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A posteriori error identities for nonlinear variational problems*

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Dedicated to the memory of Prof. Dr. Viorel Arnăutu

Abstract

A posteriori error estimation methods are usually developed in the context of upper and lower bounds of errors. In this paper, we are concerned with a posteriori analysis in terms of identities, i.e., we deduce error relations, which holds as equalities. We discuss a general form of error identities for a wide class of convex variational problems. The left hand sides of these identities can be considered as certain measures of errors (expressed in terms of primal/dual solutions and respective approximations) while the right hand sides contain only known approximations. Finally, we consider several examples and show that in some simple cases these identities lead to generalized forms of the Prager-Synge and Mikhlin's error relations. Also, we discuss particular cases related to power growth functionals and to the generalized Stokes problem.

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1 Functional setting

In this section, we present the class of variational problems to be considered and recall several basic facts related to this class of problems.

Throughout the paper we use two pairs of mutually conjugate reflexive Banach spaces. The first pair is Y and Y^* (with the duality pairing (y^*, y), where $y^* \in Y^*$ and $y \in Y$). The norms of Y and Y^* are denoted by $\|\cdot\|$ and $\|\cdot\|_*$, respectively. Another pair of spaces is V and V^*. The product of $v \in V$ and $v^* \in V^*$ is denoted by $< v^*, v >$. We assume that

$$V \subset \mathcal{V} \subset V^*,$$

where $\mathcal{V}$ is a Hilbert space with the norm $\|\cdot\|_{\mathcal{V}}$ and scalar product $(\cdot,\cdot)_{\mathcal{V}}$, so that $< v^*, v > = (v^*, v)_{\mathcal{V}}$ for any $v^* \in \mathcal{V}$.

By $\Lambda : V \to Y$ we denote a bounded linear operator and assume that the conjugate operator $\Lambda^* : Y^* \to V^*$ satisfies the relation

$$(y^*, \Lambda w) \; = < \Lambda^* y^*, w >, \qquad \forall w \in V. \tag{1.1}$$

If y^* is more regular and belongs to the set

$$H^*_{\Lambda^*} := \{\, y^* \in Y^* \;|\; \Lambda^* y^* \in \mathcal{V} \,\},$$

then (1.1) can be rewritten in the form

$$(y^*, \Lambda w) \; = \; (\Lambda^* y^*, w)_{\mathcal{V}}, \qquad \forall w \in V. \tag{1.2}$$

We consider the following class of variational problems: find $u \in V$ such that

$$J(u) = \inf \mathcal{P} := \inf_{v \in V} J(v) \qquad (Problem\, \mathcal{P}), \tag{1.3}$$

where

$$J(v) \; = \; G(\Lambda v) \; + \; F(v), \tag{1.4}$$

the functionals $G : Y \to \mathbb{R}$ and $F : V \to \mathbb{R}$ are convex and lower semicontinuous functionals such that $J(v)$ is a proper functional (cf. [2]) and

$$J(v) \; \to \; +\infty \quad \text{as } \|v\|_V \to +\infty. \tag{1.5}$$

In addition, we assume that F is finite at zero element of V and G is coercive on Y.

As usual, the functionals dual to F and G are defined by the relations

$$F^*(v^*) = \sup_{v\in V} \{ < v^*, v > -F(v)\}$$

and

$$G^*(y^*) = \sup_{y\in Y} \{ \, (y^*, y) \; - \; G(y) \, \} \, ,$$

respectively.

If $v^* \in H^*_{\Lambda^*}$, then the first relation admits another form

$$F^*(v^*) = \sup_{v\in V} \{(v^*, v)_{\mathcal{V}} - F(v)\} \, .$$

Existence of a minimizer u to Problem $\mathcal{P}$ follows from standard arguments of the variational calculus (see, e.g., [1, 2]).

Problem $\mathcal{P}$ has a saddle point formulation associated with the Lagrangian

$$L(v, y^*) := \; F(v) + (\, y^*, \Lambda v) \; - G^*(y^*),$$

which is convex and lower semicontinuous with respect to the variable v and concave and upper semicontinuous with respect to the variable y^*.

