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# A Theory of Electrical Circuits with Resistively Coupled Distributed Structures: Delay Time Predicting

CORNELIU A. MARINOV AND PEKKA NEITTAANMÄKI

**Abstract**—This paper deals with qualitative properties (existence, uniqueness, and especially, stability) and numerical solution of a circuit consisting of a resistive multiport with  $r$ - $c$ - $g$  “exactly modeled” distributed elements connected to its terminals. This kind of results are useful, for instance, when we study the effect of interconnections on the speed of transient process from an integrated structure. A formula to evaluate the delay time as a global parameter of the circuit is given and verified by numerical calculus.

## I. INTRODUCTION

THE technology of integrated circuits is structurally involved in dealing with structures with distributed parameters. For instance, in MOS integrated circuits, a given inverter or logic node may drive several gates through polysilicon thin-film wires. This thin-film technology results in interconnection lines with parameters distributed over their entire length. Also, in monolithic structures a resistor or a capacitor appears unavoidably as a component presenting as well a distributed resistance ( $r$ ), a distributed capacitance ( $c$ ), and a distributed conductance ( $g$ ). With the advances in technology, the delay time associated with interconnections becomes significant compared to the device delay time and in certain cases dominates the chip performance [1]. To predict this extra-devices delay for an integrated structure, we can replace the devices with their resistive models and thus we have to study a network composed by a resistive multiport with  $r$ - $c$ - $g$  distributed structures (“lines”) connected to its terminals (see Fig. 1).

The mathematical model of this class of circuits (formulated in Section II) consists of a parabolic system of equations with nonstandard boundary conditions. The correctness of this model is given by existence, uniqueness and stability theorems enounced in Section III where the restrictions of hypotheses are also commented.

Our theorem regarding exponential asymptotic stability (the main result of this paper) suggests a definition and

allows an evaluation of this delay time of the circuit under study. This parameter characterizes the rate of the evolution of the state vector to the stationary regime. As the upper bound of this characteristic depends upon all the circuit parameters and is easy to compute, such an approach seems to be quite useful in circuit design.

The proofs (presented in Section IV) use essentially the dissipative operator approach (see [3], [4]) and its relations to abstract differential equation arising from our model.

In Section V a numerical algorithm of the type finite elements in space, finite differences in time is formulated in order to obtain the solution of the problem under study. This algorithm is then used in Section VI in validating the evaluation of the response time computed for several examples.

The main mathematical notations and definitions are relegated to Appendix where a useful lemma is also proved.

If we try to establish the relations of this paper with mathematical and technical literature, let us mention at the beginning that the nonclassical kind of boundary conditions appearing in our model were considered in [5]–[8] but for hyperbolic problems which cannot come to our one.

The stability of the circuits with distributed structures was studied by using different methods. In the remarkable paper [9], for instance, the authors use a Lyapunov theory for a functional-differential equation of retarded type that describe a large class of such circuits. A small signal stability criterion (in terms of the roots of characteristic equation) is derived in [10] and input–output stability results are given in [11] for circuits with parasitic elements, by applying the theory of singular perturbations.

Analytical estimations of the transient response of circuits containing distributed structures ( $r$ - $c$  lines) were obtained especially for the case of a single such distributed element. A pioneering work in this field is [12]. For a nonuniform open circuited  $r$ - $c$  line the paper [13] shows the step response to be monotonously increasing. In [14] the bounds of this response were obtained.

The engineering literature shows lately an intense preoccupation regarding the estimation of signals delay in MOS integrated circuits (see [2], [15]–[19] and the references

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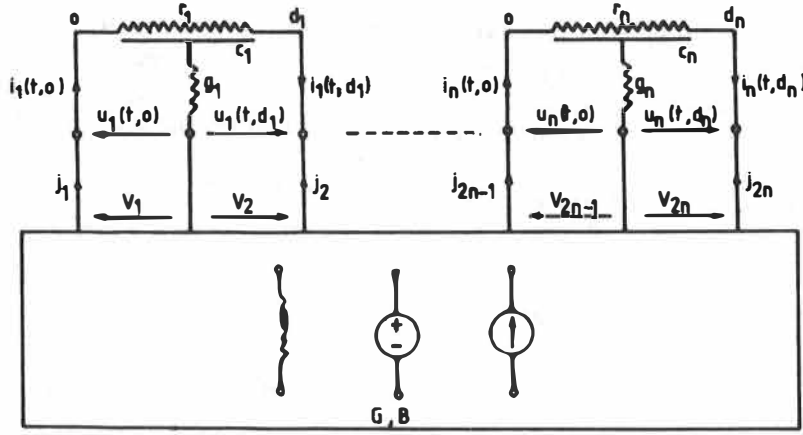


Fig. 1. The circuit under study.

quoted therein); the approximation of distributed elements by RC ladder networks (even infinite RC networks [20]) is used throughout. The accuracy of such an approximation is rarely studied and then for a single distributed element [15].

The delay estimation in lumped parameter RC circuits allows efficient using of CAD programs ([1], [17], [18] and references quoted therein). A fundamental paper in this respect is [2], where bounds of the transient response at a step-variation input are derived for each node of a lumped RC fanout circuit. The methods and results of this paper were then extended to mesh-type networks, [18], and to RC ladder networks with nonlinearities, [19]. Another extension which takes into account the effect of initial condition is presented in [17].

There are two main directions of study that arise naturally; first, to consider circuits whose distributed elements are "exact" models ( $r$ - $c$ - $g$  lines) and, second, to consider the dynamic behavior excited by step-variations simultaneously applied to more than one input. With such significant extensions of the model, the present paper gives the estimation of the delay time defined as a global parameter of the circuit.

We shall report elsewhere the extension of below results to the circuits supplementary containing lumped capacitors. A preliminary restricted version of this paper were [21], [30].

## II. THE CIRCUIT MODEL

The circuit under study consists of a lumped parameter resistive multiport with any kind of sources as well as  $r$ - $c$ - $g$  distributed elements connected to its terminals, as shown in Fig. 1.

In Fig. 1,  $u_k(t, x)$  and  $i_k(t, x)$  is, respectively, the voltage and the current at the moment  $t \in [0, \infty)$  in the point  $x \in [0, d_k]$  of the distributed structure,  $k \in \overline{1, n}$ . This structure is mathematically described by the well-known telegraph equations:

$$\begin{cases} \frac{\partial u_k(t, x)}{\partial x} = -r_k i_k(t, x) \\ \frac{\partial i_k(t, x)}{\partial x} = -c_k \frac{\partial u_k(t, x)}{\partial t} - g_k u_k(t, x). \end{cases} \quad (2.1)$$

In the following we shall consider  $r_k$ ,  $c_k$ , and  $g_k$  constant parameters and we assume  $r_k, c_k > 0$ ,  $g_k \geq 0$ .

