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Mathematics Learning through Arts and Collaborative Problem-Solving: the Princess and the Diamond-Problem

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Abstract

In this paper, we examine a mathematics education course which focused on collaborative mathematical problem solving in art topics. The students of the course were Finnish primary school pre-service teachers. During the course students investigated tessellations, reflected on their experiences and designed workshops for pupils. The course meetings were video recorded. Students' work contained elements of mathematical problem solving and gave them experiences of working collaboratively. The topic of tessellation was found suitable for this purpose.

Introduction

Many researchers and mathematics educators emphasize the need to include problem solving activities into mathematics teaching [e.g., 1]. Usually, a problem means a task in which the solver does not directly know how to proceed in solving the tasks. Problem solving in this sense is not so often implemented in schools and teachers may view routine tasks as problems [2, 3]. Thus, teacher education programs need to be developed to better prepare teachers to use problem solving in mathematics teaching. This need has been recognized for long time but pedagogical practices seem to change slowly.

Using art as a context for mathematical problem solving could be a fruitful starting point as art is usually thought to include creative thinking and finding one's own way. This may help to recognize that also real mathematics is creative thinking and doing your own mathematics. However, also in arts education sometimes too much emphasis is placed on following direct instructions on how to create a piece of art. Thus, there is a need to design activities which focus on the creative process instead of emphasizing the product which is created by following a certain plan. These kinds of activities emphasize the process aspect of mathematics [4]. Nowadays problem solving is not thought to be an individual work but a collaborative effort [5]. Integrating mathematics and art could also help students to recognize the importance of collaboration as different people's strengths in different areas are helping the group.

In this study, we investigate how a mathematics education course for primary school pre-service teachers can be enriched by integrating art into the course in order to create a context for learning about utilizing collaborative problem solving in teaching. For this purpose we designed a new course in which prospective teachers solve math-art problems in groups, reflect on their experiences and design workshops for pupils. In the area of problem solving, we decided to focus on the topic of tessellations.

Tessellations

Tessellation Education in Primary School Context. In recent decades, mathematics educators have made lots of efforts to increasing the emphasis on problem solving, conceptual understanding and a greater appreciation of mathematics as a cultural phenomenon [6]. The topic of tessellation opens rich opportunities to facilitate collaborative problem solving activities in the manner of "adventurous teaching" [7] and to show the great extent to which mathematics is embedded in culture, art and real life. Tessellation or tiling means an arrangement of figures (tiles) so that they cover the plane with no gaps or overlaps. Examples of tessellations can be found, for example, in art and architecture.

In primary school, tessellations provide a good context to investigate patterns. The development of patterning abilities can be supported by using pattern blocks and tessellation activities [8]. Different playful activities with tiles and with tiling patterns can provide opportunities for introducing basic phenomena and concepts of tessellations and symmetry [9–11]. Eberle [12] found that mathematical esthetics played an important role in primary school students' mathematical work with tessellations. Although computer assisted tessellation activities can be very effective in symmetry education and in learning the basics of tile design, using these opportunities are not typical in primary school practice. It is probably because the use of computer requires not only skills related to patterning and tiling, but also the skills of handling the software. Nevertheless, there exist some examples of using dynamic geometry software, such as *GeoGebra*, in teaching tessellation in primary school teacher education [13].

Tessellations and *GianTiles* in *Experience Workshop's* Activities for Primary School Students. Right from the start in 2008 of the *Experience Workshop International Math-Art Movement* (EW, www.experienceworkshop.hu), EW's tessellations activities have integrated mathematics, art, culture, and real life. EW's repertoire also includes a growing collection of associated educational tools as well as experience-oriented pedagogical methods [14] for teaching tessellations in a transdisciplinary STEAM framework [15]. EW's tools are designed to support the development of patterning abilities, the experiential understanding of symmetry and related transformations, and their transposition into problem-solving in mathematics, in arts and design, in real-life situations, and in intercultural topics.

