Jyväskylän yliopiston julkaisuarkisto
Jyväskylä University Digital Archive

UNIVERSITY OF JYVÄSKYLÄ

This is an electronic reprint of the original article.
This reprint *may differ* from the original in pagination and typographic detail.

Author(s): Lasiecka, I.; Sokołowski, J.; Neittaanmäki, Pekka

Title: Regularization and finite element approximation of the wave equation with Dirichlet boundary data

Year: 1990

Version:

Please cite the original version:

Lasiecka, I., Sokołowski, J. & Neittaanmäki, P. (1990). Regularization and finite element approximation of the wave equation with Dirichlet boundary data. Banach Center Publications, 24 (1), 329-354. Retrieved from https://eudml.org/doc/267771

All material supplied via JYX is protected by copyright and other intellectual property rights, and duplication or sale of all or part of any of the repository collections is not permitted, except that material may be duplicated by you for your research use or educational purposes in electronic or print form. You must obtain permission for any other use. Electronic or print copies may not be offered, whether for sale or otherwise to anyone who is not an authorised user.

REGULARIZATION AND FINITE ELEMENT APPROXIMATION OF THE WAVE EQUATION WITH DIRICHLET BOUNDARY DATA

I. LASIECKA

Applied Mathematics Department, University of Virginia, Charlottesville, USA

J. SOKOŁOWSKI

Systems Research Institute, Polish Academy of Sciences, Warsaw, Poland

P. NEITTAANMÄKI

Department of Mathematics, University of Jyväskylä, Jyväskylä, Finland

A numerical method for solving the wave equation with nonhomogenuous, nonsmooth Dirichlet boundary condition is proposed. Convergence of the method is proved and some error estimates are derived [L-S-2]. The method is based on the regularization technique [L-1], [L-S-1] of the wave equation with Dirichlet boundary data. Several numerical results are provided in two dimensional case.

1. Introduction

The paper is devoted to the approximation of the wave equation with nonhomogeneuous Dirichlet boundary conditions. We recall several theoretical results presented in [L-S-1] and [L-S-2]. We present the numerical results in the case of two-dimensional problem with respect to space variables [L-S-N].

Let Ω be an open bounded domain in $\boldsymbol{R}^n$ with a smooth boundary Γ. Consider the following second order scalar hyperbolic equation

$$
\begin{aligned}
&\ddot{u}(t, x) = \Delta u(t, x) + f(t, x) && (t, x) \in (0, T] \times \Omega \equiv Q, \\
(1.1)\qquad &u(0, x) = \dot{u}(0, x) = 0 && \text{in } \Omega, \\
&u(t, x)|_\Gamma = g(t, x) && \text{on } (0, T) \times \Gamma \equiv \Sigma.
\end{aligned}
$$

The following question may be asked: how to construct a numerical algorithm

in order to compute effectively u from the boundary data g? It is well known that the "best" numerical approximations of various p.d.e. problems are based on a certain variational formulation of the original equation. The problem in our case is, however, that due to the Dirichlet nature of the nonhomogeneous boundary condition problem (1.1) does not admit a natural variational formulation. (In contrast, a natural variational formulation is standard in the case of Neumann or Robin boundary conditions.) In this context, the idea of Lions [L-1] is to "approximate" the solution $u(t)$ of (1.1) by a sequence of functions $u_\varepsilon(t)$ which are determined as solutions of the following problems

$$
\begin{aligned}
&\ddot{u}_\varepsilon(t, x) = \Delta u_\varepsilon(t, x) + f(t, x) && \text{in } Q,\\
&u_\varepsilon(0, x) = \dot{u}_\varepsilon(0, x) = 0 && \text{in } \Omega,\\
&\varepsilon \frac{\partial u_\varepsilon}{\partial \eta} + \beta u_\varepsilon = \beta g && \text{on } \Sigma,
\end{aligned}
\tag{1.2}
$$

where β is a selfadjoint second order elliptic operator defined on the variety Γ (1).

The advantage of introducing (1.2) is — of course — that (1.2) admits a natural variational formulation:

$$
\begin{aligned}
&(\ddot{u}_\varepsilon, \phi)_\Omega + (\nabla u_\varepsilon, \nabla \phi)_\Omega + \frac{1}{\varepsilon}\langle \beta u_\varepsilon, \phi\rangle_\Gamma = \frac{1}{\varepsilon}\langle \beta g, \phi\rangle_\Gamma + (f, \phi)_\Omega \qquad \forall \phi \in C^2(\bar{\Omega}),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } t \in (0, T).\\
&u_\varepsilon(0) = \dot{u}_\varepsilon(0) = 0
\end{aligned}
\tag{1.3}
$$

In [L-1] it was shown that $u_\varepsilon(t)$ approximates $u(t)$ in the following sense: for any $g \in L_2(\Sigma)$ and $f \in L_1[0, T; L_2(\Omega)]$

$$
u_\varepsilon \to u \quad \text{in } L_\infty[0, T; L_2(\Omega)] \quad \text{weak star.}
\tag{1.4}
$$

In view of the above, one can think of (1.1) as a limit problem for (1.2). Therefore, in order to find an effective numerical approximation of (1.1), the natural idea to pursue is to look for numerical algorithms (Ritz–Galerkin, finite element, etc.) of the variational equality (1.3). However, in order to establish the convergence or even more — the rates of the convergence — of these approximations, a necessary prerequisite is to know more about the regularity properties of the solutions to (1.2) as well as their convergence to $u(t)$. Thus, in the paper [L-S-1] we study regularity (more precisely uniform differentiability) properties of the solutions $u_\varepsilon(t)$ along with the convergence of $u_\varepsilon(t)$ to $u(t)$. In particular, we prove in [L-S-1] that the convergence in (1.4) is, in fact, the strong. We also establish in [L-S-1] a number of regularity results for $u_\varepsilon(t)$, which are reminiscent of those valid for the limit solution $u(t)$. These results, besides being of interest in their own, are of fundamental importance in the

(1) This, in particular, implies that β: $H^{s+2}(\Gamma) \to H^s(\Gamma)$ is an isomorphism.

study of numerical schemes approximating (1.2). In fact, they are used crucially in [L-S-2] where finite element techniques are developed to approximate $u_\varepsilon(t)$ and hence $u(t)$.

In the paper [L-S-2] under minimal regularity assumptions imposed on the boundary term g, the finite element approximation of (1.1) is introduced and the convergence and the rates of convergence of the algorithm in $L_2(\Omega)$ norms is established. The motivation for studying approximations of second order hyperbolic equations with nonsmooth boundary data comes from problems arising in numerical considerations related to a variety of boundary control problems where the solutions are definitely nonsmooth – for example: optimization problems with boundary controls, time-optimal boundary control problem, Riccati equations arising from boundary control problems. In order to construct and to prove related convergence of numerical algorithms for these problems, a preliminary step is to establish appropriate approximation of problem (1.1) with nonsmooth boundary data g – say $g \in L_2(\Sigma)$ or $g \in H^1[0, T; H^{-1/2}(\Gamma)]$. To the authors knowledge, the literature on finite element methods for the second order hyperbolic equation with Dirichlet boundary conditions deals only with homogenous boundary data i.e. $g = 0$ in (1.1). This is not surprising, also in view of the fact that the maximal regularity of problem (1.1) with nonhomogenous boundary data has been established only recently (see [L-T-1], [L-2], [L-L-T]). As we indicated the presence of the nonhomogenous and nonsmooth Dirichlet boundary data is responsible for two immediate difficulties

(i) Dirichlet problem (1.1) does not admit a natural variational formulation which could then be taken as a basis for a numerical approximation, furthermore,

(ii) low regularity of the boundary data g (hence of the solution) rules out the usual technique for proving stability and convergence of the numerical scheme which is based on $H^1(\Omega) \times L_2(\Omega)$ energy estimates.

While the first difficulty can be handled by selecting an appropriate approximation of the elliptic operator which would take into account the nonhomogenous terms on the boundary (see, for example, [B-2], [B-7], [N-1], [S-1]), the second difficulty becomes crucial when it comes to the derivation of stability estimates for the sought after numerical algorithm. Let us elaborate more on this point. A standard finite element approximation approach in hyperbolic (as well as parabolic) case is to define a semidiscrete algorithm by taking an appropriate space – approximation of the underlined elliptic operator. The estimates on the rate of convergence – which of course depend on the smoothness of the solutions – can be obtained by taking the difference of the two solutions and by using results on elliptic approximations. It is known, however [R-1], that even if the elliptic approximations yield the optimal rates of convergence, nevertheless the rates for hyperbolic problems

are nonoptimal as they require one extra time-derivative of the solution. Since we cannot obtain optimal *convergence rates*, one would at least like to obtain *convergence* of the numerical algorithm in the "right topologies", i.e. where the maximal regularity of the map $g \to u$ takes place. To accomplish this one needs to establish stability estimates for numerical schemes in precisely the same topologies (in fact, for the homogenous boundary data, this can be done by using the mentioned earlier $H^1(\Omega) \times L_2(\Omega)$ energy methods). This issue however raises another question. What is the maximal regularity of the map $g \to u$. As we have noted, this seemingly innocent question was answered in optimal way only recently (see [L-T-1], [L-2], [L-L-T]). In the above references it was shown in particular that the map $g \to u$ is bounded from

$$L_2(\Sigma) \to C[0, T; L_2(\Omega)] \tag{1.5}$$

or, more generally,

$$H^{s,s}(\Sigma) \to H^{s,s}(Q) \cap C[0, T; H^s(\Omega)], \quad s \geqslant 0, \tag{1.6}$$

where in (1.6) we have to assume that g satisfies, for $s > \frac{1}{2}$, some appropriate compatability conditions at the origin. The results (1.5) and (1.6) improve by $\frac{1}{2}$ derivative the previous results on regularity of solutions to (1.1) given in [L-M]. Equipped with maximal regularity results [L-S-1] for the original problem, we devise in [L-S-2] the numerical algorithm which provides (i) the best possible rates of convergence (we are resigned in [L-S-2] to "loose" one derivative), (ii) stability estimates reconstructing as much as possible the regularity properties of the original solution. Since the prime interest is to consider nonsmooth boundary data, it is precisely the second point mentioned above which limits the choice of elliptic approximations in [L-S-2]. The reason for this is twofold: first the available elliptic estimates deal with more regular in space boundary data – typically $g \in H^p(\Gamma)$, $p > 3/2$ (see [B-2], [B-7], [N-1], [S-1]). Second, standard techniques of proofs based on H^1-coercitivity of the elliptic problems are not applicable as we consider boundary data which do not yield $H^1(\Omega)$ solutions. Thus the sought after elliptic approximation should allow for the treatment of nonsmooth boundary data g and moreover should be suitable to yield hyperbolic estimates in lower norms.

On the other hand, let us notice that if one takes in (1.2) $\beta \equiv I$ then the projection of (1.2) onto finite dimensional subspaces of $H^1(\Omega)$ would be a hyperbolic counterpart to the Penalty Method introduced by Babuska [B-2] for elliptic problems. However, with $\beta = I$ in (1.2), the solution $u_\varepsilon(t)$ is *not bounded* in $L_2(\Omega)$ (uniformly with respect to the parameter $\varepsilon > 0$) by $|g|_{L_2(\Sigma)}$. This shows that (1.2) with $\beta = I$ is *not a good* "approximation" of the original hyperbolic problem as it does not reconstruct the regularity properties of the original solutions. The presence of the Laplace's Beltrami operator on the boundary forces stronger convergence of the traces of u_ε which in turn is necessary to obtain the appropriate stability of the solution (see [L-S-1] [L-1]).

