MAPPINGS OF L^p-INTEGRABLE DISTORTION: REGULARITY OF THE INVERSE

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Abstract. Let $\mathbb{X}$ be an open set in $\mathbb{R}^n$ and suppose that $f\colon \mathbb{X} \to \mathbb{R}^n$ is a Sobolev homeomorphism. We study the regularity of f^{-1} under the L^p-integrability assumption on the distortion function K_f. First, if $\mathbb{X}$ is the unit ball and $p > n-1$, then the optimal local modulus of continuity of f^{-1} is attained by a radially symmetric mapping. We show that this is not the case when $p \leqslant n-1$ and $n \geqslant 3$, and answer a question raised by S. Hencl and P. Koskela. Second, we obtain the optimal integrability results for $|Df^{-1}|$ in terms of the L^p-integrability assumptions of K_f.

1. Introduction

Let $\mathbb{X}$ be an open set in $\mathbb{R}^n$, $n \geqslant 2$. A Sobolev homeomorphism $f \in W^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{R}^n)$ has *finite distortion* if there is a measurable function $K(x) \geqslant 1$, finite a.e., such that

$$|Df(x)|^n \leqslant K(x) J_f(x) , \qquad J_f(x) := \det Df(x) . \tag{1.1}$$

Hereafter $|A|$ is the operator norm of a linear map A. The smallest function $K(x) \geqslant 1$ for which the distortion inequality (1.1) holds is denoted by K_f. The target of $\mathbb{X}$ under f will be denoted by $\mathbb{Y}$, i.e. $f\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$. The theory of mappings of finite distortion [2, 10, 11] has arisen from a need to extend the ideas and applications of the classical theory of quasiconformal homeomorphisms ($K \in L^\infty(\mathbb{X})$) to the degenerate elliptic setting. However, some bounds on the distortion are needed to obtain a viable theory. There one finds applications in materials science, particularly nonlinear elasticity. The mathematical models of nonlinear elasticity have been pioneered by Antman [1], Ball [3], and Ciarlet [4]. The general law of hyperelasticity tells us that there exists an energy integral functional with given stored-energy function characterizing elastic properties of the material. If the reference configuration is the unit ball one of important problems in nonlinear elasticity is whether or not the radially symmetric minimizers are indeed global minimizers of the given energy. Here we first study optimal modulus of continuity for the inverse in the class of homeomorphisms with L^p-integrable distortion. And, bring into question whether or not such regularity is already obtained within the class of radially symmetric homeomorphisms.

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1.1. **Modulus of continuity of the inverse.** If $K_f \in L^p(\mathbb{X})$ for some $p > n-1$, then the inverse $f^{-1} \colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ enjoys locally α-logarithmic modulus of continuity with $\alpha = \frac{p(n-1)}{n}$; that is, for $B(y_\circ, R) := \{y \in \mathbb{R}^n \colon |y - y_\circ| < R\} \subset\subset \mathbb{Y}$ and all points $y \in B(y_\circ, R)$ we have

$$|f^{-1}(y) - f^{-1}(y_\circ)| \log^\alpha \left(R/|y-y_\circ|\right) \leqslant C \,, \tag{1.2}$$

where C depends only on p, n and the integral of K_f^p. This optimal modulus of continuity was established by Clop and Herron [5] and earlier in dimension two by Koskela and Takkinen [13]. Regardless the optimality, it is natural to examine radially symmetric mappings $f \colon \mathbb{B} \to \mathbb{R}^n$,

$$f(x) = \begin{cases} F(|x|)\, \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \,, \end{cases} \tag{1.3}$$

where $F \colon [0,1] \to [0, \infty)$ is a strictly increasing continuous function with $F(0) = 0$. Indeed, if we take $F(t) = \exp(-t^{-\frac{1}{\beta}})$ in (1.3), then the corresponding radially symmetric homeomorphism shows the exponent α in (1.2) cannot be replaced by $\beta > \alpha$.

Next, we consider the case $p = n-1$. On the one hand, the exponent α in (1.2) approaches $(n-1)^2/n$ as $p \to n-1$. In the class of radially symmetric mappings we have such a modulus of continuity.

Proposition 1.1. *Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$. Suppose that $f \colon \mathbb{B} \to \mathbb{R}^n$ is a radially symmetric homeomorphism with $K_f \in L^p(\mathbb{B})$, for some $p \geqslant 1$. Then the inverse of f has $\frac{p(n-1)}{n}$-logarithmic modulus of continuity.*

Again the radial mapping (1.3) with $F(t) = \exp(-t^{-\frac{1}{\beta}})$ and $\beta > \frac{p(n-1)}{n}$ shows that this result is sharp. On the other hand, if we relax the symmetric assumption and just require that $K_f \in L^{n-1}(\mathbb{X})$, then the inverse $f^{-1} \in W^{1,n}_{\text{loc}}(\mathbb{Y}, \mathbb{R}^n)$, see [6, 9]. It follows from the Sobolev embedding theorem on spheres [8, 15] that any $W^{1,n}$-homeomorphism has locally $1/n$-logarithmic modulus of continuity. Therefore, the inverse f^{-1} satisfies (1.2) with the exponent $1/n$. These two exponents only coincide in the planar case. In their recent monograph Hencl and Koskela raised the question [10, Open problem 13.] as what is the optimal exponent in the borderline case, $p = n-1$. Next result answers to this question.

Theorem 1.2. *Let $n \geqslant 3$ and $\beta > 1/n$. Then there exists a Sobolev homeomorphism $f \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ with $K_f \in L^{n-1}(\mathbb{X})$ such that for some $y_\circ \in \mathbb{Y}$ we have*

$$\limsup_{y \to y_\circ} \Big[|f^{-1}(y) - f^{-1}(y_\circ)| \, \log^\beta (1/|y - y_\circ|) \Big] = \infty \,. \tag{1.4}$$

In summary, let $f\colon \mathbb{X} \to \mathbb{R}^n$, $n \geqslant 2$, be a Sobolev homeomorphism with $K_f \in L^p(\mathbb{X})$. Then the inverse f^{-1} satisfies (1.2) with

$$\alpha = \begin{cases} \frac{p(n-1)}{n}, & \text{when } p > n-1 \\ \frac{1}{n}, & \text{when } p = n-1 \,. \end{cases}$$

Moreover, this exponent is optimal; that is, α in (1.2) cannot be replaced by any $\beta > \alpha$. Furthermore, if $p < n-1$ such an α does not exist. Even more,

