Author(s): Tengvall, Ville

Title: Differentiability in the Sobolev space W1,n-1

Year: 2014

Version:

DIFFERENTIABILITY IN THE SOBOLEV SPACE $W^{1,n-1}$

VILLE TENGVALL

Abstract   Let $\Omega \subset \mathbb{R}^n$ be a domain, $n \geq 2$. We show that a continuous, open and discrete mapping $f \in W_{\mathrm{loc}}^{1,n-1}(\Omega, \mathbb{R}^n)$ with integrable inner distortion is differentiable almost everywhere on Ω. As a corollary we get that the branch set of such a mapping has measure zero.

1   Introduction

Suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a domain and $f : \Omega \to \mathbb{R}^n$ a continuous, discrete and open mapping in the Sobolev space $W_{\mathrm{loc}}^{1,n-1}(\Omega, \mathbb{R}^n)$. A theorem of Gehring and Lehto asserts that if $n = 2$, then f is differentiable at almost every point [7]. For planar homeomorphisms the result was established earlier by Menchoff [29]. These results are false in higher dimensions. Indeed, if $n \geq 3$, one can construct a nowhere differentiable homeomorphism in $W^{1,n-1}(\Omega, \mathbb{R}^n)$, see [2, Example 5.2].

In this paper we study sufficient conformality conditions that guarantee differentiability almost everywhere for discrete and open mappings in $W_{\mathrm{loc}}^{1,n-1}(\Omega, \mathbb{R}^n)$. Such conformality conditions are usually given in terms of a distortion function. There are several distortion functions, each having considerable interest in geometric function theory, see [16, §6.4]. The principal feature of these distortions is that they provide some control on the lower order minors of the differential matrix in terms of the determinant.

The most flexible distortion inequality is given in terms of the $(n-1) \times (n-1)$-minors of the differential matrix. Precisely, we say that $f \in W_{\mathrm{loc}}^{1,1}(\Omega, \mathbb{R}^n)$ is a mapping of finite inner distortion if

(1) $J(\cdot, f) \in L_{\mathrm{loc}}^1(\Omega)$,

(2) $J(\cdot, f) \geq 0$ almost everywhere and

Ville Tengvall
University of Jyväskylä, Department of Mathematics and Statistics
P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland
e-mail: ville.tengvall@jyu.fi

(3) adj $Df(\cdot)$ vanishes almost everywhere in the zero set of the Jacobian $J(\cdot, f)$.

With such mappings we associate a measurable function $K_I : \Omega \to [0, \infty]$ as follows

$$K_I(x, f), = \begin{cases} \frac{|\text{adj } Df(x)|^n}{J(x,f)^{n-1}}, & \text{if } J(x, f) > 0 \\ 1, & \text{otherwise.} \end{cases}$$

Here adj A stands for the adjugate of a matrix A which is the transpose of the cofactor matrix, denoted by cof A. For a regular matrix A we have

$$A \text{ adj } A = I \det A,$$

where $\det A$ denotes the determinant of matrix A and I is the identity matrix. We also denote the adjungate matrix of $Df(x)$ by $D^\sharp f(x)$.

The most restrictive and most studied distortion function is the outer distortion. Precisely, we say that $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ is a mapping of finite outer distortion if

(1) $J(\cdot, f) \in L^1_{\text{loc}}(\Omega)$,

(2) $J(\cdot, f) \geq 0$ almost everywhere and

(3) $Df(\cdot)$ vanishes almost everywhere in the zero set of the Jacobian $J(\cdot, f)$.

With such mappings we associate a measurable function $K_O : \Omega \to [0, \infty]$ by the rule

$$K_O(x, f) = \begin{cases} \frac{|Df(x)|^n}{J(x,f)}, & \text{if } J(x, f) > 0 \\ 1, & \text{otherwise.} \end{cases}$$

We call $K_I(\cdot, f)$ the inner distortion of f and $K_O(\cdot, f)$ the outer distortion of f. Note that a mapping of finite inner distortion does not have to be a mapping of finite outer distortion. For instance, the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$,

$$f(x_1, \ldots, x_n) = (x_1, 0, \ldots, 0)$$

is a mapping of finite inner distortion, but not a mapping of finite outer distortion. However, we are able to show that a discrete and open mapping with $K_I(\cdot, f) \in L^1_{\text{loc}}(\Omega)$ has a finite outer distortion almost everywhere. Our main result reads as follows.

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a domain. Let $f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ be a continuous, discrete and open mapping of finite inner distortion with $K_I(\cdot, f) \in L^1_{\text{loc}}(\Omega)$. Then f is differentiable almost everywhere and has finite outer distortion.*

Theorem 1.1 is new even when f is assumed to be a homeomorphism. Moreover, we are able to relax the integrability assumption on the inner distortion, see Theorem 5.1. The sharpness of this refinement is given in Example 5.5. We show as a corollary of Theorem 1.1 that the branch set of f, i.e. the set of points where f is not a local homeomorphism, has measure zero, see Corollary 5.4. This leads to a natural generalization of theorem by Hencl and Koskela [10, Theorem 1.3] in higher dimensions for open and discrete mappings.

Theorem 1.2. *Suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a domain. Let $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a continuous, discrete and open mapping of finite inner distortion with $K_I(\cdot, f) \in L^1_{\mathrm{loc}}(\Omega)$. Then for almost every $x \in \Omega$ there is an open neighborhood $U_x \subset\subset \Omega$ of x such that the restriction map $f\,|_{U_x} : U_x \to f(U_x)$ is a homeomorphism. For the inverse mapping we have $(f\,|_{U_x})^{-1} \in W^{1,n}_{\mathrm{loc}}(f(U_x), \mathbb{R}^n)$ and $(f\,|_{U_x})^{-1}$ is a mapping of finite distortion. Moreover, f and $(f\,|_{U_x})^{-1}$ are differentiable almost everywhere.*

Our results have a strong connection to the study of the regularity of the inverse of a Sobolev homeomorphism. Under the minimal conformality assumptions these questions can be traced back to the works of Iwaniec and Šverák [17], Koskela and Onninen [21] and Astala, Iwaniec, Martin and Onninen [1]. The study of the regularity of the inverse mapping under the natural Sobolev setting $W^{1,n}_{\mathrm{loc}}$ goes back to the pioneering work of Hencl and Koskela [10], and has recently been futher developed by several authors, see [11], [12], [30] and [31]. Regularity questions in the Sobolev space $W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ were first studied by Csörnyei, Hencl and Malý [2]. For our purposes, one of the important observations that Csörnyei, Hencl and Malý made on their paper was that a homeomorphism in the Sobolev space $W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ satisfies Lusin's condition (N) on almost every hyperplane. By modifying their proof we are able to show that this is true also for a discrete and open mapping in the Sobolev space $W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.

It is important to notice that it is possible to study the regularity properties of mappings without assuming injectivity. For instance, by applying modulus and capacity methods, see [35], [40] and [32], it is possible to answer to many regularity questions in terms of inner distortion without using any invertibility of mapping.

We say that a mapping $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is a mapping of bounded distortion, or a quasiregular mapping, if it satisfies distortion inequality

$$(1) \qquad\qquad |Df(x)|^n \leq K J(x, f) \quad \text{a.e.}$$

with some global constant $1 \leq K < \infty$. There has been a lot of study on mappings of bounded distortion, see [34] and [35], and mappings of finite distortion are natural generalizations of these mappings.

By the fundamental theorem of Reshetnyak [34], every mapping of bounded distortion is continuous, i.e. has a continuous representative, and is either constant or discrete and open. However, this is not always the case with mappings of finite distortion. Especially, a mapping $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ of finite inner distortion with $K_I(\cdot, f) \in L^1_{\mathrm{loc}}(\Omega)$ does not have to be continuous, discrete or open. Thus, it is justifiable to assume these properties from f in the Theorem 1.1. For more details about the sharp analytic assumptions that guarantee continuity, discreteness and openness for mappings of finite distortion, see [9], [13], [15], [17], [18], [19], [25] and [26].