The Lagrangian yields a dual variational functional defined by the relation

$$\begin{aligned} I^*(y^*) = \inf_{v\in V} L(v, y^*) &= - \, G^*(y^*) + \inf_{v\in V} ((y^*, \Lambda v) + F(v)) \\ &= - \, G^*(y^*) - \sup_{v\in V} (< -\Lambda^* y^*, v > -F(v)) \\ &= - \, G^*(y^*) - F^*(-\Lambda^* y^*) \end{aligned}$$

and a new (dual) variational problem: find $p^* \in Y^*$ such that

$$I^*(p^*) = \sup_{y^*\in Y^*} \{- \, G^*(y^*) - F^*(-\Lambda^* y^*)\} \qquad (Problem \, \mathcal{P}^*).$$

It is not difficult to show that under the above made assumptions

$$\inf \mathcal{P} \; = \; \sup \mathcal{P}^* := \sup_{y^*\in Y^*} \inf_{v\in V} L(v, y^*) \tag{1.6}$$

and Problem $\mathcal{P}^*$ also has a solution.

2 General form of error identities for convex variational problems

Since both primal and dual problems are well posed and have solutions u^* and p^*, respectively, the pair (u, p^*) is a saddle point of L on $V \times Y^*$, i.e.

$$L(u, y^*) \leq L(u, p^*) \leq L(v, p^*), \quad \forall v \in V,\ y^* \in Y^*, \tag{2.1}$$

The left–hand side of the inequality yields the relation

$$(y^* - p^*, \Lambda u) \leq G^*(y^*) - G^*(p^*), \quad \forall y^* \in Y^*,$$

which means that

$$\Lambda u \in \partial G^*(p^*) \quad \Leftrightarrow \quad p^* \in \partial G(\Lambda u). \tag{2.2}$$

Analogously, the right–hand side of (2.1) yields the relation

$$F(v) - F(u) \geq (p^*, \Lambda(u - v)) = < -\Lambda^* p^*, v - u >, \tag{2.3}$$

which means that

$$-\Lambda^* p^* \in \partial F(u) \quad \Leftrightarrow \quad u \in \partial F^*(-\Lambda^* p^*). \tag{2.4}$$

In general, the relations (2.2) and (2.4) present necessary conditions for the solution pair (u, p^*) and have the form of differential inclusions. However, there is another equivalent way to present these conditions, which is more convenient for our purposes. It is well known (see, e.g., [2, 5]) that (2.2) and (2.4) are equivalent to the relations

$$D_G(\Lambda u, p^*) := G(\Lambda u) + G^*(p^*) - (p^*, \Lambda u) = 0, \tag{2.5}$$

and

$$D_F(u, -\Lambda^* p^*) := F(u) + F^*(-\Lambda^* p^*) + < \Lambda^* p^*, u > = 0, \tag{2.6}$$

respectively.

The functionals $D_G(y^*, y) : Y^* \times Y \to \mathbb{R}$ and $D_F(v^*, v) : V^* \times V \to \mathbb{R}$ (in the literature, they are often called compound functionals) vanish if and only if the arguments satisfy (2.2) and (2.4). In all other cases, they are positive.

Let $q^* \in Y^*$ and $v \in V$ be the functions compared with p^* and u. We introduce the following nonlinear measure of the distance between $\{u, p^*\}$ and $\{v, y^*\}$:

$$\begin{aligned}&\mathbb{M}(\{u,p^*\},\{v,y^*\}) := \\ &\quad = D_F(u, -\Lambda^* y^*) + D_G(\Lambda u, y^*) + D_F(v, -\Lambda^* p^*) + D_G(\Lambda v, p^*). \quad (2.7)\end{aligned}$$

It is easy to see that $\mathbb{M}(\{u, p^*\}, \{v, y^*\})$ is nonnegative and vanishes if and only if

$$\begin{aligned}\Lambda v \in \partial G^*(p^*), \qquad & y^* \in \partial G(\Lambda u), \\ -\Lambda^* y^* \in \partial F(u), \qquad & v \in \partial F^*(-\Lambda^* p^*).\end{aligned}$$

In other words, $\mathbb{M}(\{u, p^*\}, \{v, y^*\})$ vanishes if and only if all the necessary saddle point conditions are satisfied. Moreover, it was proved (see [5], Section 7.2 and [10]) that

$$\mathbb{M}(\{u,p^*\},\{v,y^*\}) = J(v) - I^*(y^*). \quad (2.8)$$

We see that $\mathbb{M}\{(u, p^*), (v, y^*)\} = 0$ if and only if $J(v) = I^*(y^*)$ what is possible only if v is a minimizer of the problem $\mathcal{P}$ and y^* is a maximizer of the problem $\mathcal{P}^*$. In view of this fact, in [11] the functional $\mathbb{M}$ was introduced as the right error measure for the class of variational problems (1.3)–(1.4). Since any numerical procedure is focused (explicitly or implicitly) on minimization of the duality gap $J(v) - I^*(y^*)$, it automatically minimizes the distance between $\{u, p^*\}$ and $\{v, y^*\}$ in terms of the measure $\mathbb{M}$.