Proposition 1.3. *Let $\varphi\colon [0,1] \to \mathbb{R}$ be a strictly increasing continuous function with $\varphi(0) = 0$. Then there exists a homeomorphism $f\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ with $K_f \in L^p(\mathbb{X})$ for every $1 \leqslant p < n-1$ and for some $y_\circ \in \mathbb{Y}$ we have*

$$\limsup_{y \to y_\circ} \frac{|f^{-1}(y) - f^{-1}(y_\circ)|}{\varphi(|y - y_\circ|)} = \infty \,. \tag{1.5}$$

TABLE 1. Modulus of Continuity of f^{-1}

$K_f \in L^p$	α in (1.2)	
		$f(x) = F(\lvert x\rvert)\frac{x}{\lvert x\rvert}$
$p > n-1$	$\frac{p(n-1)}{n}$	$\frac{p(n-1)}{n}$
$p = n-1$	$\frac{1}{n}$	$\frac{p(n-1)}{n}$
$p < n-1$	Does not exist	$\frac{p(n-1)}{n}$

1.2. **Integrability of $|Df^{-1}|$.** As we pointed out earlier the inverse map f^{-1} enjoys the $W^{1,n}$-regularity provided K_f is L^{n-1}-integrable [6, 9]. Here we obtain a L^p-counterpart of this result.

Theorem 1.4. *Let $\mathbb{X}$ be an open set in $\mathbb{R}^n$. Suppose that $f\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is a homeomorphism of finite distortion with $K_f \in L^p(\mathbb{X})$ for some $p \geqslant n-1$. Then*

$$\int_{\mathbb{Y}} |Df^{-1}(y)|^n \log^q (e + |Df^{-1}(y)|) \, \mathrm{d}y \leqslant C \int_{\mathbb{X}} K_f^p(x) \, \mathrm{d}x \,, \tag{1.6}$$

where $q = (n-1)(p-n+1)$ and the finite constant C depends only on n and p.

The simple example $f\colon \mathbb{B} \to \mathbb{R}^n$, $f(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n|x_n|^s)$, $0 < s < 1/(n-1)$ shows that for $f^{-1} \in W^{1,n}(f(\mathbb{B}), \mathbb{R}^n)$ the L^{n-1}-integrability of K_f cannot be relaxed. Even in the class of radially symmetric homeomorphisms this is not possible.

Proposition 1.5. *There exists a radially symmetric homeomorphism $f\colon \mathbb{B} \to \mathbb{R}^n$ such that $K_f \in L^p(\mathbb{B})$ for every $1 \leqslant p < n-1$ and*

$$\int_{f(\mathbb{B})} |Df^{-1}(y)|^n \,\mathrm{d}y = \infty \,.$$

Next, we compare Theorem 1.4 with the modulus of continuity results for f^{-1}. It follows from the Sobolev embedding theorem on spheres [8, 15] that a homeomorphism $g\colon \mathbb{X} \to \mathbb{R}^n$ whose differential belongs to the Lorentz class $L^n \log^q(\mathbb{X})$ for some $q > -1$; that is, $|Dg|^n \log^q(e+|Dg|) \in L^1(\mathbb{X})$ has locally $(q+1)/n$-logarithmic modulus of continuity. Especially, Theorem 1.4 gives that the inverse has locally γ-logarithmic modulus of continuity provided $K_f \in L^p(\mathbb{X})$ for some $p \geqslant n-1$. Here

$$\gamma = \frac{p(n-1)}{n} - (n-2)\,.$$

Therefore, Theorem 1.4 gives the optimal modulus of continuity for the inverse mapping when $p = n-1$ or $n = 2$. Otherwise, however, the optimal modulus of continuity of f^{-1} does not follow from the integrability estimate (1.6). As we pointed out earlier, the radial mapping (1.3) with $F(t) = \exp(-t^{-\frac{1}{\beta}})$ reaches this optimal modulus of continuity. We further note that the mapping in question has L^p-integrable distortion if and only if $|Df^{-1}|$ belongs to the Lorentz class $L^n \log^{p(n-1)-1}(\mathbb{X})$. In spite of this, we prove that Theorem 1.4 is optimal.

Theorem 1.6. *Let $n \geqslant 2$, $p \geqslant n-1$ and $s > (n-1)(p-n+1)$. Then there exists open subsets $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$, and a Sobolev homeomorphism $f\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$, such that $K_f \in L^p(\mathbb{X})$ and*

$$\int_{\mathbb{Y}} |Df^{-1}|^n \log^s(e + |Df^{-1}|) = \infty\,.$$

2. Modulus of continuity of the inverse

2.1. **Proof of Theorem 1.2.** Fix $\beta > 1/n$ and choose α so that $\beta > \alpha > 1/n$. We consider the cylindrical domain

$$\mathbb{X} := \{x = (\omega, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \colon |\omega| < \delta \text{ and } |x_n| < 1\},$$

where $0 < \delta < \exp\big(-\frac{(n-1)!}{2}(\alpha^{n-1}+1)\big)$ is fixed, and define

$$L(r) := \begin{cases} \log^{-\alpha} 1/r, & \text{if } \ 0 < r < \delta \\ 0, & \text{if } \ r = 0\,. \end{cases}$$

The mapping we are referring takes the form

$$f(\omega, x_n) = \left(x_1\, L(|\omega|), \dots, \, x_{n-1}\, L(|\omega|), \, x_n |\omega| + \frac{x_n}{|x_n|} \int_0^{|x_n|} \exp\big(-t^{-1/\alpha}\big)\,\mathrm{d}t \right),$$

where $\omega = (x_1, \dots, x_{n-1})$. Then, for $x_n > 0$, we have

$$|f(0,x_n) - f(0,0)| \leqslant x_n \exp\left(-x_n^{-1/\alpha}\right) .$$

Therefore, (1.4) follows with $y_\circ = 0$. Indeed, let $C > 0$ and $0 < \epsilon < 1$ be given. Set

$$\tau := (2C)^{-1/(\alpha^{-1} - \beta^{-1})} \quad \text{and} \quad \lambda := (\log \epsilon^{-2})^{-\alpha}.$$

Then for $0 < x_n < \min\{\tau, \lambda\}$,

$$\log \frac{1}{\epsilon} < \frac{1}{2x_n^{1/\alpha}} \quad \text{and} \quad C x_n^{\frac{1}{\alpha} - \frac{1}{\beta}} < \frac{1}{2} .$$

