Author(s): Aleksandrov, Konstantin D.; Kuznetsov, Nikolay; Leonov, Gennady A.; Neittaanmäki, Pekka; Yuldashev, Marat V.; Yuldashev, Renat V.

Title: Computation of the lock-in ranges of phase-locked loops with PI filter

Year: 2016

Version:

Please cite the original version:

Aleksandrov, K. D., Kuznetsov, N., Leonov, G. A., Neittaanmäki, P., Yuldashev, M. V., & Yuldashev, R. V. (2016). Computation of the lock-in ranges of phase-locked loops with PI filter. In H. Nijmeijer (Ed.), 6th IFAC Workshop on Periodic Control Systems PSYCO 2016 (pp. 36-41). International Federation of Automatic Control (IFAC). IFAC Proceedings Volumes (IFAC-PapersOnline), 49. https://doi.org/10.1016/j.ifacol.2016.07.971

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IFAC-PapersOnLine 49-14 (2016) 036–041

Computation of the lock-in ranges of phase-locked loops with PI filter

Konstantin D. Aleksandrov *,**, **Nikolay V. Kuznetsov** *,**, **Gennady A. Leonov** *,***, **Pekka Neittaanmäki** **, **Marat V. Yuldashev** *, **Renat V. Yuldashev** *

* *Faculty of Mathematics and Mechanics, Saint-Petersburg State University, Russia*
** *Dept. of Mathematical Information Technology, University of Jyväskylä, Finland (email: nkuznetsov239@gmail.com)*
*** *Institute of Problems of Mechanical Engineering RAS, Russia*

Abstract In the present work the lock-in range of PLL-based circuits with proportionally-integrating filter and sinusoidal phase-detector characteristics are studied. Considered circuits have sinusoidal phase detector characteristics. Analytical approach based on the methods of phase plane analysis is applied to estimate the lock-in ranges of the circuits under consideration. Obtained analytical results are compared with simulation results.

Keywords: phase-locked loop, nonlinear analysis, PLL, Costas loop, lock-in range, pull-out frequency, hidden oscillations

1. INTRODUCTION

The lock-in concept is widely used in engineering literature (Gardner, 2005; Best, 2007). Notion of the lock-in range can be formulated in the following way (see, e.g. (Gardner, 1966)): if the difference between reference and tunable frequencies of the circuit belongs to the lock-in range, then synchronization occurs without cycle slipping (loss of cycles). In 1979 F. Gardner (Gardner, 1979) formulated the following problem: *"There is no natural way to define exactly any unique lock-in frequency."* However, *"despite its vague reality, lock-in range is a useful concept"* (Gardner, 1979).

In the present work analytical and numerical approaches for the lock-in range estimation are presented. The analytical approach is based on the integration of the phase plane trajectories and analysis of their behaviour (Tricomi, 1933; Andronov et al., 1937). The numerical approach also can be applied for the study of PLL-based circuits. However, one has to pay the special attention to results obtained by numerical simulations. Particular examples on different qualitative behaviour for two different ODE solver's step sizes can be found in (Bianchi et al., 2015; Leonov et al., 2015a).

In the present work PLL-based circuits with sinusoidal characteristics of phase detector are considered. In Section 2 model of PLL-based circuits in the signal's phase space is described. In Section 3 rigorous mathematical definitions for lock-in range are given. Methods for verifying global stability and methods of phase plane analysis are described in Subsection 3.1. Effectiveness of obtained in Subsection 3.1 lock-in range estimations is discussed in Subsections 3.2, 3.3.

2. MODEL OF PLL-BASED CIRCUITS IN THE SIGNAL'S PHASE SPACE

For the description of PLL-based circuits, a physical model in the signals space and a mathematical model in the signal's phase space are used. Models of the PLL-based circuits in the signals space are difficult for the study (Kudrewicz and Wasowicz, 2007) since the equations, which describe these models, are nonautonomous. By contrast, equations for the models in the signal's phase space are autonomous (Viterbi, 1966; Shakhgil'dyan and Lyakhovkin, 1966; Gardner, 1966), what simplifies their study.