Now we can formulate the main result, which presents the general *a posteriori error identity* for the considered class of problems.

Theorem 2.1 *Let u be a minimizer of the Problem $\mathcal{P}$ and p^* be a maximizer of the Problem $\mathcal{P}^*$. Then, for any $v \in V$ and $y^* \in Y^*$ the following identity holds:*

$$\mathbb{M}\{(u,p^*),(v,y^*)\} = D_F(v, -\Lambda^* y^*) + D_G(\Lambda v, y^*). \quad (2.9)$$

The statement directly follows from (2.8). Indeed,

$$\begin{aligned}J(v) - I^*(y^*) &= G(\Lambda v) + F(v) + G^*(y^*) + F^*(-\Lambda^* y^*) \\ &= D_G(\Lambda v, y^*) + (y^*, \Lambda v) + D_F(v, -\Lambda^* y^*) - < \Lambda^* y^*, v > .\end{aligned}$$

We apply (1.1) and arrive at (2.9).

We note that a somewhat different notation the identity (2.9) was proved in [5] (see 7.2.14). It has a clear meaning: the distance between the pair of exact solutions and their approximations measured in terms of the measure $\mathbb{M}$ is equal to the sum of two fully computable functionals $D_G(\Lambda v, y^*)$ and $D_F(v, -\Lambda^* y^*)$ that depend only on approximate solutions and does not contain unknown exact solutions. Therefore, this relation can be viewed as the basic *a posteriori error identity* .

Remark 2.1 It is commonly accepted that errors should be measured in terms of relative (normalized) quantities, which adjust absolute values of errors to a certain measure (e.g., norm) of the exact solution. The relation (2.8) clearly suggests a proper normalization. Since the duality gap $J(v) - I^*(y^*)$ is equal to the error measure $\mathbb{M}(\{u, p^*\}, \{v, y^*\})$ and $C^* := |J(u)| = |I^*(p^*)|$ is a number inside it related to the exact values of the primal/dual energy functionals, it is natural to use the quantity

$$\mathcal{E}(v, y^*) = \frac{1}{C^*}\mathbb{M}(\{u, p^*\}, \{v, y^*\})$$

as a normalized measure of the error (trivial solutions with zero energy are excluded from this consideration). Since $J(u)$ is generally unknown, in practice it may be suggested to use the constant $\widetilde{C}^* = \frac{1}{2}(|J(v)| + |I^*(y^*)|)$ instead of C^*. Then we recall (2.9) and introduce the quantity

$$\widetilde{\mathcal{E}}(v, y^*) = \frac{1}{\widetilde{C}^*}\left(D_F(v, -\Lambda^* y^*) + D_G(\Lambda v, y^*)\right)$$

as a fully computable normalized measure that objectively quantify the accuracy of (v, y^*).

A special, but important case

$$F(v) = < \ell^*, v >, \qquad \ell^* \in V^*$$

deserves a special consideration. We have

$$F^*(-\Lambda^* y^*) = \sup_{v \in V} < -\Lambda^* y^* - \ell^*, v > = \begin{cases} 0 & \text{if } y^* \in Q^*_{\ell^*}, \\ +\infty & \text{if } y^* \notin Q^*_{\ell^*}, \end{cases}$$

where

$$Q^*_{\ell^*} := \{q^* \in Y^* \mid (q^*, \Lambda w) + \langle \ell^*, w \rangle = 0, \quad w \in V\}.$$

Hence,

$$I^*(y^*) = \begin{cases} -G^*(y^*) & \text{if } y^* \in Q^*_{\ell^*}, \\ -\infty & \text{if } y^* \notin Q^*_{\ell^*} \end{cases}$$

Problem $\mathcal{P}^*$ has the form: find $p^* \in Q^*_{\ell^*}$ such that the functional $-G^*(p^*)$ attains its supremum on $Q^*_{\ell^*}$.