So

$$\exp\left(\frac{C}{x_n^{1/\beta}} - \frac{1}{x_n^{1/\alpha}}\right) \leqslant \exp\left(-\frac{1}{2x_n^{1/\alpha}}\right) < \epsilon .$$

Thus, for any $0 < x_n < \min\{\tau, \lambda, 1\}$

$$|\, f(0,x_n) - f(0,0)\,| \leqslant x_n \exp\left(-x_n^{-1/\alpha}\right) < \epsilon \, \exp\left(-\frac{C}{x_n^{1/\beta}}\right).$$

It remains to show that $K_f \in L^{n-1}(\mathbb{X})$. We have

$$Df(x) = L(|\omega|) \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n-1} & 0 \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n-1,1} & A_{n-1,2} & \cdots & A_{n-1,n-1} & 0 \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n-1} & A_{n,n} \end{pmatrix},$$

where

$$A_{i,j} = \begin{cases} 1 + \frac{\alpha x_i^2}{|\omega|^2} \log^{-1} \frac{1}{|\omega|}, & \text{if } i = j \text{ and } 1 \leqslant i,j \leqslant n-1 \\ \alpha \frac{x_i x_j}{|\omega|^2} \log^{-1} \frac{1}{|\omega|}, & \text{if } i \neq j \text{ and } 1 \leqslant i,j \leqslant n-1 \\ \frac{x_n x_j}{|\omega| L(|\omega|)}, & \text{if } i = n \text{ and } 1 \leqslant j \leqslant n-1 \\ \frac{|\omega| + \exp(-|x_n|^{-1/\alpha})}{L(|\omega|)}, & \text{if } i = n \text{ and } j = n . \end{cases}$$

By the definition of $\delta > 0$ we may estimate the Jacobian of f in the set $\mathbb{X}$ as follows
(2.1)
$$
\begin{aligned}
J_f(x) &\geqslant A_{n,n}[L(|\omega|)]^n \left(1 - (n-1)! \sum_{k=1}^{n-1} \alpha^k \, \frac{(\max_{1\leqslant i \leqslant n-1} |x_i|)^{2k}}{|\omega|^{2k}} \, \log^{-k} \frac{1}{|\omega|} \right) \\
&\geqslant A_{n,n}[L(|\omega|)]^n \left(1 - (n-1)! (\alpha^{n-1} + 1) \, \log^{-1} \frac{1}{|\omega|} \right) \\
&\geqslant \frac{1}{2} A_{n,n}[L(|\omega|)]^n = \frac{|\omega| + \exp\big(- |x_n|^{-1/\alpha} \big)}{2 \log^{\alpha(n-1)}(1/|\omega|)} \, .
\end{aligned}
$$

Next, we observe that $|A_{i,j}| < 2$ for all $1 \leqslant i,j \leqslant n-1$. Therefore, we may conclude that

$$|Df(\omega, x_n)| \leqslant 2 \max\{L(|\omega|), |x_n|\} \quad \text{for every } (\omega, x_n) \in \mathbb{X},$$

and if we divide $\mathbb{X}$ into two sets

$$\mathbb{X}_1 := \{x \in \mathbb{X} \colon L(|\omega|) \geqslant |x_n|\} \quad \text{and} \quad \mathbb{X}_2 := \mathbb{X} \setminus \mathbb{X}_1,$$

we get

$$
\begin{aligned}
\int_{\mathbb{X}} K_f^{n-1} &= \int_{\mathbb{X}_1} K_f^{n-1} + \int_{\mathbb{X}_2} K_f^{n-1} \\
&\lesssim \int_{\mathbb{X}_1} \left(\frac{\log^{-n\alpha} \frac{1}{|\omega|}}{J_f} \right)^{n-1} + \int_{\mathbb{X}_2} \left(\frac{|x_n|^n}{J_f} \right)^{n-1} .
\end{aligned}
$$

Throughout this paper we use the symbols $\lesssim$ to indicate that the inequality holds with certain positive constant in the right hand side. This constant, referred to as *implied constant*, will vary from line to line. Here the implied constant depends only on n.

Employing the polar coordinates and (2.1), we obtain

$$
\begin{aligned}
\int_{\mathbb{X}} K_f^{n-1} \lesssim & \int_0^1 \int_{e^{-\tau^{-\frac{1}{\alpha}}}}^{e^{-1}} \frac{1}{r \log^{\alpha(n-1)}(1/r)} \, \mathrm{d}r \, \mathrm{d}\tau \\
& + \int_0^1 \left(\frac{\tau^n}{\exp(-\tau^{-\frac{1}{\alpha}})} \right)^{n-1} \int_0^{e^{-\tau^{-\frac{1}{\alpha}}}} r^{n-2} \log^{\alpha(n-1)^2}(1/r) \, \mathrm{d}r \, \mathrm{d}\tau \, ,
\end{aligned}
$$

where the implied constant depends only on n. We have to deal with two different cases:

(1) If $\alpha \neq \frac{1}{n-1}$, we have

$$\int_{\mathbb{X}} K_f^{n-1} \lesssim \int_0^1 \frac{1}{\tau^{\frac{1}{\alpha} - (n-1)}} \mathrm{d}\tau + \int_0^1 \tau^{n-1} \, \mathrm{d}\tau < \infty \, ,$$

where the implied constant depends only on n and α. The finiteness of the first integral on the right is guranteed by the assumption $\alpha > 1/n$. Thus $K_f \in L^{n-1}(\mathbb{X})$.

(2) If $\alpha = \frac{1}{n-1}$, we have

$$\int_{\mathbb{X}} K_f^{n-1} \lesssim -\int_0^1 \log\tau \,\mathrm{d}\tau + \int_0^1 \tau^{n-1}\,\mathrm{d}\tau < \infty\,,$$

where the implied constant depends only on n and α. Thus, also in this case we have $K_f \in L^{n-1}(\mathbb{X})$.

By (1) and (2) we have $K_f \in L^{n-1}(\mathbb{X})$. □

2.2. **Proof of Proposition 1.3.** Consider the cylindrical domain

$$\mathbb{X} := \{x = (\omega, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \colon |\omega| < 1 \text{ and } |x_n| < \varphi(1)\}\,,$$

and define $f : \mathbb{X} \to \mathbb{Y}$ by

$$f(x_1, \dots, x_{n-1}, x_n) = f(\omega, x_n) = \left(\omega, x_n|\omega| + \frac{x_n}{|x_n|}\int_0^{|x_n|} \varphi^{-1}(t^\alpha)\,\mathrm{d}t\right),$$

where $\alpha > 1$. Then

$$Df(x) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \frac{x_1 x_n}{|\omega|} & \frac{x_2 x_n}{|\omega|} & \cdots & \frac{x_{n-1}x_n}{|\omega|} & |\omega| + \varphi^{-1}(|x_n|^\alpha) \end{pmatrix}.$$

Thus, for every $p < n-1$, we have

$$\int_{\mathbb{X}} K_f^p(x)\,\mathrm{d}x \leqslant \int_{\mathbb{X}} \frac{1}{|\omega|^p}\,\mathrm{d}x \leqslant 2\int_0^{\varphi(1)} r^{n-2-p}\,\mathrm{d}r < \infty\,,$$

and therefore $K_f \in L^p(\mathbb{X})$.