From the numerical point of view, advantage of models in the signal's phase space is the nonexistence of high-frequency components, thus simulation in the signal's phase space allows one to consider slow varying frequency only. By contrast, the simulation of PLL-based circuits in the signals space is complicated since one has to observe simultaneously both high-frequency (fast changing of phases) and low-frequency (relatively slow changing of frequencies) oscillations. The physical models of PLL-

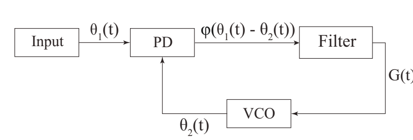

Figure 1. Model of PLL-based circuit in the signal's phase space.

based circuits can be reduced to the models in the signal's phase space (Leonov et al., 2012; Best et al., 2014, 2015;

⋆ This work was supported by Russian Science Foundation (project 14-21-00041, s. 3) Saint-Petersburg State University (project 6.38.505.2014, s. 2.)

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2016.07.971

Kuznetsov et al., 2015a; Leonov and Kuznetsov, 2014; Leonov et al., 2015b; Kuznetsov et al., 2015b) by the averaging methods (see, e.g., (Mitropolsky and Bogolubov, 1961; Samoilenko and Petryshyn, 2004)). In order to study models of PLL-based circuits in the signal's phase space (see Fig. 1) it is necessary to compute characteristic of a phase detector – nonlinear element of PLL-based circuits for matching tunable signals. The characteristic of phase detector $K_{\text{PD}}\varphi(\theta_1(t)-\theta_2(t))$ (where K_{PD} is the PD gain coefficient) is a function with respect to the difference of phases of reference and tunable oscillators (for the model in the signals space the result of the work of phase detector $\varphi(t)$ depends on time t). Further phase difference $\theta_1(t)-\theta_2(t)$ will be denoted as $\theta_\Delta(t)$.

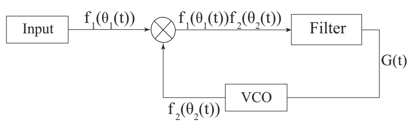

Figure 2. Model of the classical PLL in the signal space.

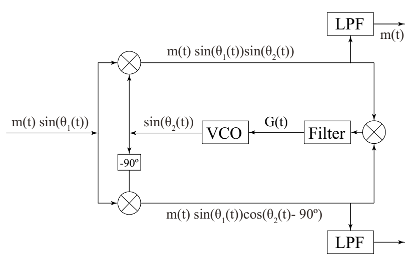

Figure 3. Realization of the BPSK Costas loop.

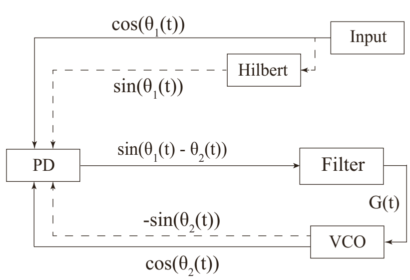

Figure 4. Two-phase PLL.

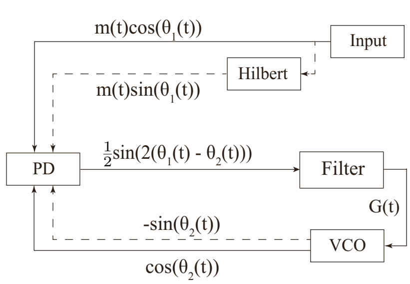

Figure 5. Two-phase Costas loop.

Let us describe a general model of PLL-based circuits in the signal's phase space (see Fig. 1). A reference oscillator and a tunable oscillator generate phases $\theta_1(t)$ and $\theta_2(t)$, respectively. The frequency of carrier signal is constant and equals ω_1:

PLL circuit	
$f_1(\theta_1)=\sin(\theta_1)$ $f_2(\theta_2)=\sin(\theta_2)$	$K_{\text{PD}}=\frac{1}{2}$
$f_1(\theta_1)=\sin(\theta_1)$ $f_2(\theta_2)=\text{sgn}(\sin(\theta_2))$	$K_{\text{PD}}=\frac{2}{\pi}$
$f_1(\theta_1)=\begin{cases}\frac{2}{\pi}\theta_1+1, \theta_1\in[0;\pi],\\ 1-\frac{2}{\pi}\theta_1, \theta_1\in[\pi;2\pi]\end{cases}$ $f_2(\theta_2)=\sin(\theta_2)$	$K_{\text{PD}}=\frac{4}{\pi^2}$
Costas loop	
$f_1(\theta_1)=\cos(\theta_1)$ $f_2(\theta_2)=\sin(\theta_2)$	$K_{\text{PD}}=\frac{1}{8}$
Two-phase PLL circuit	
$f_1(\theta_1)=\cos(\theta_1)$ $f_2(\theta_2)=\cos(\theta_2)$	$K_{\text{PD}}=1$
Two-phase Costas loop	
$f_1(\theta_1)=\cos(\theta_1)$ $f_2(\theta_2)=\cos(\theta_2)$	$K_{\text{PD}}=1$

Table 1. PD characteristics and gain coefficients of the considered circuits.

$$\frac{d\theta_1(t)}{dt}=\omega_1. \tag{1}$$

The phases $\theta_1(t)$ and $\theta_2(t)$ enter the inputs of a phase detector. A signal of phase detector output $\varphi(\theta_\Delta(t))$ is filtered by Filter. The proportionally-integrating filter with the transfer function $W(s)=\frac{1+\tau_2 s}{\tau_1 s}$, $\tau_1>0$, $\tau_2>0$ is described by the system

$$\begin{cases}\dot{x}(t)=\varphi(\theta_\Delta(t)),\\ G(t)=\frac{\tau_2}{\tau_1}K_{\text{PD}}\varphi(\theta_\Delta(t))+\frac{1}{\tau_1}K_{\text{PD}}x(t),\end{cases} \tag{2}$$

where $x(t)$ is the filter state. In the current paper only phase detectors with sinusoidal characteristic $\varphi(\theta_\Delta)=\sin\theta_\Delta$ are considered.

The output of Filter $G(t)$ serves as a control signal for VCO:

$$\dot{\theta}_2(t)=\omega_2^{\text{free}}+K_{\text{VCO}}G(t), \tag{3}$$

where $\omega_\Delta^{\text{free}}$ is the VCO free-running frequency and $K_{\text{VCO}}>0$ is a VCO gain coefficient.

Equations (1), (3) and system (2) result in autonomous system of differential equations (here and further difference of phases $\omega_1-\omega_2^{\text{free}}$ is denoted by $\omega_\Delta^{\text{free}}$)

$$\begin{cases}\dot{x}=\sin(\theta_\Delta),\\ \dot{\theta}_\Delta=\omega_\Delta^{\text{free}}-\frac{K_0}{\tau_1}\left(x+\tau_2\sin(\theta_\Delta)\right),\end{cases} \tag{4}$$

where $K_0=K_{\text{VCO}}\cdot K_{\text{PD}}$ is the loop gain coefficient.

Analytical results on the lock-in range estimation obtained for system (4) can be applied to PLL-based circuits with sinusoidal PD characteristic: classical PLL (see Fig. 2), Costas loop (see Fig. 3), two-phase PLL (see Fig. 4), and two-phase Costas loop (see Fig. 5). Corresponding PD gain coefficients for the considered PLL-based circuits are listed in Table 1.

3. ANALYTICAL APPROACH TO THE LOCK-IN RANGE ANALYSIS

In (Kuznetsov et al., 2015c; Leonov et al., 2015a) rigorous lock-in range definition for classical PLL was suggested:

Definition 1. Set of all frequency deviations $|\omega_\Delta^{\text{free}}|$ such that the mathematical model of the loop in the signal's phase space is globally asymptotically stable is called a pull-in set $\Omega_{pull-in}$. The largest interval $[0, \omega_p) \subset \Omega_{pull-in}$ is called a pull-in range.