In the theory of mappings of bounded distortion one of the powerful tools to deal with non-injective mappings is the Poletsky inequality, see [35, Theorem 8.1, Theorem 10.10]. Koskela and Onninen generalized this result in [21] by showing that a mapping of finite distortion with $K_I(\cdot, f) \in L^1_{\mathrm{loc}}(\Omega)$ and $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ enjoys a Poletsky-type inequality. It was further shown that the regularity assumption

$f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ can be slightly relaxed, say to $|Df|^n \log^{-1}(e+|Df|) \in L^1_{\text{loc}}(\Omega)$. These results were based on a duality argument, relying on integration by parts against the Jacobian determinant. This method does not work when we only assume $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$, for some $p < n$. By using the duality of p-capacity we are now able to establish Poletsky-type inequality, Lemma 4.4, for mappings in $W^{1,n-1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ with integrable inner distortion. We apply this result to prove Theorem 1.1.

2 Preliminaries

Let Ω be a domain in $\mathbb{R}^n$. We say that $f : \Omega \to \mathbb{R}^n$ belongs to the Sobolev space $W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$, $1 \le p < \infty$, if the coordinate functions of f are locally p-integrable and have locally p-integrable distributional derivatives.

For $x \in \mathbb{R}^n$ we denote by x_i, $i = 1, 2, \ldots, n$, its coordinates, i.e. $x = (x_1, x_2, \ldots, x_n)$. Fix $y \in \mathbb{R}$. Then $\{y\} \times \mathbb{R}^{n-1}$ is a copy of $\mathbb{R}^{n-1}$ and thus its Hausdorff measure coincides with $(n-1)$-dimensional Lebesgue measure and we might sometimes write dz instead of $d\mathcal{H}^{n-1}(z)$.

We denote the Euclidean norm of $x \in \mathbb{R}^n$ by $|x|$, the open ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$ by

$$B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\},$$

and its closure by $\overline{B(x, r)}$. Similarly, we denote the corresponding sphere by

$$S(x, r) = \{y \in \mathbb{R}^n : |x - y| = r\}.$$

We define an n-dimensional open cube centered at $x \in \mathbb{R}^n$ with radius $r > 0$ by

$$(2) \qquad Q(x, r) = \{y \in \mathbb{R}^n : x_i - y_i \in (-r, r) \text{ for every } i = 1, \ldots, n\}.$$

For the convenience of the reader we recall the boxing lemma, proved in [8].

Lemma 2.1 (Gustin's boxing lemma). *Every compact set $K \subset \mathbb{R}^n$ can be covered by balls $B(x_i, r_i)$, $i = 1, \ldots, p$, such a way that*

$$(3) \qquad \sum_{i=1}^{p} r_i^{n-1} \le C(n)\, \mathcal{H}^{n-1}(\partial K),$$

where the constant $C(n) > 0$ depends only on dimension n. Moreover, for every compact set K we have

$$(4) \qquad \mathcal{H}^{n-1}_{\infty}(K) \le C(n)\, \mathcal{H}^{n-1}(\partial K).$$

Here $\mathcal{H}^{n-1}_{\infty}$ stands for the Hausdorff $(n-1)$-content.

If $E \subset \mathbb{R}^n$ is a measurable set, then we denote by $|E|$ its Lebesgue measure. We say that a mapping $f : \Omega \to \mathbb{R}^n$ satisfies Lusin's condition (N) on E if $|f(A)| = 0$ for every $A \subset E$ such that $|A| = 0$.

The notation $E \subset\subset \Omega$ means that the closure of E is compact and $\overline{E} \subset \Omega$. For a mapping $f : \Omega \to \mathbb{R}^n$ and a Borel set $E \subset\subset \Omega$, the multiplicity function $N(y, f, E)$ of f is defined by

$$N(y, f, E) = \operatorname{card} f^{-1}(y) \cap E,$$

i.e. $N(y, f, E)$ is the number of preimages of y lying on E under the mapping f.

For $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^m)$, $\Omega \subset \mathbb{R}^n$, we can form the Jacobi matrix

$$\left[\frac{\partial f_i}{\partial x_j}(x) \right]_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \in \mathbb{R}^{m\times n},$$

where $f = (f_1, \ldots, f_m)$. Let $I(r, k)$ be the set of all increasing multi-indices from $\{1, \ldots, r\}^k$, i.e. $\alpha = (\alpha_1, \ldots, \alpha_k) \in I(r, k)$ if α_j are integers, $1 \le \alpha_1 < \cdots < \alpha_k \le r$. If $\alpha \in I(m, k)$ and $\beta \in I(n, k)$, we define the k-dimensional Jacobian of f at x as

$$J^k(x, f) = \left[\det\left(\frac{\partial f_{\alpha_i}}{\partial x_{\beta_j}} \right)_{i,j=1,\ldots k} \right]_{\substack{\alpha \in I(m,k) \\ \beta \in I(n,k)}}.$$

We write $J(x, f) = J^n(x, f)$.

If $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^m)$, where $\Omega \subset \mathbb{R}^n$ and $m \ge n$, we say that the area formula holds for f on $\Omega' \subset \Omega$ if for each measurable set $E \subset \Omega'$ we have

$$(5) \qquad \int_E \eta(f(x)) \,|J(x, f)|\, dx = \int_{\mathbb{R}^m} \eta(y)\, N(y, f, E)\, dy$$

for any nonnegative Borel measurable function η on $\mathbb{R}^m$. It follows from [4, 3.1.4, 3.1.8 and 3.2.5] that the area formula holds for $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^m)$ on each set on which Lusin's condition (N) is satisfied. Moreover, due to [4, 3.1.8], there exists a Borel set $\Omega' \subset \Omega$ of full measure such that the area formula holds for f on Ω'. If $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^m)$, then for every Borel set $E \subset\subset \Omega$ and for any nonnegative Borel measurable function η on $\mathbb{R}^m$ we have

$$(6) \qquad \int_E \eta(f(x)) \,|J(x, f)|\, dx \le \int_{\mathbb{R}^m} \eta(y)\, N(y, f, E)\, dy.$$

Next we collect some topological facts about open and discrete mappings. For more details we refer to [35, I.4]. A mapping $f : \Omega \to \mathbb{R}^n$ is open if it maps open sets in Ω to open sets in $\mathbb{R}^n$. A mapping f is discrete if the set $f^{-1}(y)$ of preimages does not accumulate in Ω for any $y \in \mathbb{R}^n$. Next we assume that $f : \Omega \to \mathbb{R}^n$ is a discrete and open mapping. From now on we always assume that f is continuous. A domain $U \subset\subset \Omega$ is called a normal domain of f if $f(\partial U) = \partial f(U)$. The openness of f implies that $\partial f(U) \subset f(\partial U)$ holds for every domain U. If $D \subset \mathbb{R}^n$ is a domain and U is a component of $f^{-1}(D)$ such that $U \subset\subset \Omega$, then U is a normal domain and $f(U) = D \subset\subset f(\Omega)$. In particular, $N(y, f, U)$ is a bounded function of y. If $x \in \Omega$ and U is a normal domain of f such that $U \cap f^{-1}(f(x)) = \{x\}$, then U is called a normal neighborhood of x.

Let $f : \Omega \to \mathbb{R}^n$ be a continuous mapping from a domain $\Omega \subset \mathbb{R}^n$. Then we can define the local degree $\deg(y, f, U) \in \mathbb{Z}$ for any subdomain $U \subset\subset \Omega$ and every $y \in \mathbb{R}^n \setminus f(\partial U)$, see [35]. If $y \notin f(\partial U)$, we say that y is an (f, U)-admissible point. If f and g are two homotopic mappings via homotopy h_t, $t \in [0, 1]$, and y is (h_t, U)-admissible for all $t \in [0, 1]$, then $\deg(y, f, U) = \deg(y, g, U)$.