The outline of the paper is as follows: In Section 2 we provide some material on the properties and regularity of the continuous solution $u(t)$ as well as those of the regularized solution $u_\varepsilon(t)$. In Section 3 we discuss the regularity and convergence of the steady state solutions to (1.2). In Section 4, we define semidiscrete approximating subspaces and approximations of (1.1) and we recall some of results presented in [L-S-2]. Finally, in Section 5 the case of domains $\Omega \subset \boldsymbol{R}^2$ is discussed in details. The proofs of the results presented here are given in [L-S-1], [L-S-2], [L-S-N]. In Section 6 some numerical results are provided. The following notation will be used in the paper: $(\cdot,\cdot)$, (resp. $\|\cdot\|$) denote the usual $L_2(\Omega)$ inner product (resp. the norm in $L_2(\Omega)$). $\langle .,. \rangle$ (resp. $|\cdot|$) denote $L_2(\Gamma)$ inner product (resp. — the norm in $L_2(\Gamma)$), $H^2(\Omega)$, $H^{r,s}(Q)$ for $r, s > 0$ are the usual Sobolev spaces defined as in [L-M], if $r = s$ we shall use $H^r(Q) \equiv H^{r,r}(Q)$. $H^{-s} = (H^s)'$ $s > 0$ where X' stands for the dual space to X. $\mathscr{L}(X \to Y)$ denotes the space of linear transformations from X to Y. $L_p[0, T; X]$, $1 \leqslant p \leqslant \infty$ denotes the space of $u(t) \in X$ such that $L_p[0, T]$ norm of $\|u(t)\|_x$ is well defined; we denote $u_t = \dot{u} = \partial u/\partial t$.

2. Regularization of wave equation

Let us begin by collecting regularity results available for the original problem (1.1).

THEOREM 2.1 ([L-T-1], [L-T-2], [L-L-T], [L-2], [S-1]). *Let u be the solution to* (1.1) *with $g \in L_2(\Sigma)$ and $f = 0$. Then*

$$\|u\|_{C[0,T;L_2(\Omega)]} + \|u_t\|_{C[0,T;(H_0^1(\Omega))']} + \left|\frac{\partial u}{\partial \eta}\right|_{H^{-1,-1}(\Sigma)} \leqslant C\,|g|_{L_2(\Sigma)}. \tag{2.1}$$ [2]

If in addition we assume that $g \in H^{1,1}(\Sigma)$, $g(0) = 0$ and take $f \in L_1[0, T; L_2(\Omega)]$, then

$$\|u\|_{C[0,T;H^1(\Omega)]} + \|u_t\|_{C[0,T;L_2(\Omega)]} + \left|\frac{\partial u}{\partial \eta}\right|_{L_2(\Sigma)} \leqslant C\,[|g|_{H^{1,1}(\Sigma)} + \|f\|_{L_1[0,T;L_2(\Omega)]}]. \tag{2.2}$$

More generally, with $g \in H^{s,s}(\Sigma)$, $s \geqslant 1$ where g satisfies the appropriate compatability conditions and with $f = 0$ we have

$$\|u\|_{C[0,T;H^s(\Omega)]} + \|u_t\|_{C[0,T;H^{s-1}(\Omega)]} + \left|\frac{\partial u}{\partial \eta}\right|_{H^{s-1,s-1}(\Sigma)} \leqslant C\,|g|_{H^{s,s}(\Sigma)}. \ \blacksquare \tag{2.3}$$

Notice that the regularity of the solution on the boundary does not follow from the interior regularity. In fact, the regularity of the normal derivative of the solution on the boundary is higher than the Trace Theorem combined with interior regularity would imply.

---

[2] C will stand for a generic constant.

Next, let u_ε stands for the solution to (1.2). The following results were proved in [L-1].

THEOREM 2.2 [L-1]. *Let u* (resp. u_ε) *be the solution to* (1.1) (resp. (1.2)) *with $g = 0$ and $f \in L_1[0, T; L_2(\Omega)]$. Then*

$$\|u_\varepsilon\|_{C[0,T;H^1(\Omega)]} + \|\dot{u}_\varepsilon\|_{C[0,T;L_2(\Omega)]} + \left|\frac{\partial u_\varepsilon}{\partial \eta}\right|_{L_2(\Sigma)} + |\dot{u}_\varepsilon|_{L_2(\Sigma)} \leqslant C\|f\|_{L_1[0,T;L_2(\Omega)]}, \tag{2.4}$$ (3)

$$|u_\varepsilon|_{C[0,T;H^1(\Gamma)]} \leqslant C\sqrt{\varepsilon}\,\|f\|_{L_1[0,T;L_2(\Omega)]}, \tag{2.5}$$

$$\begin{aligned} &\text{(i)} && u_\varepsilon \to u && \text{in } L^\infty[0, T; H^1(\Omega)] \text{ weak star,} \\ &\text{(ii)} && \dot{u}_\varepsilon \to \dot{u} && \text{in } L^\infty[0, T; L_2(\Omega)] \text{ weak star,} \\ &\text{(iii)} && u_\varepsilon \to 0 && \text{in } C[0, T; H^1(\Gamma)], \\ &\text{(iv)} && \frac{\partial u_\varepsilon}{\partial \eta} \to \frac{\partial u}{\partial \eta} && \text{in } L^2(\Sigma) \text{ weakly.} \end{aligned} \tag{2.6}$$

With $g \in L_2(\Sigma)$ in (1.1) (resp. (1.2)) *and $f \in L_1[0, T; L_2(\Omega)]$ we have*

$$\|u_\varepsilon\|_{C[0,T;L_2(\Omega)]} \leqslant C[|g|_{L_2(\Sigma)} + \|f\|_{L_1[0,T;L_2(\Omega)]}], \tag{2.7}$$

$$u_\varepsilon \to u \quad \text{in } L^\infty[0, T; L_2(\Omega)] \text{ weak star,} \tag{2.8}$$

where C stands for a generic constant independent on $\varepsilon > 0$.

In the paper [L-S-1] we prove the uniform (with respect to ε) differentiability of the solutions u_ε with nonhomogeneous, smooth boundary data g and we improve the convergence results of (2.6) and (2.8).

The results are:

THEOREM 2.3 (differentiability) [L-S-1]. (i) *Let u* (resp. u_ε) *be the solution to* 1.1 (resp. 1.2) *with $g \in H^{1,1}(\Sigma)$; $g(0) = 0$ and $f \equiv 0$. Then*

$$\|u_\varepsilon\|_{C[0,T;H^1(\Omega)]} + \|\dot{u}_\varepsilon\|_{C[0,T;L_2(\Omega)]} \leqslant C|g|_{H^{1,1}(\Sigma)}, \tag{2.9}$$

$$\left|\frac{\partial u_\varepsilon}{\partial \eta}\right|_{L_2(\Sigma)} + |\dot{u}_\varepsilon|_{L_2(\Sigma)} + |u_\varepsilon|_{L_2[0,T;H^1(\Gamma)]} < C|g|_{H^{1,1}(\Sigma)}, \tag{2.10}$$

$$|u_\varepsilon - g|_{L_2[0,T;H^2(\Gamma)]} \leqslant C\varepsilon|g|_{H^{1,1}(\Sigma)}. \tag{2.11}$$

(ii) *If in addition we assume that $g \in H^{s,s}(\Sigma)$, $s \geqslant 1$ and g satisfies the appropriate compatability conditions then*

$$\|u_\varepsilon\|_{C[0,T;H^s(\Omega)]} + \|\dot{u}_\varepsilon\|_{C[0,T;H^{s-1}(\Omega)]} \leqslant C|g|_{H^{s,s}(\Sigma)}, \tag{2.12}$$

$$\left|\frac{\partial u_\varepsilon}{\partial \eta}\right|_{H^{s-1,s-1}(\Sigma)} + \|\dot{u}_\varepsilon\|_{H^{s-1,s-1}(\Sigma)} + |u_\varepsilon|_{L_2[0,T;H^2(\Gamma)]} \leqslant C|g|_{H^{s,s}(\Sigma)}, \tag{2.13}$$

$$|u_\varepsilon - g|_{L_2[0,T;H^{s+1}(\Gamma)]} \leqslant C\varepsilon|g|_{H^{s,s}(\Sigma)}. \tag{2.14}$$

(3) From now on the constant C will stand for a generic constant independent of $\varepsilon > 0$.

THEOREM 2.4 (convergence) [L-S-1]. (i) *Let* u (resp. u_ε) *be the solution to* (1.1) (resp. (1.2)) *with* $f \in L_1[0, T; L_2(\Omega)]$ *and* $g = 0$. *Then*

$$(2.15)\quad \begin{aligned} &\text{(i)} \quad |u_\varepsilon|_{L_2[0,T;H^2(\Gamma)]} \leqslant C\varepsilon \|f\|_{L_1[0,T;L_2(\Omega)]}, \\ &\text{(ii)} \quad \|u_\varepsilon - u\|_{L_\infty[0,T;L_2(\Omega)]} \leqslant C\varepsilon \|f\|_{L_1[0,T;L_2(\Omega)]}, \end{aligned}$$

$$(2.16)\quad \begin{aligned} &\text{(i)} \quad u_\varepsilon \to u \quad \text{in } C[0, T; H^1(\Omega)], \\ &\text{(ii)} \quad \dot{u}_\varepsilon \to \dot{u} \quad \text{in } C[0, T; L^1(\Omega)], \end{aligned}$$

$$(2.17)\quad \begin{aligned} &\text{(i)} \quad u_{\varepsilon|\Gamma} \to 0 \quad \text{in } L_2(\Sigma), \\ &\text{(ii)} \quad \frac{\partial u_\varepsilon}{\partial \eta} \to \frac{\partial u}{\partial \eta} \quad \text{in } L_2(\Sigma). \end{aligned}$$

(ii) *Let* u (resp. u_ε) *be the solution to* (1.1) (resp. (1.2)) *with* $f = 0$. *Then*

$$(2.18)\quad \|u_\varepsilon - u\|_{C[0,T;L_2(\Omega)]} \leqslant C\varepsilon |g|_{H^{1,1}(\Sigma)},$$

$$(2.19)\quad u_\varepsilon \to u \quad \text{in } C[0, T; L_2(\Omega)] \quad \text{for any } g \in L_2(\Sigma). \ \blacksquare$$

Notice that the regularity results of u_ε stated in Theorem 2.3 are reminiscent of those in Theorem 2.1. In fact, (2.9), (2.10) reconstruct the regularity of the original solution $u(t)$ given by (2.1) and (2.2).

3. Convergence of the steady state solutions

In order to prove our regularity and convergence results for the problem (1.2) we establish in [L-S-1] the similar results for the corresponding elliptic problem which we recall here.