Experience Workshop's first tessellation-project, the *GianTile* workshop, was launched in collaboration with Dániel Erdély. The workshop was based on the tiles designed by Paul Gailiunas and Chaim Goodman-Strauss. EW has produced large size tiles for easier manipulation by children. This also facilitates cooperative group activities that can be executed on large scale and, for example, gives the opportunity to fully tessellate a classroom or even a sports hall. Based on their geometrical shape and monochrome coloring, each GianTile has the appearance of a geometrical rather than figurative image. However, students can intuitively recognize the possibility of creating not only geometrically interesting symmetrical patterns, but also figuratively recognizable patterns (such as animals). Based on the successful implementation of GianTiles, EW's tessellation inventory for different school levels was further extended. Several toolkits are also included in the collection *Adventures on Paper* [16].

Princess-tile. In this study, we decided to use Paul Gailiunas' tile-design, called *Princess* (see Figure 1). The reason for our selection was that even with three colors it is possible to create a wide range of different symmetric patterns. Princess was created as a result of Gailiunas' research on spiral tilings [17, 18]. Gailiunas' research was concentrated on tiles that are derived from regular polygons for covering the plane in ways that appear as spirals. Princess was derived from the regular decagon and as Gailiunas introduces the tile it is "very unusual, and allows a wide range of tilings" [18, p. 12]. For example, Princess allows "centrally symmetric tiling with V-shaped sections around a central core, and another that uses the same V-shaped units arranged asymmetrically"; and "shows a tiling that can be seen as spiral in two different ways" [Ibid.]. To demonstrate Princess' wide artistic potentials, tessellations based on the Princess were also on show at Bridges 2007 Art Exhibit [19].

Problem Solving

In developing tessellation activities, our aim was that students' work should include elements of mathematical problem solving. Several models have been constructed to describe mathematical problem solving [1, 20, 21]. The models emphasize moving between analyzing the starting situation, planning the solution, implementing the plan and reflection. For example, according to Mason et al. [20], the solver moves back and forth between *entry* and *attack* phases as he or she comes up with ideas, tries to implement them but gets stuck and has to begin a new entry. Thus, mathematical problem solving can hardly be considered as a straightforward linear process.

The process of solving a problem is as essential as the solution. Open problem solving emphasizes this. In open problem solving, the starting situation is not clearly defined, there are multiple correct answers to the problem, or there exist many different approaches to solving the problem [22, 23]. Open problem solving is claimed to be one way to enhance students' creativity in mathematics [24]. Thus, open problem solving fits well with the ideas of integrating mathematics and arts.

Problem solving includes also a social aspect. As defined by Hesse et al. [5], in collaborative problem solving (CPS), a group of learners accomplish together problem solving phases, such as problem identification and analysis, problem representation, planning, executing, and monitoring. Accordingly, if compared to individual problem solving, CPS is organized through the use of directly observable, verbal and nonverbal signals, which means that the participants are obliged to communicate, exchange, and share knowledge over the aforementioned problem solving phases. In this way, CPS includes not only cognitive but also social aspects of learning [5].

To support collaborative problem solving as intentional and goal-orientated collective activity, a macro-script [e.g., 25], a pedagogical model was designed [see 26]. The designed script implemented the cyclical idea of SRL learning theories [27], combined with elements of CPS process [5]. In the pedagogical model, prompting [e.g. 28] was used as targeted questions that aimed to increase solvers' awareness of their own and other group members' thoughts and understanding and in this way, to activate and guide their shared problem solving as a cognitive and metacognitive collective activity.

Context of the Study

In line with the national basic education curriculum reform, the curriculum of the Department of Teacher Education in University of Jyväskylä has just been revised. The ideology of the current curriculum emphasizes studying phenomena without restricting learners in one specific school subject. As a part of this development, we designed a new mathematics education course. After some background of the curriculum, we present the course design.

Curriculum for Teacher Education. Since August 2014 the Department of Teacher Education has implemented a phenomenon-based curriculum. When studying according to the curriculum, students are trying to understand the phenomena and problems related to learning. This requires the ability to combine various scientific theories and authentic everyday experiences because the phenomenon of learning cannot be profoundly understood from a single viewpoint. Thus, teacher education in Jyväskylä is aiming at supporting students' professional development so that they become autonomous and ethically responsible experts who are able to critically analyze and reform the culture of school and education as well as their own activities.

