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**Title:** Weak Chord-Arc Curves and Double-Dome Quasisymmetric Spheres

**Year:** 2016

**Version:**

**Please cite the original version:**
Vellis, V. (2016). Weak Chord-Arc Curves and Double-Dome Quasisymmetric Spheres. Analysis and Geometry in Metric Spaces, 4(1). https://doi.org/10.1515/agms-2016-0002

Anal. Geom. Metr. Spaces 2016; 4:54–67 **DE GRUYTER** OPEN

**Research Article** **Open Access**

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# Weak Chord-Arc Curves and Double-Dome Quasisymmetric Spheres

DOI 10.1515/agms-2016-0002
Received June 5, 2015; accepted February 14, 2016

**Abstract:** Let $\Omega$ be a planar Jordan domain and $\alpha > 0$. We consider double-dome-like surfaces $\Sigma(\Omega, t^\alpha)$ over $\overline{\Omega}$ where the height of the surface over any point $x \in \overline{\Omega}$ equals $\operatorname{dist}(x, \partial\Omega)^\alpha$. We identify the necessary and sufficient conditions in terms of $\Omega$ and $\alpha$ so that these surfaces are quasisymmetric to $\mathbb{S}^2$ and we show that $\Sigma(\Omega, t^\alpha)$ is quasisymmetric to the unit sphere $\mathbb{S}^2$ if and only if it is linearly locally connected and Ahlfors 2-regular.

**Keywords:** quasisymmetric spheres; double-dome-like surfaces; chord-arc property; Ahlfors 2-regularity

**MSC:** Primary: 30C65. Secondary: 30C62

## 1 Introduction

A metric space which is quasisymmetric to the standard $n$-sphere $\mathbb{S}^n$ is called a quasisymmetric $n$-sphere. Quasisymmetric circles were completely characterized by Tukia and Väisälä in [9]. Bonk and Kleiner [4] identified a necessary and sufficient condition for metric 2-spheres to be quasisymmetric spheres. A consequence of their main theorem is that if a metric 2-sphere is linearly locally connected (or LLC) and Ahlfors 2-regular then it is a quasisymmetric 2-sphere. Although the LLC property is necessary, there are examples of quasisymmetric 2-spheres constructed by snowflaking procedures that fail the 2-regularity [3], [5], [7].

In this paper we consider a special case of the double dome-like surfaces constructed over planar Jordan domains $\Omega$ in [12]. For a number $\alpha > 0$ and a Jordan domain $\Omega \subset \mathbb{R}^2$ consider the 2-dimensional surface in $\mathbb{R}^3$

$$\Sigma(\Omega, t^\alpha) = \{(x, z) \colon x \in \overline{\Omega}, z = \pm(\operatorname{dist}(x, \partial\Omega))^\alpha\}.$$

As it turns out, for these surfaces the 2-regularity is necessary for quasisymmetric parametrization by $\mathbb{S}^2$.

**Theorem 1.1.** *The surface $\Sigma(\Omega, t^\alpha)$ is quasisymmetric to $\mathbb{S}^2$ if and only if it is linearly locally connected and 2-regular.*

What conditions on $\Omega$ and $\alpha$ ensure that $\Sigma(\Omega, t^\alpha)$ is a quasisymmetric 2-sphere? When $\alpha > 1$, Theorem 1.1 is trivial since, for any Jordan domain $\Omega$, the surface $\Sigma(\Omega, t^\alpha)$ is not linearly locally connected and hence not quasisymmetric to $\mathbb{S}^2$. If $\alpha = 1$, $\Sigma(\Omega, t)$ is quasisymmetric to $\mathbb{S}^2$ if and only if $\partial\Omega$ is a quasicircle [12, Theorem 1.1]. This result, combined with the fact that the projection of $\Sigma(\Omega, t)$ on $\overline{\Omega}$ is a bi-Lipschitz mapping, and the fact that $\overline{\Omega}$ is 2-regular if $\partial\Omega$ is a quasicircle, gives Theorem 1.1 when $\alpha = 1$.