From (2.1) we formally derive the system:

$$\frac{\partial u_k(t, x)}{\partial t} = \frac{1}{r_k c_k} \frac{\partial^2 u_k(t, x)}{\partial x^2} - \frac{g_k}{c_k} u_k(t, x), \quad x \in [0, d_k], t \geq 0, k \in \overline{1, n}. \quad (2.2)$$

On the other hand, we shall suppose that the multiport ( $2n$ -port) is described by the following linear relation between the vector of the currents and the vector of voltages:

$$\begin{bmatrix} j_1 \\ \vdots \\ j_{2n} \end{bmatrix} = -G \begin{bmatrix} v_1 \\ \vdots \\ v_{2n} \end{bmatrix} + B(t) \quad (2.3)$$

where  $G$  is a  $2n \times 2n$  matrix of conductances and  $B$  is a  $2n$  vector depending on sources. By observing in Fig. 1 that  $i_k(t, 0) = j_{2k-1}(t)$  and  $i_k(t, d_k) = -j_{2k}(t)$  and that  $u_k(t, 0) = v_{2k-1}(t)$  and  $u_k(t, d_k) = v_{2k}(t)$  we can derive from the first relation (2.1) and from (2.3) the following system of boundary conditions:

$$\begin{bmatrix} -\frac{1}{r_1} \frac{\partial u_1}{\partial x}(t, 0) \\ \frac{1}{r_1} \frac{\partial u_1}{\partial x}(t, d_1) \\ \vdots \\ -\frac{1}{r_n} \frac{\partial u_n}{\partial x}(t, 0) \\ \frac{1}{r_n} \frac{\partial u_n}{\partial x}(t, d_n) \end{bmatrix} = -G \begin{bmatrix} u_1(t, 0) \\ u_1(t, d_1) \\ \vdots \\ u_n(t, 0) \\ u_n(t, d_n) \end{bmatrix} + B(t). \quad (2.4)$$

Of course we have to add the initial condition:

$$u_k(0, x) = u_{k0}(x), \quad x \in [0, d_k], \quad k \in \overline{1, n}$$

and let  $u_0(x)$  the vector with  $u_{k0}(x)$  as components. (2.5)

With these, our problem is the system (2.2) with boundary conditions (2.4) and with initial conditions (2.5).

### III. STATEMENT OF THE RESULTS AND COMMENTS

In this section we refer to the solutions in the classical sense of our problem. The first result deals with the sufficient conditions for existence and uniqueness of such a solution.

**Theorem 3.1:** Let us consider all independent sources having simultaneous step variation at  $t = 0$ . Then

- i) If  $G$  is a positive semidefinite matrix and  $u_0$  is a  $C^2$  function satisfying the boundary conditions, the dynamic problem (2.2), (2.4), (2.5) has a unique solution.
- ii) If  $\overline{G}$  is positive semidefinite and  $g_i > 0$  for all  $i \in \overline{1, n}$ , the steady-state solution of our problem exists and is unique.
- iii) If  $G$  is positive definite and there is  $i \in \overline{1, n}$  for which  $g_i = 0$ , there exists a unique steady-state solution.

Our second result establishes sufficient conditions for global exponential asymptotic stability of the network solutions. Before enouncing it, let us observe that if  $G$  is DRD we can find a unique  $\gamma_j \in (0, \pi/2)$  such that

$$\cos \gamma_j = \frac{S_{2j} + \sqrt{S_{2j}^2 + \frac{4}{r_j d_j} \left( G_{2j,2j} + \frac{1}{r_j d_j} \right)}}{2 \left( G_{2j,2j} + \frac{1}{r_j d_j} \right)} \quad (3.1)$$

for each  $j \in \overline{1, n}$ . For our  $2n \times 2n$  matrix  $G$  with  $G_{ij}$  elements we shall denote

$$S_i = \sum_{j=1, j \neq i}^{2n} |G_{ij}|$$

and we shall say that  $G$  is *diagonally row sum dominant* (DRD) if for each  $i$  we have  $G_{ii} > S_i$ . If  $G_{ii} \geq S_i$ ,  $G$  is said to be *weakly diagonally row sum dominant* (WDRD).

Also, if we denote

$$\omega_0 = - \min_{1 \leq j \leq n} \left( \frac{\gamma_j^2}{r_j c_j d_j^2} + \frac{g_j}{c_j} \right) \quad (3.2)$$

we have:

**Theorem 3.2:** Let  $G$  be a DRD matrix and  $u, \bar{u}$  be two solutions of the problem (2.2), (2.4), (2.5) for two different  $C^2$  initial conditions  $u_0$  and  $\bar{u}_0$ , respectively, and for the same continuous sources. Then,

$$\frac{\max_{1 \leq i \leq n} \max_{0 \leq x_i \leq d_i} |u_i(t, x_i) - \bar{u}_i(t, x_i)|}{\max_{1 \leq i \leq n} \max_{0 \leq x_i \leq d_i} |u_{i0}(x_i) - \bar{u}_{i0}(x_i)|} \leq \frac{e^{\omega_0 t}}{\min_{1 \leq i \leq n} \cos \gamma_i}$$

Moreover, the time function from the left-hand side of above inequality is strictly decreasing on  $[0, \infty)$ .

Now let us suppose that  $u_i$  are the components of the solution for step sources,  $u_{i0}$  are their initial values and  $\bar{u}_i \neq u_{i0}$  are the components of the solution of the time

independent problem with the same step sources. The above theorem gives  $\lim_{t \rightarrow \infty} u_i(t, x) = \bar{u}_i(x)$  for every  $i \in \overline{1, n}$  and  $x \in [0, d_i]$ . The evolution of all  $u_i$  functions (and, therefore, the evolution of the state of our network) from the initial state  $u_{i0}$  towards the steady state  $\bar{u}_i$  can be globally described by the function  $D: [0, \infty) \rightarrow (0, 1]$  with

$$D(t) = \frac{\max_{1 \leq i \leq n} \max_{0 \leq x_i \leq d_i} |u_i(t, x_i) - \bar{u}_i(x_i)|}{\max_{1 \leq i \leq n} \max_{0 \leq x_i \leq d_i} |u_{i0}(x_i) - \bar{u}_i(x_i)|} \quad (3.3)$$