Consider the following elliptic problems

$$(3.1)\quad \begin{aligned} \Delta v &= f \quad \text{in } \Omega, \\ v|_\Gamma &= 0 \quad \text{in } \Gamma, \end{aligned}$$

and

$$(3.2)\quad \begin{aligned} \Delta v_\varepsilon &= f \quad \text{in } \Omega, \\ \frac{\partial v_\varepsilon}{\partial \eta} + \beta v_\varepsilon &= 0 \quad \text{in } \Gamma. \end{aligned}$$

If we define the operator A: $L_2(\Omega) \to L_2(\Omega)$ by

$$Av \equiv \Delta v, \quad \forall v \in \mathscr{D}(A) \equiv H^2(\Omega) \cap H_0^1(\Omega),$$

then (3.1) is equivalent to

$$(3.1')\quad Av = f.$$

Similarly with A_ε: $L_2(\Omega) \to L_2(\Omega)$ defined as

$$A_\varepsilon v_\varepsilon \equiv \Delta v, \quad \forall v_\varepsilon \in \mathscr{D}(A_\varepsilon)$$

where

$$\mathscr{D}(A_\varepsilon) = \left\{u \in L_2(\Omega);\ \Delta u \in L_2(\Omega);\ \varepsilon \frac{\partial u}{\partial \eta} + \beta u = 0\right\},$$

(3.2) can be rewritten as

$$A_\varepsilon v_\varepsilon = f. \tag{3.2'}$$

Below we state a number of regularity and convergence results established in [L-S-1] for the problems (3.1) and (3.2). The proofs of these results are given in [L-S-1].

LEMMA 3.1 [L-S-1].

$$\|A_\varepsilon^{-1} f\|_{H^1(\Omega)} + \frac{1}{\sqrt{\varepsilon}} |A_\varepsilon^{-1} f|_{H^1(\Gamma)} \leqslant C \|f\|_{H^{-1}(\Omega)}, \tag{3.3}$$

$$\|A_\varepsilon^{-1} f\|_{H^2(\Omega)} + \frac{1}{\varepsilon} |A_\varepsilon^{-1} f|_{H^2(\Gamma)} \leqslant C \|f\|, \tag{3.4}$$

$$\|A_\varepsilon^{-1} f - A^{-1} f\|_{H^2(\Omega)} \leqslant C\varepsilon \|f\|_{H^{-1}(\Omega)}. \tag{3.5}$$

Notice that (3.5) implies that a posteriori

$$A_\varepsilon^{-1} f \to A^{-1} f \quad \text{in } H^1(\Omega) \quad \text{for any } f \in H^{-1}(\Omega).$$

Next, let us define the so-called Dirichlet map D: $L_2(\Gamma) \to L_2(\Omega)$

$$\begin{aligned} \Delta D g &= 0 \quad \text{in } \Omega, \\ D g|_\Gamma &= g \quad \text{in } \Gamma. \end{aligned} \tag{3.6}$$

It is well known [L-M] that

$$D \in \mathscr{L}\left(H^s(\Gamma) \to H^{s+1/2}(\Omega)\right) \quad \text{for all real } s > 0. \tag{3.7}$$

Similarly we define the map N_ε: $L_2(\Gamma) \to L_2(\Omega)$ by

$$\begin{aligned} \Delta N_\varepsilon g &= 0 \quad \text{in } \Omega, \\ \varepsilon \frac{\partial N_\varepsilon}{\partial \eta} g + \beta N_\varepsilon g &= \beta g \quad \text{in } \Gamma. \end{aligned} \tag{3.8}$$

LEMMA 3.2 [L-S-1].

$$\|N_\varepsilon g\|_{H^{3/2}(\Omega)} + \left|\frac{\partial}{\partial \eta} N_\varepsilon g\right| \leqslant C |g|_{H^1(\Gamma)}, \tag{3.9}$$

$$\|N_\varepsilon g\|_{H^{1/2}(\Omega)} + \left|\frac{\partial}{\partial \eta} N_\varepsilon g\right|_{H^{-1}(\Gamma)} \leqslant C |g|, \tag{3.10}$$

$$\|N_\varepsilon g\|_{H^2(\Omega)} \leqslant C |g|_{H^{3/2}(\Gamma)}, \tag{3.11}$$

$$\|N_\varepsilon g - Dg\|_{H^{3/2}(\Omega)} \leqslant C\varepsilon |g|, \tag{3.12}$$

$$\|N_\varepsilon g - Dg\|_{H^{5/2}(\Omega)} \leqslant C\varepsilon |g|_{H^1(\Gamma)}. \quad \blacksquare \tag{3.13}$$

Notice that the regularity properties (3.3)–(3.4), and (3.9)–(3.11), reconstruct (uniformly in the parameter $\varepsilon > 0$) the well-known regularity properties of the elliptic Dirichlet problems.

As it is pointed out in [L-S-1], the results of Lemma 3.2 can be easily generalized to obtain

$$N_\varepsilon \in \mathscr{L}\big(H^s(\Gamma) \to H^{s+1/2}(\Omega)\big) \quad \text{for all real } s > 0.$$

Using the above definitions of elliptic operators, we are in a position to represent the solution $u(t)$ in the semigroup form as in [L-T-1]:

$$u(t) = A\int_0^t S(t-z)\,Dg(z)\,dz \equiv (Lg)(t). \tag{3.14}$$

Theorem 2.1 gives

$$L \in \mathscr{L}\big(L_2(\Sigma) \to C[0, T; L_2(\Omega)]\big). \tag{3.15}$$

Similarily we represent the solution $u_\varepsilon(t)$ of (1.2) via the semigroup formula.

First, let us recall that the following identities are simple consequences of Green formula (see [L-S-1]):

$$\begin{aligned} N_\varepsilon^* A_\varepsilon u &= \frac{\partial}{\partial\eta} u, \qquad \forall u \in \mathscr{D}(A_\varepsilon), \\ N_\varepsilon^* A_\varepsilon u &= \frac{1}{\varepsilon}\beta u, \qquad \forall u \in C^\infty(\Omega). \end{aligned} \tag{3.16}$$

Since A_ε is selfadjoint and the spectrum of A_ε is on the real negative axis, A_ε generates cosine $C_\varepsilon(t)$ and sine $S_\varepsilon(t)$ operators on $L_2(\Omega)$. Therefore, following the same arguments as in [L-T-1], one can show that the solution $u_\varepsilon(t)$ of (1.2) can be written as

$$u_\varepsilon(t) = A_\varepsilon\int_0^t S_\varepsilon(t-z)\,N_\varepsilon g(z)\,dz \equiv (L_\varepsilon g)(t). \tag{3.17}$$

From Theorem 2.2 it follows that

$$L_\varepsilon \in \mathscr{L}\big(L_2(\Sigma) \to C[0, T; L_2(\Omega)]\big) \quad \text{with the norm independent of } \varepsilon > 0. \tag{3.18}$$

Considering L_ε as acting from $L_2(\Sigma)$ into $L_2(Q)$, we compute its adjoint L_ε^*: $L_2(Q) \to L_2(\Sigma)$

$$(L_\varepsilon^* f)(t) = N_\varepsilon^* A\int_t^T S_\varepsilon(z-t)\,f(z)\,dz,$$

by (3.16)

$$(L_\varepsilon^* f)(t) = \frac{\partial}{\partial\eta}\int_t^T S_\varepsilon(z-t)\,f(z)\,dz. \tag{3.19}$$

As a consequence of (3.18) we have

$$L_\varepsilon^* \in \mathscr{L}(L_1[0, T; L_2(\Omega)] \to L_2(\Sigma)) \tag{3.20}$$

with the norm independent of $\varepsilon > 0$.

The solution $u_\varepsilon(t)$ given by (3.17) (or equivalently by (1.2)) can also be represented as the solution of the following abstract ODE problem

$$\begin{aligned} &\ddot{u}_\varepsilon(t) = A_\varepsilon u_\varepsilon(t) - A_\varepsilon N_\varepsilon g(t), \quad \text{on } \mathscr{D}(A_\varepsilon)', \\ &u_\varepsilon(0) = \dot{u}_\varepsilon(0) = 0, \end{aligned} \tag{3.21}$$

(3.21) together with (3.16) and (3.18) lead to the variational formulation (1.3) of the problem (1.2) (see also [L-1])

$$(\ddot{u}_\varepsilon(t), \phi) + a(u_\varepsilon(t), \phi) + \frac{1}{\varepsilon}\langle \beta u_\varepsilon(t), \phi\rangle = \frac{1}{\varepsilon}\langle \beta g, \phi\rangle$$

$$\text{for all } \phi \in \mathscr{D}(A_\varepsilon^{1/2}) \equiv \{\phi \in H^1(\Omega),\ \phi|_\Gamma \in H^1(\Gamma)\}.$$

Let us point out that the semidiscrete scheme (4.1) introduced in next section can be obtained from (3.22) by restricting the test functions ϕ to lie in the finite dimensional subspace V_h.

4. Finite element approximation

In order to define a semidiscrete approximation of the original problem (1.1) a natural idea is to "project" the variational form of (1.3) onto the finite dimensional subspaces. To this end, let h be the parameter of discretization tending to zero and let V_h stand for the approximating space of $H^1(\Omega)$ with the usual approximation properties (to be specified later) and such that $\tilde{V}_h \equiv V_{h|\Gamma} \subset H^1(\Gamma)$. As an approximation of $u_\varepsilon(t)$ (solution to (1.3)) we take $u_{h,\varepsilon}(t) \in V_h$ such that

$$\begin{aligned} &(\ddot{u}_{h,\varepsilon}(t), \phi_h)_\Omega + a(u_{h,\varepsilon}(t), \phi_h) + \frac{1}{\varepsilon}\langle \beta u_{h,\varepsilon}(t), \phi_h\rangle_\Gamma \\ &\qquad = \frac{1}{\varepsilon}\langle P_h g, \beta\phi_h\rangle_\Gamma, \quad \phi_h \in V_h, \\ &u_{h,\varepsilon}(0) = \dot{u}_{h,\varepsilon}(0) = 0, \end{aligned} \tag{4.1}$$

where $a(u, v)$ is the bilinear form associated with the second order elliptic operator $A(x, \partial)$, i.e. in general

$$a(u, v) = \sum_{i,j=1}^{n} \left(a_{ij}(x)\frac{\partial u}{\partial x_i}, \frac{\partial v}{x_i}\right)_\Omega = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j}\,dx;$$

in particular, $a_{ij} = \delta_{ij}$ for $A(x, \partial) = -\Delta$, and P_h is the orthogonal projection

from $L_2(\Gamma)$ onto $\tilde{V}_h$. Later we shall use (4.1) with $\varepsilon = \varepsilon(h) = h^\gamma$ for some $\gamma > 0$. The corresponding solution will be denoted by u_h. Notice that the procedure described above:

(i) is well defined directly on $g \in L_2(\Sigma)$,

(ii) V_h are subspaces of $H^1(\Omega)$ which are not required to satisfy boundary conditions.

In the paper [L-S-2] the stability and the rates of convergence of the approximation $u_h(t)$ to the original solution $u(t)$ are established. In fact, the main results in the case when V_h consists of piecewise linear functions (see Corollary 4.2 and 4.1) establish in particular that with $\varepsilon = \varepsilon(h) = h$ in (4.1) we have

(4.2) (convergence)

(i) $\|u-u_h\|_{C[0,T;L_2(\Omega)]} \leqslant Ch\,[|\dot{g}|_{H^1(\Sigma)} + |g|_{L_2[0,T;H^1(\Gamma)]}]$,

(ii) $\|u-u_h\|_{C[0,T;H^{1/2-\varrho}(\Omega)]} \leqslant Ch\,[|\dot{g}|_{H^{3/2}(\Sigma)} + |g|_{L_2[0,T;H^{3/2}(\Gamma)]}]$,

where $\varrho > 0$ is arbitrary small;

(4.3) (stability)

(i) $\|u-u_h\|_{C[0,T;L_2(\Omega)]} \leqslant C\,[|\dot{g}|_{H^1[0,T;H^{-1/2+\varrho}(\Gamma)]} + |g|_{L_2(\Sigma)}]$,

(ii) $\|u-u_h\|_{C[0,T;H^{1/2-\varrho}(\Omega)]} \leqslant C\,[|\dot{g}|_{H^1[0,T;H^{\varrho}(\Gamma)]} + |g|_{H^{1/2-\varrho}(\Sigma)}]$,

where C stands for a generic constant which is independent on $h > 0$ and g. For boundary data which display more regularity properties and satisfy the appropriate compatability conditions, higher order rates of convergence are given in Corollary 4.2.