It is easy to see that the identity (2.8) holds in the form $+\infty = +\infty$ if $y^* \notin Q^*_{\ell^*}$ and in the form

$$D_G(\Lambda u, y^*) + D_G(\Lambda v, p^*) = J(v) - I^*(y^*) \tag{2.10}$$

for $y^* \in Q^*_{\ell^*}$. Therefore, we conclude that for $v \in V$ and $y^* \in Q^*_{\ell^*}$ the error measure is defined by the relation

$$\mathbb{M}\{(u, p^*), (v, y^*)\} \ = D_G(\Lambda u, y^*) + D_G(\Lambda v, p^*).$$

and the a posteriori error identity (2.9) holds on the affine manifold $Q^*_{\ell^*}$ in the form

$$\mathbb{M}\{(u, p^*), (v, y^*)\} \ = D_G(\Lambda v, y^*). \tag{2.11}$$

Identities (2.10) and (2.11) have been established in [9, 10] and used for the derivation of functional type a posteriori error estimates for a wide class of convex variational problems.

Now we consider particular forms of (2.8)–(2.11) related to some classes of functionals commonly used in mathematical modeling.

3 Problems with quadratic $G(y^*)$

Let U be a Hilbert space endowed with the scalar product $(\cdot, \cdot)_U$ containing the same elements as Y and Y^* and $A : U \to U$ be a bounded, linear, and positive definite operator. The spaces Y and Y^* are identified by the norms

$$\|\tau\|^2 = (A\tau, \tau)_U \quad \text{and} \quad \|\tau\|^2_* = (A^{-1}\tau, \tau)_U,$$

respectively (it is clear that these norms are equivalent to the original norm of U). We define Λ as a linear bounded operator acting from V to U. The conjugate operator $\Lambda^* : U \to V^*$ is defined by the relation

$$(y, \Lambda v)_U \ = < \Lambda^* y, v > .$$

Consider first the problem

$$\Lambda^* A \Lambda u + \alpha u = \ell^*, \tag{3.1}$$

where α is a positive constant and $\ell^* \in \mathcal{V}$. The corresponding generalized solution u is defined by the relation

$$(A\Lambda u, \Lambda w)_U + \alpha (u, w)_{\mathcal{V}} \;=\; (\ell^*, w)_{\mathcal{V}} \qquad \forall w \in V. \tag{3.2}$$

In this case,

$$G(y) = \frac{1}{2}(Ay, y)_U, \quad G^*(y^*, y^*) = \frac{1}{2}(A^{-1}y^*, y^*)_U,$$

and

$$F(v) = \frac{\alpha}{2}\|v\|^2_{\mathcal{V}} - (\ell^*, v)_{\mathcal{V}}.$$

We find that for any $v^* \in \mathcal{V}$

$$F^*(v^*) = \sup_{v \in V} \Big\{ (v^* + \ell^*, v)_{\mathcal{V}} - \frac{\alpha}{2}\|v\|^2_{\mathcal{V}} \Big\} = \frac{1}{2\alpha}\|v^* + \ell^*\|^2_{\mathcal{V}}.$$

For any $y^* \in Y^*$, we have

$$\begin{aligned} D_G(\Lambda u, y^*) = \frac{1}{2}\big((A\Lambda u, \Lambda u)_U + (A^{-1}y^*, y^*)_U - 2(\Lambda u, y^*)_U \big) \\ = \frac{1}{2}|\!|\!| A\Lambda u - y^* |\!|\!|^2_* \end{aligned} \tag{3.3}$$

and

$$D_G(\Lambda v, p^*) = \frac{1}{2}|\!|\!| A\Lambda v - p^* |\!|\!|^2_*. \tag{3.4}$$

Let $y^* \in H^*_{\Lambda^*}$. Then,

$$\begin{aligned} D_F(u, -\Lambda^* y^*) = \frac{\alpha}{2}\|u\|^2_{\mathcal{V}} + \frac{1}{2\alpha}\|\ell^* - \Lambda^* y^*\|^2_{\mathcal{V}} + (u, \Lambda^* y^* - \ell^*)_{\mathcal{V}} \\ = \frac{1}{2\alpha}\|\Lambda^* y^* + \alpha u - \ell^*\|^2_{\mathcal{V}} \end{aligned} \tag{3.5}$$

and quite analogously (note that $p^* \in H^*_{\Lambda^*}$) we obtain

$$D_F(v, -\Lambda^* p^*) = \frac{1}{2\alpha}\|\Lambda^* p^* + \alpha v - \ell^*\|^2_{\mathcal{V}}. \tag{3.6}$$