Next, we define $g : [0,\infty) \to \mathbb{R}$ by

$$g(t) = |f(0, \dots, 0, t)|\,.$$

Then it suffices to show that

$$\limsup_{s\to 0} \frac{g^{-1}(s)}{\varphi(s)} = \infty\,.$$

We prove this by contradiction. Suppose that there exists constants $C > 0$ and $\delta > 0$ such that

$$\frac{g^{-1}(s)}{\varphi(s)} < C\,,$$

for all $0 < s < \delta$. Then for all sufficiently small $t > 0$ we have $\frac{g^{-1}(\varphi^{-1}(t))}{t} < C$. Especially

$$\varphi^{-1}(t) < g(Ct)$$

for all small enough $t > 0$. Therefore, for every $0 < s < \min\{ C^{-1}, C^{-\frac{\alpha}{\alpha-1}} \}$ we have

$$\begin{aligned}\varphi^{-1}(s) < g(Cs) = \int_0^{Cs} \varphi^{-1}(t^\alpha)\,\mathrm{d}t \leqslant Cs\,\varphi^{-1}([Cs]^\alpha) \\ < \varphi^{-1}([Cs]^\alpha) < \varphi^{-1}(s)\,,\end{aligned}$$

which leads to a contradiction. $\square$

3. Integrability of Df^{-1}

There is a close connection between the integral of $|Df^{-1}|^n \log^\delta(e + |Df^{-1}|)$ and the inner distortion of the mapping $f\colon \mathbb{X} \to \mathbb{Y}$. To see this, we define the co-differential $D^\sharp f$ via the Cramer's rule $D^\sharp f \circ Df = J_f(x)\mathbf{I}$, where $\mathbf{I}$ stands for the identity matrix. The *inner distortion* of a Sobolev mapping $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{X}, \mathbb{R}^n)$ is the smallest function $K_I(x, f) \geqslant 1$ such that

$$|D^\sharp f(x)|^n \leqslant K_I(x, f) J_f(x)^{n-1}.$$

Lemma 3.1. *Let $f \in W^{1,n-1}_{\mathrm{loc}}(\mathbb{X}, \mathbb{R}^n)$ be a homeomorphism with integrable inner distortion. Then for $\delta \geqslant 0$ we have*

(3.1)

$$\int_{f(\mathbb{X})} |Df^{-1}|^n \log^\delta(e + |Df^{-1}|) = \int_{\mathbb{X}} K_I(x, f) \log^\delta \left(e + \left[\frac{K_I(x, f)}{J_f(x)} \right]^{\frac{1}{n}} \right) \mathrm{d}x\,.$$

Proof. By [16, Theorem 1.2] (see also [6]) both the mapping f and its inverse f^{-1} have finite distortion. Furthermore, f and f^{-1} are differentiable almost everywhere. Therefore, the measure of the set

$$F' = \{ y \in f(\mathbb{X})\colon J_{f^{-1}}(y) > 0 \text{ and } f^{-1} \text{ is differentiable at } y \}$$

equals $|\{y \in f(\mathbb{X})\colon Df^{-1}(y) \neq 0\}|$ and so,

$$\int_{f(\mathbb{X})} |Df^{-1}|^n \log^\delta(e + |Df^{-1}|) = \int_{F'} |Df^{-1}|^n \log^\delta(e + |Df^{-1}|)\,. \tag{3.2}$$

For every $y \in F'$, f^{-1} is differentiable at y and $J_{f^{-1}}(y) > 0$. Therefore, f is differentiable at $f^{-1}(y)$ and

$$J_f(f^{-1}(y)) = \frac{1}{J_{f^{-1}}(y)}\,.$$

At this point we recall a version of the area formula. Let $g\colon \Omega \to \mathbb{R}^n$ be a homeomorphims and $G \subset \Omega$ a measurable set. Suppose that g is differentiable at every point of G and u is a non-negative Borel-measurable function in $\mathbb{R}^n$. Then

$$\int_G u\big(g(x)\big)|J_g(x)|\,\mathrm{d}x \leqslant \int_{\mathbb{R}^n} u(y)\,\mathrm{d}y\,. \tag{3.3}$$

This follows from [7, Theorem 3.1.8] together with the the area formula for Lipschitz mappings. Now, applying (3.3) for f^{-1}, we have

$$\begin{aligned}&\int_{F'}|Df^{-1}|^n\log^\delta(e+|Df^{-1}|)\\&\leqslant\int_{\mathbb{X}}|Df^{-1}(f(x))|^n\log^\delta(e+|Df^{-1}(f(x))|)J_f(x)\,\mathrm{d}x\,.\end{aligned}$$

On the other hand, the change of variable (3.3) for f gives

$$\begin{aligned}&\int_{f(\mathbb{X})}|Df^{-1}|^n\log^\delta(e+|Df^{-1}|)\\&\geqslant\int_{\mathbb{X}}|Df^{-1}(f(x))|^n\log^\delta(e+|Df^{-1}(f(x))|)J_f(x)\,\mathrm{d}x\,.\end{aligned}$$

Above estimates guarantee that

$$\begin{aligned}&\int_{f(\mathbb{X})}|Df^{-1}|^n\log^\delta(e+|Df^{-1}|)\\&=\int_{\mathbb{X}}|Df^{-1}(f(x))|^n\log^\delta(e+|Df^{-1}(f(x))|)J_f(x)\,\mathrm{d}x\,.\end{aligned}$$

By [12, Theorem 1.2], the Jacobian determinant of f is strictly positive almost everywhere. Here we also used the pointwise inequality $K_f(x)\leqslant K_I^{n-1}(x,f)$ for f with finite distortion. The familiar Cramer's rule gives

$$\begin{aligned}&\int_{\mathbb{X}}|Df^{-1}(f(x))|^n\log^\delta(e+|Df^{-1}(f(x))|)J_f(x)\,\mathrm{d}x\\&=\int_{\mathbb{X}}K_I(x,f)\log^\delta\left(e+\left[\frac{K_I(x,f)}{J_f(x)}\right]^{\frac{1}{n}}\right)\,\mathrm{d}x\,.\end{aligned}$$