Notice that in view of the optimal results of convergence for the wave equation with homogeneous boundary conditions, where an extra derivative in the solution is necessary (see [R-1] and also [B-1], [D-1], [B-3]) and optimal regularity of the solutions to Dirichlet problems (see 1.5), the estimates (4.2) are optimal. Although the stability results in (4.3) improve "almost" by 1/2 derivative the stability estimates implied by the convergence result in (4.2), they are, however, still nonoptimal with respect to the sharp regularity of the solution u. In fact, $g \in L_2(\Sigma)$ will produce the solution $u \in C[0, T; L_2(\Omega)]$ (see [L-1], [L-T-2], [L-L-T]), thus we would expect that the stability estimate (4.3) (i) should hold for any $g \in L_2(\Sigma)$ (instead of $\dot{g} \in L_2[0, T; H^{-1/2}(\Gamma)]$).

In the analysis of the error of the approximation, a crucial role is played by the very special behavior of the traces of hyperbolic solutions. In fact, the solutions to wave equations are shown [L-L-T] to have better regularity on the boundary than interior regularity and the trace theorem would imply. This fact is used an essential way in [L-S-2] in the process of proving (4.2) and (4.3).

4.1. Approximating subspaces. Let V_h be a family of $N(h)$ finite dimensional subspaces of $H^1(\Omega)$, of the order $r \geqslant 2$ satisfying the local and inverse

assumption (see [B-1], p. 98) in addition to the following properties:

(4.4) (a) $V_{h|\Gamma} \subset H^1(\Gamma)$;

(b) $\forall u \in H^s(\Omega)$:

$$\inf_{\phi_h \in V_h} [\|u-\phi_h\| + h\,\|u-\phi_h\|_{H^1(\Omega)} + h^{1/2}\,|u-\phi_h| + h^{3/2}\,|u-\phi_h|_{H^1(\Gamma)}] \leqslant Ch^s\,\|u\|_{H^s(\Omega)};$$

(c) $\|u-P_h u\|_{H^l(\Omega)} \leqslant Ch^{s-l}\,\|u\|_{H^s(\Omega)}, \quad 2 \leqslant s \leqslant r,\ 0 \leqslant s \leqslant r,\ 0 \leqslant l < 3/2,$

where P_h stands for the orthogonal projection in $L_2(\Omega)$ (with respect to $L_2(\Omega)$ inner product) onto V_h.

(d) for any $\tilde{\phi}_h \subset V_h \equiv \tilde{V}_h|_\Gamma$ (4), there exists $\phi_h \in V_h$ such that

$$\phi_h|_\Gamma = \tilde{\phi}_h \quad \text{and} \quad \|\phi_h\|_{H^s(\Omega)} \leqslant C\,\|\tilde{\phi}_h\|_{H^{s-1/2}(\Gamma)},\ 0 < s \leqslant 1.$$

It is well known that the properties (3.1) a, b, c are standard and they are satisfied by piecewise polynomials defined on the uniform mesh. The property (3.1.(d)) with $s = 1$ was shown to be true in [B-4] for polynomials defined on triangles. Similar arguments to those in [B-4] were used in [L-S-N] to prove that the inequality in (4.4.d) can be extended to negative norms (i.e. $0 < s < 1$).

4.2. Semidiscrete approximation. Below we recall the results on stability and the rate of convergence of the solution $u_{h,\varepsilon}(t)$ to $u_\varepsilon(t)$ in $H^s(\Omega)$ topology $0 < s < \frac{1}{2}$.

THEOREM 4.1 (stability) [L-S-2]. *Let u_ε be the weak solution of the problem* (1.2) *and let $u_{h,\varepsilon}$ be the approximate solution of the problem* (1.3) *defined by* (4.1). *Then with $\varrho > 0$ arbitrary small, we have*

(i) $\|u_{h,\varepsilon}\|_{C[0,T;H^{1/2-\varrho}(\Omega)]} \leqslant C\,|g|_{H^1[0,T;H^\varrho(\Gamma)]}$;

(ii) $\|u_{h,\varepsilon}\|_{C[0,T;L_2(\Omega)]} \leqslant C\left[1+\dfrac{h^{\sigma/2-1}}{\sqrt{\varepsilon}}\right]|g|_{H^1[0,T;H^{-1/2+\varrho}(\Gamma)]}$;

where $\sigma \geqslant 1$ and $r \geqslant 1+(\sigma-1)/2\varrho$.

(iii) *if in addition we assume that $V_h^0 \subset V_h$* (5)

then

$$\|u_{h,\varepsilon}\|_{C[0,T;L_2(\Omega)]} \leqslant C\,|g|_{H^1[0,T;H^{-1/2+\varrho}(\Gamma)]}.$$

Here C is independent on h, ε, and g.

THEOREM 4.2 (convergence) [L-S-2]. *Let u_ε* (resp. $u_{h,\varepsilon}$) *be the solution of* (1.3) (resp. (4.1)). *Assume that g satisfies the appropriate compatibility conditions*

---

(4) It is well known that $\tilde{V}_h \subset H^1(\Gamma)$ is an approximating subspace of $L_2(\Gamma)$ of the same order r as V_h (see [B-2] Theorem 4.22).

(5) V_h^0 stands for the subspace of $H_0^1(\Omega) \cap V_h$ with approximating properties (3.1.b).

at the origin, in order to guarantee that the $u_\varepsilon \in C[0, T; H^s(\Omega)]$. *Then for any* $\varrho > 0$ *arbitrary small,* $\delta \geqslant 1$, $s \geqslant 1$ *there exists a constant* C *independent of* h, ε *and* g *such that*

(i) $$\|u_\varepsilon - u_{h,\varepsilon}\|_{C[0,T;L_2(\Omega)]} \leqslant Ch^{s-\varrho}\left[1+\frac{h^{\sigma/2-1}}{\sqrt{\varepsilon}}\right]^2 [|\dot{g}|_{H^s(\Sigma)} + |g|_{L_2[0,T;H^s(\Gamma)]}]$$

where $r \geqslant 1+(s-1)(\sigma-1)/\varrho$ (6);

(ii) $$\|u_\varepsilon - u_{h,\varepsilon}\|_{C[0,T;H^{1/2-\varrho}(\Omega)]} \leqslant Ch^{s-1/2-2\varrho} f(h)\,[|\dot{g}|_{H^s(\Sigma)} + |g|_{L_2[0,T;H^s(\Gamma)]}]$$

$$\text{where } f(h) \equiv \begin{cases} h^\varrho & \text{if } s < 3/2, \\ \left(1+\dfrac{h^{\sigma/2-1}}{\sqrt{\varepsilon}}\right) & \text{if } r \geqslant 1+\dfrac{(s-1)(\sigma-1)}{2\varrho}; \end{cases}$$

if in addition we assume that $V_h^0 \subset V_h$ *then for* $1 \leqslant s \leqslant r$, *we have*

(iii) $$\|u_\varepsilon - u_{h,\varepsilon}\|_{C[0,T,L_2(\Omega)]} \leqslant Ch^s [|\dot{g}|_{H^s(\Sigma)} + |g|_{L_2[0,T;H^s(\Gamma)]}],$$

(iv) $$\|u_\varepsilon - u_{h,\varepsilon}\|_{C[0,T;H^{1/2-\varrho}(\Omega)]} \leqslant Ch^{s-1/2} [|\dot{g}|_{H^s(\Sigma)} + |g|_{L_2[0,T;H^s(\Gamma)]}].$$

Let us set in (1.5) $\varepsilon = \varepsilon(h) = h^\gamma$ for some $\gamma > 1$ and denote the corresponding solution $u_{h,\varepsilon(h)}$ by u_h. After combining the results of Theorems 1.1, 3.1 and 3.2 of [L-S-2] we obtain

COROLLARY 4.1 (stability) [L-S-2].

(i) $$\|u_h - u\|_{C[0,T;H^{1/2-\varrho}(\Omega)]} \leqslant C[|\dot{g}|_{H^1[0,T;H^\varrho(\Gamma)]} + |g|_{H^{1/2-\varrho}(\Sigma)}],$$

(ii) $$\|u_h - u\|_{C[0,T;L_2(\Omega)]} < C[|\dot{g}|_{H^1[0,T;H^{1/2+\varrho}(\Gamma)]} + |g|_{L_2(\Sigma)}],$$

where $r \geqslant 1+(1+\gamma)/2\varrho$;

(iii) *if* $V_h^0 \subset V_h$ *then*

$$\|u_h - u\|_{C[0,T;L_2(\Omega)]} \leqslant C[|\dot{g}|_{H^1[0,T;H^{1/2+\varrho}(\Gamma)]} + |g|_{L_2(\Sigma)}].$$

COROLLARY 4.2 (convergence) [L-S-2]. *Let* u (resp. u_h) *be the solution of problem* (1.1) (resp. (4.1) *with* $\varepsilon(h) = h^\gamma$ *for some* $\gamma > 0$). *Then for any* $\varrho > 0$ *arbitrary small,* $s \geqslant 1$ *we have*

(i) $$\|u - u_h\|_{C[0,T;L_2(\Omega)]} \leqslant C[h^{s-\varrho} + h^\gamma]\,[|\dot{g}|_{H^s(\Sigma)} + |g|_{L_2[0,T;H^s(\Gamma)]}]$$

where $r \geqslant 1+(s-1)(1+\gamma)/\varrho$;

(ii) $$\|u - u_h\|_{C[0,T;H^{1/2-\varrho}(\Omega)]} \leqslant C[h^{s-1/2-\varrho} + h^\gamma]\,[|\dot{g}|_{H^s(\Sigma)} + |g|_{L_2[0,T;H^s(\Gamma)]}]$$

where $r \geqslant 1+(s-1)(1+\gamma)/\varrho$ *if* $s \geqslant 3/2$;

(iii) *If in addition we assume that* $V_h^0 \subset V_h$ *then* (i) *and* (ii) *holds for any* $1 \leqslant s \leqslant r$ *and* $\varrho = 0$. ∎

---

(6) If $s = 1$ then we can take $\varrho = 0$, $\sigma > 1$ arbitrary and $r \geqslant 1$.

Notice that the rates of convergence established in part (iii) (resp. (i), (ii)) are optimal (resp. quasioptimal) in the following sense — they reconstruct the optimal regularity of the solution (compare Theorem 2.1 in [L-S-2]), (modulo the usual loss of one derivative). The estimates of the error given in part (iii) under the additional assumption that $V_h^0 \subset V_h$ reconstruct also the best approximation properties of the underlined approximating subspaces. If condition $V_h^0 \subset V_h$ fails, then for $s > 1$ one needs to use higher order polynomials to obtain the quasioptimal error reflecting the optimal regularity of the solutions.