Now we recall (2.2) and (2.4). Since the functionals G and G^* are Gateaux differentiable, the relations (2.2) have the form

$$p^* = G'(\Lambda u) = A\Lambda u \quad \text{and} \quad \Lambda u = (G^*)'(p^*) = A^{-1}p^*. \tag{3.7}$$

The functionals F, and F^* are also differentiable. Therefore, (2.4) have the form

$$u = (F^*)'(-\Lambda^* p^*) = \frac{1}{\alpha}(\ell^* - \Lambda^* p^*) \tag{3.8}$$

and

$$-\Lambda^* p^* = F'(u) = \alpha u - \ell^*. \tag{3.9}$$

By (3.7)–(3.9) we conclude that the components of the measure $\mathbb{M}$ are as follows:

$$D_G(\Lambda u, y^*) = \frac{1}{2}|\!|\!|p^* - y^*|\!|\!|_*^2, \tag{3.10}$$

$$D_G(\Lambda v, p^*) = \frac{1}{2}|\!|\!|\Lambda(u - v)|\!|\!|^2, \tag{3.11}$$

$$D_F(u, -\Lambda^* y^*) = \frac{1}{2\alpha}\|\Lambda^*(y^* - p^*)\|_{\mathcal{V}}^2, \tag{3.12}$$

$$D_F(v, -\Lambda^* p^*) = \frac{\alpha}{2}\|v - u\|_{\mathcal{V}}^2. \tag{3.13}$$

Thus, for this class of linear problems the measure $\mathbb{M}\{(u, p^*), (v, y^*)\}$ is defined by the sum of above presented four norms of two error functions $e := u - v$ and $\eta^* := p^* - y^*$.

It is easy to see that $\mathbb{M}\{(u, p^*), (v, y^*)\}$ is equivalent to the sum of two norms associated with the primal and dual errors:

$$|\!|\!|e|\!|\!|_\alpha^2 := \frac{1}{2}\left(|\!|\!|\Lambda e|\!|\!|^2 + \alpha\|e\|_{\mathcal{V}}^2\right). \tag{3.14}$$

and

$$\|\eta^*\|_{H^*,\frac{1}{\alpha}}^2 := \frac{1}{2}\left(|\!|\!|\eta^*|\!|\!|_*^2 + \frac{1}{\alpha}\|\Lambda^*\eta^*\|_{\mathcal{V}}^2\right) \tag{3.15}$$

Here the first norm is the energy norm associated with the primal variational functional J and the second one can be viewed a norm of the space $H^*_{\Lambda^*}$. We see that

$$\mathbb{M}(\{u, p^*\}, \{v, y^*\}) = |\!|\!|e|\!|\!|_\alpha^2 + \|\eta^*\|_{H^*,\frac{1}{\alpha}}^2$$

and the identity (2.8) reads

$$\|\!|e|\!\|_\alpha^2 + \|\eta^*\|^2_{H^*,\frac{1}{\alpha}} = J(v) - I^*(y^*). \tag{3.16}$$

In other words, for this class of variational problems the difference between the primal and dual functionals measured in terms of $\mathbb{M}$ is equal to the sum of specially selected norms.

Theorem 2.1 implies the following a posteriori error identity:

$$\|\!|e|\!\|_\alpha^2 \;+\; \|\eta^*\|^2_{H^*,\frac{1}{\alpha}} \;=\; \frac{1}{2}\|\!|A\Lambda v \,-\, y^*|\!\|_*^2 \,+\, \frac{1}{2\alpha}\|\!|\Lambda^* y^* \,+\, \alpha v \,-\, \ell^*|\!\|_{\mathcal{V}}^2. \tag{3.17}$$

The right hand side of this identity contains only known functions and vanishes if and only if

$$\begin{aligned} &A\Lambda v - y^* = 0,\\ &\Lambda^* y^* + \alpha v - \ell^* = 0, \end{aligned}$$

i.e., if $v = u$ (cf. (3.1)) and $y^* = p^*$. In all other cases the right hand side is positive and equals to the combined primal–dual measure of the error presented by the left hand side. We note that such type identity (both sides of which are expressed in terms of squared norms) takes place only for this class of linear problems.