The desired identity follows. □

Since the inner distortion function $K_I(\cdot,f)$ is at least 1, Lemma 3.1 gives

(3.4)
$$\int_{f(\mathbb{X})}|Df^{-1}|^n\log^\delta(e+|Df^{-1}|)\geqslant n^{-\delta}\int_{\mathbb{X}}K_I(x,f)\log^\delta\left(e+\frac{1}{J_f(x)}\right)\,\mathrm{d}x\,.$$

On the other hand, we have

(3.5)
$$\begin{aligned}\int_{f(\mathbb{X})}|Df^{-1}|^n\log^\delta(e+|Df^{-1}|)&\leqslant 2^\delta\int_{\mathbb{X}}K_I(x,f)\log^\delta K_I(x,f)\,\mathrm{d}x\\&\quad+4^\delta\int_{\mathbb{X}}K_I(x,f)\log^\delta\left(e+\frac{1}{J_f(x)}\right)\,\mathrm{d}x\,.\end{aligned}$$

3.1. **Proof of Theorem 1.4.** Denote $q=(n-1)(p-n+1)$. By applying (3.5) and the pointwise inequality $K_I(x,f)\leqslant K_f^{n-1}(x)$, we have

$$\int_{f(\mathbb{X})}|Df^{-1}|^n\log^q(e+|Df^{-1}|)\lesssim\int_{\mathbb{X}}K_f^p+\int_{\mathbb{X}}K_I\log^q\left(e+\frac{1}{J_f}\right)\,,\tag{3.6}$$

where the implied constant depends only on n and p.

On the other hand, Hölder's inequality and [14, Theorem 1.1] gives us

(3.7)
$$\begin{aligned}\int_{\mathbb{X}} K_I \log^q\left(e+\frac{1}{J_f}\right) &\leqslant \left(\int_{\mathbb{X}} K_I^{\frac{p}{n-1}}\right)^{\frac{n-1}{p}} \left(\int_{\mathbb{X}} \log^{(n-1)p}\left(e+\frac{1}{J_f}\right)\right)^{\frac{p-n+1}{p}} \\ &\leqslant \left(\int_{\mathbb{X}} K_f^p\right)^{\frac{n-1}{p}} \left(\int_{\mathbb{X}} K_f^p\right)^{\frac{p-n+1}{p}} = \int_{\mathbb{X}} K_f^p(x)\,.\end{aligned}$$

The claim follows by combining (3.6) and (3.7). □

Hereafter, the symbol $a \approx b$ means that $b \lesssim a \lesssim b$. In the proof of Theorem 1.6 we will apply the following technical lemma.

Lemma 3.2. *Suppose that a, b, c are given positive numbers such that $a \geqslant c$ and $a \geqslant bx_i$ for all $i = 1, \dots, n-1$, where $0 \leqslant x_i < 1$, $i = 1, \dots, n-1$, are some given numbers. Define*

$$D = \begin{pmatrix} a & 0 & 0 & \cdots & 0 & x_1 b \\ 0 & a & 0 & \cdots & 0 & x_2 b \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & a & x_{n-1} b \\ 0 & \cdots & \cdots & \cdots & 0 & c \end{pmatrix},$$

and suppose that $\lambda_1^2 \geqslant \lambda_2^2 \geqslant \cdots \geqslant \lambda_n^2 > 0$ are the eigenvalues of the symmetric matrix $A = DD^T$. Then $|\lambda_n| \approx c$, and $|\lambda_1| \approx |\lambda_2| \approx \cdots \approx |\lambda_{n-1}| \approx a$, where the implied constants depends only on n.

Proof. By applying Gaussian elimination, we may show that

(3.8)
$$\det(A - \lambda I) = (a^2 - \lambda)^{n-2} \left(\lambda^2 - \lambda\big(a^2 + b^2\big[\sum_{i=1}^{n-1} x_i^2\big] + c^2\big) + a^2 c^2 \right).$$

It follows from (3.8) that at least $n-2$ of the eigenvalues $\lambda_1^2, \dots, \lambda_n^2$ equals to a^2. Next, fix $\hat{x} = (x_1, \dots, x_{n-1})$ such that for every $i = 1, \dots, n-1$ we have $0 \leqslant x_i < 1$ and $a \geqslant bx_i$. Then for the polynomial

$$P_{\hat{x}}(\lambda) := \lambda^2 - \lambda\big(a^2 + b^2\big[\sum_{i=1}^{n-1} x_i^2\big] + c^2\big) + a^2 c^2\,,$$

we have

$$0 \leqslant P_{\hat{x}}(c^2/(n+1)) \quad \text{and} \quad P_{\hat{x}}(c^2) \leqslant 0\,.$$

Therefore, by applying the continuity of $P_{\hat{x}}$ we have that the equation $P_{\hat{x}}(\lambda) = 0$ has a solution $\lambda \in [c^2/(n-1), c^2]$. Especially, this implies that $\lambda_n^2 \approx c^2$ where the implied constants depends only on n.

We know now the size of $n-1$ of the eigenvalues. The size of the last eigenvalue can be solved by applying the fact $|\lambda_1 \cdots \lambda_n| = \det(A) = a^{n-1}c$, and thus the claim follows. $\square$

3.2. **Proof of Theorem 1.6.** First we suppose that $p = n-1$. Let $s > 0$ and $0 < \epsilon < s$. We define a smooth function $h : (0,1) \to (0,\infty)$ by

$$h(t) = t^{\frac{n}{n-1}} \log^{\frac{1+\epsilon}{n-1}}(1/t)\,,$$

and our domain takes the form

$$\mathbb{X} = \left\{ x \in \mathbb{R}^n : 0 < x_n < c,\ 0 < x_i < 1 \text{ for all } i = 1, \dots, n-1 \right\},$$

where the constant $c > 0$ is so small that the function h is strictly increasing on the interval $(0, c)$. We define a mapping $f : \mathbb{X} \to \mathbb{R}^n$ by

$$f(x) = (x_1, \dots, x_{n-1}, h(x_n))\,.$$

Then

$$\int_{\mathbb{X}} K_f^{n-1} = \int_0^c \left[\frac{1}{h'(x_n)}\right]^{n-1} \mathrm{d}x_n \lesssim \int_0^c \frac{\mathrm{d}x_n}{x_n \log^{1+\epsilon}(1/x_n)} < \infty\,,$$

where the implied constant depends only on n, and therefore $K_f \in L^{n-1}(\mathbb{X})$.