If points y and z belong to the same component of $\mathbb{R}^n \setminus f(\partial U)$, then $\deg(y, f, U) = \deg(z, f, U)$. Moreover, if U is a normal domain of f, then $f(U) \cap f(\partial U) = \emptyset$ and hence $\deg(y, f, U)$ is a constant for $y \in f(U)$.

By a condenser in $\Omega \subset \mathbb{R}^n$ we understand a pair (E, G) of sets with $G \subset\subset \Omega$ open and E compact in $\mathbb{R}^n$ and with $E \subset G$. The p-capacity of a condenser (E, G) is defined as

$$(7) \qquad \mathrm{cap}_p(E, G) = \inf_u \int_G |\nabla u(x)|^p \, dx,$$

where the infimum is taken over all smooth functions $u \geq 0$ with support in G such that $u \geq 1$ on E. We call such functions admissible for $\mathrm{cap}_p(E, G)$. The n-capacity is also called conformal capacity. A condenser (E, G) is ringlike if E and the complement of G are connected.

We will say that a set σ separates a set A from a set B if σ is a compact set in $\mathbb{R}^n$ and if there are disjoint open sets $\mathcal{A}$ and $\mathcal{B}$ such that $\mathbb{R}^n \setminus \sigma = \mathcal{A} \cup \mathcal{B}$, $A \subset \mathcal{A}$ and $B \subset \mathcal{B}$. Let Λ denote the class of all sets that separate A from B. For every $\sigma \in \Lambda$ we associate a complete measure μ as follows. For a $\mathcal{H}^{n-1}$-measurable set $D \subset \mathbb{R}^n$, we define

$$\mu(D) = \mathcal{H}^{n-1}(D \cap \sigma).$$

It follows from the properties of Hausdorff measure that the Borel sets of $\mathbb{R}^n$ are μ-measurable. The modulus of Λ is defined as follows.

$$M_p^S(\Lambda) = \inf \Big\{ \int_{\mathbb{R}^n} \rho(x)^p \, dx : \rho : \mathbb{R}^n \to [0, +\infty] \text{ is Borel measurable},$$
$$\int_\sigma \rho(x) \, d\mathcal{H}^{n-1}(x) \geq 1 \text{ for every } \sigma \in \Lambda \Big\},$$

and for $h \in L^1_{\mathrm{loc}}(\Omega)$

$$M_{p,h}^S(\Lambda) = \inf \Big\{ \int_{\mathbb{R}^n} \rho(x)^p \, h(x) \, dx : \rho : \mathbb{R}^n \to [0, +\infty] \text{ is Borel measurable},$$
$$\int_\sigma \rho(x) \, d\mathcal{H}^{n-1}(x) \geq 1 \text{ for every } \sigma \in \Lambda \Big\}.$$

If now (E, G) is a condenser in $\mathbb{R}^n$ and Λ is the class of all the sets separating E form $\mathbb{R}^n \setminus G$ then, by [42], we have the following duality

$$(8) \qquad \mathrm{cap}_n(E, G) = M_{n'}^S(\Lambda)^{-(n-1)},$$

where $n' = \frac{n}{n-1}$. More generally, for $1 < p < \infty$

$$\text{(9)} \qquad \qquad \text{cap}_p(E, G) = M_{p'}^S(\Lambda)^{-(p-1)},$$

where $p' = \frac{p}{p-1}$. One can find more information about p-capacity and the relations (8) and (9) from [35], [42] and [43].

3   Lusin's condition (N) on hyperplanes

In this section we show that a discrete and open mapping $f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ satisfies Lusin's condition (N) on almost every hyperplane. This fact has been proven for homeomorphisms by Csörnyei, Hencl and Malý in [2], and we use methods similar to theirs.

Let φ be the standard smooth mollification kernel on $\mathbb{R}^n$ i.e.

$$\varphi(x) = \begin{cases} C \exp[-1/(1 - |x|^2)], & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1 \end{cases}$$

where the constant C is chosen such that $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. Then for $\delta > 0$ we define the family of mollifiers φ_δ by the formula

$$\text{(10)} \qquad \qquad \varphi_\delta(x) = \delta^{-n}\,\varphi\!\left(\frac{x}{\delta}\right).$$

This means that φ_δ is a smooth nonnegative function supported in $\overline{B(0,\delta)}$ and $\int_{\mathbb{R}^n} \varphi_\delta(x)\,dx = 1$. Moreover, $\varphi_\delta > 0$ on $B(0,\delta)$.

We also introduce a "crude family" of mollification kernels

$$\tilde{\varphi}_\delta = (2\delta)^{-n}\chi_{Q(0,\delta)}.$$

Here Q is a cube defined in (2) and χ_E denotes the characteristic function of E i.e.

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Suppose that $f : \Omega \to \mathbb{R}^n$ is a discrete and open mapping. Let $U \subset\subset \Omega$ be a normal domain of f, and let $D \subset\subset U$ be an open n-interval. If the restriction of f to ∂D belongs to the Sobolev space $W^{1,n-1}$, then*

$$\mathcal{H}^{n-1}_\infty(f(D)) \leq C(n) \int_{\partial D} |\text{cof } Df(x)\, \nu_D(x)|\, d\mathcal{H}^{n-1}(x),$$

where ν_D is the outer normal of D.

Proof. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of smooth approximations of f such that $f_k \to f$ in $W^{1,n-1}(\partial D, \mathbb{R}^n)$ and $f_k \to f$ uniformly in $\overline{D}$. Since U is a normal domain of f, the degree of f is constant in $f(U)$. We set $m = \deg(\cdot, f, U)$.

Because f is a continuous, discrete and open mapping, it is either sense-preserving or sense-reversing, see [28, p. 151]. Without loss of generality we may assume that f is sense-preserving. Then $m \geq 1$, and we have $1 \leq \deg(y, f, D) \leq m$ for every $y \in f(D) \setminus f(\partial D)$. Set

$$G_k = \{ y \in f(U) : 1 \leq \deg(y, f_k, D) \leq m \}.$$

Then by Gustin's boxing lemma we have

$$\mathcal{H}_{\infty}^{n-1}(G_k) \leq \mathcal{H}_{\infty}^{n-1}(\overline{G_k}) \leq C(n) \, \mathcal{H}^{n-1}(\partial \overline{G_k}).$$

Now $f_k(\partial D)$ divides $\mathbb{R}^n$ to open components. Since $\deg(\cdot, f_k, D)$ is constant on each component of $\mathbb{R}^n \setminus f_k(\partial D)$, $\partial \overline{G_k}$ is contained in $f_k(\partial D)$, and thus by area formula for smooth mappings [24, Theorem 4.6], we have

$$(11) \qquad \mathcal{H}_{\infty}^{n-1}(G_k) \leq C(n) \, \mathcal{H}^{n-1}(f_k(\partial D)) \leq C(n) \int_{\partial D} |\operatorname{cof} Df_k(x) \, \nu_D(x)| \, d\mathcal{H}^{n-1}(x).$$

Now the integrand above is continuous in $W^{1,n-1}$, and thus

$$\int_{\partial D} |\operatorname{cof} Df_k(x) \, \nu_D(x)| \, d\mathcal{H}^{n-1}(x) \to \int_{\partial D} |\operatorname{cof} Df(x) \, \nu_D(x)| \, d\mathcal{H}^{n-1}(x),$$

as $k \to \infty$. Moreover, because $f_k \to f$ uniformly in $\overline{D}$, for any $y \in f(D)$ we can choose k_y such that y is (h_t^k, D)-admissible for all $t \in [0, 1]$ and for every homotopy

$$h_t^k(x) = t f_k(x) + (1 - t) f(x), \quad k \geq k_y.$$

Thus, by [35, Proposition 4.4] $\deg(y, f_k, D) = \deg(y, f, D)$ for every $k \geq k_y$, and we have

$$(12) \qquad f(D) \subset \bigcup_{k=1}^{\infty} \bigcap_{j \geq k} G_j.$$

By using (12) and [4, Theorem 2.10.22], we have

$$\mathcal{H}_{\infty}^{n-1}(f(D)) \leq \mathcal{H}_{\infty}^{n-1}\left(\bigcup_{k=1}^{\infty} \bigcap_{j \geq k} G_j \right) = \lim_{k \to \infty} \mathcal{H}_{\infty}^{n-1}\left(\bigcap_{j \geq k} G_j \right) \leq \limsup_{j \to \infty} \mathcal{H}_{\infty}^{n-1}(G_j).$$