Definition 2. The lock-in range $[0, \omega_l)$ is a subset of the pull-in range $[0, \omega_p)$ such that for each corresponding frequency deviation $|\omega_\Delta^{\text{free}}|$ the lock-in domain (i.e., a domain of the loop states, where fast acquisition without cycle slipping is possible) contains both symmetric locked states (i.e., locked states for the positive and negative value of the difference between the reference frequency and the VCO free-running frequency).

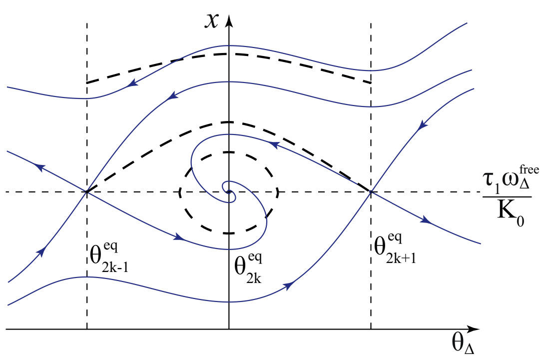

Figure 6. Periodic trajectories on the phase plane of system (4).

Since the lock-in range is the subset of the pull-in range $[0, \omega_p)$, before studying the lock-in range it is necessary to estimate the pull-in set. In (Viterbi, 1966), by the methods of phase plane analysis, it is explained that the pull-in range of the classical PLL with PI filter and sinusoidal characteristic of the phase detector is infinite. However to complete rigorously the explanations given in (Viterbi, 1966), one has to prove the nonexistence of a heteroclitic trajectory and limit cycles of the first kind (see Fig. 6). In the general case, applying the phase plane analysis for proving global stability of the system under consideration (which is necessary for estimating lock-in range) leads to lots of new cumbersome integrations for any new type of PD characteristic. To overcome these difficulties, global stability can be proved using another method, based on the Lyapunov function construction (Lyapunov, 1892). Required modifications of the classical global stability criteria for the cylindrical phase space have been developed in (Gelig et al., 1978; Leonov and Kuznetsov, 2014) and allow one to prove global stability for PLL-based systems with periodic PD characteristic $\varphi(\theta_\Delta)$. In (Bakaev, 1963) to prove global stability of PLL with PI filter and sinusoidal PD characteristic $\varphi(\theta_\Delta) = \sin(\theta_\Delta)$, Lyapunov function

$$V(x, \theta_\Delta) = \frac{1}{2}\left(x - \frac{\omega_\Delta^{\text{free}}}{K_0}\right)^2 + \frac{2\tau_1}{K_0}\sin^2\left(\frac{\theta_\Delta}{2}\right)$$

was suggested (see, also discussion in (Leonov et al., 2015a)).

To prove the global stability of PLL-based systems with proportionally-integrating filter with periodic PD characteristic $\varphi(\theta_\Delta)$ in general form, one may use the following Lyapunov function (Alexandrov et al., 2015):

$$V(x, \theta_\Delta) = \frac{K_0}{2\tau_1}\left(x - \frac{\tau_1\omega_\Delta^{\text{free}}}{K_0}\right)^2 + \int\limits_0^{\theta_\Delta}\varphi(s)ds,$$

$$\dot{V}(x, \theta_\Delta) = -\frac{K_0\tau_2}{\tau_1}\varphi^2(\theta_\Delta).$$

Despite of the global stability property of system (4) phase trajectories may have cycle slips. To determine if cycle slips occur, it is necessary to study behaviour of separatrices on the phase plane (see Fig. 7).

3.1 Phase plane analysis for the lock-in range estimation

Let us consider the analytical approach for the lock-in range estimation of system (4), based on the phase plane analysis. Equilibria points of system (4) can be found from the following system of equations

$$\begin{cases}\sin(\theta_\Delta) = 0,\\ \omega_\Delta^{\text{free}} - \frac{K_0}{\tau_1}x = 0.\end{cases}$$