On the other hand, for the inverse we have

$$f^{-1}(y) = (y_1, \dots, y_{n-1}, h^{-1}(y_n))\,.$$

We observe that $h^{-1}(s) \approx s^{\frac{n-1}{n}} \log^{-\frac{1+\epsilon}{n}}(1/s)$, and thus

$$\begin{aligned} 1 = \frac{\mathrm{d}}{\mathrm{d}s} h(h^{-1}(s)) &\lesssim (h^{-1})'(s)\, (h^{-1}(s))^{\frac{1}{n-1}} \log^{\frac{1+\epsilon}{n-1}} \left(1/h^{-1}(s)\right) \\ &\approx (h^{-1})'(s)\, s^{\frac{1}{n}} \log^{\frac{1+\epsilon}{n}}(1/s)\,, \end{aligned}$$

where all the implied constants depends only on n. Especially, we get

$$(h^{-1})'(s) \gtrsim s^{-\frac{1}{n}} \log^{-\frac{1+\epsilon}{n}}(1/s)\,, \tag{3.9}$$

where the implied constant depends only on n. By applying (3.9) and the assumption $0 < \epsilon < s$ we have

$$\int_{\mathbb{Y}} |Df^{-1}|^n \log^s(e + |Df^{-1}|) = \infty\,.$$

This gives the proof in the case $p = n-1$.

Next we assume that $p > n-1$. Let us denote denote $\beta = \frac{1}{(n-1)(p-1)}$. Fix $\epsilon > 0$ small enough such that $(1+\epsilon)\beta - 1 < 0$. We define a smooth function $\psi : (0,1) \to (0,\infty)$ by

$$\psi(t) = t^{n\beta} \log^{(1+\epsilon)\beta}(1/t)\,,$$

and smooth functions $g, h : (0,\infty) \to \mathbb{R}$ by

$$g(t) = \exp(-1/\psi(t)) \quad \text{and} \quad h(t) = \int_0^t g(s)\psi(s)^{n-1}\,\mathrm{d}s\,.$$

Suppose that $c_1 > 0$ is so small that $\log^{(1+\epsilon)\beta-1}(1/t) < \frac{n\beta}{2}$ for all $0 < t < c_1$. We may assume that c_1 does not depend on ϵ by assuming $\epsilon > 0$ to be sufficiently small. Then we have

$$g'(t) \approx \frac{g(t)}{t\psi(t)}\,, \tag{3.10}$$

for all $0 < t < c_1$, where the implied constants depends only on n and p.

We fix a constant $0 < c < e^{-1}$ so small that functions ψ, g and h are strictly increasing and the condition (3.10) holds on the interval $(0, c)$. Then our domain $\mathbb{X}$ takes the form

$$\mathbb{X} = \left\{ x \in \mathbb{R}^n : 0 < x_n < c,\, 0 < x_i < g(x_n)/g'(x_n) \text{ for } i = 1, \dots, n-1 \right\},$$

and we define a mapping $f : \mathbb{X} \to \mathbb{Y}$ as

$$f(x) = (x_1 g(x_n), \dots, x_{n-1} g(x_n), h(x_n))\,.$$

Then we have

$$Df(x) = \begin{pmatrix} g(x_n) & 0 & \cdots & 0 & x_1\, g'(x_n) \\ 0 & g(x_n) & \cdots & 0 & x_2\, g'(x_n) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & g(x_n) & x_{n-1}\, g'(x_n) \\ 0 & 0 & \cdots & 0 & h'(x_n) \end{pmatrix}$$

Next, at each point $x \in \mathbb{X}$ we define a symmetric matrix $S(x) = Df(x)\, Df(x)^T$, and denote its eigenvalues as

$$\lambda_1^2(x) \geqslant \lambda_2^2(x) \geqslant \cdots \geqslant \lambda_n^2(x) > 0\,.$$

By applying Lemma 3.2, we get

$$|\lambda_1(x)| \approx \cdots \approx |\lambda_{n-1}(x)| \approx g(x_n), \text{ and } |\lambda_n(x)| \approx h'(x_n)\,,$$

where the implied constants depends only on n. Therefore

$$|Df(x)| \approx g(x_n), \quad J_f(x) = g(x_n)^{n-1}\, h'(x_n)\,,$$

where the implied constants depends only on n. Especially, for the distortions we have

$$K_f(x) \approx \frac{g(x_n)}{h'(x_n)} \quad \text{and} \quad K_I(x, f) \approx \left(\frac{g(x_n)}{h'(x_n)} \right)^{n-1},$$

with implied constants depending only on n. By Fubini's theorem and (3.10) we get

$$\begin{aligned} \int_{\mathbb{X}} K_f^p(x)\, \mathrm{d}x &\approx \int_0^c \int_0^{g(x_n)/g'(x_n)} \cdots \int_0^{g(x_n)/g'(x_n)} \left(\frac{g(x_n)}{h'(x_n)} \right)^p \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \int_0^c \left(\frac{g(x_n)}{g'(x_n)} \right)^{n-1} \left(\frac{g(x_n)}{h'(x_n)} \right)^p \mathrm{d}x_n \approx \int_0^c \frac{\mathrm{d}x_n}{x_n \log^{1+\epsilon}(1/x_n)} < \infty\,, \end{aligned}$$

where the implied constants depends only on n and p. Thus $K_f \in L^p(\mathbb{X})$. On the other hand, by (3.4) we have

$$\begin{aligned}
&\int_{\mathbb{Y}} |Df^{-1}|^n \log^s(e+|Df^{-1}|) \geqslant n^{-s} \int_{\mathbb{X}} K_I(x,f) \log^s\left(e+\frac{1}{J_f(x)}\right) \mathrm{d}x \\
&= n^{-s} \int_0^c \int_0^{g(x_n)/g'(x_n)} \cdots \int_0^{g(x_n)/g'(x_n)} K_I(x,f) \log^s\left(e+\frac{1}{J_f(x)}\right) \mathrm{d}x \\
&\gtrsim n^{-s} \int_0^c \int_0^{g(x_n)/g'(x_n)} \cdots \int_0^{g(x_n)/g'(x_n)} \left(\frac{g(x_n)}{h'(x_n)}\right)^{n-1} \left(\frac{1}{|\psi(x_n)|}\right)^s \mathrm{d}x \\
&= n^{-s} \int_0^c \left(\frac{g(x_n)}{g'(x_n)}\right)^{n-1} \left(\frac{g(x_n)}{h'(x_n)}\right)^{n-1} \left(\frac{1}{|\psi(x_n)|}\right)^s \mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&\approx n^{-s} \int_0^c \frac{\mathrm{d}x_n}{x_n^{s_1} \log^{s_2}(1/x_n)},
\end{aligned}$$

