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ON THE VALIDITY OF FRIEDRICHS' INEQUALITIES

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Abstract.

A standard proof of Friedrich's second inequality is based on contradiction argumentation. In this paper a direct proof is presented. Moreover, necessary and sufficient conditions for the validity of Friedrichs' first and second inequality are given for plane domains.

1. Introduction.

In the Sobolov space solution theory of certain partial differential equations the following three types of inequalities play an essential role: in the potential theory the Poincaré inequality

$$\|\varphi\|_{1,\Omega} \leqq C_1 \|\operatorname{grad} \varphi\|_{0,\Omega} \quad \text{for all } \varphi \in H_0^1(\Omega) \tag{1.1}$$

(or its variants in $H^1(\Omega)$; often also called Poincaré–Friedrichs' inequalities, [12]), in the theory of elasticity Korn's inequality

$$\|\varphi\|_{1,\Omega} \leqq C_2 \sum_{i,j=1}^{d} \|\partial_i \varphi_j + \partial_j \varphi_i\|_{0,\Omega} \tag{1.2}$$

for all $\varphi \in (H_0^1(\Omega))^d$, $d=2,3$, (or its variant in $(H^1(\Omega))^d$; the so-called Korn's second inequality [4], [13]) and in electromagnetism (magnetohydrodynamics) the Friedrichs first inequality

$$\|u\|_{1,\Omega} \leqq C_3(\|\operatorname{div} u\|_{0,\Omega} + \|\operatorname{rot} u\|_{0,\Omega} + \|u\|_{0,\Omega}) \tag{1.3}$$

for all $u \in H_0(\operatorname{div}; \Omega) \cap H(\operatorname{rot}; \Omega)$ or $u \in H(\operatorname{div}; \Omega) \cap H_0(\operatorname{rot}; \Omega)$, and the Friedrichs second inequality

$$\|u\|_{1,\Omega} \leqq C_4(\|\operatorname{div} u\|_{0,\Omega} + \|\operatorname{rot} u\|_{0,\Omega}) \tag{1.4}$$

for all $u \in H_0(\operatorname{div}; \Omega) \cap H(\operatorname{rot}; \Omega)$ or $u \in H(\operatorname{div}; \Omega) \cap H_0(\operatorname{rot}; \Omega)$ (see [5], [10], [18], [20]).

The validity of Poincaré's inequalities for different types of domains is well studied (see [1], [12] for bounded and [14] for unbounded domains). This is roughly speaking true as well for Korn's inequalities (see [4], [13] for bounded and [15] for unbounded domains). In recent years relatively much attention has been given to the proof of Friedrichs' inequalities. In [3], [10], and [18], (1.3) has been proved for bounded smooth domains and in [20] a proof of (1.3) and (1.4) is given to bounded convex domains of $\mathbf{R}^d$, $d=2,3$. These considerations conclude also inhomogenities in div-term corresponding to the anisotropic media.

The typical method for the proof of inequality (1.4) is based on the use of contradiction argumentation together with inequality (1.3) and with suitable imbedding results (see [16], [17], [20]). On the other hand, in the case of smooth domains of the plane, the inequality (1.4) can be obtained from general results for elliptic systems.

In this work necessary and sufficient conditions for the validity of Friedrichs' inequalities (1.3) and (1.4) are given. We shall prove that (1.3) holds, if and only if the variational (weak) solution v of the Poisson equation

$$-\Delta v = f \quad \text{in } \Omega \tag{1.5}$$

has $H^2(\Omega)$-regularity with Dirichlet and simultaneously with Neumann homogeneous boundary condition (see Theorem 4.1). If, furthermore, Ω is simply connected, the validity of (1.4) is also equivalent to $H^2(\Omega)$-regularity of the above boundary value problem for (1.5) (see Theorem 4.3). In this work we restrict to the two dimensional case, where the sharpest results of the regularity for v in (1.5) can be achieved (see [7], [8], [11], [22], and [23]). This $H^2(\Omega)$-regularity is true e.g. for all bounded Lipschitz domains (not necessarily simply connected) with piecewise continuously differentiable boundary, which have no "concave angles". In the planar case also certain equivalent imbedding results can be proved (see Theorem 3.2). The main advantage of our way in the proof of Theorem 4.3 is that we have concrete information on the constant C_4 in inequality (1.4). In contradiction argumentation this type of information vanishes.