The claim follows now from (11) when $k \to \infty$. $\qquad\square$

Let $U \subset\subset \Omega$ be a domain. Then for every $t \in \mathbb{R}$ we denote

$$U_t = \{ x \in U : x_n = t \}.$$

For $\delta > 0$ small enough, we define

$$U^{\delta} = \{ x \in U : \inf_{y \in \partial U} |x - y| > \delta \}.$$

Lemma 3.2. *Suppose that $f : U \to \mathbb{R}^n$ is a discrete and open mapping of the class $W^{1,n-1}$, where U is a normal domain of f. Let $t \in \mathbb{R}$ be such that $f \in W^{1,n-1}(U_t, \mathbb{R}^n)$ and the condition*

$$(13) \qquad \liminf_{\delta \to 0+} \int_{(U^\delta)_t} \varphi_\delta * |Df|^{n-1}(x) \, d\mathcal{H}^{n-1}(x) \le \int_{U_t} |Df(x)|^{n-1} \, d\mathcal{H}^{n-1}(x)$$

is satisfied. Then

$$\mathcal{H}_\infty^{n-1}(f(Q \times \{t\})) \le C(n) \int_{Q \times \{t\}} |Df(x)|^{n-1} \, d\mathcal{H}^{n-1}(x)$$

for each $(n-1)$-dimensional closed cube $Q \times \{t\} \subset U_t$.

Proof. For simplicity, let us assume that $t = 0$ and $Q = [-R, R]^{n-1}$. We denote

$$J = Q \times \{0\}$$
$$I_r = (-R - r, R + r)^{n-1} \times (-r, r).$$

Consider $\rho > 0$ such that $I_{2\rho n} \subset U$. Because $f_{|\partial I_r} \in W^{1,n-1}$ for almost every $r \in (0, \rho)$, by Lemma 3.1 we have

$$\mathcal{H}_\infty^{n-1}(f(J)) \le C(n) \int_{\partial I_r} |\operatorname{cof} Df(x) \, \nu_{I_r}(x)| \, d\mathcal{H}^{n-1}(x)$$

$$\le C(n) \int_{\partial I_r} |Df(x)|^{n-1} \, d\mathcal{H}^{n-1}(x)$$

for almost every $r \in (0, \rho)$. We integrate over the interval $(0, \rho)$ with respect to r and use Fubini's theorem to obtain

$$\rho \, \mathcal{H}_\infty^{n-1}(f(J)) \le C(n) \int_0^\rho \int_{\partial I_r} |Df(x)|^{n-1} \, d\mathcal{H}^{n-1}(x) \, dr$$

$$= C(n) \int_{I_\rho} |Df(x)|^{n-1} \, dx.$$

Let $x \in I_\rho$. Then

$$\mathcal{H}^{n-1}(J \cap Q(x, 2\rho)) \ge \rho^{n-1}.$$

Hence, using Fubini's theorem and the fact that $y \in Q(x, 2\rho)$ if and only if $x \in Q(y, 2\rho)$ we obtain

$$\rho^{n-1} \int_{I_\rho} |Df(x)|^{n-1} \, dx \le \int_{I_\rho} \left(\int_{J \cap Q(x,2\rho)} |Df(x)|^{n-1} \, d\mathcal{H}^{n-1}(y) \right) dx$$

$$= \int_J \left(\int_{Q(y,2\rho) \cap I_\rho} |Df(x)|^{n-1} \, dx \right) d\mathcal{H}^{n-1}(y)$$

$$\le \int_J \left(\int_{Q(y,2\rho)} |Df(x)|^{n-1} \, dx \right) d\mathcal{H}^{n-1}(y)$$

$$= 4^n \, \rho^n \int_J (\tilde{\varphi}_{2\rho} * |Df|^{n-1})(y) \, d\mathcal{H}^{n-1}(y).$$

9

Since $\tilde{\varphi}_{2\rho} \le C(n)\, \varphi_{2\rho n}$, we have

$$\mathcal{H}^{n-1}_\infty(f(J)) \le C(n) \int_J (\varphi_{2\rho n} * |Df|^{n-1})(x)\, d\mathcal{H}^{n-1}(x).$$

Now the claim follows from the condition (13). □

Now we are ready to prove that a discrete and open mapping $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ satisfied Lusin's condition (N) on almost every hyperplane inside a normal domain U of f.

Proposition 3.3. *Let $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be an open and discrete mapping, and let $U \subset\subset \Omega$ be a domain. Then for almost every t the mapping $f|_{U_t}$ satisfies Lusin's condition (N), i.e., for every $A \subset U_t$ with $\mathcal{H}^{n-1}(A) = 0$, also $\mathcal{H}^{n-1}(f|_{U_t}(A)) = 0$.*

Proof. We may assume that U is a normal domain of f. By using a translation, if necessary, we may assume that $U_t \ne \emptyset$ if and only if $t \in (-u, u)$. First we show that f satisfies condition (13) for almost every $t \in (-u, u)$. For this purpose, we denote

$$\Phi_\delta(t) = \int_{(U^\delta)_t} \left| |Df|^{n-1} - \varphi_\delta * |Df|^{n-1} \right|.$$

By L^1-convergence of the mollifications, Fatou's lemma and Fubini's theorem, we have

$$\int_{-u}^{u} \liminf_{\delta \to 0+} \Phi_\delta(t)\, dt \le \liminf_{\delta \to 0+} \int_{-u}^{u} \Phi_\delta(t)\, dt$$

$$= \liminf_{\delta \to 0+} \int_{U^\delta} \left| |Df|^{n-1} - \varphi_\delta * |Df|^{n-1} \right| = 0.$$

This means that $\liminf_{\delta \to 0+} \Phi_\delta(t) = 0$ for almost every $t \in (-u, u)$, and each such t satisfies (13).

Next we fix $t \in (-u, u)$ in such a way that (13) is satisfied and $\int_{U_t} |Df(x)|^{n-1}\, d\mathcal{H}^{n-1}(x) < \infty$. This is true for almost every $t \in (-u, u)$. By Lemma 3.2

$$\mathcal{H}^{n-1}_\infty(f(Q \times \{t\})) \le C(n) \int_{Q \times \{t\}} |Df(x)|^{n-1}\, d\mathcal{H}^{n-1}(x)$$

holds for every closed cube $Q \times \{t\} \subset U_t$. Let $E \subset U_t$ be a set of $(n-1)$-measure zero. Given $\epsilon > 0$, we find an open set $G \subset U_t$ such that $E \subset G$ and

$$\int_{G \times \{t\}} |Df(x)|^{n-1}\, d\mathcal{H}^{n-1}(x) < \epsilon.$$

Let $\{Q_j\}_j$ be a sequence of non-overlapping cubes in U_t such that $G = \bigcup_j Q_j$. Then

$$\mathcal{H}^{n-1}_\infty(f(E)) \le \mathcal{H}^{n-1}_\infty(f(G)) \le \sum_{j=1}^{\infty} \mathcal{H}^{n-1}_\infty(f(Q_j \times \{t\}))$$

$$\le C(n) \sum_j \int_{Q_j \times \{t\}} |Df|^{n-1} = C(n) \int_{G \times \{t\}} |Df|^{n-1} < C(n)\,\epsilon.$$

Letting $\epsilon \to 0$ the claim follows. □

The following corollary follows easily from Proposition 3.3.