Course Design. As a consequence of our new teacher education curriculum and ideology behind it, we designed a new mathematics education course for prospective primary school teachers. The course is compulsory, but students can choose from different groups which may emphasize different things. The course that we designed focuses on collaborative mathematical problem solving in art related topics. This course is one of the two compulsory mathematics education courses for primary school teachers. Before this 2-credit course, the students have studied a 3-credit course about basics of mathematics education. These two mathematics education courses also cover all the compulsory mathematics required for their master's degree in education (300 credits). 15 students participated in the new course.

The course consisted of seven group meetings (á 90 min). The general flow of the course was following:

- 1st meeting: Introduction to the experience-oriented mathematics education, mathematics-art connections and collaborative mathematical problem solving.
- 2nd meeting: Designing tessellations with GeoGebra software.
- 3rd meeting: Designing giant tessellations using tiles made from cardboard.
- 4th meeting: Discussing and reflecting on the experiences of the second and third meeting.
- 5th and 6th meeting: Developing and sharing ideas for the forthcoming workshops.
- Teamwork to design, implement and report a workshop in school.
- 7th meeting: Presenting and discussing the implemented workshops.

In the 2^{nd} and 3^{rd} meeting students got some experience of what it could mean to combine collaborative mathematical problem solving and art. Students reflected on these experiences which gave them a basis for designing their own workshop and implementing it in school. In such a short course we decided to work on one topic (tessellation) in depth. Then, the responsibility was left to the students to design their own workshop. As during the students' own problem solving in the 2^{nd} and 3^{rd} meeting, also designing the workshops was guided only in a general level. Students problem solving was supported by scripts which included prompts. The prompts were provided in the form of cards in three phases: (1) orientation card at the beginning of the problem solving session, (2) check-up card after about 30 minutes of working, and (3) reflection card in the end of the session.

Description of the Research. We were interested in understanding how prospective teachers themselves solve collaboratively mathematical problems, how the art-topic enriches their problem solving, and how they reflect on their experiences. For these purposes we collected several kinds of data. The data comprised of video recordings of the course sessions (26 h), screen capture recordings of students' computer screens, photographs and written reports. One of the 15 students did not take part on the research. In this paper, we focus on students' work in the 2^{nd} and 3^{rd} meetings.

Using GeoGebra to Design Tessellation

In the second course meeting, the groups were given the following problem:

> Design with GeoGebra a pattern that covers the plane. Describe the pattern in terms of translations, rotations and reflections.

The students were asked to use the Princess-tile and rotation, translation and reflection about a line -tools of GeoGebra. They were given the following applet: http://ggbtu.be/m2601303.

Although working with GeoGebra is not as fast as using physical tiles, it forces learners to think about geometric transformations. Students can just place the physical tiles but with GeoGebra they have to think how the tile is translated, rotated or reflected. For example, one group was thinking how to rotate the tile:

STUDENT A: So this should basically be rotated to this direction from this point.
STUDENT B: Yes. […]
STUDENT A: Is it this point?
STUDENT B: Yes. Clockwise. How much would it be? 90.
STUDENT C: Try 90.

STUDENT A: Let's start with that ((laughing)). We go with that. (Students rotated the tile 90° as shown in Figure 1.)
STUDENT C: No.
STUDENT B: [Not] quite.
STUDENT C: Less.
STUDENT B: Would it be 45 then?

After this, the group tried other angles (Figure 1). Trying with GeoGebra is easy and helps to figure out what kind of angle they are looking for. After trying some angles, the group noticed that they need the angle in the tile:

STUDENT A: [I'm thinking] how can you take all the angles? Because then you could, in principle, see the angle.
(The teacher says how to measure angles with GeoGebra and the students measure all the angles in the tile.)
STUDENT A: Okay. Then it is the 72.
STUDENT C: So then this should have been 72 too. Because we put 75 there, it went too much.
STUDENT A: Mm-m, yeah. Okay. So in principle-
STUDENT C: Because it is the same angle that we sort of transl-, sort of turned, rotated.
STUDENT A: [Yes because it] can be seen that these are always 72, 72 and then this is incomplete.
STUDENT C: Yes because this is the same angle which was the rotation center.