Stability estimates provided by Corollary 4.1 improve by $\frac{1}{2}$ derivative the stability results implied by the convergence results. Nevertheless, the stability estimates are still nonoptimal as we are loosing $\frac{1}{2}$ derivative with respect to the optimal regularity of the solutions (see Theorem 2.1 in [L-S-2]).

One can, of course, interpolate between the results of Corollary 4.1 and 4.2. For example; interpolation between the $L_2(\Omega)$-estimates of Corollary 4.1 and 4.2 applied with $s = \gamma = 1$ yields

$$\|u-u_h\|_{C[0,T;L_2(\Omega)]} \leqslant Ch^{(1-Q)}\,[|\dot{g}|_{H^{1-3/2Q+Q\varrho,1-Q(\Sigma)}} + |g|_{L_2[0,T;H^{1-3/2Q+Q\varrho(\Gamma)}]}]$$

where $\varrho > 0$ is arbitrary small and $0 \leqslant Q \leqslant 1$.

5. Finite element approximation in R^2

Let $\Omega \subset R^2$ be given domain. Let us consider the special case when the approximating subspaces V_h are the spaces of piecewise polynomials defined on two-dimensional domain Ω. Let $V_h \subset H^1(\Omega)$ be a family of $N(h)$ dimensional subspaces of algebraic polynomials of degree $p > 1$ defined on each triangle of the uniform triangulation of Ω. We show in [L-S-N] using the method proposed in [B-S] that we can extend the inequality (4.4d) to negative norms. Actually we proved in [L-S-N] that the following assumption holds true.

For any $\tilde{\Phi}_h \in \tilde{V}_h \equiv V_{h|\Gamma}$, there exists an element $\Phi_h \in V_h$ such that $\Phi_{h|\Gamma} = \tilde{\Phi}_h$ and

$$\|\Phi_h\| \leqslant C\,|\tilde{\Phi}_h|_{-1/2+\varepsilon,\Gamma}, \tag{5.1}$$

$$\|\Phi_h\|_{1,\Omega} \leqslant C\,|\tilde{\Phi}_h|_{1/2,\Gamma}, \tag{5.2}$$

where $\varepsilon > 0$ is arbitrary small.

Therefore we can extend the inequality (4.4d) to negative norms.

Remark. Notice that (5.1) and (5.2) represent the surjectivity of the trace operator, but restricted to the finite dimensional subspaces V_h and $\tilde{V}_h$.

To prove (5.1), (5.2), it is enough [L-S-N] to establish these inequalities for a single triangle (generalization to curvilinear element is straightforward). To accomplish this let T be an equilateral triangle (see Fig. 1)

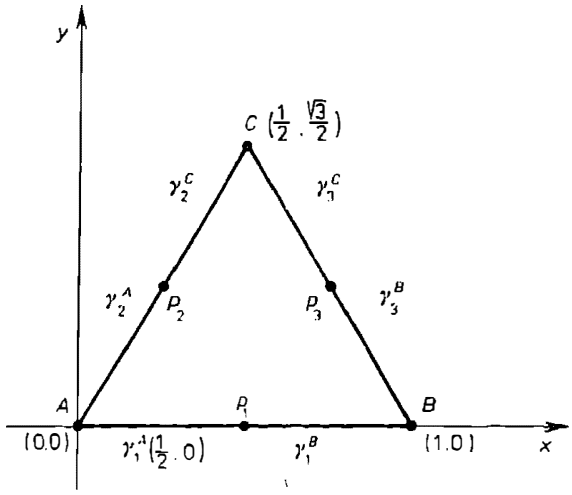

Fig. 1. Triangle T

$$T = \{(x, y) \in \boldsymbol{R}^2 \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant \sqrt{3}x,\ y \leqslant \sqrt{3} - \sqrt{3}x\}$$

with the boundary $\partial T = \gamma_1 \cup \gamma_2 \cup \gamma_3$ where

$$\gamma_1 = \gamma_1^A \cup \gamma_1^B = \{(x, 0) \mid 0 \leqslant x \leqslant 1\},$$

$$\gamma_2 = \gamma_2^A \cup \gamma_2^C = \{(x, y) \mid y = \sqrt{3}x,\ 0 \leqslant x \leqslant \tfrac{1}{2}\},$$

$$\gamma_3 = \gamma_3^B \cup \gamma_3^C = \{(x, y) \mid y = -\sqrt{3}x + \sqrt{3},\ \tfrac{1}{2} \leqslant x \leqslant 1\}.$$

We denote by $V(T)$ (resp. $V(\gamma)$) the space of polynomials of order p on T (resp. $\gamma = \partial T$).

The estimates (5.1) and (5.2) will follow from the following theorem:

THEOREM 5.1 [L-S-N]. *Let there be given an element $\tilde{\Phi} \in V(\gamma)$. There exists an element $\Phi \in V(T)$ such that $\Phi|_\gamma = \tilde{\Phi}$ and*

$$\|\Phi\|_{0,T} \leqslant C\,\|\tilde{\Phi}\|_{-1/2+\varepsilon,\gamma}, \tag{5.3}$$

$$\|\Phi\|_{1,T} \leqslant C\,\|\tilde{\Phi}\|_{1/2,\gamma}, \tag{5.4}$$

where C does not depend on ϕ.

The inequality (5.4) has been proved in [B-S] in the case of h-p version of the finite elements. Thus in particular it is valid in our case.

To prove (5.3) we shall follow the conceptual framework of the proof of (5.4) given in [B-S]. However, because of the presence of "negative" norms in (5.3) (rather unnatural for the trace theorem), there will be a number of technical differences with respect to the ideas presented in [B-S].

Following [B-S] we define the linear mapping F_1: $V(\gamma_1) \to V(T)$ where

$$F_1^{[f]}(x, y) = \frac{\sqrt{3}}{1y} \int_{x - y/\sqrt{3}}^{x + y/\sqrt{3}} f(t)\,dt.$$

The mappings F_i: $V(\gamma_i) \to V(T)$, $i = 2, 3$ are defined in the similar way.

The proof of (5.3) is based on the following three Lemmas:

LEMMA 5.1 [L-S-N]. *Let* $f \in V(\gamma_1)$. *Then*

(a) $$F_1^{[f]}(x, y) \in V(T),$$

(b) $$F_1^{[f]}(x, 0) = f(x),$$

(c) $$\|F_1^{[f]}\|_{0,T} \leqslant C|f|_{-1/2,\gamma_1},$$

(d_1) $$|F_1^{[f]}|_{k,\gamma_2^A} \leqslant C|f|_{k,\gamma_1^A}, \quad -\tfrac{1}{2}+\varepsilon \leqslant k \leqslant 0,$$

(d_2) $$|F_1^{[f]}|_{k,\gamma_3^B} \leqslant C|f|_{k,\gamma_1^B}, \quad -\tfrac{1}{2}+\varepsilon \leqslant k \leqslant 0,$$

(d_3) $$|F_1^{[f]}|_{0,\gamma_2^C} \leqslant C|f|_{-1/2,\gamma_1},$$

(d_4) $$|F_1^{[f]}|_{0,\gamma_3^C} \leqslant C|f|_{-1/2,\gamma_1}.$$

LEMMA 5.2 [L-S-N]. *Let* T *be the triangle as in* Fig. 1 *and let* f *be a continuous function on* ∂T, *such that* $f_i = f|_{\gamma_i} \in V(\gamma_i)$, $i = 1, 2, 3$. *Then there exists* $\Phi_i \in V(\gamma_i)$, $i = 1, 2$, *such that*

(a) $$U = F_1^{[\Phi_1]} + F_2^{[\Phi_2]} \in V(T),$$

(b) $$U = f_i \quad \text{on } \gamma_i, \quad i = 1, 2,$$

(c) $$\|U\|_{0,T} \leqslant C(|f_1|_{-1/2,\gamma_1} + |f_2|_{-1/2,\gamma_2}),$$

(d_1) $$|\Phi_i|_{k,\gamma_i} \leqslant C \sum_{j=1}^{2} |f_j|_{k,\gamma_j}, \quad -\tfrac{1}{2}+\varepsilon \leqslant k \leqslant 0, \ i = 1, 2,$$

(d_2) $$|\Phi_1|_{k,\gamma_1^B} \leqslant C\Big[|f_1|_{k,\gamma_1^B} + \sum_{j=1}^{2} |f_j|_{-1/2,\gamma_j}\Big], \quad -\tfrac{1}{2}+\varepsilon \leqslant k \leqslant 0,$$

(d_3) $$|\Phi_2|_{k,\gamma_2^C} \leqslant C\Big[|f_2|_{k,\gamma_2^C} + \sum_{j=1}^{2} |f_j|_{-1/2,\gamma_j}\Big], \quad -\tfrac{1}{2}+\varepsilon \leqslant k \leqslant 0.$$

LEMMA 5.3 [L-S-N]. *Let* T *be the triangle as before,* f *be a continuous function on* ∂T, $f_2 = f_3 = 0$ *and* $f_1 \in V(\gamma_1)$. *Then there exists a polynomial* $v \in V(T)$ *such that*

$$\|v\|_{0,T} \leqslant C|f_1|_{0,\gamma_1},$$

$$v = f_1 \quad \text{on } \gamma_1,$$

$$v = 0 \quad \text{on } \gamma_2 \cup \gamma_3,$$

where C *is a constant independent of* f *and of the order* p *of polynomials in* $V(T)$.

Lemmas 5.1, 5.2 and 5.3 are the "negative norm" counterparts of the results given by Lemmas 7.1, 7.2 and 7.3 in [B-S]. The proof of Lemma 5.3 follows via straightforward modification of the arguments given in [B-3]. More technical proofs of Lemma 1 and 2 are given in [L-S-N].