Corollary 3.4. *Let $f \in W_{\text{loc}}^{1,n-1}(\Omega, \mathbb{R}^n)$ be a discrete and open mapping. Let $U \subset\subset \Omega$ be a normal domain of f and $B(x, r) \subset\subset U$. Then for almost every $t \in (0, r)$ the mapping $f\,|_{S(x,t)} \colon S(x, t) \to \mathbb{R}^n$ satisfies Lusin's condition (N) i.e. for almost every $t \in (0, r)$*

$$\mathcal{H}^{n-1}(f\,|_{S(x,t)}(A)) = 0 \text{ for every } A \subset S(x, t) \text{ such that } \mathcal{H}^{n-1}(A) = 0.$$

4 Poletsky's inequality for spherical rings

In view of Corollary 3.4 every discrete and open mapping $f \in W_{\text{loc}}^{1,n-1}$ satisfies Lusin's condition (N) on almost every $(n-1)$-dimensional sphere. Using this fact we are able to show that if $f \in W_{\text{loc}}^{1,n-1}$ has integrable inner distortion, then it satisfies a Poletsky type inequality, see Lemma 4.4. The importance of this result is based on the fact that it allows us to use modulus and capacity methods in the study of non-homeomorphic mappings. For more details, see [35].

To prove Theorem 1.1 it would be enough to use only the duality of conformal capacity, introduced in (8). However, to prove Theorem 5.1 in Chapter 5 we have to use a more general duality of p-capacity, introduced in (9). The duality of capacity is also a strong tool in the applications that use modulus and capacity methods. Gehring [6] showed that the conformal capacity is related to the extremal length of a family of surfaces that separate the boundary components of a ring. Also other authors have dealt with the extremal length of separating surfaces, see [6], [5], [36], [14], [42] and [43]. In this paper we follow closely the work of Ziemer, see [42] and [43].

We give next a change of variables -type formula for subsets of spheres. This result follows from [24, Theorem 9.2], see also [33].

Lemma 4.1. *Let $f \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$, be a continuous mapping, and $B(x, r) \subset\subset \Omega$. If f satisfies condition (N) on almost every $(n-1)$-dimensional sphere $S(x, t)$, $0 < t < r$, with respect to measure $\mathcal{H}^{n-1}$, then for almost every $t \in (0, r)$ we have*

$$(14) \qquad \int_E u(x) |D^\sharp f(x)| \, d\mathcal{H}^{n-1}(x) \geq \int_{\mathbb{R}^n} \left(\sum_{x \in f^{-1}(y) \cap E} u(x) \right) d\mathcal{H}^{n-1}(y),$$

for all $\mathcal{H}^{n-1}$-measurable subsets $E \subset S(x, t)$ and for all $\mathcal{H}^{n-1}$-measurable functions $u : E \to \mathbb{R}$.

The next lemma is crucial in our proof of Poletsky type -inequality for rings. This lemma is a modified version of [33, Theorem 3.2].

Lemma 4.2. *Let $f : \Omega \to \mathbb{R}^n$ be a continuous, discrete and open mapping in $W_{\text{loc}}^{1,n-1}(\Omega, \mathbb{R}^n)$, and $p > 1$. Assume that U is a normal domain and $B(x_0, r) \subset\subset U$. Suppose that $I \subset (0, r)$ is a Borel set. Let*

$$\Lambda := \{ S(x_0, t) : t \in I \}.$$

11

Let $\rho : \mathbb{R}^n \to [0, +\infty]$ be a Borel function with the property

$$\int_{f(S_t)} N(y, f, S_t) \rho(y) \, d\mathcal{H}^{n-1}(y) \geq 1 \quad \text{for all } t \in I,$$

where $S_t := S(x_0, t)$. Then

$$M^S_{p', A_{p'}}(\Lambda) \leq m \int_{\mathbb{R}^n} \rho(y)^{p'} \, dy,$$

where the constant m is the degree of mapping f on normal domain U, $p' = \frac{p}{p-1}$ and

$$A_{p'} := A_{p'}(x) = \begin{cases} \frac{J(x,f)}{|D^\sharp f(x)|^{p'}}, & \text{if } |D^\sharp f(x)| > 0 \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Let ρ be a test function as in the statement of the lemma. Define a Borel function $\tilde{\rho} : B(x_0, r) \to [0, +\infty]$ by setting

$$\tilde{\rho}(x) = \begin{cases} \rho(f(x)) |D^\sharp f(x)|, & \text{if } D^\sharp f(x) \text{ is well-defined} \\ 0, & \text{otherwise.} \end{cases}$$

By Corollary 3.4 the restriction of f to the sphere S_t satisfies condition (N) with respect to $\mathcal{H}^{n-1}$ for almost every $t \in (0, r)$. Lemma 4.1 shows that

$$\int_{S_t} \tilde{\rho}(x) \, d\mathcal{H}^{n-1}(x) = \int_{S_t} \rho(f(x)) |D^\sharp f(x)| \, d\mathcal{H}^{n-1}(x)$$

$$\geq \int_{\mathbb{R}^n} N(y, f, S_t) \rho(y) \, d\mathcal{H}^{n-1}(y) \geq 1$$

for almost every $t \in I$. On the other hand

$$\int_{B_r} \tilde{\rho}(x)^{p'} A_{p'}(x) \, dx = \int_{B_r} \rho(f(x))^{p'} |D^\sharp f(x)|^{p'} A_{p'}(x) \, dx = \int_{B_r} \rho(f(x))^{p'} J(x, f) \, dx$$

$$\leq \int_{\mathbb{R}^n} N(y, f, B_r) \rho(y)^{p'} \, dy \leq m \int_{\mathbb{R}^n} \rho(y)^{p'} \, dy.$$

This completes the proof. $\qquad\qquad\square$

Next we will prove our duality result, Lemma 4.3. To formulate this result, we have to define some new concepts that are not in common use in literature. For this purpose, let $U \subset\subset \Omega$ be a normal domain of f and $B(x_0, 2r) \subset\subset U$. Let $M : \Omega \to \mathbb{R}$ be a weight function. Then we can define the capacity of the condenser $(\overline{B(x_0, r)}, B(x_0, 2r))$ with respect to symmetric test functions and weight M as

$$\text{cap}^{sym}_{p, M}(\overline{B(x_0, r)}, B(x_0, 2r)) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u(x)|^p M(x) \, dx : u \in Ad(sym) \right\},$$

where with $Ad(sym)$ we denote the set of admissible Borel functions for capacity $\text{cap}_p(\overline{B(x_0, r)}, B(x_0, 2r))$ with the additional assumption that for every $s \in (0, 1)$ the set $u^{-1}(s)$ is a sphere $S(x_0, t)$, for some $r < t < 2r$.

Lemma 4.3. *Let $f \in W^{1,n-1}_{loc}(\Omega, \mathbb{R}^n)$ be a discrete and open mapping of finite inner distortion and $1 < p \leq n$. Assume that U is a normal domain of f and $B(x_0, 2r) \subset\subset U$. Let*

$$\Lambda = \{ S(x_0, t) : r < t < 2r \}.$$

Then

$$\mathrm{M}^S_{p', A_{p'}}(\Lambda) \geq \mathrm{cap}^{sym}_{p, B_p}(\overline{B(x_0, r)}, B(x_0, 2r))^{\frac{-1}{p-1}},$$

where $A_{p'}$ is as in Lemma 4.2 and

$$B_p := B_p(x) = \begin{cases} K_I(x, f)^{p/n} J(x, f)^{(n-p)/n}, & \text{if } J(x, f) > 0 \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Choose $\epsilon > 0$ and let ρ be any Borel function in U with property

$$\int_\sigma \rho(x) \, d\mathcal{H}^{n-1}(x) \geq 1 \quad \text{for every } \sigma \in \Lambda.$$