One can show that equilibria points

$$\left(\theta_{2k}^{eq}, x_{2k}^{eq}(\omega_\Delta^{\text{free}})\right) = \left(2\pi k, \frac{\omega_\Delta^{\text{free}}\tau_1}{K_0}\right)$$

are stable equilibria points and

$$\left(\theta_{2k+1}^{eq}, x_{2k+1}^{eq}(\omega_\Delta^{\text{free}})\right) = \left(2\pi k + \pi, \frac{\omega_\Delta^{\text{free}}\tau_1}{K_0}\right)$$

are unstable saddle points for $\forall k \in \mathbb{Z}$. To estimate the

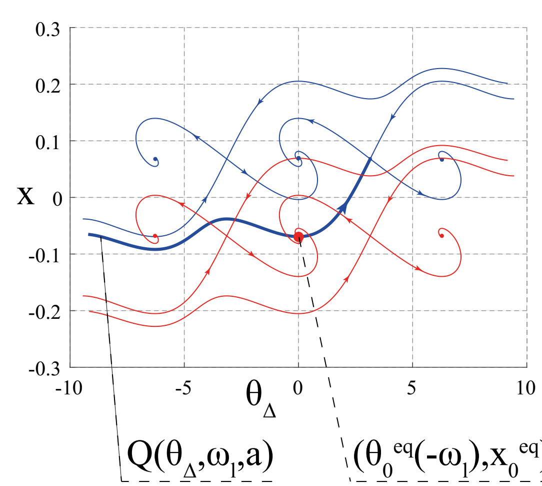

Figure 7. Phase portrait of system (4).

lock-in range of system (4) one needs to study behaviour of the lower separatrix $Q(\theta_\Delta,\omega_\Delta^{\text{free}})$ (see Fig. 7), which tends to the saddle point $\left(\theta_1^{eq}, x_1^{eq}(\omega_\Delta^{\text{free}})\right) = \left(\pi, \frac{\omega_\Delta^{\text{free}}\tau_1}{K_0}\right)$ as $t \to +\infty$ (due to symmetry of the lower and the upper half-planes, the study of the the upper separatrix is also acceptable). Since system (4) is 2π-periodic, corresponding separatrix of any saddle point on the phase plane has the same behaviour.

The parameter $\omega_\Delta^{\text{free}}$ shifts the phase plane vertically. To verify this, one can perform linear change $x \to x + \frac{\omega_\Delta^{\text{free}}\tau_1}{K_0}$. Thus, to compute the lock-in range of system (4), one needs to find $\omega_\Delta^{\text{free}} = \omega_l$ (where ω_l is called a lock-in frequency) such that

$$x_0^{eq}(-\omega_l) = Q(\theta_0^{eq},\omega_l). \tag{5}$$

Using relation (5), one can obtain precise formula for the ω_l:

$$-\frac{\omega_l}{K_0/\tau_1} = \frac{\omega_l}{K_0/\tau_1} + Q(\theta_0^{eq},0).$$
$$\omega_l = -\frac{K_0 Q(\theta_0^{eq},0)}{2\tau_1}. \tag{6}$$

For system (4) with varying parameter $0 < \tau_2/\tau_1 \ll 1$ and one can expand the separatrix $Q(\theta_\Delta,0,\tau_2/\tau_1)$ in a Taylor series in variable τ_2/τ_1. The consideration of a filter with small parameter (Alexandrov et al., 2014) allows one to integrate separatrices and to estimate the lock-in range. For this purpose approximations of the separatrix $Q(\theta_\Delta,0,\tau_2/\tau_1)$ on interval $0 \le \theta_\Delta < \pi$ are used.

Approximations obtained below are computed by consideration of the system, which is equivalent system (4):

$$\begin{cases} \dot\theta_\Delta(t) = y(t), \\ \dot y(t) = -\frac{K_0\tau_2}{\tau_1}\dot\varphi(\theta_\Delta(t))y(t) - \frac{K_0}{\tau_1}\varphi(\theta_\Delta(t)), \end{cases} \tag{7}$$

where $y(t) = \omega_\Delta^{\text{free}} - \frac{K_0}{\tau_1}\left(x(t) + \tau_2\varphi(\theta_\Delta(t))\right)$. The first approximation of the lower separatrix $Q(\theta_\Delta,0,\tau_2/\tau_1)$ has the form

$$\hat Q_1(\theta_\Delta,0,\tau_2/\tau_1) = -2\sqrt{K_0/\tau_1}\cos\frac{\theta_\Delta}{2} - \frac{\tau_2}{\tau_1}\frac{K_0\left(\frac{2}{3} - \sin\frac{\theta_\Delta}{2} - \frac{1}{3}\sin\frac{3\theta_\Delta}{2}\right)}{\cos\frac{\theta_\Delta}{2}}. \tag{8}$$