2. Preliminaries.

Let $\Omega \subset \mathbf{R}^2$ be a nonempty bounded domain with a Lipschitz boundary $\partial\Omega$ (see [13]). The outward unit normal to $\partial\Omega$ is always denoted by $n=(n_1, n_2)$.

Notations $H^k(\Omega)$ ($k \geq 0$, integer) are used for classical Sobolev spaces of functions, the generalized derivatives of which up to the order k are square integrable in Ω; especially let $L^2(\Omega) := H^0(\Omega)$. We write $(H^k(\Omega))^2 = H^k(\Omega) \times H^k(\Omega)$. The norm (scalar product) of $H^k(\Omega)$ and of $(H^k(\Omega))^2$ will be denoted by $\|\cdot\|_{k,\Omega}((\cdot,\cdot)_{k,\Omega})$. When no confusion will arise, we write $\|\cdot\|_k$ $((\cdot,\cdot)_k)$ instead of $\|\cdot\|_{k,\Omega}$ $((\cdot,\cdot)_{k,\Omega})$. The subspace $H_0^1(\Omega)$ of $H^1(\Omega)$ characterizes the homogeneous Dirichlet boundary condition $u=0$ on $\partial\Omega$. Furthermore, $H^s(\partial\Omega)$

($s \in \mathbf{R}$) denotes the Sobolev space ("trace space"), for definition see [12, chapter 2]. By $C_0^\infty(\Omega)$ we mean the set of all infinite times continuously differentiable functions, which vanish in some neighbourhood of $\partial\Omega$.

The following standard notation will be used:

$$\begin{cases} \operatorname{grad} v = (\partial_1 v, \partial_2 v) & \text{for } v \in H^1(\Omega)\,, \\ \operatorname{curl} v = (\partial_2 v, -\partial_1 v) & \text{for } v \in H^1(\Omega)\,, \end{cases}$$

where $\partial_i \varphi = \partial\varphi/\partial x_i$. Let

$$\text{(2.1)} \qquad \begin{aligned} H(\text{div}; \Omega) := \{q \in (L^2(\Omega))^2 \mid \exists f \in L^2(\Omega) :& \\ (q, \operatorname{grad} v)_0 = -(f, v)_0 \,\forall v \in C_0^\infty(\Omega)\}& \\ = \{q \in (L^2(\Omega))^2 \mid \operatorname{div} q \in L^2(\Omega)\}\,,& \end{aligned}$$

$$\text{(2.2)} \qquad \begin{aligned} H(\text{rot}; \Omega) := \{q \in (L^2(\Omega))^2 \mid \exists g \in L^2(\Omega) :& \\ (q, \operatorname{curl} v)_0 = (g, v)_0 \,\forall v \in C_0^\infty(\Omega)\}& \\ = \{q \in (L^2(\Omega))^2 \mid \operatorname{rot} q \in L^2(\Omega)\}\,.& \end{aligned}$$

Here the functions f and g are called divergence and rotation, respectively. For $q = (q_1, q_2) \in (H^1(\Omega))^2$, it can be seen by (2.1) and (2.2) that

$$\begin{cases} \operatorname{div} q = \partial_1 q_1 + \partial_2 q_2\,, \\ \operatorname{rot} q = \partial_1 q_2 - \partial_2 q_1\,. \end{cases}$$

Let us introduce Green's formulas (see [6])

$$\text{(2.3)} \qquad (q, \operatorname{grad} v)_0 + (\operatorname{div} q, v)_0 = \langle n \cdot q, v \rangle_{\partial\Omega} \qquad \forall q \in H(\text{div}: \Omega),\ v \in H^1(\Omega)\,,$$

$$\text{(2.4)} \qquad -(q, \operatorname{curl} v)_0 + (\operatorname{rot} q, v)_0 = \langle n \wedge q, v \rangle_{\partial\Omega} \qquad \forall q \in H(\text{rot}; \Omega),\ v \in H^1(\Omega)\,,$$

where $n \cdot q \in H^{-1/2}(\partial\Omega)$, $n \wedge q \in H^{-1/2}(\partial\Omega)$, and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ is the usual dual pairing. Let us note that for smooth $q = (q_1, q_2)$ it is $n \cdot q = n_1 q_1 + n_2 q_2$ and $n \wedge q = q_2 n_1 - q_1 n_2$ on $\partial\Omega$.