where $s_1 = \frac{(n-1)(p-1+n(n-1-p))+ns}{(n-1)(p-1)}$ and $s_2 = \frac{(1+\epsilon)((n-1)(n-2)+s)}{(n-1)(p-1)}$. Above, the implied constants depends on n. Therefore, this integral can be finite only if $s_1 \leqslant 1$, which is equivalet with the condition

$$s \leqslant (n-1)(p-n+1)\,,$$

and the claim follows. □

4. Radially symmetric homeomorphisms

4.1. Proof of Proposition 1.1. When $n=2$ the claim follows from [13].

Next, assume $n \geqslant 3$, and denote $\alpha := p(n-1)-1 \geqslant 1$. We follow the notation in (1.3) and write

$$f(x) = F(|x|)\frac{x}{|x|}\,,$$

where F is a strictly increasing, continuous funcion such that $F(0)=0$. Then $f^{-1}(y) = F^{-1}(|y|)\frac{y}{|y|}$. Let us denote $H := F^{-1}$. It suffices to prove that

$$\int_{\mathbb{B}} \frac{H^n(|y|)}{|y|^n} \log^\alpha\left(1/\sqrt{|y|}\right) \mathrm{d}y < \infty\,. \tag{4.1}$$

Indeed, suppose that (4.1) is true, and let $0<r<1$. Then Hölder's inequality implies that

$$\begin{aligned}
H^n(r)\log^{\alpha+1}(1/r) &\leqslant H^n(r)\left(\int_r^1 \frac{1}{t}\,\mathrm{d}t\right)^{1-\alpha} \left(\int_r^1 \frac{1}{t}\,\log(1/t)\,\mathrm{d}t\right)^\alpha \\
&\leqslant H^n(r)\int_r^1 \frac{1}{t}\,\log^\alpha(1/t)\,\mathrm{d}t \leqslant \int_{\mathbb{B}} \frac{H^n(|y|)}{|y|^n}\,\log^\alpha\left(1/\sqrt{|y|}\right)\,\mathrm{d}y < \infty\,,
\end{aligned}$$

and the claim will follow.

For verifying (4.1) we define the sets

$$B_1 = \{\, y \in \mathbb{B} : H^n(|y|) \leqslant |y|^{n-\frac{1}{2}} \,\}, \quad B_2 = \left\{ y \in \mathbb{B}\backslash B_1 : \frac{H(|y|)}{|y|} \leqslant H'(|y|) \right\},$$
$$B_3 = \left\{ y \in \mathbb{B}\backslash B_1 : \frac{H(|y|)}{|y|} > H'(|y|) \right\}.$$

We have

$$\int_{B_1} \frac{H^n(|y|)}{|y|^n} \log^\alpha \left(\frac{1}{\sqrt{|y|}} \right) \mathrm{d}y \leqslant \int_{\mathbb{B}} \frac{1}{\sqrt{|y|}} \log^\alpha \left(\frac{1}{\sqrt{|y|}} \right) \mathrm{d}y \lesssim 1 \,,$$

where the implied constant depends only on n. On the other hand, it follows from [14] that

$$\begin{aligned}
\int_{B_2} \frac{H^n(|y|)}{|y|^n} \log^\alpha \left(\frac{1}{\sqrt{|y|}} \right) \mathrm{d}y &\leqslant \int_{B_2} \frac{H^n(|y|)}{|y|^n} \log^\alpha \left(e + \frac{H^n(|y|)}{|y|^n} \right) \mathrm{d}y \\
&= \int_{\{\, F(|x|)/|x| \geqslant F'(|x|) \,\}} \frac{|x| F'(|x|)}{F(|x|)} \log^\alpha \left(e + \frac{|x|^n}{F^n(|x|)} \right) \mathrm{d}x \\
&\leqslant \int_{\mathbb{B}} \log^\alpha \left(e + \frac{1}{J_f(x)} \right) \mathrm{d}x \lesssim \int_{\mathbb{B}} K_f^{\frac{\alpha}{n-1}}(x) \, \mathrm{d}x \,,
\end{aligned}$$

and

$$\begin{aligned}
\int_{B_3} \frac{H^n(|y|)}{|y|^n} \log^\alpha \left(\frac{1}{\sqrt{|y|}} \right) \mathrm{d}y &\leqslant \int_{B_3} \frac{H^n(|y|)}{|y|^n} \log^\alpha \left(e + \frac{H^n(|y|)}{|y|^n} \right) \mathrm{d}y \\
&= \int_{\{\, F(|x|)/|x| < F'(|x|) \,\}} \frac{|x| F'(|x|)}{F(|x|)} \log^\alpha \left(e + \frac{|x|^n}{F^n(|x|)} \right) \mathrm{d}x \\
&\lesssim \int_{\mathbb{B}} K_f^{\frac{1}{n-1}} \log^\alpha \left(e + \frac{K_f}{J_f} \right) \lesssim \int_{\mathbb{B}} K_f^p \,,
\end{aligned}$$

where the implied constant depends on n and p. Above, the last inequality follows from the Hölder's inequality and [14]. By combining the estimates above (4.1) follows. This ends the proof. $\square$

4.2. **Proof of Proposition 1.5.** For a given $i \in \mathbb{N}$ and $\alpha_i > 1$, we set $I_i = [2^{-i}, 2^{-(i-1)})$ and define a function $P_{\alpha_i} \colon I_i \xrightarrow{\text{onto}} I_i$ as

$$P_{\alpha_i}(s) = (2^{-(i-1)} - 2^{-i}) \left(\frac{s - 2^{-i}}{2^{-(i-1)} - 2^{-i}} \right)^{\alpha_i} + 2^{-i} \,.$$

Then

$$P'_{\alpha_i}(s) = \alpha_i \left(\frac{s - 2^{-i}}{2^{-(i-1)} - 2^{-i}} \right)^{\alpha_i - 1} .$$

Next, suppose that $s_i \in I_i$ is the unique solution of the equation

$$P'_{\alpha_i}(s) = 1.$$

More precisely, $s_i = 2^{-i}\big(\alpha_i^{-\frac{1}{\alpha_i-1}}+1\big)$. Define a linear function $L_{i+1}\colon [2^{-(i+1)}, s_i) \xrightarrow{\text{onto}} [2^{-(i+1)}, P_{\alpha_i}(s_i))$ as

$$L_{i+1}(s) = (s-2^{-(i+1)})S_i + 2^{-(i+1)}\,,$$

where

$$S_i = \frac{P_{\alpha_i}(s_i) - P_{\alpha_{i+1}}(2^{-(i+1)})}{s_i - 2^{-(i+1)}}\,.$$

Then $\frac{1}{3} \leqslant S_i \leqslant 1$, for all $i \in \mathbb{N}$.

