	·5121525

In Table 5.2 we find that the results are analogous to the first test example. Due to (3.20), the norms $|||\cdot|||$ and $\|\cdot\|_1$ are equivalent in both the test examples.

Example 5.3. Consider the problem

$$\Delta p = f \quad \text{in} \quad \Omega\,, \qquad p = 0 \quad \text{on} \quad \partial\Omega\,, \tag{5.1}$$

where $\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\} \setminus \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 \geqq 0,\ x_2 \leqq 0\}$,

$$f(x_1, x_2) = \bar{f}(r, \varphi) = \tfrac{1}{9}(77r - 32)\sin\tfrac{2}{3}\varphi \in L^2(\Omega)\,, \tag{5.2}$$

and where (r, φ) are the usual polar coordinates. The weak solution $p \in H^1(\Omega)$ of (5.1) is of the form

$$p(x_1, x_2) = \bar{p}(r, \varphi) = (r^3 - r^2)\sin\tfrac{2}{3}\varphi\,.$$

It is easy to verify that $\boldsymbol{u} = \operatorname{grad} p \in V$,

$$\boldsymbol{u}(x_1, x_2) = \bar{\boldsymbol{u}}(r, \varphi) = \Big(\tfrac{2}{3}(r - r^2)\sin\frac{\varphi}{3} + \frac{r}{3}(7r - 4)\sin\tfrac{2}{3}\varphi\cos\varphi\,,$$

$$\tfrac{2}{3}(r^2 - r)\cos\frac{\varphi}{3} + \frac{r}{3}(7r - 4)\sin\tfrac{2}{3}\varphi\sin\varphi\Big)\,,$$

is the solution of the system (1.1)–(1.2) for $\Gamma_2 = \partial\Omega$, $g = 0$ and for f given by (5.2).

In the calculation of the approximation for solution $\boldsymbol{u}$ of (1.1)–(1.2), the initial triangulation of Ω with $h = 1/2$ containing 12 elements was chosen. Refinements were carried out three times. The finite element mesh of Ω for $h = 1/8$ and the corresponding solution $\boldsymbol{u}_h$ can be seen in Figure 5.3.

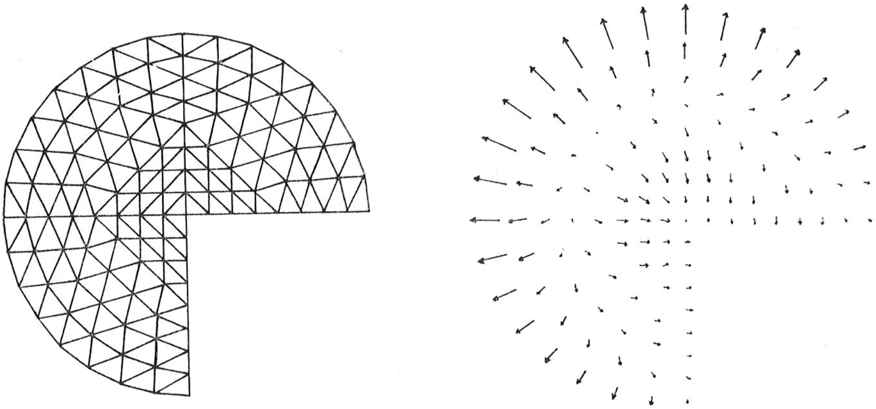

Fig. 5.3. Finite element mesh of Ω and the corresponding solution u_h for $h = 1/8$.

The values of errors in different norms are listed in Table 5.4.

Table 5.4. Errors in Example 5.3.

h	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0$	$\|\|\|\boldsymbol{u} - \boldsymbol{u}_h\|\|\|$	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1$
1/2	·1472717	1·961094	3·054758
1/4	·0319256	1·142105	1·775036
1/8	·0097791	·622665	·962657
1/16	·0036453	·330447	·510507

The above results confirm the theoretical accuracy.

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Souhrn

APROXIMACE KONEČNÝMI PRVKY DIV-ROT SYSTÉMU S KOMBINOVANÝMI OKRAJOVÝMI PODMÍNKAMI NA NEHLADKÝCH OBLASTECH

MICHAL KŘÍŽEK, PEKKA NEITTAANMÄKI

Na rovinných omezených oblastech s po částech hladkou hranicí je vyšetřována metoda konečných prvků pro řešení div-rot systému (1.1) s kombinovanými okrajovými podmínkami (1.2). Je dokázána jednoznačná řešitelnost variační úlohy (1.1) až (1.2) i její diskrétní aproximace opírající se o lineární prvky. Dále jsou odvozeny aproximační vlastnosti této metody, které jsou ilustrovány třemi testovacími příklady.

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