Now, let us introduce the spaces (see also [6], [9], [21])

$$\text{(2.5)} \qquad \begin{aligned} H_0(div; \Omega) := \{q \in H(\text{div}; \Omega) \mid (q, \operatorname{grad} v)_0 + (\operatorname{div} q, v)_0 = 0 \quad \forall v \in H^1(\Omega)\}& \\ = \{q \in H(\text{div}; \Omega) \mid n \cdot q = 0 \text{ on } \partial\Omega\}\,,& \end{aligned}$$

$$\text{(2.6)} \qquad \begin{aligned} H_0(\text{rot}; \Omega) := \{q \in H(\text{rot}; \Omega) \mid (q, \operatorname{curl} v)_0 = (\operatorname{rot} q, v)_0 \quad \forall v \in H^1(\Omega)\}& \\ = \{q \in H(\text{rot}; \Omega) \mid n \wedge q = 0 \text{ on } \partial\Omega\}\,.& \end{aligned}$$

The divergence-free and rotation-free vector spaces are defined by

$$\text{(2.7)} \qquad H(\text{div}^0; \Omega) := \{q \in H(\text{div}; \Omega) \mid \operatorname{div} q = 0\}$$

and

(2.8) $$H(\mathrm{rot}^0;\, \Omega) := \{q \in H(\mathrm{rot};\, \Omega) \mid \mathrm{rot}\, q = 0\}\,,$$

respectively.

3. Some imbedding theorems.

Obviously, it holds

$$H(\mathrm{div};\, \Omega) \cap H(\mathrm{rot};\, \Omega) \supset (H^1(\Omega))^2\,.$$

Let us note that the equality in this relation does not hold for any non-empty bounded Lipschitz domain Ω. To see this, it is sufficient to consider a harmonic function h in Ω with the trace $h/\partial\Omega \in H^{1/2}(\partial\Omega) \setminus H^{3/2}(\partial\Omega)$. Then evidently $\mathrm{grad}\, h \in H(\mathrm{div};\, \Omega) \cap H(\mathrm{rot};\, \Omega)$, while $\mathrm{grad}\, h$ cannot be in $(H^1(\Omega))^2$, due to the choice of the trace of h. Thus we shall add further some conditions on traces.

First of all, we give the necessary and sufficient conditions for the following algebraic imbedding

(3.1) $$H(\mathrm{div};\, \Omega) \cap H_0(\mathrm{rot}\colon \Omega) \subset (H^1(\Omega))^2\,,$$

which slightly generalizes the recent result of [20, Theorem 4.2].

Define the subsets $\mathscr{D}$ and $\mathscr{N}$ of the set of all bounded Lipschitz domains in R^2 as follows:

Definition 3.1. $\Omega \in \mathscr{D}$ if the *Dirichlet problem*

(3.2) $$\begin{cases} -\Delta z = f & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega \end{cases}$$

has the variational solution in the space $H^2(\Omega)$ for every $f \in L^2(\Omega)$ and

(3.3) $$\|z\|_2 \leqq C_1 \|f\|_0\,,$$

where $C_1 > 0$ depends only on Ω.

$\Omega \in \mathscr{N}$ if the *Neumann problem*

(3.4) $$\begin{cases} -\Delta z = f & \text{in } \Omega \\ \partial_n z = 0 & \text{on } \partial\Omega \end{cases}$$

($\partial_n z = n \cdot \mathrm{grad}\, z$) has the variational solution in the space $H^2(\Omega) \cap L_0^2(\Omega)$ for every $f \in L_0^2(\Omega) := \{s \in L^2(\Omega) \mid (s, 1)_{0,\Omega} = 0\}$ and

(3.5) $$\|z\|_2 \leqq C_2 \|f\|_0\,,$$

where $C_2 > 0$ depends only on Ω.