We define $f^{-1}\colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}$ by

$$f^{-1}(y) = \frac{y}{|y|}\rho^{-1}(|y|)\,,$$

where the function $\rho^{-1} : [0,1) \to [0,1)$ is defined as

$$\rho^{-1}(s) = \begin{cases} P_{\alpha_i}(s), & \text{if } s \in [s_i, 2^{-(i-1)}) \text{ and } i \text{ is odd} \\ L_{i+1}(s), & \text{if } s \in [2^{-(i+1)}, s_i) \text{ and } i \text{ is odd} \\ 0, & \text{if } s = 0\,. \end{cases}$$

Then

$$|Df^{-1}(y)| \approx \begin{cases} P'_{\alpha_i}(|y|), & \text{if } |y| \in [s_i, 2^{-(i-1)}) \text{ and } i \text{ is odd} \\ 1, & \text{if } |y| \in [2^{-(i+1)}, s_i) \text{ and } i \text{ is odd}\,, \end{cases}$$

where the implied constants depends only on n. Therefore, by using the change of variables with $h = \frac{s-2^{-i}}{2^{-(i-1)}-2^{-i}}$, and then applying the binomial expansion, we get

$$\begin{aligned} \int_{\mathbb{B}} |Df^{-1}(y)|^n \,\mathrm{d}y &\gtrsim \sum_{i \text{ odd}} \int_{I_i} s^{n-1}\, P'_{\alpha_i}(s)^n \,\mathrm{d}s \approx \sum_{i \text{ odd}} \sum_{k=0}^{n-1} \frac{\alpha_i^n}{2^{in}} \int_0^1 h^{n(\alpha_i-1)+k}\,\mathrm{d}h \\ &= \sum_{i \text{ odd}} \sum_{k=0}^{n-1} \frac{\alpha_i^n}{2^{in}\,(n(\alpha_i-1)+k+1)} \gtrsim \sum_{i \text{ odd}} \frac{\alpha_i^{n-1}}{2^{in}}\,, \end{aligned} \tag{4.2}$$

where the implied constants depends only on n.

Next, we approximate the integral of the distortion K_f. For this, we notice that $f : \mathbb{B} \to \mathbb{B}$ can be written as

$$f(x) = \frac{x}{|x|}\,\rho(|x|)\,,$$

where the function $\rho : [0,1) \to [0,1)$ is defined by

$$\rho(t) = \begin{cases} P_{\alpha_i}^{-1}(t), & \text{if } t \in [P_{\alpha_i}(s_i), 2^{-(i-1)}) \text{ and } i \text{ is odd} \\ L_{i+1}^{-1}(t), & \text{if } t \in [2^{-(i+1)}, P_{\alpha_i}(s_i)) \text{ and } i \text{ is odd} \\ 0, & \text{if } s = 0\,. \end{cases}$$

We recall that

$$K_f(x) = \max\left\{ \frac{\rho(|x|)}{|x|\rho'(|x|)}, \left[\frac{\rho'(|x|)|x|}{\rho(|x|)}\right]^{n-1} \right\},$$

see [10, Lemma 2.1]. Therefore, for each $x \in B(0, 2^{-(i-1)}) \setminus \overline{B(0, P_{\alpha_i}(s_i))}$, $i \in \mathbb{N}$ odd, we have

$$K_f(x) = \max\left\{ \frac{P_{\alpha_i}^{-1}(|x|)}{|x|(P_{\alpha_i}^{-1})'(|x|)}, \left[\frac{(P_{\alpha_i}^{-1})'(|x|)|x|}{P_{\alpha_i}^{-1}(|x|)}\right]^{n-1} \right\} = \frac{P_{\alpha_i}^{-1}(|x|)}{|x|(P_{\alpha_i}^{-1})'(|x|)}$$

$$= \frac{\alpha_i}{|x|}\left[|x| - 2^{-i} + (2^{-i})^{\frac{1}{\alpha_i}} \left[\, |x| - 2^{-i} \,\right]^{\frac{\alpha_i - 1}{\alpha_i}} \right] \leqslant \alpha_i \,, \tag{4.3}$$

and for each $x \in B(0, P_{\alpha_i}(s_i)) \setminus \overline{B(0, 2^{-(i+1)})}$, $i \in \mathbb{N}$ odd, we have

$$K_f(x) = \max\left\{ \frac{L_{i+1}^{-1}(|x|)}{|x|(L_{i+1}^{-1})'(|x|)}, \left[\frac{(L_{i+1}^{-1})'(|x|)|x|}{L_{i+1}^{-1}(|x|)}\right]^{n-1} \right\} \leqslant 3^{n-1} \,. \tag{4.4}$$

If we now set $\alpha_i = \left[\frac{2^{in}}{i}\right]^{\frac{1}{n-1}}$, then (4.2) implies

$$\int_{\mathbb{B}} |Df^{-1}(y)|^n \,\mathrm{d}y \gtrsim \sum_{i \text{ odd}} \frac{1}{i} = \infty \,. \tag{4.5}$$

Moreover, it follows from (4.3) and (4.4) that for every $0 < p < n-1$ we have

$$\begin{aligned} \int_{\mathbb{B}} K_f^p(x) \,\mathrm{d}x &\leqslant \sum_{i \text{ odd}} \int_{B(0,2^{-i})} (\, \alpha_i^p + 3^{p(n-1)} \,) \,\mathrm{d}x \\ &\lesssim \sum_{i=1}^{\infty} \left[2^{-in} \left[\frac{2^{in}}{i} \right]^{\frac{p}{n-1}} + \frac{3^{p(n-1)}}{2^{in}} \right] < \infty \,, \end{aligned} \tag{4.6}$$

where the implied constant depends only on n. The claim follows from (4.5) and (4.6). □

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