In the assumptions of Theorem 3.2 it holds

$$D(t) \leq \bar{D}(t) = \frac{e^{\omega_0 t}}{\min_{1 \leq i \leq n} \cos \gamma_i} \quad (3.4)$$

If we fix  $\lambda \in (0, 1)$  let us define the " $\lambda$  delay time" of the circuit as the last instant when  $D(t)$  passes through  $\lambda$  value, i.e.,

$$T_\lambda = \sup_{t \geq 0} \{ t; D(t) = \lambda \} \quad (3.5)$$

Theorem 3.2 allows us to write  $T_\lambda = D^{-1}(\lambda)$  and, more important, provides us an upper bound  $\bar{T}_\lambda$  of this parameter, namely,

$$T_\lambda \leq \bar{T}_\lambda = \frac{\ln \lambda \min_{1 \leq i \leq n} \cos \gamma_i}{\omega_0} \quad (3.6)$$

It is clear from its definition that  $T_\lambda$  is a measure for the speed of signals propagation in our network. As its upper bound  $\bar{T}_\lambda$  can be easily computed from circuit parameters (see (3.1), (3.2), (3.6)) and is sufficiently tight (see examples in Section VI), it can be useful for initial design stages.

In order to apply the previous results for circuit models validation, one has to check the assumptions used in obtaining those results. Consider the usual case of the MOS interconnections analysis [2], when:

- a) one of the multiport terminals is common for all external elements (this is the common "ground" of the  $r$ - $c$ - $g$  lines)
- b) all sources are independent.

A straightforward analysis of the resistive multiports having the properties a) and b), [22], shows that:

- 1) The existence of the matrix  $G$  is ensured if and only if the following hold:
  - c) none of the 0 or  $d_i$ -terminals is connected directly (i.e., through a zero-resistance branch) to the ground
  - d) there is no direct connection between 0 and/or  $d_i$ -terminals.
- 2) If  $G$  does exist, then it is symmetric and WDRD, hence it is positively semidefinite.

All of above facts imply that if, besides the assumptions a)–d), is required that  $g_i > 0$  for all  $i$  then, according to Theorem 3.1, we obtain a broad class of circuits—let us



By checking the boundary conditions (4.2) for the functions (4.3) and using the notations  $w_{2k-1} = B_k$  and  $w_{2k} = A_k d_k + B_k$  for  $k \in \overline{1, k_0}$  and also  $w_{2k-1} = A_k + B_k$  and  $w_{2k} = A_k e^{\sqrt{a_k} d_k} + B_k e^{-\sqrt{a_k} d_k}$  for  $k \in \overline{k_0+1, n}$ , we obtain a linear equation in  $\mathbf{R}^{2n}$  of the form

$$Pw = -Gw + B \quad (4.4)$$

where  $w$  is the  $2n$  vector with components  $w_k$ . The following estimation is easy to be derived for the matrix  $P$ :

$$w^T P w \geq M_1 \sum_{k=1}^{k_0} (w_{2k-1} - w_{2k})^2 + M_2 \sum_{k=k_0+1}^n (w_{2k-1} + w_{2k})^2$$

where  $M_1$  and  $M_2$  are strictly positive constants. Hence, the matrix  $P + G$  is positive definite under each of our hypotheses ii) or iii), so that (4.4) has a solution which fixes the constants in (4.3).

The uniqueness of steady-state solution results from the dissipativity properties that follow. Let us define the operator  $A: \mathcal{D}(A) \rightarrow H_c^0$  with

$$\mathcal{D}(A) = \{ u = (u_1, \dots, u_n) \in H_c^2 \text{ and satisfies (2.4)} \}$$

and

$$Au = \begin{bmatrix} \frac{1}{r_1 c_1} \frac{\partial^2 u_1}{\partial x^2} - \frac{g_1}{c_1} u_1 \\ \dots \\ \frac{1}{r_n c_n} \frac{\partial^2 u_n}{\partial x^2} - \frac{g_n}{c_n} u_n \end{bmatrix}, \quad \text{where } u \in \mathcal{D}(A) \subset H_c^0.$$

If we take  $\bar{u}, \bar{\bar{u}} \in \mathcal{D}(A)$ , then an elementary calculation shows, that

$$\begin{aligned} & \langle \bar{u} - \bar{\bar{u}}, A\bar{u} - A\bar{\bar{u}} \rangle_{H_c^0} \\ & \leq \sum_{k=1}^n \left( -\frac{1}{r_k} \frac{\partial u_k}{\partial x}(0) \cdot u_k(0) + \frac{1}{r_k} \frac{\partial u_k}{\partial x}(d_k) \cdot u_k(d_k) \right) \\ & = -[(u_1(0), u_1(d_1), \dots, u_n(d_n)) \\ & \quad \cdot G(u_1(0), u_1(d_1), \dots, u_n(d_n))]^T \end{aligned} \quad (4.5)$$

where  $u_k$  are the components of vector  $u = \bar{u} - \bar{\bar{u}}$ . When  $G$  is positive semidefinite (4.5) implies that  $A$  is dissipative. The notation  $H_c^l(0, d_k)$ ,  $l \geq 0$ , is used for the usual Sobolev space (with a weight  $c_k$ ) of real functions defined on  $(0, d_k)$ . Particularly  $L_c^2(0, d_k) = H_c^0(0, d_k)$ . Consider also  $H_c^l = \prod_{k=1}^n H_c^l(0, d_k)$  and  $H^l = \prod_{k=1}^n H_1^l(0, d_k)$ . The inner product in  $H_c^0$  will be  $\langle u, v \rangle_{H_c^0} = \sum_{k=1}^n c_k \int_0^{d_k} u(x) v(x) dx$ .