We note that (3.3) and (3.5) are in fact consequences of the assumptions $z \in H_0^1(\Omega) \cap H^2(\Omega)$ for all $f \in L^2(\Omega)$ and $z \in \{s \in H^2(\Omega) \mid \partial_n s|_{\partial\Omega} = 0\} \cap L_0^2(\Omega)$ for all $f \in L_0^2(\Omega)$, respectively (cf. the closed graph theorem).

THEOREM 3.2. *Let $\Omega \subset \mathrm{R}^2$ be a bounded Lipschitz domain. Then the following three assertions are equivalent:*

(i) $\Omega \in \mathscr{D} \cap \mathscr{N}$,
(ii) $H_0(\mathrm{div}; \Omega) \cap H(\mathrm{rot}; \Omega) \subset (H^1(\Omega))^2$,
(iii) $H(\mathrm{div}; \Omega) \cap H_0(\mathrm{rot}; \Omega) \subset (H^1(\Omega))^2$.

PROOF. 1° (i) → (ii). Let $\Omega \in \mathscr{D} \cap \mathscr{N}$ and let $u \in H_0(\mathrm{div}; \Omega) \cap H(\mathrm{rot}; \Omega)$. By [6, Theorem I.3.3], there exist ψ, $\varphi \in H^1(\Omega)$ such that

$$u = \mathrm{grad}\,\psi + \mathrm{curl}\,\varphi \quad \text{in } (L^2(\Omega))^2, \tag{3.6}$$

$$n \cdot \mathrm{grad}\,\psi = 0 \quad \text{in } H^{-1/2}(\partial\Omega)\,. \tag{3.7}$$

The (stream) function φ in (3.6) is unique apart from a constant.

Taking into account that $\mathrm{div}\,\mathrm{curl} \equiv 0$ and that $\mathrm{rot}\,\mathrm{grad} \equiv 0$ we have by (3.6) that

$$\mathrm{div}\,u = \mathrm{div}\,\mathrm{grad}\,\psi + \mathrm{div}\,\mathrm{curl}\,\varphi = \Delta\psi\,, \tag{3.8}$$

$$\mathrm{rot}\,u = \mathrm{rot}\,\mathrm{grad}\,\psi + \mathrm{rot}\,\mathrm{curl}\,\varphi = -\Delta\varphi\,. \tag{3.9}$$

As $n \cdot u = 0$ on $\partial\Omega$, it holds by the Green formula (2.3) that

$$(\mathrm{div}\,u, 1)_0 = 0\,,$$

that is, $\mathrm{div}\,u \in L_0^2(\Omega)$. Since $\Omega \in \mathscr{N}$, we obtain by (3.7) and (3.8) that $\psi \in H^2(\Omega)$.

Next, as $n \cdot u = 0$, by (3.6) and (3.7) we also have

$$n \cdot \mathrm{curl}\,\varphi = 0 \quad \text{in } H^{-1/2}(\partial\Omega)\,.$$

According to [6, Remark I.3.1], the function φ is constant on any component of the boundary $\partial\Omega$. Thus there evidently exists a function $\theta \in C^\infty(\bar{\Omega})$ such that

$$\theta|_{\partial\Omega} = \varphi|_{\partial\Omega}.$$

Putting

$$\chi = \varphi - \theta\,, \tag{3.10}$$

we see that

$$\chi = 0 \quad \text{on } \partial\Omega\,. \tag{3.11}$$

Now, (3.9) and (3.10) imply that

$$(3.12) \qquad -\Delta\chi = \operatorname{rot} u + \Delta\theta \in L^2(\Omega) .$$

As $\Omega \in \mathscr{D}$, (3.11) and (3.12) yield $\chi \in H^2(\Omega)$. So, finally, we have

$$(3.13) \qquad u = \operatorname{grad} \psi + \operatorname{curl} (\chi + \theta) ,$$

where $\psi, \chi, \theta \in H^2(\Omega)$. Thus, $u \in (H^1(\Omega))^2$.

2° (ii) → (iii). If $u = (u_1, u_2) \in H(\operatorname{div}; \Omega) \cap H_0(\operatorname{rot}; \Omega)$, then by (2.1), (2.2), (2.5), and (2.6) we have $u^* := (u_2, -u_1) \in H(\operatorname{rot}; \Omega) \cap H_0(\operatorname{div}; \Omega)$. According to (ii), we get $u^* \in (H^1(\Omega))^2$, that is, $u \in (H^1(\Omega))^2$.