Let u be any admissible function for $C := \mathrm{cap}^{sym}_{p, B_p}(\overline{B(x_0, r)}, B(x_0, 2r))$ such that

$$\int_{\mathbb{R}^n} |\nabla u|^n B_p(x) \, dx < C + \epsilon.$$

It is clear that $u^{-1}(s) \in \Lambda$ for every $0 < s < 1$. Hence, by Hölder's inequality and coarea formula [41], we have

$$\left(\int_{\mathbb{R}^n} \rho^{p'}(x) A_{p'}(x) \, dx \right)^{\frac{1}{p'}} (C + \epsilon)^{\frac{1}{p}}$$

$$\geq \left(\int_{\mathbb{R}^n} \rho(x)^{p'} A_{p'}(x) \, dx \right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p B_p(x) \, dx \right)^{\frac{1}{p}}$$

$$\geq \int_{\mathbb{R}^n} \rho(x) |\nabla u(x)| A_{p'}(x)^{\frac{1}{p'}} B_p(x)^{\frac{1}{p}} \, dx$$

$$= \int_{\mathbb{R}^n} \rho(x) |\nabla u(x)| \, dx$$

$$= \int_0^1 \int_{u^{-1}(s)} \rho \, d\mathcal{H}^{n-1}(x) \, ds \geq 1.$$

Since $\epsilon > 0$ was arbitary,

$$\int_{\mathbb{R}^n} \rho(x)^{p'} A_{p'}(x) \, dx \geq C^{-1/(p-1)}$$

which is also true when the infimum is taken on the left hand side over all admissible test functions ρ. $\square$

We could have formulated Lemma 4.3 also for more general weights. However, for our purpose it is enough to use the weight function B_p defined above.

The following Poletsky-type inequality is crucial for our proof of Theorem 1.1. It would be interesting to know if it is possible to prove this result also for more general condensers.

Lemma 4.4. *Let $f \in W_{\mathrm{loc}}^{1,n-1}(\Omega, \mathbb{R}^n)$ be a discrete and open mapping of finite inner distortion. Assume that $\overline{U} \subset\subset \Omega$ is a normal domain of f and $B(x_0, 2r) \subset\subset U$. Let $E = (\overline{B(x_0, r)}, B(x_0, 2r))$ be a condenser in U. Then*

$$\mathrm{cap}_p(f(\overline{B(x_0, r)}), f(B(x_0, 2r))) \le m^{p-1} \, \mathrm{cap}_{p, B_p}^{sym}(\overline{B(x_0, r)}, B(x_0, 2r)),$$

where $C(n, m) > 0$ is a constant depending only on dimension n and on the degree m of the mapping f on normal domain U, and B_p is defined as in Lemma 4.3.

Proof. Now $(f(\overline{B(x_0, r)}), f(B(x_0, 2r)))$ is a condenser in U and we can define Λ' to be the family of sets separating $\mathbb{R}^n \setminus f(B(x_0, 2r))$ from $f(\overline{B(x_0, r)})$). Notice that because f is an open mapping, we have $\Lambda' \ne \emptyset$. Moreover, openness of f implies that $f(B(x_0, 2r))$ is an open set and continuity of f impliest that $f(\overline{B(x_0, r)})$ is a compact set. Thus $(f(\overline{B(x_0, r)}), f(B(x_0, 2r)))$ is a condenser in $f(U)$. We set

$$\Lambda_L := \{ S_t : t \in (r, 2r) \}$$

and

$$\Lambda_l := \{ S_t' : t \in (r, 2r) \},$$

where

$$S_t' = \{ x \in S_t : f(x) \in \partial f(S_t) \},$$

where $S_t := S^{n-1}(x_0, t)$. Because every admissible test function for the modulus $\mathrm{M}_{p'}^S(f(\Lambda_l))$ is also admissible for the modulus $\mathrm{M}_{p'}^S(f(\Lambda_L))$, we have that $\mathrm{M}_{p'}^S(f(\Lambda_L)) \le \mathrm{M}_{p'}^S(f(\Lambda_l))$. Moreover, we also have $\Lambda' \supset f(\Lambda_l)$. Thus we have by (8), using the monotonicity of modulus and by Lemma 4.2 and Lemma 4.3

$$\begin{aligned}
\mathrm{cap}_p(f(\overline{B(x_0, r)}), f(B(x_0, 2r))) = \mathrm{M}_{p'}^S(\Lambda')^{-(p-1)} &\le \mathrm{M}_{p'}^S(f(\Lambda_l))^{-(p-1)} \\
&\le \mathrm{M}_{p'}^S(f(\Lambda_L))^{-(p-1)} \\
&\le m^{p-1} \, \mathrm{M}_{p', A_{p'}}^S(\Lambda_L)^{-(p-1)} \\
&\le m^{p-1} \, \mathrm{cap}_{p, B_p}^{sym}(\overline{B(x_0, r)}, B(x_0, 2r)),
\end{aligned}$$

where $A_{p'}$ is as in Lemma 4.2 and Lemma 4.3. $\qquad\square$

5 Differentiability in the Sobolev space $W^{1,n-1}$

In this section we prove our main result which says that a discrete and open mapping $f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ with integrable inner distortion is differentiable almost everywhere in Ω. The integrability assumption of the distortion function can be relaxed if we assume higher integrability of the Jacobian determinant. Indeed, we will prove the following result.

Theorem 5.1. *Suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a domain. Let $f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ be a continuous, discrete and open mapping of finite inner distortion with $K_I^{p/n}(\,\cdot\,, f) J(\cdot, f)^{(n-p)/n} \in L^1_{\text{loc}}(\Omega)$, for some $n - 1 < p \leq n$. Then f is differentiable almost everywhere. Moreover, f is a mapping of finite outer distortion.*

Theorem 1.1 follows from Theorem 5.1 when we choose $p = n$. In the proof of Theorem 5.1 we apply the Rademacher-Stepanov theorem [38], see also [23].

Lemma 5.2 (Rademacher-Stepanov). *A mapping $f : \Omega \to \mathbb{R}^n$ is differentiable almost everywhere if and only if*

$$\limsup_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} < \infty \quad a.e.$$

The following lemma can be originally found from [22, Proposition 6], see also [27, Lemma 5.9]. A big part of the geometric structure of the proof of Theorem 5.1 and Theorem 1.1 is in Lemma 5.3. Indeed, in the case where f is a homeomorphism of finite outer distortion it is possible to prove Theorem 1.1 by only combining [31, Lemma 2.1] and [30, Theorem 2.1]. Proof of [31, Lemma 2.1] uses the same techniques that are used in the proof of Lemma 5.3 given in [22].

Lemma 5.3. *Let (E, G) be a condenser in domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$. If $E \subset \mathbb{R}^n$ is connected, then for $n - 1 < p \leq n$*

$$(15) \qquad \text{cap}_p^{n-1}(E, G) \geq C(n, p) \frac{(\text{diam } E)^p}{|G|^{1-n+p}}.$$

Proof. See [22, Proposition 6]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We are now ready to prove our main result Theorem 5.1. We will follow the idea of the proof of Theorem 1 by Salimov and Sevost'yanov in [37], see also [28, §4.3].

Proof of Theorem 5.1. If $n = 2$, the result follows from the theorem of Gehring and Lehto [6], and from the fact that in the planar case every mapping of finite inner distortion is always a mapping of finite outer distortion.