In order to prove the  $m$ -dissipativity we have to find a function  $u \in \mathcal{D}(A)$  such as for each component

$$\frac{\partial^2 u_k}{\partial x^2} - a_k u_k = v_k \quad (4.6)$$

where  $v_k \in H_c^0(0, d_k)$  and  $a_k > 0$ . This problem splits into

two simpler problems:

$$\begin{cases} \frac{\partial^2 \bar{u}_k}{\partial x^2} - a_k \bar{u}_k = v_k, & \bar{u}_k \in H_c^2(0, d_k) \\ \bar{u}_k(0) = \bar{u}_k(d_k) = 0 \end{cases} \quad (4.7)$$

and

$$\begin{cases} \frac{\partial^2 \tilde{u}_k}{\partial x^2} - a_k \tilde{u}_k = 0, & \tilde{u}_k \in H_c^2(0, d_k) \\ \begin{bmatrix} -\frac{1}{r_1} \frac{\partial \tilde{u}_1}{\partial x}(0) \\ \frac{1}{r_1} \frac{\partial \tilde{u}_1}{\partial x}(d_1) \\ \vdots \\ \frac{1}{r_n} \frac{\partial \tilde{u}_n}{\partial x}(d_n) \end{bmatrix} = -G \begin{bmatrix} \tilde{u}_1(0) \\ \tilde{u}_1(d_1) \\ \vdots \\ \tilde{u}_n(0) \\ \tilde{u}_n(d_n) \end{bmatrix} - \begin{bmatrix} -\frac{1}{r_1} \frac{\partial \bar{u}_1}{\partial x}(0) \\ \frac{1}{r_1} \frac{\partial \bar{u}_1}{\partial x}(d_1) \\ \vdots \\ \frac{1}{r_n} \frac{\partial \bar{u}_n}{\partial x}(d_n) \end{bmatrix} + B. \end{cases} \quad (4.8)$$

Solving problem (4.7) is trivial; upon introducing its solution  $\bar{u}$  in the boundary condition from (4.8), a problem of the type (4.1), (4.2) (with  $k_0 = 0$ ) is obtained. With similar arguments we conclude that, for semi-positive  $G$ , the operator  $A$  is  $m$ -dissipative in  $H_c^0$ . On the other hand, our dynamic problem (2.2), (2.4), (2.5) is equivalent with a Cauchy problem on  $H_c^0$ , namely,

$$\begin{cases} \frac{du(t, \cdot)}{dt} = Au(t, \cdot) \\ u(0, \cdot) = u_0(\cdot). \end{cases} \quad (4.9)$$

The  $m$ -dissipativity implies that  $A$  generates a contraction semigroup of class  $C_0$ , [3], and hence, [23] for any  $u_0 \in \mathcal{D}(A)$  there exists a unique solution of (4.9) belonging to class  $C([0, \infty), H_c^0) \cap C^1([0, \infty), \mathcal{D}(A))$ . Finally, from the well-known *a priori* estimates for parabolic problems combined with Sobolev's imbedding theorem, the desired differentiability properties follow (see [24] and [25] for more classical approach). For recent study of Cauchy problem we refer to [29].

#### B. Proof of Theorem 3.2

Let  $u = (u_1, \dots, u_n)$  be the classical solution of our problem (2.2), (2.4), (2.5). For every  $k \in \overline{1, n}$  we take

$$u_k(t, x_k) = v_k(t, x_k) \cos \alpha_k \frac{x_k}{d_k} \quad (4.10)$$

where  $\alpha_k$  will be conveniently chosen. We obtain a new problem

$$\begin{cases} \frac{\partial v_k}{\partial t} = \frac{1}{r_k c_k} \frac{\partial^2 v_k}{\partial x^2} - \left( \frac{2}{r_k c_k} \frac{\alpha_k}{d_k} t g \frac{\alpha_k x_k}{d_k} \right) \frac{\partial v_k}{\partial x} \\ - \left[ \left( \frac{\alpha_k}{d_k} \right)^2 \frac{1}{r_k c_k} + \frac{g_k}{c_k} \right] v_k, \\ x \in [0, d_k], \quad t \geq 0, \quad k \in \overline{1, n} \end{cases} \quad (4.11)$$

with boundary conditions: (we leave out the argument "t" of  $v_k$  and  $\partial v_k/\partial x$ )

$$\begin{bmatrix} -\frac{1}{r_1} \frac{\partial v_1}{\partial x}(0) \\ \frac{1}{r_1} \frac{\partial v_1}{\partial x}(d_1) \cos \alpha_1 \\ \vdots \\ -\frac{1}{r_n} \frac{\partial v_n}{\partial x}(0) \\ \frac{1}{r_n} \frac{\partial v_n}{\partial x}(d_n) \cos \alpha_n \end{bmatrix} = -G \begin{bmatrix} v_1(0) \\ v_1(d_1) \cos \alpha_1 \\ \vdots \\ v_n(0) \\ v_n(d_n) \cos \alpha_n \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\alpha_1 v_1(d_1)}{r_1 d_1} \sin \alpha_1 \\ 0 \\ \vdots \\ 0 \\ \frac{\alpha_n v_n(d_n)}{r_n d_n} \sin \alpha_n \end{bmatrix} + B(t) \quad (4.12)$$

and with initial conditions

$$v(0, x) = u_{k0}(x) / \cos \alpha_k \frac{x}{d_k}. \quad (4.13)$$

The problem (4.11)–(4.13) can be seen as an abstract Cauchy problem on the space

$$Z = \left\{ f: \prod_{k=1}^n [0, d_k] \rightarrow \mathbf{R}^n, \text{ with components } f_k: [0, d_k] \rightarrow \mathbf{R} \text{ continuous} \right\}.$$

For this we put

$$\mathcal{D}(\Lambda) = \{ f \in Z, \text{ with components twice differentiable and satisfying (4.12)} \}$$

and we define  $\Lambda: \mathcal{D}(\Lambda) \rightarrow Z$  with components given by the right-hand side of (4.11). Then (4.11)–(4.13) is equivalent with

$$\begin{cases} \frac{d}{dt} v(t, \cdot) = \Lambda v(t, \cdot) \\ v(0, \cdot) = \text{a vector with components given by (4.13).} \end{cases} \quad (4.14)$$

The following result is essential for our aims:

**Lemma 4.1:** We suppose that  $G$  is DRD and for every  $j \in \overline{1, n}$  we denote  $\alpha_j = \gamma_j - \epsilon$  where  $\gamma_j \in (0, \pi/2)$  is the solution of (3.1) and  $\epsilon > 0$  is such that  $\alpha_j \in (0, \pi/2)$ . If we take

$$\omega_\epsilon = - \min_{1 \leq j \leq n} \left[ \frac{(\gamma_j - \epsilon)^2}{d_j^2 r_j c_j} + \frac{g_j}{c_j} \right]$$

then, the operator  $\Lambda - \omega_\epsilon I$  constructed with these  $\alpha_j$  is dissipative on  $Z$ .