The second approximation of the $Q(\theta_\Delta,0,\tau_2/\tau_1)$ has the form

$$\begin{aligned}\hat Q_2(\theta_\Delta,0,\tau_2/\tau_1) &= -2\sqrt{K_0/\tau_1}\cos\frac{\theta_\Delta}{2} - \\ &- \frac{\tau_2}{\tau_1}\frac{K_0\left(\frac{2}{3} - \sin\frac{\theta_\Delta}{2} - \frac{1}{3}\sin\frac{3\theta_\Delta}{2}\right)}{\cos\frac{\theta_\Delta}{2}} - \\ &- \left(\frac{\tau_2}{\tau_1}\right)^2 \frac{K_0^2(6\frac{1}{2} - 4\ln 2)}{6\sqrt{K_0/\tau_1}\cos\frac{\theta_\Delta}{2}} + \\ &+ \frac{K_0^2\left(\frac{2}{3} - \sin\frac{\theta_\Delta}{2} - \frac{1}{3}\sin\frac{3\theta_\Delta}{2}\right)^2}{4\sqrt{K_0/\tau_1}\cos^3\frac{\theta_\Delta}{2}} + \\ &+ \left(\frac{\tau_2}{\tau_1}\right)^2 \frac{K_0^2\left(8\sin(\frac{\theta_\Delta}{2}) - 4\ln\left|\sin\frac{\theta_\Delta}{2} + 1\right|\right)}{6\sqrt{K_0/\tau_1}\cos\frac{\theta_\Delta}{2}} + \\ &+ \left(\frac{\tau_2}{\tau_1}\right)^2 \frac{K_0^2\left(\frac{1}{2}\cos 2\theta_\Delta + 2\cos\theta_\Delta\right)}{6\sqrt{K_0/\tau_1}\cos\frac{\theta_\Delta}{2}}.\end{aligned} \tag{9}$$

For approximations $\hat Q_1(\theta_\Delta,0,\tau_2/\tau_1)$ and $\hat Q_2(\theta_\Delta,0,\tau_2/\tau_1)$, which are finite sums of Taylor series terms, the following relations are valid:

$$Q(\theta_\Delta,0,\tau_2/\tau_1) = \hat Q_1(\theta_\Delta,0,\tau_2/\tau_1) + O\left((\tau_2/\tau_1)^2\right),$$
$$Q(\theta_\Delta,0,\tau_2/\tau_1) = \hat Q_2(\theta_\Delta,0,\tau_2/\tau_1) + O\left((\tau_2/\tau_1)^3\right).$$

The first and the second approximations of the $Q(\theta_\Delta,0,\tau_2/\tau_1)$ in $\theta_\Delta = \theta_0^{eq}$ are equal to the following values:

$$\hat Q_1(\theta_0^{eq},0,\tau_2/\tau_1) = -2\sqrt{K_0/\tau_1} - \frac{2K_0\tau_2}{3\tau_1},$$
$$\hat Q_2(\theta_0^{eq},0,\tau_2/\tau_1) = -2\sqrt{K_0/\tau_1} - \frac{2K_0\tau_2}{3\tau_1} - \frac{K_0\tau_2^2(5 - 6\ln 2)}{9\tau_1}\sqrt{K_0/\tau_1}.$$

Using formula (6) approximations for the lock-in frequency ω_l can be found:

$$\omega_l = \frac{K_0\sqrt{K_0/\tau_1}}{\tau_1} + \frac{K_0^2\tau_2}{3\tau_1^2} + O\left((\tau_2/\tau_1)^2\right), \tag{10}$$

$$\omega_l = \frac{K_0\sqrt{K_0/\tau_1}}{\tau_1} + \frac{K_0^2\tau_2}{3\tau_1^2} + \frac{K_0^2\tau_2^2(5 - 6\ln 2)}{18\tau_1^2}\sqrt{K_0/\tau_1} + O\left((\tau_2/\tau_1)^3\right). \tag{11}$$