3° (iii) → (i). Let $\Omega \notin \mathscr{D}$. Then there exists $f \in L^2(\Omega)$ such that the Dirichlet problem (3.2) has the variational solution $z \in H_0^1(\Omega) \setminus H^2(\Omega)$. Putting $u = \operatorname{grad} z$, we see that $u \in H(\operatorname{div}; \Omega)$, since

$$\operatorname{div} u = \operatorname{div} \operatorname{grad} z = -f \in L^2(\Omega) .$$

As $\operatorname{grad} H_0^1(\Omega) \subset H_0(\operatorname{rot}; \Omega) \cap H(\operatorname{rot}^0; \Omega)$, it also holds that $u \in H_0(\operatorname{rot}; \Omega)$. On the other hand, by the assumption $\Omega \notin \mathscr{D}$, we have $u \notin (H^1(\Omega))^2$; i.e. (iii) cannot hold.

Let $\Omega \notin \mathscr{N}$. Then there exists $f \in L_0^2(\Omega)$ such that the Neumann problem (3.4) has the variational solution $z \in L_0^2(\Omega)$ in $H^1(\Omega) \setminus H^2(\Omega)$. Putting $u = \operatorname{curl} z$, we see that $u \in H(\operatorname{rot}; \Omega)$, since

$$\operatorname{rot} u = \operatorname{rot} \operatorname{curl} z = -\Delta z = f \in L_0^2(\Omega) .$$

As $\operatorname{curl} H^1(\Omega) \subset H(\operatorname{div}^0; \Omega)$, it also holds that $u \in H(\operatorname{div}; \Omega)$. Moreover, we see that

$$0 = \partial_n z = n \cdot \operatorname{grad} z = -n \wedge \operatorname{curl} z = -n \wedge u ,$$

that is $u \in H(\operatorname{div}; \Omega) \cap H_0(\operatorname{rot}; \Omega)$. On the other hand, by the assumption $\Omega \notin \mathscr{N}$, we get $u \notin (H^1(\Omega))^2$. This is in contradiction to (iii).

As a consequence of the above theorem we obtain the following characterizations.

Corollary 3.3. *For $\Omega \in \mathscr{D} \cap \mathscr{N}$ it is*

$$H_0(\operatorname{div}; \Omega) \cap H(\operatorname{rot}; \Omega) = \{u \in (H^1(\Omega))^2 \mid n \cdot u = 0 \text{ on } \partial\Omega\} ,$$

$$H(\operatorname{div}; \Omega) \cap H_0(\operatorname{rot}; \Omega) = \{u \in (H^1(\Omega))^2 \mid n \wedge u = 0 \text{ on } \partial\Omega\} .$$

Corollary 3.4. *For any bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ it is*

$$H_0(\operatorname{div}; \Omega) \cap H_0(\operatorname{rot}; \Omega) = (H_0^1(\Omega))^2 .$$

Proof. Let $u \in H_0(\text{div}; \Omega) \cap H_0(\text{rot}; \Omega)$ and let B be a sufficiently large open ball such that $\bar{\Omega} \subset B$. Let $\bar{u}$ be the extension of u in B defined by $\bar{u}(x)=0$ for $x \in B \setminus \Omega$. With regard to (2.5) and (2.6), it is easy to show that $\bar{u} \in H_0(\text{div}; B)$ and $\bar{u} \in H_0(\text{rot}; B)$. Since $B \in \mathscr{D} \cap \mathscr{N}$, we get by Theorem 3.2 that $\bar{u} \in (H^1(B))^2$, that is $u \in (H^1(\Omega))^2$. As now $n \cdot u = 0$ and $n \wedge u = 0$ in $L^2(\partial\Omega)$, we obtain $u \in (H_0^1(\Omega))^2$. The converse inclusion is trivial.

4. Proof of Friedrichs' inequalities.

By using the results of the previous section we are now able to prove Friedrichs' inequalities. We first give necessary and sufficient conditions for the validity of Friedrichs' first inequality.