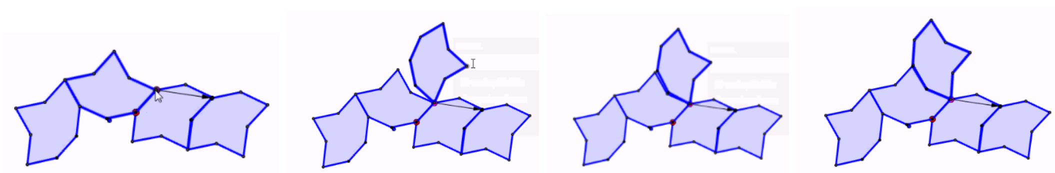

Figure 1: *Students tried to rotate a tile by 90°, 75° and 70° before measuring the angle (72°).*

Building Giant Tessellations

In the third course meeting, the groups were given the following problem:

Plan and build a giant pattern from the physical tiles. Use colors as you like. Illustrate the used translations, rotations and reflections with the physical tiles.

The students were given a set of red, blue and yellow Princess-tiles. They had lots of space in a sport hall. The groups built several different tessellations. Some of the tessellations are presented in Figure 2.

Figure 2: *Some of the tessellations produced by the students.*

Designing the tessellations included many creative ideas and students also challenged themselves. For example, one group of students decided to build a tessellation which "has a sort of logic". The students got the idea of repeating a shape composed of several tiles. In Figure 3 they have three yellow shapes composed of five tiles.

Figure 3: *Trying to repeat the yellow shape composed of five Princess-tiles.*

When trying to build a tiling this way, they noticed that there are small holes in the tiling (see Figure 3, one hole is pointed by a student):

STUDENT B: But if we think that this figure would translate to here, then how would it look like?
STUDENT D: Yeah. You cannot get it, there will be, if you want to do it like that, there will be some kind of small mid-things.
STUDENT C: Oh it would?
STUDENT B: Yes.

When these holes appeared after many different tiling trials, they named it "the diamond problem":

STUDENT A: We have got one good figure which proceeds logically as a sequence.
STUDENT D: ((Laughing)) then we would only need these little diamond shapes so that we could get this work.
STUDENT A: ((Laughing)) yes.
STUDENT B: Is this really not working in any way?
STUDENT D: No.
STUDENT A: No, it is not.
STUDENT D: No because you need those diamonds. Then it would be like that.
STUDENT C: Argyle.
STUDENT D: Or argyle here, and then it would be like that.
STUDENT A: Yes. […]
STUDENT B: But there appears then the-
STUDENT C: [What was it?] Did the diamond problem appear here again?
STUDENT B: The diamond problem appeared there ((laughing)).
STUDENT C: Yes it is appearing.

The group noticed that to overcome the "diamond problem", they would need two different kinds of tiles: Princess and a diamond shaped tile. The diamond problem illustrates how the journey is more important than the outcome. The group did not manage to use their idea of repeating the same shape. However, they did mathematics during their solution trials. And, after all, they did find the diamond problem. The group was also able to build other kinds of tessellations.

When working with the physical tiles, the groups also talked about using geometric transformations and spontaneously compared the work with the physical tiles to the use of GeoGebra. For example, one group of students was thinking about a certain translation using the physical tiles (see Figure 4).

Figure 4: *Translating the physical tile.*

After this they started to use GeoGebra to check whether the transformation really can be made (see Figure 5). This shows that they were aware of the possibility that with the physical tiles it only looks like working but actually is really not precise. This is one of the key ideas in mathematics.