*Proof:* If we take  $v, \bar{v} \in \mathcal{D}(\Lambda)$  and suppose the non-trivial case  $w = v - \bar{v} \neq 0$ , according with Lemma A (see the Appendix) we have to prove

$$\sup_{(j; x) \in M(w)} [\Lambda v - \Lambda \bar{v}]_j(x) \operatorname{sgn} w_j(x) \leq \omega_\epsilon \|w\|_Z. \quad (4.15)$$

Let us take the pair  $(p; y) \in M(w)$ , i.e.,

$$\max_{1 \leq i \leq n} \max_{0 \leq x_i \leq d_i} |w_i(x_i)| = |w_p(y)| = \|w\|_Z \quad (4.16)$$

and for which clearly we have  $w_p(y) \neq 0$ . If we suppose already proved that  $y \neq 0, d_p$ , two cases may appear. For the first one, i.e.,  $w_p(y) > 0$ ,  $y$  is a maximum point for  $w_p$  in  $(0, d_p)$  and then, from (4.11) we have

$$\begin{aligned} & [\Lambda v - \Lambda \bar{v}]_p(y) \operatorname{sgn} w_p(y) \\ &= \frac{1}{r_p c_p} \frac{\partial^2 w_p}{\partial x^2}(y) - \left( \frac{2}{r_p c_p} \frac{\alpha_p}{d_p} \operatorname{tg} \frac{\alpha_p x_p}{d_p} \right) \frac{\partial w_p}{\partial x}(y) \\ &\quad - \left( \frac{\alpha_p^2}{d_p^2 r_p c_p} + \frac{g_p}{c_p} \right) |w_p(y)| \\ &\leq - \left( \frac{\alpha_p^2}{d_p^2 r_p c_p} + \frac{g_p}{c_p} \right) |w_p(y)| \leq \omega_\epsilon \|w\|_Z. \end{aligned}$$

The same inequality can be obtained in the second case  $w_p(y) < 0$ , when  $y$  is a minimum point for  $w_p$ . Thus we have shown (4.15). It remains to prove that  $y \neq 0, d_p$ . Let us suppose  $y = 0$ . From odd rows of (4.12), following (4.16) we easily derive

$$-\frac{1}{r_p} \frac{\partial w_p}{\partial x}(0) w_p(0) \leq (-G_{2p-1, 2p-1} + S_{2p-1}) |w_p(0)|^2$$

and from DRD property of  $G$  it results

$$-\frac{1}{r_p} \frac{\partial w_p}{\partial x}(0) w_p(0) < 0. \quad (4.17)$$

But if  $w_p(0) > 0$ , then  $w_p(0) \geq w_i(x)$  for any  $i$  and  $x$  and

$$\left[ -\frac{1}{r_p} \lim_{x_p \rightarrow 0^+} \frac{w_p(x_p) - w_p(0)}{x_p} \right] w_p(0) \geq 0$$

that contradicts (4.17). Also if  $w_p(0) < 0$ , then  $w_p(0) \leq w_i(x)$  for any  $i$  and  $x$  and we obtain again a contradiction. Hence  $y \neq 0$ .

Let us suppose now  $y = d_p$ . Also, from boundary conditions (4.12) (namely from even rows) and from (4.16) we may derive the following inequality:

$$\begin{aligned} & \frac{1}{r_p} \frac{\partial w_p}{\partial x}(d_p) w_p(d_p) \cos \alpha_p \\ &\leq \left( -G_{2p, 2p} \cos \alpha_p + S_{2p} + \frac{\alpha_p}{r_p d_p} \sin \alpha_p \right) |w_p(d_p)|^2. \end{aligned} \quad (4.18)$$

On the other hand, we have  $\cos \alpha_p > \cos \gamma_p =$  the right-

hand side member of (3.1) (with  $j = p$ ) and this implies

$$-\left(G_{2p,2p} + \frac{1}{r_p d_p}\right) \cos^2 \alpha_p + S_{2p} \cos \alpha_p + \frac{1}{r_p d_p} < 0.$$

From here, with the inequality  $\alpha_p < \tan \alpha_p$  we derive that

$$-G_{2p,2p} \cos \alpha_p + S_{2p} + \frac{\alpha_p}{r_p d_p} \sin \alpha_p < 0$$

and (4.18) gives us

$$\frac{1}{r_p} \frac{\partial w_p}{\partial x}(d_p) w_p(d_p) \cos \alpha_p < 0.$$

But, as above, is easy to see that we have obtained a contradiction with  $(p; d_p) \in M(w)$ . Then  $y \neq d_p$  and the proof of Lemma 4.1 is finished.

Now we are in position to obtain the result of Theorem 3.2. If we denote by  $v$  and  $\bar{v}$  the correspondent functions of  $u$  and  $\bar{u}$ , respectively, via (4.10) with  $\alpha_k = \gamma_k - \epsilon$  and  $\gamma_k$  given by (3.1), these functions satisfy (4.14) resulting in

$$\frac{d}{dt} [v(t, \cdot) - \bar{v}(t, \cdot)] = \Lambda v(t, \cdot) - \Lambda \bar{v}(t, \cdot). \quad (4.19)$$

On the other hand, the norm  $\|v(t, \cdot) - \bar{v}(t, \cdot)\|_Z$  is differentiable on  $\mathcal{M} = [0, \infty) - \mathcal{N}$  where  $\mathcal{N}$  is an at most countable set. A known result of Coppel [27] combined with (4.18) gives us

$$\begin{aligned} \frac{d}{dt} \|v(t, \cdot) - \bar{v}(t, \cdot)\|_Z \\ = \sigma(v(t, \cdot) - \bar{v}(t, \cdot), \Lambda v(t, \cdot) - \Lambda \bar{v}(t, \cdot)) \end{aligned}$$

for any  $t \in \mathcal{M}$ . From here and from the previous lemma we obtain on  $\mathcal{M}$ ,

$$\frac{d}{dt} \|v(t, \cdot) - \bar{v}(t, \cdot)\|_Z \leq \omega_\epsilon \|v(t, \cdot) - \bar{v}(t, \cdot)\|_Z. \quad (4.20)$$

Hence  $\|v(t, \cdot) - \bar{v}(t, \cdot)\|_Z \leq \|v(0, \cdot) - \bar{v}(0, \cdot)\|_Z e^{\omega_\epsilon t}$  for every  $t \geq 0$ . From this relation, with (4.10) and  $\epsilon \rightarrow 0$ , we easily derive the inequality from theorem. Also, because  $\omega_\epsilon < 0$ , (4.20) gives the desired monotony.

## V. APPROXIMATION OF THE SOLUTION

We shall approximate the solution  $u$  of the problem (2.2), (2.4), (2.5) by the finite difference method in time and the finite element method in space. If  $\phi = (\phi_1, \dots, \phi_n) \in H^1 = \prod_{k=1}^n H_{c_k}^1(0, d_k)$ , then by (4.9),

$$\left\{ \begin{aligned} \left\langle \frac{d}{dt} u(t), \phi \right\rangle_{H^0} &= \langle Au, \phi \rangle_{H^0} \\ \langle u(0), \phi \rangle_{H^0} &= \langle u_0, \phi \rangle_{H^0} \end{aligned} \right. \quad (5.1)$$

where  $\langle \cdot, \cdot \rangle_{H^0}$  denotes the inner product in  $H^0 = \prod_{k=1}^n L_{c_k}^2(0, d_k)$ .