Theorem 4.1. *Let $\Omega \subset \mathrm{R}^2$ be a bounded Lipschitz domain. Then there exists a constant $C>0$ such that*

$$\|u\|_1 \leqq C(\|\text{div}\, u\|_0 + \|\text{rot}\, u\|_0 + \|u\|_0) \tag{4.1}$$

for all $u \in H_0(\text{div}; \Omega) \cap H(\text{rot}; \Omega)$ if and only if $\Omega \in \mathscr{D} \cap \mathscr{N}$.

Proof. 1° (necessity). If $\Omega \notin \mathscr{D} \cap \mathscr{N}$, then by Theorem 3.2 there exists $u \in H_0(\text{div}; \Omega) \cap H(\text{rot}; \Omega)$ such that $u \notin (H^1(\Omega))^2$; i.e. (4.1) cannot hold.

2° (sufficiency). Let $\Omega \in \mathscr{D} \cap \mathscr{N}$ and let $\partial\Omega_1, \ldots, \partial\Omega_k$ be the components of the boundary $\partial\Omega$. Let $\theta_i \in C^\infty(\bar{\Omega})$, $i=1,\ldots,k$, be arbitrary functions such that

$$\theta_i|_{\partial\Omega_j} = \delta_{ij}, \qquad i,j=1,\ldots,k\,.$$

The linear hull of $\theta_1, \ldots, \theta_k$ is denoted by M. If $u \in H_0(\text{div}; \Omega) \cap H(\text{rot}; \Omega)$, we know by Theorem 3.2 that $u \in (H^1(\Omega))^2$. Hence by using (3.13), (3.8), (3.12), (3.3), and (3.5), we obtain

$$\begin{aligned} \|u\|_1 &\leqq \|\text{grad}\, \psi\|_1 + \|\text{curl}\, \chi\|_1 + \|\text{curl}\, \theta\|_1 \\ &\leqq C(\|\text{div}\, u\|_0 + \|\text{rot}\, u\|_0 + \|\Delta\theta\|_0) + \|\text{curl}\, \theta\|_1 \\ &\leqq C(\|\text{div}\, u\|_0 + \|\text{rot}\, u\|_0) + (1+C)\|\text{curl}\, \theta\|_1\,, \end{aligned} \tag{4.2}$$

where $C=\max\{C_1, C_2\}$ and where θ can be chosen in the finite-dimensional space M. Since all norms in a finite-dimensional space (curl M) are equivalent, it holds

$$\|\text{curl}\, \theta\|_1 \leqq C'\|\text{curl}\, \theta\|_0\,, \tag{4.3}$$

where the constant $C'>0$ does not depend upon $\theta \in M$.

By applying (3.13) together with (3.3) and (3.5) we obtain

$$\|\operatorname{curl}\theta\|_0 \leqq \|\operatorname{grad}\psi\|_0 + \|\operatorname{curl}\chi\|_0 + \|u\|_0 \leqq C_1\|\operatorname{div} u\|_0 + C_2\|\operatorname{rot} u\|_0 + \|u\|_0 . \tag{4.4}$$

A combination of estimates (4.2), (4.3), and (4.4) yields the assertion (4.1).

COROLLARY 4.2. *Let $\Omega \subset \mathrm{R}^2$ be a bounded Lipschitz domain. Then there exists a constant $C>0$ such that*

$$\|u\|_1 \leqq C(\|\operatorname{div} u\|_0 + \|\operatorname{rot} u\|_0 + \|u\|_0) \tag{4.5}$$

holds for all $u \in H(\mathrm{div};\,\Omega) \cap H_0(\mathrm{rot};\,\Omega)$ if and only if $\Omega \in \mathscr{D} \cap \mathscr{N}$.

PROOF. Use the mapping $u=(u_1,u_2) \mapsto u^*=(u_2,-u_1)$ and Theorem 4.1.

Finally we shall consider the validity of Friedrichs' second inequality

$$\|u\|_1 \leqq C(\|\operatorname{div} u\|_0 + \|\operatorname{rot} u\|_0) , \tag{4.6}$$

where the constant $C>0$ does not depend upon u.