3.2 Verification of obtained results

To verify obtained analytical results, numerical simulations are performed. In Fig. 10 dependency of $\frac{\omega_l\tau_1}{K_0}$ on parameter $\frac{K_0}{\tau_1}$ is presented. The dependency is obtained numerically using relation (6). To compute the value of ω_l for fixed values of K_0, τ_1, τ_2, one has to follow the steps described below. First, choose the curve in Fig. 10, which corresponds to the value of τ_2. On that curve choose point for which X-axis value is equal to the value K_0/τ_1, and get the Y-axis value of the chosen point. Multiply this value by K_0/τ_1. The result is the desired value of ω_l.

In Fig. 8 three curves are shown for the fixed $\tau_2 = 0.1$. Values of ω_l, obtained numerically using relation (6) (blue curve), are estimated from below by values of (10) (red

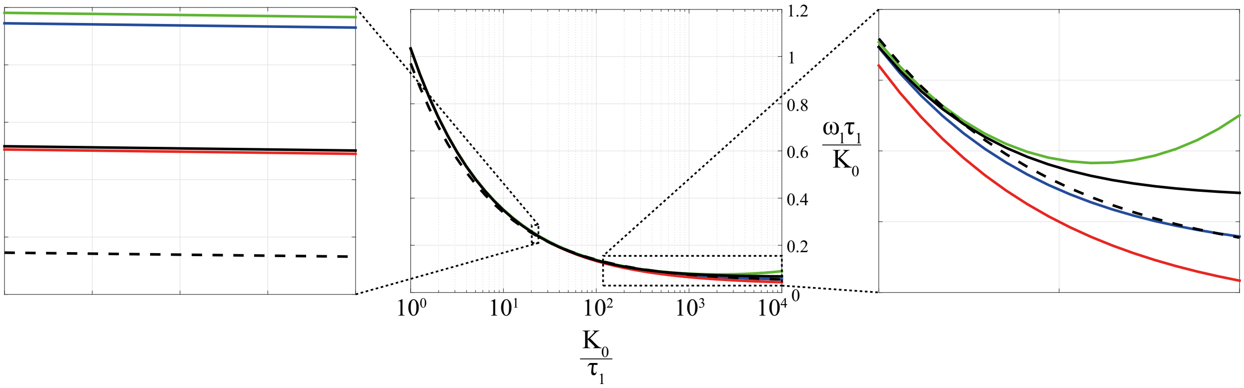

Figure 9. Estimations on separatrix $S(\theta_\Delta, 0, \tau_2/\tau_1)$.

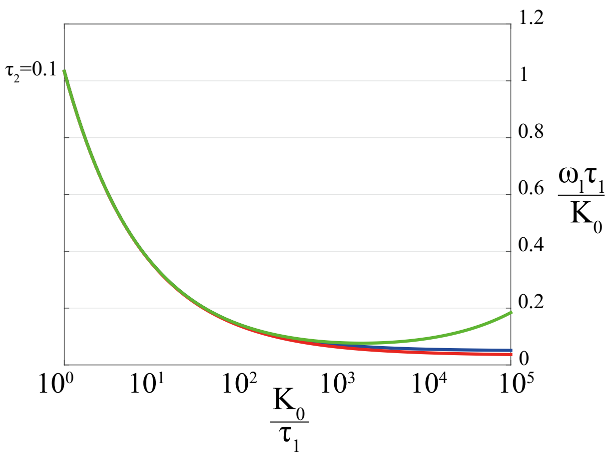

Figure 8. Values of $\frac{\omega_l}{K_0/\tau_1}$ for various K_0, τ_1, τ_2.