If we formally integrate by parts, then according with (2.5) we get that

$$\frac{d}{dt} \langle u(t), \phi \rangle_{H^0} + P(u(t), \phi) = R(B(t), \phi) \quad (5.2)$$

for all  $\phi \in H^1$ , where

$$\begin{aligned} P(u(t), \phi) &= \sum_{k=1}^n \frac{1}{c_k} \left\{ \int_0^{d_k} \left[ \frac{1}{r_k} \frac{\partial u_k}{\partial x}(t, x) \frac{\partial \phi_k}{\partial x}(x) \right. \right. \\ &\quad \left. \left. + g_k u_k(t, x) \phi_k(x) \right] dx \right. \\ &\quad \left. + \phi_k(0) \cdot \sum_{j=1}^n [G_{2k-1,2j} u_j(0) + G_{2k-1,2j-1} u_j(d_j)] \right. \\ &\quad \left. + \phi_k(d_k) \cdot \sum_{j=1}^n [G_{2k,2j} u_j(0) + G_{2k-1,2j-1} u_j(d_j)] \right\} \end{aligned}$$

and

$$R(B(t), \phi) = \sum_{k=1}^n \frac{1}{c_k} [b_{2k-1}(t) \phi_k(0) + b_{2k}(t) \phi_k(d_k)].$$

To approximate (5.2) we suppose that a family of finite dimensional subspaces  $\{S_h\}_{h>0}$  of  $H^1$  is given such that for a small positive  $h$  there is a constant  $c$ , independent of  $h$  and  $v$  for which

$$\inf_{\phi \in S_h} \{ \|v - \phi\|_{H^0} + h \|v - \phi\|_{H^1} \} \leq ch^s \|v\|_{H^s}$$

where  $v \in H^s$  for  $s = 2$ . For examples of such subspaces, which include the usual piecewise-linear finite element spaces see [28].

We may now pose the semi-discrete problem corresponding to (5.2). Find  $u_h(t) \in S_h$  for  $t > 0$ , such that

$$\left\{ \begin{aligned} \frac{d}{dt} \langle u_h(t), \phi \rangle_{H^0} + P(u_h(t), \phi) &= R(B(t), \phi) \\ \langle u_h(0), \phi \rangle_{H^0} &= \langle u_0, \phi \rangle_{H^0}, \\ \text{for all } \phi \in S_h. \end{aligned} \right. \quad (5.3)$$

Once a basis for the finite-dimensional space

$$S_h = S_h^1 \times \dots \times S_h^n, \quad \{\phi_1^1, \dots, \phi_{N_1}^1, \dots, \phi_1^n, \dots, \phi_{N_n}^n\}$$

is given ( $N_i = N_i(h)$  is the dimension of  $S_h^i$ ), we can write

$$u_h(t) = \sum_{i=1}^n \sum_{j=1}^{N_i} Q_j^i(t) \phi_j^i.$$

In this way, the problem (5.3) comes to a system of ordinary differential equations of the form:

$$\left\{ \begin{aligned} MQ'(t) + KQ(t) &= F(t) \\ Q(0) &= Q_0 \end{aligned} \right. \quad (5.4)$$

where the mass and the stiffness matrices are, respectively,

$$M = \{ \langle \phi_i^j, \phi_k^l \rangle_{H^0} \} \text{ and } K = \{ P(\phi_i^j, \phi_k^l) \}.$$

The system (5.4) can be solved numerically, for example, by implicit Euler difference scheme. If  $\Delta t$  denotes the time mesh size, and  $Q^k = Q(k\Delta t)$ ,  $F^k = F(k\Delta t)$ , then

$$\left\{ \begin{aligned} \left( \frac{M}{\Delta t} + K \right) Q^{k+1} &= F^{k+1} + \frac{M}{\Delta t} Q^k, \quad k = 0, 1, \dots \\ Q^0 &= Q_0 \end{aligned} \right. \quad (5.5)$$



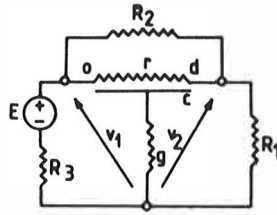


Fig. 3. The network for Examples I, II, and III.

TABLE I

Ex.	$r(\Omega\text{m}^{-1})$	$c(\text{Fm}^{-1})$	$g(\Omega^{-1}\text{m}^{-1})$	$d(\text{m})$
6.1	4000	9000	0.001	0.001
6.2	4000	9	0.001	0.001
6.3	400	0.9	0.0001	0.01

## VI. NUMERICAL EXAMPLES

Let us consider a very simple network (Fig. 3) with the resistive part having the values  $R_1 = 5 \Omega$ ,  $R_2 = 5 \Omega$ ,  $R_3 = 1 \Omega$ ,  $E = 1 \text{ V}$  and with the distributed parameters from Table I (three variants, Examples I–III).

The matrix  $G$  and the vector  $B$  have the following terms:  $G_{11} = R_2^{-1} + R_3^{-1} = 1.2$ ,  $G_{22} = R_1^{-1} + R_2^{-1} = 0.4$ ,  $G_{12} = G_{21} = -R_2^{-1} = -0.2$ ,  $b_1 = E/R_3 = 1$ ,  $b_2 = 0$ . The matrix  $G$  is symmetric and DRD. This is why our network has unique dynamic and steady-state solutions and by (3.1), (3.2), (3.4), (3.6) we can calculate  $\bar{D}(t)$ ,  $\bar{T}_{0.1}$ ,  $\bar{T}_{0.5}$ .

On the other hand, we have computed numerically our problem by the method presented in Section V. In Fig. 4 we give the space variation of the solution for the different values  $t_i = i\Delta t$  of the time in the case of Example I.

The discretization parameters were  $h = 0.0202 \cdot d$  and  $\Delta t = 8.52$ . For the same example, in Fig. 5 the numerically computed  $D(t)$ ,  $T_{0.1}$ ,  $T_{0.5}$  together with their *a priori* upper bounds are given.

The results for the delay time in Examples I–III are summarized in Table II.