The identity (3.17) is not valid for $\alpha = 0$. In this case, then we must use (2.10) and (2.11) and use introduce the condition

$$\Lambda^* y^* = \ell^*. \tag{3.18}$$

We find that

$$\begin{aligned} \mathbb{M}\{(u,p^*),(v,y^*)\} \;&= \frac{1}{2}\|\!|A\Lambda u - y^*|\!\|_*^2 + \frac{1}{2}\|\!|A\Lambda v - p^*|\!\|_*^2\\ &= \frac{1}{2}\|\!|\eta^*|\!\|^2 + \frac{1}{2}\|\!|\Lambda e\,|\!\|^2. \end{aligned}$$

Then, (2.10) yields the identity

$$\frac{1}{2}\|\!|\eta^*|\!\|_*^2 + \frac{1}{2}\|\!|\Lambda e\,|\!\|^2 = J(v) - I^*(y^*). \tag{3.19}$$

Set here $y^* = p^*$. Since

$$I^*(p^*) = J(u) \quad \text{and} \quad p^* = A\Lambda u,$$

we obtain

$$\frac{1}{2}\|\Lambda e\|^2 \;=\; J(v) - J(u). \tag{3.20}$$

This is a generalized form of the Mikhlin's error identity (see, e.g., [4, 3]), which was derived for variational functionals defined by quadratic forms.

By (2.11), we find that

$$\|A\Lambda u - y^*\|_*^2 + \|A\Lambda v - p^*\|_*^2 = \|A\Lambda v - y^*\|_*^2. \tag{3.21}$$

We can rewrite it in an equivalent form

$$\|\eta^*\|_*^2 \;+\; \|\Lambda e\|^2 \;=\; \|A\Lambda v - y^*\|_*^2, \quad \forall y^* \in Q^*_{\ell^*}. \tag{3.22}$$

The latter identity can be viewed as a generalization of the Prager–Synge error relation derived in [7] for linear elasticity problems.

Remark 3.1 We see that error measures arising in the estimates are generated by nonnegative compound functionals. In the linear case (i.e., for quadratic type functionals), they are equivalent to norms. However in general, $\mathbb{M}$ consists of nonlinear terms that jointly form a proper measure of the accuracy (see [11]).

4 Particular cases

Now we briefly discuss applications of the above presented error identities to particular classes variational problems.

4.1 Quadratic growth problems with $\Lambda = \mathrm{grad}$

Let $V = H_0^1(\Omega)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^d$ $(d \geq 1)$. We set $U = L^2(\Omega, \mathbb{R}^d)$, $\mathcal{V} = L^2(\Omega)$, and identify A with a symmetric real matrix in $\mathbb{M}^{d\times d}_{sym}$. Then $\Lambda^* = -\mathrm{div}$ and (3.1) is the equation

$$\mathrm{div}\, A\nabla u - \alpha u + \ell^* = 0. \tag{4.1}$$

In this case,

$$\|e\|_\alpha^2 = \frac{1}{2}\int\limits_\Omega \left(A\nabla e\cdot\nabla e + \alpha|e|^2\right)dx,$$

$$\|\eta^*\|^2_{H^*,\frac{1}{\alpha}} = \frac{1}{2}\int\limits_\Omega \left(A^{-1}\eta^*\cdot\eta^* + \frac{1}{\alpha}|\mathrm{div}\,\eta^*|^2\right),$$

and the a posteriori error identity (3.17) has the form

$$\|\!|e|\!\|_\alpha^2 + \|\!|\eta^*|\!\|^2_{H^*,\frac{1}{\alpha}} = \frac{1}{2}\int\limits_\Omega \big(A\nabla v\cdot\nabla v + A^{-1}y^*\cdot y^* - y^*\cdot\nabla v\big)\,dx + \frac{1}{2\alpha}\|\operatorname{div} y^* - \alpha v + \ell^*\|^2. \quad (4.2)$$

If $\alpha = 0$, then we use (3.18) and (3.19) and obtain the identities

$$\frac{1}{2}\int\limits_\Omega A^{-1}\eta^*\cdot\eta^*\,dx + \frac{1}{2}\int\limits_\Omega A\nabla e\cdot\nabla e\,dx = J(v) - I^*(y^*) \quad (4.3)$$

and the Mikhlin's identity

$$\frac{1}{2}\int\limits_\Omega A\nabla e\cdot\nabla e\,dx \;=\; J(v) - J(u). \quad (4.4)$$