THEOREM 4.3. *Let $\Omega \subset \mathrm{R}^2$ be a bounded Lipschitz domain. Then the following three assertions are equivalent*

(i) *$\Omega \in \mathscr{D} \cap \mathscr{N}$ and Ω is simply connected,*
(ii) *the relation* (4.6) *holds for all $u \in H_0(\mathrm{div};\,\Omega) \cap H(\mathrm{rot};\,\Omega)$,*
(iii) *the relation* (4.6) *holds for all $u \in H(\mathrm{div};\,\Omega) \cap H_0(\mathrm{rot};\,\Omega)$.*

PROOF. **1°** (i) → (ii). Let (i) be satisfied and let $u \in H_0(\mathrm{div};\,\Omega) \cap H(\mathrm{rot};\,\Omega)$. Since Ω is simply connected, the function φ in decomposition (3.6) can be chosen in the space $H_0^1(\Omega)$. Consequently, we have in (3.10) $\theta=0$. This implies (cf. (4.2))

$$\|u\|_1 \leqq C(\|\operatorname{div} u\|_0 + \|\operatorname{rot} u\|_0) ,$$

where $C=\max\{C_1, C_2\}$.

2° (ii) → (iii). Use again the mapping $u=(u_1,u_2) \mapsto u^*=(u_2,-u_1)$ and the validity of (4.6) in $H_0(\mathrm{div};\,\Omega) \cap H(\mathrm{rot};\,\Omega)$.

3° (iii) → (i). Let $\Omega \notin \mathscr{D} \cap \mathscr{N}$. Then by Theorem 3.2 there exists $u \in H(\mathrm{div};\,\Omega) \cap H_0(\mathrm{rot};\,\Omega)$ such that $u \notin (H_1(\Omega))^2$; i.e., (4.6) cannot hold. So, let $\Omega \in \mathscr{D} \cap \mathscr{N}$ and assume that Ω is not simply connected. Then consider the problem

$$\begin{cases} \Delta z = 0 & \text{in } \Omega \\ z = 1 & \text{on } \Gamma \\ z = 0 & \text{on } \partial\Omega \setminus \Gamma \end{cases}$$

where Γ is an arbitrary fixed component of the set $\partial\Omega$. Putting $u = \operatorname{grad} z$, we find that $\operatorname{rot} u = 0$, $\operatorname{div} u = 0$, $n \wedge u = 0$, but evidently $\|u\|_1 \neq 0$, i.e. (4.6) cannot hold.

THEOREM 4.4. *Let $\Omega \subset \mathrm{R}^2$ be a simply connected bounded Lipschitz domain. Then there exists a constant $C > 0$ such that*

$$\|u\|_0 \leqq C(\|\operatorname{div} u\|_0 + \|\operatorname{rot} u\|_0) \tag{4.7}$$

for all $u \in H_0(\operatorname{div}; \Omega) \cap H(\operatorname{rot}; \Omega)$ *and for all* $u \in H(\operatorname{div}; \Omega) \cap H_0(\operatorname{rot}; \Omega)$, *respectively.*

PROOF. Let $u \in H_0(\operatorname{div}; \Omega) \cap H(\operatorname{rot}; \Omega)$. As Ω is simply connected, it holds (cf. (3.13), [6, Theorem I.3.3])

$$u = \operatorname{grad} \psi + \operatorname{curl} \chi , \tag{4.8}$$

where $\psi \in H_1(\Omega) \cap L_0^2(\Omega)$, $\chi \in H^1(\Omega)$ are weak solutions of problems $-\Delta\psi = \operatorname{div} u$, $(\partial/\partial n)\psi|_{\partial\Omega} = 0$ and $\Delta\chi = \operatorname{rot} u$, $\chi|_{\partial\Omega} = 0$. Hence

$$\|\operatorname{grad} \psi\|_0 \leqq C'\|\operatorname{div} u\|_0 \tag{4.9}$$

and

$$\|\operatorname{curl} \chi\|_0 \leqq C''\|\operatorname{rot} u\|_0 . \tag{4.10}$$

A combination of (4.8), (4.9), and (4.10) yields (4.7) for $u \in H_0(\operatorname{div}; \Omega) \cap H(\operatorname{rot}; \Omega)$.

By using again the mapping $(u_1, u_2) \mapsto (u_2, -u_1)$ the second assertion of Theorem 4.4 follows.

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