To continue with the proof of (5.3), following [B-S] we denote (see Fig. 1)

$$\gamma_1 = \gamma_1^A \cup \gamma_1^B \quad \text{where } \gamma_1^A = \overline{AP_1},\ \gamma_1^B = \overline{P_1 B},$$

$$\gamma_2 = \gamma_2^A \cup \gamma_2^C \quad \text{where } \gamma_2^A = \overline{AP_2},\ \gamma_2^C = \overline{P_2 C},$$

$$\gamma_3 = \gamma_3^B \cup \gamma_3^C \quad \text{where } \gamma_3^B = \overline{BP_3},\ \gamma_3^C = \overline{P_3 C},$$

P_1, P_2, P_3 are the midpoints of line segments $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$, respectively. Without loss of generality we may assume that

$$\tilde{\phi}|_{\gamma_1} = \tilde{\phi}_1 \not\equiv 0, \tag{5.5}$$

$$\tilde{\phi}|_{\gamma_2} = \tilde{\phi}_2 \equiv 0, \tag{5.6}$$

$$\tilde{\phi}|_{\gamma_3} = \tilde{\phi}_3 \equiv 0. \tag{5.7}$$

We apply Lemma 5.2 to $f_1 = \tilde{\phi}_1$, $f_2 \equiv 0$. Thus there exist elements $\Phi_1 \in V(\gamma_1)$, $\Phi_2 \in V(\gamma_2)$ such that

$$U = F_1^{[\Phi_1]} + F_2^{[\Phi_2]} \in V(T), \tag{5.8}$$

$$U|_{\gamma_1} = \tilde{\phi}_1, \quad U|_{\gamma_2} = \tilde{\phi}_2 \equiv 0, \tag{5.9}$$

$$U|_{\gamma_3} = g_3 \tag{5.10}$$

and

$$\|U\|_{0,T} \leqslant C|\tilde{\phi}_1\|_{-1/2,\gamma_1}. \tag{5.11}$$

We estimate the norm $|g_3|_{-1/2,\gamma_3}$ as follows

$$|g_3|_{-1/2,\gamma_3} \leqslant C_1 |g_3|_{k,\gamma_3} \leqslant C_1 (|F_1^{[\Phi]}|_{k,\gamma_3} + |F_2^{[\Phi]}|_{k,\gamma_3}),$$

by Lemma 5.1 (d_2), (d_3)

$$\leqslant C_2 (|\Phi_1|_{k,\gamma_1} + |\Phi_2|_{k,\gamma_2}) \leqslant C|\tilde{\phi}_1|_{k,\gamma_1}, \quad -\tfrac{1}{2}+\varepsilon \leqslant k \leqslant 0, \text{ since } \tilde{\phi}_2 \equiv 0. \tag{5.12}$$

Now we extend g_3 to T, using again Lemma 5.2 for $f_1 \equiv 0, f_3 = g_3$. By Lemma 5.2 there exist elements

$$\Phi_1' \in V(\gamma_1), \quad \Phi_3' \in V(\gamma_3) \tag{5.13}$$

such that

$$U_1 = F_1^{[\Phi_1']} + F_3^{[\Phi_3']} \in V(T), \tag{5.14}$$

$$\|U_1\|_{0,T} \leqslant C_1 |g_3|_{-1/2,\gamma_3} \leqslant C|g_3|_{k,\gamma_3}, \quad -\tfrac{1}{2}+\varepsilon \leqslant k \leqslant 0, \tag{5.15}$$

$$U_1|_{\gamma_1} = 0, \quad U_1|_{\gamma_2} = g_2, \quad U_1|_{\gamma_3} = g_3. \tag{5.16}$$

Finally we extend an element f of the form

$$f_1 \equiv 0, \quad f_2 = g_2, \quad f_3 \equiv 0 \tag{5.17}$$

to the triangle T using Lemma 5.3. By Lemma 5.3 there exists an element $U_2 \in V(T)$ such that

$$U_2|_{\gamma_1} = 0, \quad U_2|_{\gamma_2} = g_2, \quad U_2|_{\gamma_3} = 0 \tag{5.18}$$

and

$$\|U_2\|_{0,T} \leqslant C\,|g_2|_{0,\gamma_2}. \tag{5.19}$$

We estimate the norm $|g_2|_{0,\gamma_2}$ in terms of the norm $|\tilde{\phi}_1|_{-1/2,\gamma_1}$. Since $g_2 = U_1|_{\gamma_2}$, it follows that

$$\begin{aligned} |g_2|_{0,\gamma_2} = |U_1|_{0,\gamma_2} &\leqslant (|F_3^{[\Phi_3']}|_{0,\gamma_2^A} + |F_3^{[\Phi_3']}|_{0,\gamma_2^C} + |F_1^{[\Phi_1']}|_{0,\gamma_2^A} + |F_1^{[\Phi_1']}|^2_{0,\gamma_2^C}) \\ &\leqslant C_1\,(\Phi_3'|_{-1/2,\gamma_3} + |\Phi_3'|_{0,\gamma_2^C} + |\Phi_1'|_{0,\gamma_1^A} + |\Phi_1'|_{-1/2,\gamma_1}) \end{aligned}$$

by Lemma 5.2 (d_1), (d_3)

$$\leqslant C_2\,(|g_3|_{-1/2,\gamma_3} + |g_3|_{0,\gamma_3^C}). \tag{5.20}$$

On the other hand,

$$|g_3|_{0,\gamma_3^C} \leqslant |F_1^{[\Phi_1]}|_{0,\gamma_3^C} + |F_2^{[\Phi_2]}|_{0,\gamma_3^C} \leqslant$$

by Lemma 5.1

$$\leqslant C_1\,(|\Phi_1|_{-1/2,\gamma_1} + |\Phi_2|_{0,\gamma_2^C}) \leqslant$$

by Lemma 5.2 (d_1), (d_3), since $f_2 = U|_{\gamma_2} \equiv 0$

$$\leqslant C\,|\tilde{\phi}_1|_{-1/2,\gamma_1}$$

Finally let

$$\phi = U - U_1 + U_2 \tag{5.21}$$

then in view of (5.11), (5.12), (5.13), (5.16), (5.19)

$$\|\phi\|_{0,T} \leqslant C\,|\tilde{\phi}_1|_{k,\gamma_1}, \qquad -\tfrac{1}{2}+\varepsilon \leqslant k \leqslant 0,$$

and

$$\phi|_{\gamma_1} = \tilde{\phi}_1, \quad \phi|_{\gamma_2} = 0, \quad \phi|_{\gamma_3} = 0,$$

which completes the proof of Theorem 5.1, hence of (5.3). ∎

6. Numerical tests

Let in (4.1) $A(x, \partial) = -\Delta$, $\beta = -\alpha_1 \Delta_\Gamma + \alpha_2 I$, where $\alpha_1, \alpha_2 > 0$. We denote by $U(t) = (U_1(t), \ldots, U_{n(h)})$, $n(h) = \dim V_h$ the vector of nodal points values of $u_{h,\varepsilon}$. We write (4.1) in the matrix setting:

$$\begin{cases} M_\Omega U + \left(K_\Omega + \dfrac{1}{\varepsilon}(\alpha_1 K_\Gamma + \alpha_2 M_\Gamma)\right) U = \dfrac{1}{\varepsilon}(\alpha_1 K_\Gamma + \alpha_2 M_\Gamma)\, G, \\ U(0) = \bar{U}^1, \\ U(0) = \bar{U}^0, \end{cases} \tag{6.1}$$

where we have used the following notations:

(6.2) $K_\Omega = (\int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, dx)_{i,j\in I_\Omega}$ stiffness matrix, where ϕ_i are linear shape functions,

(6.3) $M_\Omega = (\int_\Omega \phi_i \, \phi_j \, dx)_{i,j\in I_\Omega}$ mass matrix,

(6.4) $$K_\Gamma = \begin{cases} \int_\Gamma \frac{\partial}{\partial\tau}\phi_i \Big|_\Gamma \frac{\partial}{\partial\tau}\phi_j \Big|_\Gamma d\Gamma & i, j \in I_\Gamma, \\ 0 & \text{otherwise}, \end{cases}$$

(6.5) $$M_\Gamma = \begin{cases} \int_\Gamma \phi_i|_\Gamma \, \phi_j|_\Gamma \, d\Gamma & i, j \in I_\Gamma, \\ 0 & \text{otherwise}, \end{cases}$$

(6.6) $$G = \begin{cases} g(p_i) & i \in I_\Gamma \text{ the vector of boundary data}, \\ 0 & \text{otherwise}, \end{cases}$$

$$I_{\bar\Omega} = \text{indices of nodal points } P_i,$$

$$I_\Gamma = \text{indices of nodal points lying on } \Gamma.$$

One can use a class of single step methods for discretizing the semidiscrete system (6.1) (see [B-2]). We have applied the scheme proposed in [D-2]. It is of order $O(h^2)+O((\Delta t)^2)$. We first shortly summarize the method. Let N be a positive number and let $\Delta t = T/N$. We denote by u^k any function u at the time levels $t = t_k = k\Delta t$, $k = 0, \ldots, N$. Furthermore we use the notations

$$u^{n,\theta} = \theta u^{k+1} + (1-2\theta)\, u^k + \theta u^{k-1},$$

$$\partial_t^2 u^k = \frac{u^{k+1} - 2u^k + u^{k-1}}{(\Delta t)^2}.$$

We define the fully discrete approximation for (1.3) to be a sequence $(U^k)_{k=0}^N$ such that

(6.7) $$M_\Omega \partial_t^2 U^k + \left(K_\Omega + \frac{1}{\varepsilon}(\alpha_1 K_\Gamma + \alpha_2 M_\Gamma)\right) U^{k,1/4}$$
$$= \frac{1}{\varepsilon}(a_1 K_\Gamma + \alpha_2 M_\Gamma)\, G^{k,1/4} \quad \text{for } k = 1, \ldots, N-1.$$

To start the scheme (6.7) we need the solution vector at two levels, say $t = 0$; $t = \Delta t$. By (6.1)

$$U^0 = \bar U^0.$$

We shall present a numerical example. There the data is differentiable. Other cases will be considered in other connections.

EXAMPLE 6.1. Let $\Omega = (0, 1)\times(0, 1)$ and $T = 1$. Consider the problem

$$(6.8)\qquad \begin{cases} \ddot{u} = \Delta u & \text{in } \Omega\times(0, 1), \\ u(x, y, t) = \sin\left(\pi(x+y+\sqrt{2}t)\right) & \text{on } \Gamma\times(0, 1), \\ u(x, y, 0) = \operatorname{son}\left(\pi(x+y)\right), \\ \dot{u}(x, y, 0) = \sqrt{2}\pi\cos\left(\pi(x+y)\right). \end{cases}$$

The exact solution of (6.8) is $u(x, y, t) = \sin\left(\pi(x+y+\sqrt{2}t)\right)$ (see Fig. 2.).