By (3.22) we obtain a version of the Prager-Synge identity

$$\int\limits_\Omega (A^{-1}\eta^*\cdot\eta^* \;+ A\nabla e\cdot\nabla e)dx = \int\limits_\Omega \big(A\nabla v\cdot\nabla v + A^{-1}y^*\cdot y^* - \nabla v\cdot y^*\big)\,dx. \quad (4.5)$$

4.2 Problems with the operator $\Lambda = \operatorname{Sym}\nabla$

Problems of this type arise in continuum media problems, where Λ is a symmetric part of the tensor ∇u and u is a vector field. In this case, the error identities are quite similar to (4.2)–(4.5). The reader can find a sistematic discussion of them and respective error majorants in [5, 9, 10, 12]).

4.3 Generalized Stokes problem

If V coincides with the space $S_0^{1,2}(\Omega,\mathbb{R}^d)$ that is the closure of smooth solenoidal fields with respect to the norm of $H^1(\Omega,\mathbb{R}^d)$ and $\Lambda v = \nabla v$, where v is the velocity vector field, then we arrive at a class of variational problems generated by incompressible media. The generalized Stokes problem is one of the most known problems in this class. It often arises in time

discretization of the parabolic Stokes problem. It is related to the system

$$-\nu\Delta u + \alpha u = \ell^* \qquad \text{in } \Omega, \tag{4.6}$$

$$\operatorname{div} u = 0, \tag{4.7}$$

$$u = u_0 \qquad \text{on } \partial\Omega, \tag{4.8}$$

where u_0 is a divergence free field in $H^1(\Omega, \mathbb{R}^d)$ and $\nu > 0$ is the viscosity.

In this case, $\Lambda^* = -\mathrm{Div}$ (i.e., the conjugate operator is formed by the divergence of a tensor field), $U = L^2(\Omega, \mathbb{R}^d)$,

$$G(\Lambda v) = \int\limits_\Omega \frac{\nu}{2}|\nabla v|^2 dx, \quad \text{and} \quad G^*(y^*) = \int\limits_\Omega \frac{1}{2\nu}|y^*|^2 dx.$$

Here $|y^*|$ denotes the Euclidean norm of the tensor y^* ($|y^*|^2 := y^* : y^*$).

Let $v \in S_0^{1,2}(\Omega, \mathbb{R}^d)$ and $y^* \in L^2(\Omega, \mathbb{M}^{d\times d}_{sym})$ be approximations of the exact velocity u and exact stress σ^*, respectively. Then the general method exposed in Sect. 2 suggests to measure the errors $e = u - v$ and $\eta^* = \sigma^* - y^*$ (for the velocity and stress) in terms of the integral type measures

$$|\!|\!| e |\!|\!|_\alpha^2 = \frac{1}{2}\int\limits_\Omega \left(\nu|\nabla e|^2 + \alpha|e|^2\right) dx$$

and

$$|\!|\!| \eta^* |\!|\!|^2_{H^*,\frac{1}{\nu}} = \frac{1}{2}\int\limits_\Omega \left(\frac{1}{\nu}|\eta^*|^2 + \frac{1}{\alpha}|\mathrm{Div}\,\eta^*|^2\right) dx,$$

respectively.

We conclude that the a posteriori error identity (3.17) has the form

$$|\!|\!| e |\!|\!|_\alpha^2 + |\!|\!| \eta^* |\!|\!|^2_{H^*,\frac{1}{\nu}} = \frac{1}{2}\int\limits_\Omega \left(\nu|\nabla v|^2 + \frac{1}{\nu}|y^*|^2 - y^* : \nabla v\right) dx + \frac{1}{2\alpha}\|\mathrm{Div}\, y^* - \alpha v + \ell^*\|^2. \tag{4.9}$$

4.4 Nonlinear problem

Finally, we consider an example of highly nonlinear problem, where G is a power growth functional and F has linear growth with respect to v. Let

$$J(v) = \frac{1}{q}\int\limits_\Omega |\nabla v|^q\, dx + \alpha\int\limits_\Omega |v|\, dx + \int\limits_\Omega f v dx.$$

We assume that $0 < q < +\infty$, $\alpha > 0$, and f is a bounded real valued function. Existence of the minimizer u is obvious because $J(v)$ is coercive of the reflexive space $V = \overset{\circ}{W}{}^{1,q}(\Omega)$. In this case, $\Lambda = \nabla$, $\Lambda^* = -\mathrm{div}$,

$$G^*(y^*) = \frac{1}{q^*} \int\limits_{\Omega} |y^*|^{q^*}\, dx, \quad \frac{1}{q} + \frac{1}{q^*} = 1,$$

and

$$D_G(y, y^*) = \int\limits_{\Omega} \left(\frac{1}{q}|y|^q + \frac{1}{q^*}|y^*|^{q^*} - yy^* \right) dx.$$