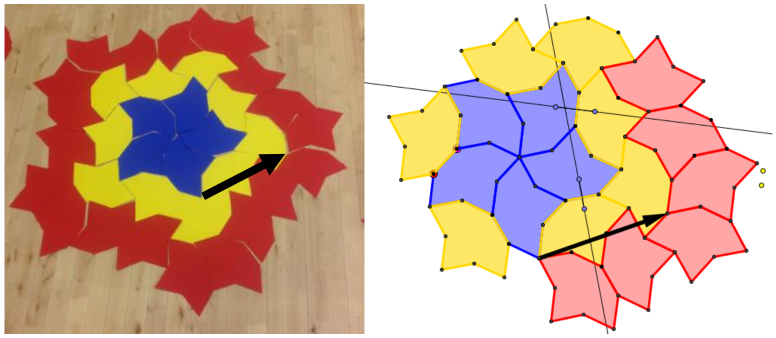

Figure 5: *Using GeoGebra to check whether the proposed translation really is a translation.*

Discussion

Several aims of the course were successfully implemented. First of all, the students really solved problems. They analyzed the situation, explored and planned solution, were stuck, reflected on their work, got ideas and started over. All these are features of mathematical problem solving [1, 20, 21]. Most importantly, they have built their own solution paths without strict instructions. Thus, the students got first-hand experience about open problem solving [22, 23]. This inspired them to create pedagogical innovations for mathematical problem solving activities to various ages of pupils. Tessellation as a topic was suitable for these purposes as students were able to build many different kinds of tessellations depending on their decisions, and they also thought about geometric transformations which are basic concepts in primary school mathematics. Thus, this study strengthens the claim that tessellation is a good topic for mathematical investigation [8, 9, 10, 16]. Combination of digital and hands-on material did work in getting students to think about transformations more strictly and use more mathematical language as well as to focus more on the patterns without technical demands and express themselves more freely paying attention to aesthetical issues. With both tools the students needed to collaborate to achieve their collective aim which is one feature of collective problem solving [25]. Also the scripts [26] helped them in controlling and reflecting on their problem solving. Our research will continue by examining how the students reflected on the learning opportunities that are created by integrating art into mathematics and what kind of workshops they designed.

References

[1] A. Schoenfeld, *Mathematical Problem Solving*, London: Academic Press. 1985.

[2] J. Jacobs, J. Hiebert, K. Givvin, H. Hollingsworth, H. Garnier, & D. Wearne, Does Eight-grade Mathematics Teaching in the United States Align with the NCTM Standards? Results from TIMSS 1995 and 1999 Video Studies, *Journal for Research in Mathematics Education, 37,* pp. 5-32. 2006.

[3] M. Stein, B. Grover, & M. Henningsen, Building Student Capacity for Mathematical Thinking and Reasoning: An Analysis of Mathematical Tasks Used in Reform Classrooms. *American Educational Research Journal, 33,* pp. 455-488. 1996.

[4] P. Ernest, Philosophy, Mathematics and Education, *International Journal of Mathematical Education in Science and Technology, 20*(4), pp. 555-559. 1989.

[5] H. Hesse, E. Care, J. Buder, J. Sassenberg, & P. Griffin, Framework for Teachable Collaborative Problem Solving Skills, in P. Griffin & E. Care (eds.), *Assessment and Teaching of 21st Century Skills. Methods and Approach*, Dortdrecht: Springer, pp. 37-56. 2015.

[6] R. Borasi, J. Fonzi, C. Smith, & B. Rose, Beginning the Process of Rethinking Mathematics Instruction: A Professional Development Program, *Journal of Mathematics Teacher Education, 2*(1), pp. 49–78. 1999.