Next, assume that $n \geq 3$. Let B_p be as in Lemma 4.3 and Lemma 4.4. Then, because $K_I^{p/n}(\,\cdot\,, f) J(\cdot, f)^{(n-p)/n} \in L^1_{\text{loc}}(\Omega)$, we have $B_p \in L^1_{\text{loc}}(\Omega)$. First we show that f is differentiable almost everywhere. Let us define

$$(16) \qquad \mu'_f(x) := \limsup_{r \to 0} \frac{|f(B(x,r))|}{|B(x,r)|}.$$

15

By differentiating the measure

$$E \mapsto \int_{\mathbb{R}^n} N(y, f, E) \, dy$$

and applying [3, Theorem 1 p.38], we see that $\mu'_f(x) < \infty$ for almost every $x \in \Omega$. Assume that $U \subset\subset \Omega$ is a normal domain of f and let $x_0 \in U$. Let $r > 0$ be so small that $B(x_0, 2r) \subset\subset U$. Then $E = (\overline{B(x_0, r)}, B(x_0, 2r))$ is a condenser in U and $E' = (f(\overline{B(x_0, r)}), f(B(x_0, 2r)))$ is a condenser in $f(U)$. We define a Borel function $u : \mathbb{R}^n \to [0, +\infty]$ by setting

$$u(y) = \begin{cases} 1, & \text{if } |x_0 - y| \leq r \\ \frac{2r - |x_0 - y|}{r}, & \text{if } r < |x_0 - y| < 2r \\ 0, & \text{if } |x_0 - y| \geq 2r. \end{cases}$$

Then

$$|\nabla u(y)| = \begin{cases} 0, & \text{if } |x_0 - y| < r \\ r^{-1}, & \text{if } r < |x_0 - y| < 2r \\ 0, & \text{if } |x_0 - y| > 2r. \end{cases}$$

Let $n - 1 < p \leq n$. By Lemma 4.4 we get

(17) $\qquad \mathrm{cap}_p(f(\overline{B(x_0, r)}), f(B(x_0, 2r))) \leq C(n, m) \, \mathrm{cap}^{sym}_{p, B_p}(\overline{B(x_0, r)}, B(x_0, 2r))$

$$\leq C(n, m) \int_{B(x_0, 2r)} |\nabla u(x)|^p B_p(x) \, dx \leq \frac{C(n, m)}{r^p} \int_{B(x_0, 2r)} B_p(x) \, dx.$$

On the other hand, by Lemma 5.3

(18) $\qquad \mathrm{cap}_p(f(\overline{B(x_0, r)}), f(B(x_0, 2r))) \geq C(n, p) \left(\frac{(\mathrm{diam}\,(f(\overline{B(x_0, r)})))^p}{|f(B(x_0, 2r))|^{1-n+p}} \right)^{\frac{1}{n-1}}.$

Combining (17) and (18) we have

(19) $\qquad \dfrac{\mathrm{diam}\,(f(\overline{B(x_0, r)}))}{r} \leq C'(n, m, p) \left(\dfrac{|f(B(x_0, 2r))|}{|B(x_0, 2r)|} \right)^{\frac{1-n+p}{p}} \left(\fint_{B(x_0, 2r)} B_p(y) \, dy \right)^{\frac{n-1}{p}}.$

By using Lebesgue differentiation theorem to estimate (19) and (16) we have

(20) $\qquad \limsup_{y \to x_0} \dfrac{|f(x_0) - f(y)|}{|x_0 - y|} \leq C(n, p, m) \, [\mu'_f(x_0)]^{\frac{1-n+p}{p}} \, [B_p(x_0)]^{\frac{n-1}{p}} < \infty$

for almost every $x_0 \in U$, whenever $B_p \in L^1_{\mathrm{loc}}(\Omega)$. Thus, by Lemma 5.2 f is differentiable almost everywhere.

Next we show that f is a mapping of finite outer distortion. Because f is a mapping of finite inner distortion, it is enough to show that $J(x, f) = 0$ implies

16

$Df(x) = 0$ for almost every $x \in \Omega$. Let $U \subset\subset \Omega$ be a normal domain of f with $m = \deg(\cdot, f, U)$. By using the proof of Lemma II. 2.9 in [34], we have that

$$\mu'_f(x) = |J(x, f)|$$

in the points of differentiability of f. Thus, by using the estimate (20) and the fact that $J(x, f) \geq 0$ almost everywhere, we have

$$|Df(x)| \leq C(n, p, m)\, \mu'_f(x)^{\frac{1-n+p}{p}} B_p(x)^{\frac{n-1}{p}} \leq C(n)\, J(x, f)^{\frac{1-n+p}{p}} B_p(x)^{\frac{n-1}{p}}$$

for almost every $x \in U$. Thus $J(x, f) = 0$ implies $Df(x) = 0$ for almost every $x \in \Omega$. Therefore f is a mapping of outer distortion. $\square$

Corollary 5.4. *Let $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $n \geq 2$, be a continuous, open and discrete mapping of finite inner distortion with locally integrable inner distortion. Then for the branch set B_f of f, i.e. for the set of points where f is not a local homeomorphism, we have*

$$|B_f| = 0.$$

Proof. By [20, Theorem 1.2] $J(x, f) > 0$ for almost every $x \in \Omega$ and by [35, I.4.11] $J(x, f) = 0$ for every point $x \in B_f$ of differentiability. Because by Theorem 1.1 f is differentiable almost everywhere, we have $|B_f| = 0$. $\square$

By using Theorem 1.1 and Corollary 5.4 we can now prove Theorem 1.2.

Proof of Theorem 1.2. By Theorem 1.1 we already know that f is differentiable almost everywhere and that it is a mapping of finite distortion. Moreover, by Corollary 5.4 we know that for almost every $x \in \Omega$ we can find a neighborhood U_x such that the restriction map $f\,|_{U_x} : U_x \to f(U_x)$ is a homeomorphism.

If $n = 2$, the result follows now from the theorem of Hencl and Koskela [10, Theorem 1.3]. Thus, we may assume that $n \geq 3$. Because $f\,|_{U_x} \in W^{1,n-1}_{\mathrm{loc}}(U_x, \mathbb{R}^n)$ is a homeomorphism of finite distortion, by [2, Theorem 1.2] $(f\,|_{U_x})^{-1}$ is a mapping of finite distortion. Moreover, by [30, Theorem 1.2] we have that $(f\,|_{U_x})^{-1} \in W^{1,n}_{\mathrm{loc}}(f(U_x), \mathbb{R}^n)$ with

$$\int_{f(U_x)} |D(f\,|_{U_x})^{-1}(y)|^n \, dy = \int_{U_x} K_I(x, f\,|_{U_x}) \, dx.$$

Because $(f\,|_{U_x})^{-1} \in W^{1,n}_{\mathrm{loc}}(f(U_x), \mathbb{R}^n)$ is a homeomorphism, it is differentiable almost everywhere by the differentiability result of Väisälä [39]. $\square$

Theorem 5.1 is sharp in the sense that one cannot allow $p = n - 1$. To see this, we follow an idea by Csörnyei, Hencl and Malý [2, Example 5.2].

Example 5.5. *There is a homeomorphism $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $n \geq 3$, with $K_I(\cdot, f) \in L^{(n-1)/n}_{\mathrm{loc}}(\Omega)$ such that f is nowhere differentiable.*

Proof. It is possible to construct a nowhere differentiable continuous function $\varphi :$ $(-1,1)^{n-1} \to \mathbb{R}^n$ such that $\varphi \in W^{1,n-1}((-1,1)^{n-1})$. For more details, see [2, Example 5.2]. Next we define $f : (-1,1)^n \to \mathbb{R}^n$ by setting

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, x_n + \varphi(x_1, \ldots, x_{n-1})).$$

One can check that f is a homeomorphism in $W^{1,n-1}$. Analogously we obtain that $f^{-1} \in W^{1,n-1}_{\text{loc}}(f((-1,1)^n), (-1,1)^n)$

$$f^{-1}(y_1, \ldots, y_n) = (y_1, \ldots, y_{n-1}, y_n - \varphi(y_1, \ldots, y_{n-1})).$$

It is also easy to see that $J(x,f) = 1$ for every $x \in (-1,1)^n$ and $J(y, f^{-1}) = 1$ for every $y \in f((-1,1)^n)$. Moreover, we notice that f and f^{-1} are nowhere differentiable. Next we calculate for any compact set $F \subset\subset (-1,1)^n$

$$\int_F K_I^{\frac{n-1}{n}}(x,f) J(x,f)^{\frac{1}{n}} dx = \int_F \left(\frac{|D^{\sharp}f(x)|}{J(x,f)} \right)^{n-1} J(x,f) dx$$
$$= \int_F |Df^{-1}(f(x))|^{n-1} J(x,f) dx$$
$$\leq \int_{f(F)} |Df^{-1}(y)|^{n-1} dy < \infty,$$

and the claim follows. $\qquad\square$

To construct our mapping in Example 5.5 we had to assume that $n \geq 3$. Indeed, every continuous mapping $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$, $\Omega \subset \mathbb{R}^2$, has finite partial derivatives almost everywhere, because it is absolutely continuous on almost every line parallel to coordinate axes, see [35, I.1.2]. Therefore, as a consequence of theorem by Gehring and Lehto [7, Theorem 2], we have that every planar homeomorphism $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$, $\Omega \subset \mathbb{R}^2$, is differentiable almost everywhere.