Next,

$$F(v) = \alpha \int\limits_{\Omega} |v| dx + \int\limits_{\Omega} fv dx,$$

For any real valued function $v^* \in V^*$, we have

$$F^*(v^*) = \sup_{v \in V} \int\limits_{\Omega} ((v^* - f)v - \alpha|v|) dx$$
$$= \begin{cases} 0 & \text{if } |v^* - f| \leq \alpha, \\ +\infty & \text{if } |v^* - f| > \alpha \end{cases} \tag{4.10}$$

and, therefore,

$$F^*(-\Lambda^* y^*) = \begin{cases} 0 & \text{if } |\mathrm{div}\, y^* + f| \leq \alpha, \\ +\infty & \text{if } |\mathrm{div}\, y^* + f| > \alpha. \end{cases}$$

Hence

$$D_F(v, -\Lambda^* y^*) = \begin{cases} \int\limits_{\Omega} (\alpha|v| + v(\mathrm{div}\, y^* + f)) dx & \text{if } |\mathrm{div}\, y^* + f| \leq \alpha, \\ +\infty & \text{if } |\mathrm{div}\, y^* + f| > \alpha. \end{cases}$$

We see that the measure $\mathbb{M}$ is finite only if

$$y^* \in Q^*_\alpha := \{y^* \in Y^* \mid |\mathrm{div}\, y^*(x) + f| \leq \alpha \quad \text{for a.a. } x \in \Omega\}. \tag{4.11}$$

This condition plays the same role as (3.18) for variational problems with $F(v) =< \ell^*, v >$. However, there is an essential difference. Now the error

identities are finite not on the set $Q^*_{\ell^*}$ (which is an affine manifold defined by (3.18)) but in a "strip" Q^*_α and width of this strip depends on the parameter α.

Note that $p^* = |\nabla u|^{q-2}\nabla u$ and $\nabla u = |p^*|^{\alpha^*-2}p^*$. Another duality relation reads

$$\operatorname{div} p^* + f = \begin{cases} -\alpha\frac{u}{|u|} & \text{if } u \neq 0, \\ -\alpha\zeta \quad \text{where } |\zeta| \leq 1 & \text{if } u = 0 \end{cases}$$

and we conclude that $p^* \in Q^*_\alpha$. In view of (2.7), for any $y^* \in Q^*_\alpha$ and $v \in V$, the measure $\mathbb{M}$ is defined by the relation

$$\begin{aligned} &\mathbb{M}(\{u, p^*\}, \{v, y^*\}) \\ &\quad = \int_\Omega (\alpha|u| + u(\operatorname{div} y^* + f) + \alpha|v| + v(\operatorname{div} p^* + f))\, dx \\ &\qquad + \int_\Omega \left(\frac{1}{q}|p^*|^q + \frac{1}{q^*}|y^*|^{q^*} - p^* \cdot y^* |p^*|^{\alpha^*-2}\right) dx \\ &\qquad\quad + \int_\Omega \left(\frac{1}{q}|\nabla v|^q + \frac{1}{q^*}|\nabla u|^q - \nabla v \cdot \nabla u |\nabla u|^{q-2}\right) dx. \end{aligned} \tag{4.12}$$

It is easy to see that the measure vanishes if $u = v$ and $p^* = y^*$. Now Theorem 2.1 yields the following error identity for this variational problem:

$$\begin{aligned} \mathbb{M}(\{u, p^*\}, \{v, y^*\}) &= \int_\Omega (\alpha|v| + v(\operatorname{div} y^* + f))\, dx \\ &\quad + \int_\Omega \left(\frac{1}{q}|\nabla v|^q + \frac{1}{q^*}|y^*|^{q^*} - \nabla v \cdot y^*\right) dx. \end{aligned} \tag{4.13}$$

Finally, we note that the problem considered above generates an elliptic variational inequality of the second kind. Analysis of suitable error measures (and corresponding error majorants) for variational inequalities of the first kind is presented in [13] for obstacle type problems and in [6] for problems with nonlinear boundary conditions.

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