[7] J. Frykholm, Teacher's Tolerance for Discomfort: Implications for Curricular Reform in Mathematics, *Journal for Curriculum and Supervision*, *19*(2), pp. 125-149. 2004.
[8] J. Waters (Fox), A Study of Mathematical Patterning in Early Childhood Settings, in I. Putt, R. Faragher, & M. MacLean (eds.), *Mathematics education for the 3rd millennium: Towards 2010. Proceedings of the 27th Annual Conference of the Mathematics Education Research Group of Australasia* . Townsville, Queensland, Australia, pp. 321-328. 2004.
[9] L. Cooper, S. Spitzer, & M. Tomayko, Become an Escher Sleuth, *Mathematics Teaching in the Middle School, 18*(6), pp. 378-385. 2013.
[10] R. Reys, R. Reys, & B. Reys, Sport Courts and Fields: A Context for Estimation and Tessellation, *Mathematics Teaching in the Middle School, 18*(9), pp. 566-570. 2013.
[11] M. Civil, Chapter 4: Everyday Mathematics, Mathematicians' Mathematics, and School Mathematics: Can We Bring Them Together?, *Journal for Research in Mathematics Education, Monograph, Vol. 11, Everyday and Academic Mathematics in the Classroom*, pp. 40-62. 2002.
[12] S. Eberle, The Role of Children's Mathematical Aesthetics: The Case of Tessellations, *The Journal of Mathematical Behavior*, 35, pp. 129–43. 2014.
[13] L. Rumanová, & E. Smiešková, Creativity And Motivation For Geometric Tasks Designing In Education, *Acta Didactica Napocensia, 8*(1), pp. 49-56. 2015.
[14] K. Fenyvesi, R. Koskimaa, & Zs. Lavicza, Experiential Education of Mathematics: Art and Games for Digital Natives. *Kasvatus & Aika*, 2015/1, pp. 107-134. 2015.
[15] X. Ge, D. Ifenthaler, & J. M. Spector (eds.), *Emerging Technologies for STEAM Education*, Springer International Publishing. 2015
[16] K. Fenyvesi, I. Oláh, & I. Szilágyi (eds.), *Adventures on Paper. Exercise Book for Experience-centered Education of Mathematics*, Eger: Eger University. 2014.
[17] P. Gailiunas, Spiral Tilings, in R. Sarhangi (ed.), *BRIDGES Mathematical Connections in Art, Music, and Science*, pp. 133-140. 2000.
[18] P. Gailiunas, Some Monohedral Tilings Derived From Regular Polygons, in R. Sarhangi & J. Barrallo (eds.), *BRIDGES Mathematical Connections in Art, Music, and· Science*, pp. 9-14. 2007.
[19] P. Gailiunas, Two Spirals for the Price of One and Asymmetric Tiling, Bridges 2007 Art Exhibit. URL: http://www.bridgesmathart.org/art-exhibits/bridges2007/gailiunas.html
[20] J. Mason, L. Burton, & K. Stacey, *Thinking mathematically*, Bristol: Addison–Wesley. 1982.
[21] G. Pólya, *How to Solve It? A New Aspect of Mathematical Method*, Princeton University Press. 1945.
[22] E. Pehkonen, Introduction to the Concept 'Open-Ended Problem', in E. Pehkonen (ed.), *Use of Open-Ended Problems on Mathematics Classroom*, University of Helsinki, Department of Teacher Education, Research Report 176, Helsinki, Finland, pp. 7-11. 1997.
[23] N. Nohda, Teaching by Open-Approach Method in Japanese Mathematics Classroom, in T. Nakahara & M. Koyama (eds.), *Proceedings of the 24th Conference of theIInternational Group for the Psychology of Mathematics Education*, Hiroshima, Japan: PME, Vol. 1, pp. 39–53. 2000.
[24] O. N. Kwon, J. S. Park, & J. H. Park, Cultivating Divergent Thinking in Mathematics through an Open-Ended Approach, *Asia Pacific Educational Review*, *7*(1), pp. 51–61. 2006.
[25] P. Dillenbourg, & F. Hong, The Mechanics of CSCL Macro Scripts, *International Journal of Computer-Supported Collaborative Learning*, *3*(1), pp. 5-23. 2008.
[26] P. Näykki, J. Pöysä-Tarhonen, S. Järvelä, & P. Häkkinen, Enhancing Teacher Education Students' Collaborative Problem Solving and Shared Regulation of Learning, in O. Lindwall, P. Häkkinen, T. Koschmann, P. Tchounikine & S. Ludvigsen (eds.), *Exploring the Material Conditions of Learning: The Computer Supported Collaborative Learning (CSCL) Conference 2015*, Gothenburg, Sweden: The International Society of the Learning Sciences, Vol. 2, pp. 514-517. 2015.
[27] B. Zimmerman, A Social Cognitive View of Self-Regulated Academic Learning, *Journal of Educational Psychology*, *81*(3), pp. 329-339. 1989.
[28] M. Bannert, & P. Reimann, Supporting Self-Regulated Hypermedia Learning through Prompts, *Instructional Science*, *40*(1), pp. 193-211. 2012.