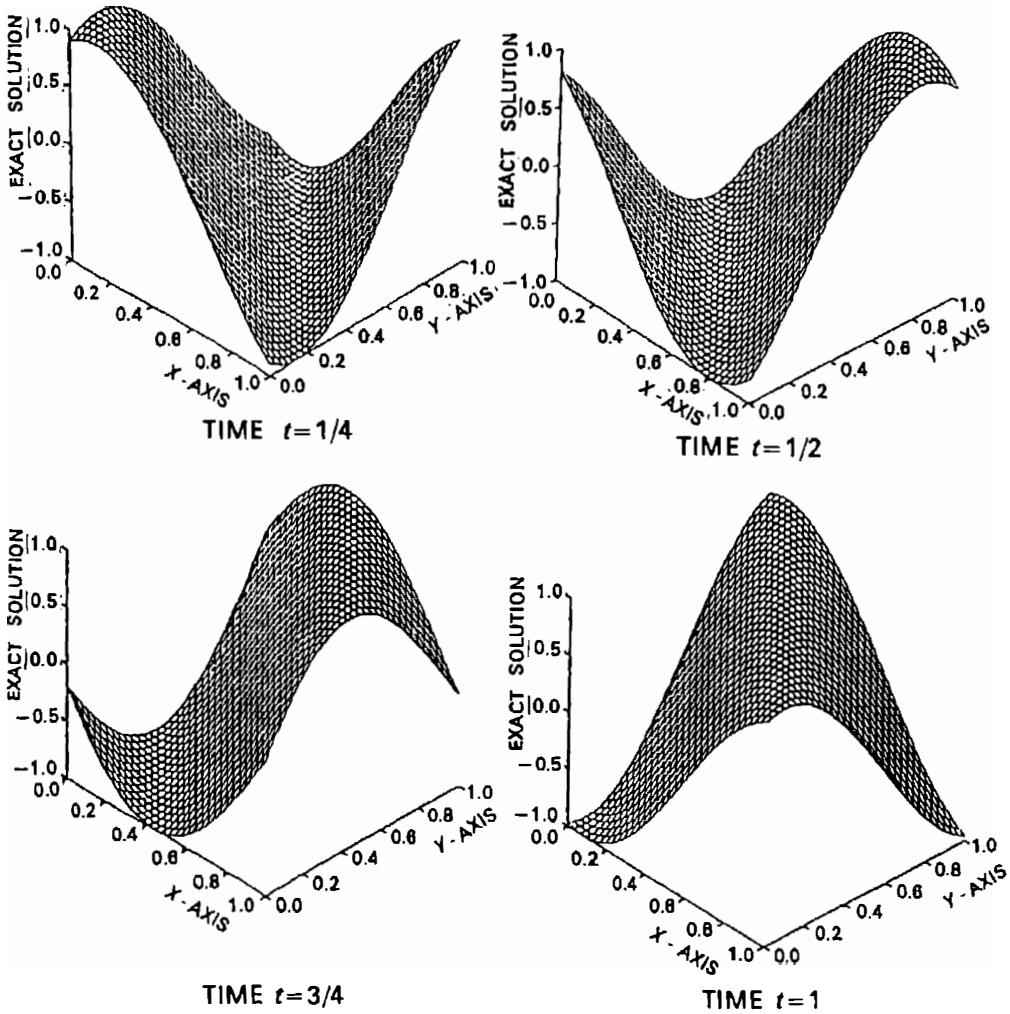

Fig. 2. The exact solution of (6.8) for $t = 1/4$, $1/2$, $3/4$ and 1

In numerical tests we have chosen $\alpha_1 = \alpha_2 = 1$, $\varepsilon = 10^{-5}$, $h = 1/18$, $1/16$, $1/32$, $\Delta t = 1/32$, $1/64$, $1/128$. In Table 1 we see the $L^\infty(L^\infty)$ errors. We see that the method gives $O(h^2)+O\left((\Delta t)^2\right)$ in $L^\infty(L^\infty)$-norm (see the diagonal of Table 1).

In Figures 3–5 we see $u(t)-u_{\varepsilon h\Delta t}(t)$ for different time levels $t = 1/4$, $t = 1/2$, $t = 3/4$, $t = 1$; $\varepsilon = 2^{-50}$ and with $h = 1/8$, $1/16$, $1/32$, $\Delta t = 1/32$, $1/64$, $1/128$.

Table 1

Δt / h	1/32	1/64	1/128
1/8	0.02531	0.03027	0.03145
1/16	0.003239	0.006714	0.007717
1/32	0.005869	0.0008219	0.001572