As expected by intuition, the delay time  $T_\lambda$  is (approximately) the same in Examples II and III, where the global parameters  $rd$ ,  $cd$ , and  $gd$  have been chosen to be the same. A high mark of our bound  $\bar{T}_\lambda$  is the fact that it is also the same for the two cases.

Let us consider now the simplest amplifier with one bipolar transistor and with two distributed elements (Fig. 6).

In Fig. 7 the same circuit is plotted with the incremental equivalent model for the transistor.

The physical parameters are:

- for the resistances:  $R_1 = 50 \Omega$ ,  $R_2 = 10^3 \Omega$ ,  $R_C = 10^3 \Omega$ ,  $R_e = 12235 \Omega$ .
- for the small signal model:  $\beta = 50$ ,  $h = 10^3 \Omega$ .
- for the distributed elements:  $r_1 = 2 \times 10^7 \Omega \cdot \text{m}^{-1}$ ,  $c_1 = 2 \times 10^{-8} \text{ F} \cdot \text{m}^{-1}$ ,  $g_1 = 0$ ,  $d_1 = 5 \times 10^{-3} \text{ m}$ ,  $r_2 = 5 \times 10^5 \Omega \cdot \text{m}^{-1}$ ,  $c_2 = 5 \times 10^{-6} \text{ F} \cdot \text{m}^{-1}$ ,  $g_2 = 0$ ,  $d_2 = 2 \times 10^{-3} \text{ m}$ .

With these parameters  $G$  is a nonsymmetric but positive definite and DRD matrix:  $G_{11} = 2 \times 10^{-2} \Omega^{-1}$ ,  $G_{22} = 1.6 \times$

Time dependent solutions

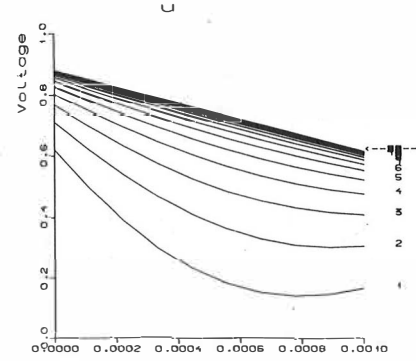


Fig. 4. Numerically computed solutions in Example I.

The delay time

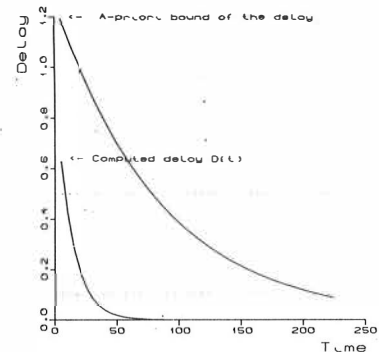


Fig. 5.  $D(t)$  and  $\bar{D}(t)$  in Example I.

TABLE II

Ex.	$\omega_0 (\text{s}^{-1})$	$\cos \gamma$	$T_{0.1} (\text{s})$	$\bar{T}_{0.1} (\text{s})$	$T_{0.5} (\text{s})$	$\bar{T}_{0.5} (\text{s})$
6.1	-0.0119	0.7928	~ 43	213	~ 17	77.758
6.2	-11.9429	0.7928	~ 0.043	0.213	~ 0.017	0.077
6.3	-11.9429	0.7928	~ 0.043	0.213	~ 0.017	0.077

$10^{-6} \Omega^{-1}$ ,  $G_{33} = 10^{-3} \Omega^{-1}$ ,  $G_{44} = 10^{-3} \Omega^{-1}$ ,  $G_{32} = 8 \times 10^{-5} \Omega^{-1}$  and other components are  $= 0$ . For  $B$ , only the first element is nonzero:  $b_1 = 2 \times 10^{-2} \text{ A}$ .

After a calculus with formulas from Section III we obtain

$$\omega_0 = -14490 \text{ s}^{-1} \text{ and } \min \{ \cos \gamma_1, \cos \gamma_2 \} = 0.707.$$

These give *a priori* bounds of response time:

$$\bar{T}_{0.1} = 182.8 \mu\text{s} \text{ and } \bar{T}_{0.5} = 71.76 \mu\text{s}.$$

On the other hand, the application of the numerical method of Section V provides the results shown in Fig. 8 for  $u_{1,h}(t, x)$  and in Fig. 9 for  $u_{2,h}(t, x)$ . The discretization parameters are  $h_1 = 0.0202 \cdot d_1$ ,  $h_2 = 0.0202 \cdot d_2$ ,  $\Delta t = 1.8 \times 10^{-6}$ . Numerical calculus gives  $T_{0.1} = 14.4 \mu\text{s}$  and  $T_{0.5} = 7.2 \mu\text{s}$ .

In Fig. 10 we can see the computed  $D(t)$  and its *a priori* bound.

All above examples show that our bounds are sufficiently tight for a first evaluation, if we take into account how simple they are to calculate. Certainly, a similar lower bound of the delay time is desirable.

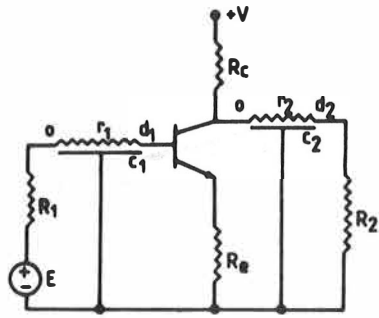


Fig. 6. A simple transistor network.

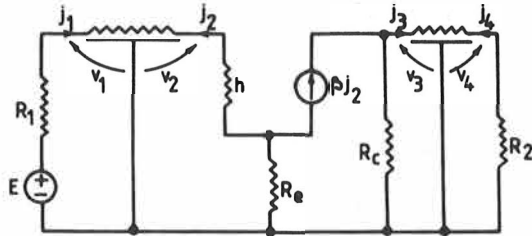


Fig. 7. The equivalent circuit.

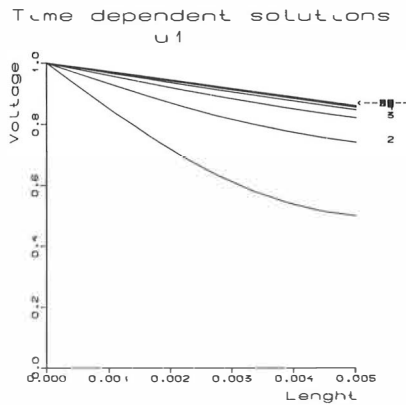


Fig. 8. Numerically computed  $u_1(t, x)$ .

## VII. CONCLUSIONS

The complexity of circuits models whose consistency is assured by our theorems is limited by the existence of the description matrix of the resistive multiport and by its properties needed as hypotheses. As we saw, the area of applicability of the results is large enough.