We do not know what is the sharp integrability assumption for $K_I(\cdot, f)$ that guarantees differentiability at almost every point in Theorem 1.1. However, we believe that the correct assumption should be $K_I(\cdot, f) \in L^1_{\text{loc}}(\Omega)$.

References

[1] K. Astala, T. Iwaniec, G.J. Martin and J. Onninen: *Extremal mappings of finite distortion*, Proc. London Math. Soc. **91** no. 3, 655–702 (2005).

[2] M. Csörnyei, S. Hencl and J. Malý: *Homeomorphisms in the Sobolev space $W^{1,n-1}$*, J. Reine Angew. Math. **644**, 221–235 (2010).

[3] L.C. Evans and R.F. Gariepy: *Measure theory and Fine Properties of Functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL (1992).

[4] H. Federer: *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag, New York, Second edition (1996).

[5] B. Fuglede: *Extremal length and functional completion*, Acta Math. **98**, 171–219 (1957).

[6] F.W. Gehring: *Extremal length definitions for the conformal capacity in space*, Michigan Math J. **9**, 137–150 (1962).

[7] F.W. Gehring and O. Lehto: *On the total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fennn. Ser. A. I No. **272**, 1–9 (1959).

[8] W. Gustin: *Boxing inequalities*, J. Math. Mech. **9**, 229–239 (1960).

[9] V. Gol'dstein and S. Vodop'yanov: *Quasiconformal mappings and spaces of functions with generalized first derivatives*, Sibirsk. Mat. Z. **17**, 515–531 (1976).

[10] S. Hencl and P. Koskela: *Regularity of the inverse of a planar Sobolev homeomorphism*, Arch. Rational Mech. Anal. **180**, 75–95 (2006).

[11] S. Hencl, P. Koskela and J. Malý: *Regularity of the inverse of a Sobolev homeomorphism in space*, Proc. Roy. Soc. Edinburgh Sect. **136A**, no. 6, 1267–1285 (2006).

[12] S. Hencl, P. Koskela and J. Onninen: *Homeomorphism of bounded variation*, Arh. Rational Mech. Anal. **186**, 351–360 (2007).

[13] S. Hencl and K. Rajala: *Optimal assumptions for discreteness*, Preprint, (2011).

[14] J. Hersch: *Longeurs extremales dans l'espace, resistance electrique et capacité*, C. R. Acad. Sci. Paris **238**, 1639–1641 (1954).

[15] T. Iwaniec, P. Koskela and J. Onninen: *Mappings of finite distortion: Monotonicity and continuity*, Invent. Math. **144**, 507–531 (2001).

[16] T. Iwaniec and G.J. Martin: *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs, Clarendon Press, Oxford (2001).

[17] T. Iwaniec and V. Šverák: *On mappings with integrable dilation*, Proc. Amer. Math. Soc. **118**, 181–188 (1993).

[18] J. Kauhanen, P. Koskela and J. Malý: *Mappings of finite distortion: Discreteness and openness*, Arch. Ration. Mech. Anal. **160**, 135–151 (2001).

[19] J. KAUHANEN, P. KOSKELA, J. MALÝ, J. ONNINEN AND X. ZHONG: *Mappings of fnite distortion: Sharp Orlicz-conditions, ,* REV. MAT. IBEROAMERICANA **19**, 857–872 (2003).

[20] P. KOSKELA AND J. MALÝ: *Mappings of finite distortion: The zero set of the Jacobian,* J. EUR. MATH. SOC. **5** NO. 2, 95–105 (2003).

[21] P. KOSKELA AND J. ONNINEN: *Mappings of finite distortion: Capacity and modulus inequalities,* J. REINE ANGEW. MATH. **599**, 1–26 (2006).

[22] V.I. KRUGLIKOV: *Capacity of condensers and spatial mappings quasiconformal in the mean,* J. MATH. USSR-SB. **58**, 185–205 (1987).

[23] J. MALÝ: *A simple proof of the Stepanovs theorem on differentiability almost everywhere,* EXPOSITION. MATH. **17**, 59–61 (1999).

[24] J. MALÝ: *Lectures on change of variables in integral,* PREPRINT 305, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HELSINKI, (2001).

[25] J. MANFREDI AND E. VILLAMOR: *Mappings with integrable dilatation in higher dimensions,* BULL. AMER. MATH. SOC. **32** NO. 2, 235–240 (1995).

[26] J. MANFREDI AND E. VILLAMOR: *An extension of Reshetnyak's theorem,* INDIANA UNIV. MATH. J. **47** NO. 3, 1131–1145 (1998).

[27] O. MARTIO, S. RICKMAN AND J. VÄISÄLÄ: *Definitions for quasiregular mappings,* ANN. ACAD. SCI. FENN. SER. A1 MATH., **448**, 1–40 (1969).

[28] O. MARTIO, V. RYAZANOV, U. SREBRO AND E. YAKUBOV: *Moduli in modern mapping theory,* SPRINGER MONOGRAPHS IN MATHEMATICS. SPRINGER, NEW YORK (2009).

[29] D. MENCHOFF: *Sur les différentielles totales des fonctions univalentes,* MATH. ANN., **105** NO. 1, 75–85 (1931).

[30] G. MOSCARIELLO AND A. PASSARELLI DI NAPOLI: *The regularity of the inverse of Sobolev homeomorphism with finite distortion,* J. GEOM. ANAL., **24**, NO. 1, 571–594 (2014).

[31] J. ONNINEN: *Regularity of the inverse of spatial mappings with finite distortion,* CAL. VAR. PARTIAL DIFFERENTIAL EQUATIONS **26** NO. 3, 331–341 (2006).

[32] E.A. POLETSKY: *The modulus method for non-homeomorphic quasiconformal mappings,* MAT. SB. **83** NO. 125, 261–272 (1970).

[33] K. RAJALA: *Local homeomorphism property of spatial quasiregular mappings with distortion close to one,* GEOM. FUNCT. ANAL., **15** NO. 5, 1100–1127 (2005).

[34] Y. G. RESHETNYAK: *Space Mappings with Bounded Distortion,* TRANSL. MATH. MONOGRAPHS **73**, AMS (1989).

[35] S. Rickman: *Quasiregular mappings*, Springer-Verlag, Berlin (1993).

[36] B.B. Šabat: *The modulus method in space*, Dokl. Akad. Nauk. SSSR **130** (1960), 1210–1213 = Soviet Math. Dokl. **1**, 165–168 (1960).

[37] R.R. Salimov and E.A. Sevost'yanov: ACL *and differentiability of open discrete ring* (p, Q)*-mappings*, Mat. Stud. **35** no. 1, 28–36 (2011).

[38] V. Stepanov: *Sur les conditions de l'existence de la differentielle totale*, Mat. Sb., **32**, 487–489 (1924) [in Russian].

[39] J. Väisälä: *Two new characterizations for quasiconformality*, Ann. Acad. Sci. Fenn. Ser. A I No. **362**, 1–12 (1965).

[40] J. Väisälä: *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math. **229**. Springer-Verlag, Berlin (1971).

[41] W.P. Ziemer: *Some lower bounds for Lebesgue area*, Pacific J. Math **19** (1966), 381–390.

[42] W.P. Ziemer: *Extremal length and conformal capacity*, Trans. Amer. Math. Soc., **126** no. 3, 460–473 (1967).

[43] W.P. Ziemer: *Extremal length and p-capacity*, Michigan Math. J. **16**, 43–51 (1969).