$DX = 1/8 \quad DY = 1/8 \quad DT = 1/32$

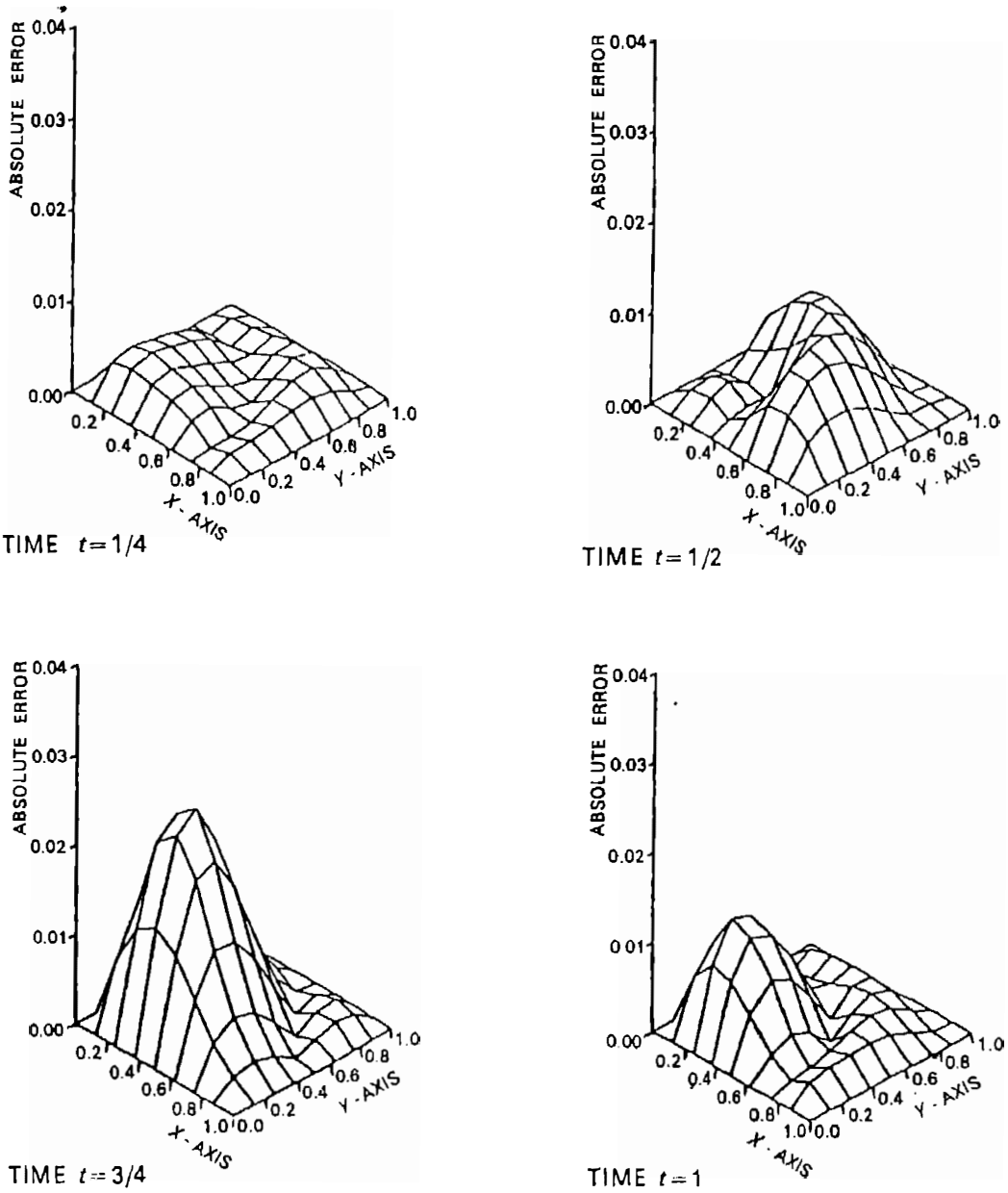

Fig. 3. $|u(t)-u_{rh\Delta t}(t)|$, $h = 1/8$, $\Delta t = 1/32$

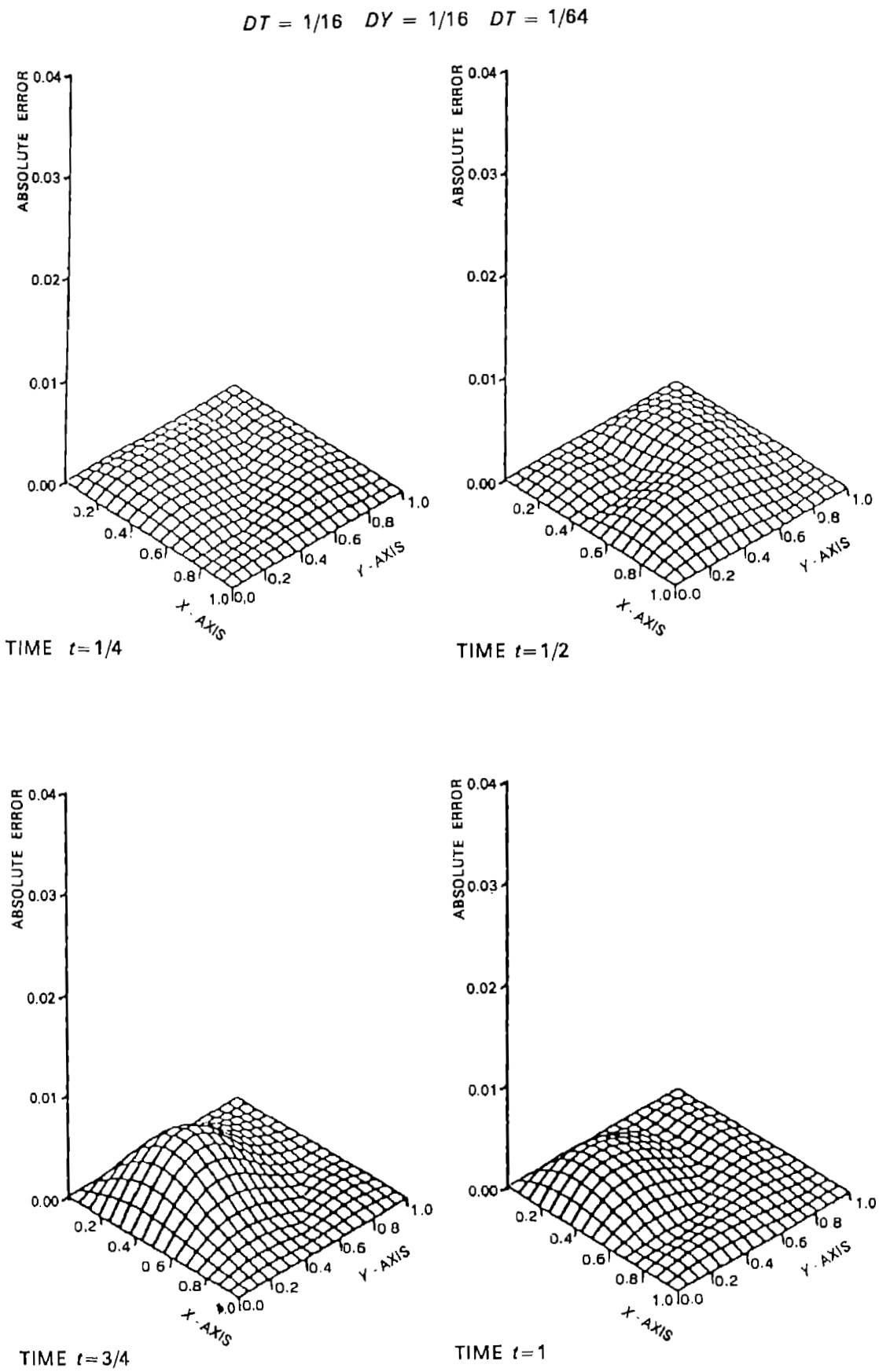

Fig. 4. $|u(t)-u_{zh\Delta t}(t)|$, $h = 1/16$, $\Delta t = 1/64$

$DT = 1/32 \quad DY = 1/32 \quad DT = 1/128$

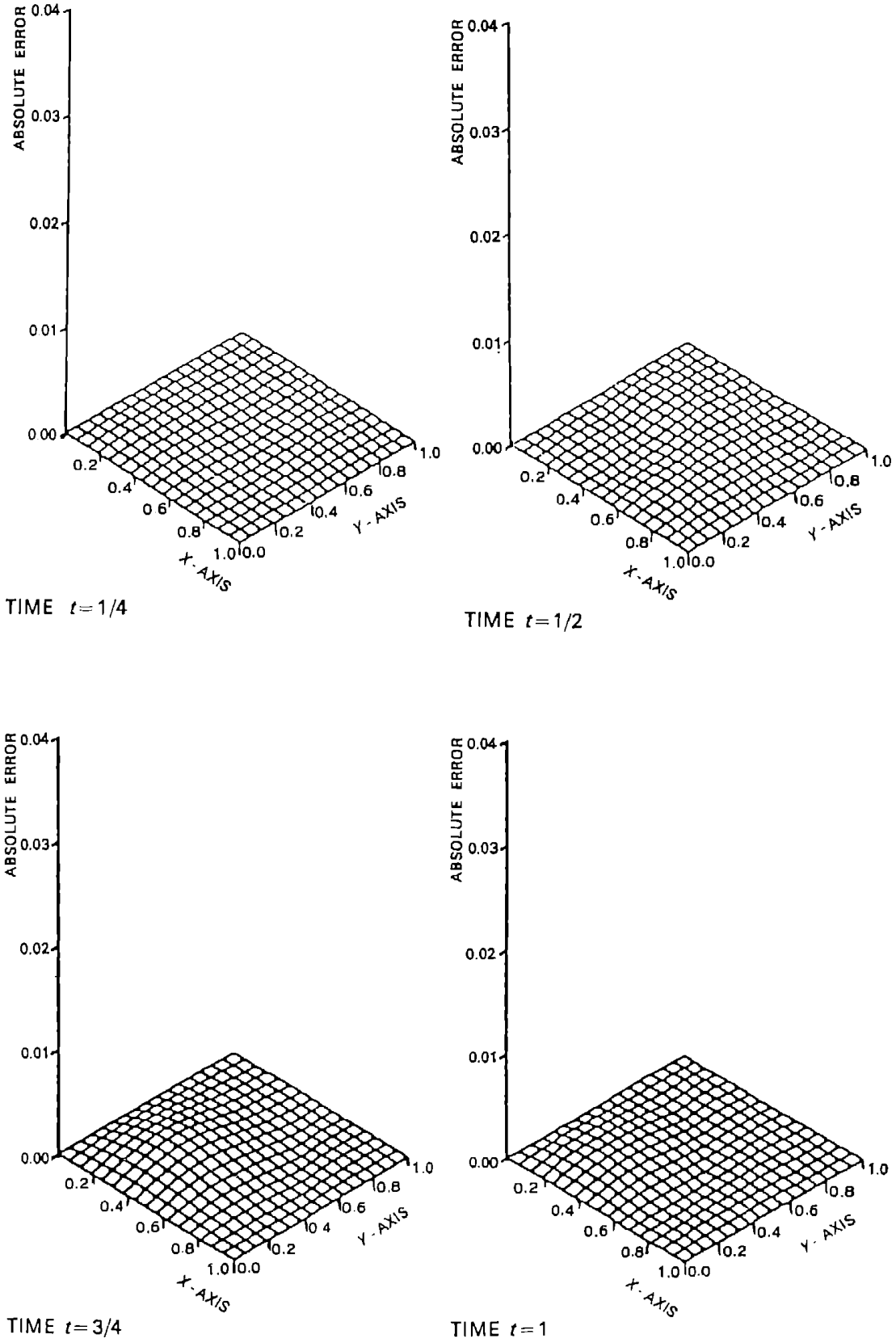

Fig. 5. $|u(t)-u_{ch\Delta t}(t)|$, $h = 1/32$, $\Delta t = 1/128$

In Fig. 6 a–c we see the evolution of $u-u_{\varepsilon h\Delta t}$ difference in L^∞-norm for $h = 1/8$, $\Delta t = 1/32$, $h = 1/16$, $\Delta t = 1/64$ and $h = 1/32$, $\Delta t = 1/128$ (piecewise linear interpolation is applied for $u(t)-u_{\varepsilon h\Delta t}(t)$ for $t \neq k\Delta t$).

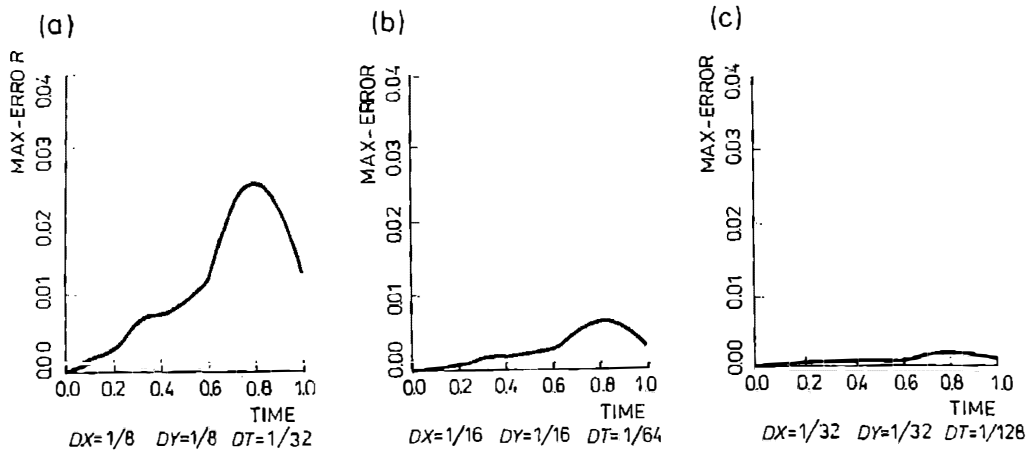

Fig. 6. $\|u(t)-u_{\varepsilon h\Delta t}(t)\|_{L^\infty(\Omega)}$, $t \in [0, 1]$

Table 2 presents L^∞ (L^2) errors.

Table 2

h	Δt 1/32	1/64	1/128
1/8	0.00946	0.01147	0.0119
1/16	0.00087	0.00245	0.00289
1/32	0.00267	0.00040	0.00058

We find that $O(h^2)+O((\Delta t)^2)$ for L^∞ (L^2) norm (see the diagonals of Table 2). In Fig. 7 a–c we see the evolution of $u-u_{\varepsilon h\Delta t}$ in L^2-norm for $h = 1/8$. $\Delta t = 1/32$, $h = 1/16$, $\Delta t = 1/64$ and $h = 1/32$, $\Delta t = 1/128$ (as above piecewise linear interpolation for error at time $t \neq k\Delta t$ is applied).

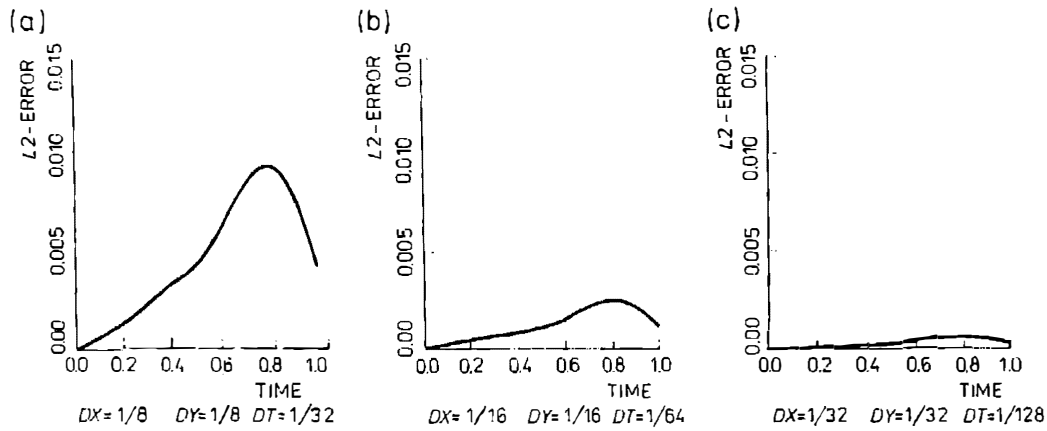

Fig. 7. $\|u(t)-u_{\varepsilon h\Delta t}(t)\|_{L^2(\Omega)}$, $t \in [0, 1]$

Acknowledgements

The authors are indebted to J. Katainen for his assistance in numerical tests.

References

[B-1] I. Babuska and A. Aziz, *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, Academic Press, New York 1972.

[B-2] I. Babuska, *The finite element method with penalty*, Math. Comp. 27 (122) (1973), 221–228.

[B-3] —, *Approximation by Hill functions II*, Comment. Math. Univ. Carolinae 13 (1972), 1–22.

[B-4] I. Babuska and M. Suri, *The h-p version of the finite element method with quasiuniform meshes*, Institute for Physical Science and Technology, University of Maryland, Technical Report BN-1046, (1986).

[B-5] G. Baker, *Error estimates for finite element methods for second order hyperbolic equations*, SIAM J. Numer. Anal. 13 (1976), 564–576.

[B-6] L. Bales, *Higher order single step fully discrete approximations for second order hyperbolic equations with time dependent coefficients*, SIAM J. Numer. Anal. 23 (1) (1986), 27–43.

[B-7] J. J. Blair, *Higher order approximations to the boundary conditions for the finite element method*, Math. Comp. 30 (134) (1976), 250–262.

[D-1] J. Douglas and T. Dupont, *Galerkin methods for parabolic equations*, SIAM J. Numer. Anal. 7 (1970), 575–626.

[D-2] T. Dupont, *L^2-estimates for Galerkin methods for second order hyperbolic equations*, SIAM J. Numer. Anal. 10 (5) (1973), 880–889.

[F-1] H. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, North-Holland 1985.

[G-1] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York 1984.

[L-1] J. L. Lions, *A remark on the approximation of nonhomogeneous hyperbolic boundary value problems — paper dedicated to G. I. Marchuk on his sixtieth anniversary.*

[L-2] —, *Contrôle des systèmes distribués singuliers*, Dunod, Paris 1983.

[L-M] J. L. Lions and E. Magenes, *Nonhomogeneous Boundary Value Problems and Applications*, Vol. I, II. Springer Verlag 1972.

[L-L-T] I. Lasiecka, J. L. Lions and R. Triggiani, *Nonhomogeneous boundary value problems for second order hyperbolic operators*, to appear in J. Math. Pures Appl.

[L-T-1] I. Lasiecka and R. Triggiani, *A cosine operator approach to modelling $L_2(0, T; L_2(\Gamma))$ boundary input hyperbolic equations*, Appl. Math. Optimiz. 7 (1981), 35–83.

[L-T-2] —, *Regularity of hyperbolic equations under $L_2(0, T; L_2(\Gamma))$ Dirichlet boundary terms*, Appl. Math. Optimiz. 10 (1983), 275–286.

[L-S-1] I. Lasiecka and J. Sokołowski, *Regularity and strong convergence of variational approximation to nonhomogeneous Dirichlet hyperbolic boundary value problems*, SIAM J. Math. Anal. 19 (3) (1988), 528–540.

[L-S-2] —, *Semidiscrete approximations of hyperbolic boundary value problems with nonhomogeneous Dirichlet boundary conditions*, to appear SIAM J. Math. Anal. (1989).

[L-S-N] I. Lasiecka, J. Sokołowski and P. Neittaanmäki, *Finite element approximations of the wave equation with Dirichlet boundary data on a bounded domain in $\mathbf{R}^2$*, in: Proc. Conf. on Control of Distributed Parameter Systems, Vorau (Austria), July 6–12, 1986.

[P-1] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag 1983.

[R-1] J. Rauch, *On convergence of the finite element method for the wave equation*, SIAM J. Numer. Anal. 22 (2) 1985, 245–250.

[N-1] J. A. Nitsche, *Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind*, Abh. Math. Sem. Univ. Hamburg 36 (1971), 9–15.

[S-1] R. Scott, *Interpolated boundary conditions in the finite element method*, SIAM J. Numer. Anal. 12 (3) (1975), 404–427.

Presented to the Semester
Numerical Analysis and Mathematical Modelling
February 25 – May 29, 1987