As regards the delay time, in many cases this quantity is useful as a comparison criterium. For instance, in circuit design, in view to obtain from a circuit  $C_1$  "a two times faster" circuit  $C_2$ , we must change the parameters such that  $T_\lambda^2/T_\lambda^1 = 2$ . If the equality  $\bar{T}_\lambda^2/\bar{T}_\lambda^1 = T_\lambda^2/T_\lambda^1$  is (at least approximately) satisfied, then  $\bar{T}_\lambda$  itself can be considered as "the delay time" of the network. Examples I, II, III (regarding the same circuit) show that this is really the case.

## APPENDIX

If  $X$  is a real Banach space with the norm  $\|\cdot\|_X$ , let us consider the functional  $\sigma: X \times X \rightarrow \mathbf{R}$  with  $\sigma(a, b) = \lim_{h \rightarrow 0^+} (\|a + hb\|_X - \|a\|_X)/h$ . We shall say that  $f: \mathcal{D} \subset$

Time dependent solutions

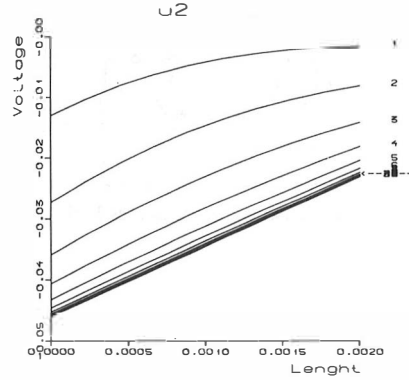


Fig. 9. Numerically computed  $u_2(t, x)$ .

The delay time

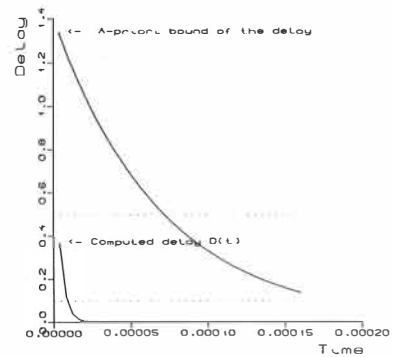


Fig. 10.  $D(t)$  and  $\bar{D}(t)$ .

$X \rightarrow X$  is dissipative if  $\sigma(x - y, f(x) - f(y)) \leq 0$  for any  $x, y \in \mathcal{D}$ . If in addition  $\text{Range}(I - f) = X$ , then  $f$  is said to be  $m$ -dissipative. If  $\alpha$  is a real number, it is easily to show that  $f - \alpha I$  is dissipative iff for every  $x, y \in \mathcal{D}$ ,

$$\sigma(x - y, f(x) - f(y)) \leq \alpha \|x - y\|_X.$$

For a Hilbert space  $X$  with the inner product  $\langle \cdot, \cdot \rangle_X$ , we find  $\|a\|_X \sigma(a, b) = \langle a, b \rangle_X$ , so that  $f$  is dissipative iff  $\langle x - y, f(x) - f(y) \rangle_X \leq 0$  for any  $x, y \in \mathcal{D}$ .

Finally, if  $Y$  is a compact space, let us consider the Banach space  $C = C(Y, \mathbf{R}^n) = \{f: X \rightarrow \mathbf{R}^n \text{ with continuous } f_i \text{ components}\}$  with the usual supremum norm. In order "to calculate" the  $\sigma$  functional in this space, we shall give a result which is an extension of a theorem of Sato [26], with a simpler proof.

*Lemma A:* For  $f, g \in C$ ,  $f \neq 0$ , it holds

$$\sigma(f, g) = \sup_{(p; x) \in M(f)} g_p(x) \text{sgn } f_p(x)$$

where

$$M(f) = \left\{ (p; \bar{x}) \mid \max_{1 \leq i \leq n} \max_{x \in Y} |f_i(x)| = |f_p(\bar{x})| = \|f\|_C \right\}.$$

*Proof:* Let  $\{\epsilon_k\}_{k=1}^\infty$  with  $\epsilon_k \rightarrow 0^+$  for  $k \rightarrow \infty$ . For every pair  $\{k, i\}$  of indexes we choose  $x_{k,i} \in Y$  such that

$$\max_{x \in Y} |f_i(x) + \epsilon_k g_i(x)| = |f_i(x_{k,i}) + \epsilon_k g_i(x_{k,i})|.$$

Consequently,

$$\begin{aligned} \|f + \epsilon_k g\|_C &= \max_i |f_i(x_{k,i}) + \epsilon_k g_i(x_{k,i})| \\ &= |f_p(x_{k,p}) + \epsilon_k g_p(x_{k,p})|. \end{aligned} \quad (\text{A.1})$$

We choose from  $\{x_{k,p}\}_{k=1}^\infty$  a convergent subsequence (keeping the same notation)  $\{x_{k,p}\}_{k=1}^\infty \rightarrow x_p$ , for  $k \rightarrow \infty$ . By (A.1),  $(p, x_p) \in M(f)$ . On the other hand, it is easy to see that there exists an index  $N$  such that for  $k > N$  we have  $\text{sgn}[f_p(x_{k,p}) + \epsilon_k g_p(x_{k,p})] = \text{sgn} f_p(x_{k,p}) = \text{sgn} f_p(x_p)$ . Hence, taking also into account the relation (A.1) we obtain for  $k > N$ :

$$\begin{aligned} \frac{\|f + \epsilon_k g\|_C - \|f\|_C}{\epsilon_k} &\leq \frac{|f_p(x_{k,p}) + \epsilon_k g_p(x_{k,p})| - |f_p(x_{k,p})|}{\epsilon_k} \\ &= g_p(x_{k,p}) \text{sgn} f_p(x_{k,p}). \end{aligned}$$

From here we find

$$\sigma(f, g) \leq g_p(x_p) \text{sgn} f_p(x_p) \leq \sup_{(p; x) \in M(f)} g_p(x) \text{sgn} f_p(x).$$

In order to give the converse part of the proof, we take  $(p, x_p) \in M(f)$  and we observe that

$$\frac{\|f + \epsilon g\|_C - \|f\|_C}{\epsilon} \geq |f_p(x_p)| \frac{\left| \frac{1 + \epsilon g_p(x_p)}{f_p(x_p)} - 1 \right|}{\epsilon}.$$

Due to the fact that the right-hand side tends to  $|f_p(x_p)|(g_p(x_p)/f_p(x_p))$  when  $\epsilon$  tends to  $0^+$ , the inequality follows.

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