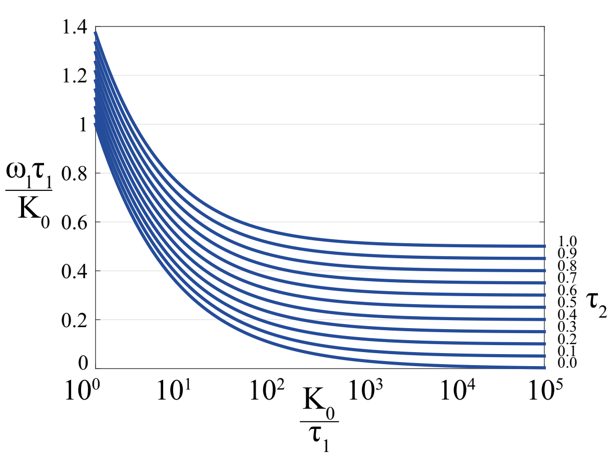

Figure 10. Values of $\frac{\omega_l}{K_0/\tau_1}$ for various K_0, τ_1, τ_2.

curve) and from above by values of (11) (green curve). Axis in Fig. 8 are chosen in such a way that obtained plot is universal, i.e. for fixed value of τ_2 one is able to get the value of ω_l for any values of K_0, τ_1.

Remark 1. Approximations of $Q(\theta_\Delta, 0, \tau_2/\tau_1)$ are obtained under the condition that parameter τ_2/τ_1 is small, thus for the large values of K_0/τ_1 obtained estimations give less precise result.

3.3 Pull-out frequency and lock-in range

While lock-in range is useful for phase acquisition, there exist another concept of pull-out frequency, which is used for frequency tracking (Gardner, 2005). In (Gardner, 2005) pull-out frequency $\omega_{\rm po}$ is defined as frequency-step limit, below which the loop does not skip cycles but remains in lock. However, in contrast to Definition 2 of the lock-in range, notion of the pull-out frequency has not been generalized (Huque and Stensby, 2013) (in the general case $\omega_{\rm po}$ depends on $\omega_\Delta^{\rm free}$, see (Leonov et al., 2015a, Fig. 10)). In case of PI filter both Definition 2 and notion (Gardner, 2005) of the pull-out frequency needs separatrices of system (4) to be computed. Results for pull-out frequency estimation (approximations of the upper separatrix $S(\theta_\Delta, 0, \tau_2/\tau_1)$) obtained in (Gardner, 2005; Huque and Stensby, 2013) are compared with results obtained in this paper (see Fig. 3.1).

Values of the $S(\theta_0^{eq}, 0, \tau_2/\tau_1)$, obtained numerically are drawn in blue color. The black curve is the estimation of $S(\theta_0^{eq}, 0, \tau_2/\tau_1)$ from (Huque and Stensby, 2013). Values of

$$\hat{S}_1(\theta_0^{eq}, 0, \tau_2/\tau_1) = -\hat{Q}_1(\theta_0^{eq}, 0, \tau_2/\tau_1)$$

are drawn in red color. Values of

$$\hat{S}_2(\theta_0^{eq}, 0, \tau_2/\tau_1) = -\hat{Q}_2(\theta_0^{eq}, 0, \tau_2/\tau_1)$$

are drawn in green color. Dashed curve corresponds to the empirical estimation (Gardner, 1979; Stensby, 1997) :

$$S(\theta_0^{eq}, 0, \tau_2/\tau_1) = 1.85\left(\frac{1}{2} + \frac{\tau_1}{K_0\tau_2^2}\right), \qquad (12)$$

In practice, for K_0/τ_1 not large $\hat{S}_2(\theta_0^{eq}, 0, \tau_2/\tau_1)$ gives the most precise result compared to presented estimations. However, for K_0/τ_1 large relation (12) is the most precise estimation.

4. CONCLUSION

Combination of different analytical approaches to the lock-in range estimation is discussed. Considered methods are based on the integration of trajectories and on Lyapunov functions construction. Validity of obtained analytical estimations is verified by numerical simulations.

ACKNOWLEDGMENTS

The authors would like to thank Roland E. Best, the founder of the Best Engineering Company, Oberwil, Switzerland and the author of the bestseller on PLL-based circuits Best (2007) for valuable discussion.

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