In the case $\alpha \in (0, 1)$, the part of $\Sigma(\Omega, t^\alpha)$ near $\partial\Omega \times \{0\}$ resembles the product of $\partial\Omega$ with an interval. Väisälä [10] has shown that the product $\gamma \times I$ of a Jordan arc $\gamma$ and an interval $I$ is quasisymmetric embeddable in $\mathbb{R}^2$ if and only if $\gamma$ satisfies the chord-arc condition (2.2). In view of this result, it is expected that the chord-arc property of $\partial\Omega$ is necessary for $\Sigma(\Omega, t^\alpha)$ to be a quasisymmetric sphere. Moreover, as the double-dome-like

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surface envelops the interior $\Omega$ above and below, the following condition on $\Omega$ is needed to ensure the linear local connectedness of the surfaces $\Sigma(\Omega, t^\alpha)$. We say that $\Omega$ has the *level quasicircle property* (or LQC property) if there exist $\epsilon_0 > 0$ and $C > 1$ such that for all $\epsilon \in [0, \epsilon_0]$, the *$\epsilon$-level set* of $\Omega$

$$\gamma_\epsilon = \{x \in \overline{\Omega} \colon \mathrm{dist}(x, \partial\Omega) = \epsilon\}$$

is a $C$-quasicircle. A consequence of Theorem 1.2 in [12] is that *if a planar Jordan domain $\Omega$ has the LQC property and $\partial\Omega$ is a chord-arc curve then $\Sigma(\Omega, t^\alpha)$ is quasisymmetric to $\mathbb{S}^2$ for all $\alpha \in (0, 1)$.*

For these surfaces, the LQC property of $\Omega$ is essential: if a Jordan domain $\Omega$ does not have the LQC property then $\Sigma(\Omega, t^\alpha)$ is not quasisymmetric to $\mathbb{S}^2$ for any $\alpha \in (0, 1)$; see Lemma 4.6. However, the chord-arc condition of $\partial\Omega$ is not necessary. Contrary to the intuition based on Väisälä's result [10], we construct in Section 5.1 a Jordan domain $\Omega$ whose boundary is a non-rectifiable curve and $\Sigma(\Omega, t^\alpha)$ is a quasisymmetric sphere for all $\alpha \in (0, 1)$.

Instead, only a weak form of the chord-arc condition is needed: a metric circle $\Gamma$ is said to have the *weak chord-arc condition* if there exists $N_0 > 1$ such that every subarc $\Gamma' \subset \Gamma$ with $\mathrm{diam}\, \Gamma' \le 1$ can be covered by at most $N_0(\mathrm{diam}\, \Gamma')^{-1}$ subarcs $\Gamma_i \subset \Gamma'$ of diameter at most $(\mathrm{diam}\, \Gamma')^2$. Under this terminology we have the following.

**Theorem 1.2.** *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain and $\alpha \in (0, 1)$. The following are equivalent.*

1. *The surface $\Sigma(\Omega, t^\alpha)$ is a quasisymmetric sphere.*
2. *The surface $\Sigma(\Omega, t^\alpha)$ is LLC and 2-regular.*
3. *$\Omega$ has the LQC property and $\partial\Omega$ is a weak chord-arc curve.*

An immediate consequence of Theorem 1.2 is that the existence of a quasisymmetric parametrization of $\Sigma(\Omega, t^\alpha)$ by $\mathbb{S}^2$ does not depend on $\alpha \in (0, 1)$. In particular, if $\Sigma(\Omega, t^\alpha)$ is quasisymmetric to $\mathbb{S}^2$ for some $\alpha \in (0, 1)$ then it is quasisymmetric to $\mathbb{S}^2$ for all $\alpha \in (0, 1)$.

Although not neccessarily rectifiable, any weak chord-arc quasicircle has Hausdorff dimension equal to 1. In fact, a slightly stronger result holds true.

**Theorem 1.3.** *If a quasicircle satisfies the weak chord-arc property then it has Assouad dimension equal to* 1.

The example in Section 5.2, however, shows that the converse is false. Moreover, since the Assouad dimension is larger than the Hausdorff, upper box counting and lower box-counting dimensions, the conclusion of Theorem 1.3 holds for any of these dimensions. A consequence of Theorem 1.3 is that if $\Omega$ is a Jordan domain and $\partial\Omega$ has Assouad dimension greater than 1, then the surface $\Sigma(\Omega, t^\alpha)$ is not quasisymmetric to $\mathbb{S}^2$ for any $\alpha \in (0, 1)$.

In Section 3 we define an index that measures how much a curve $\Gamma$ deviates from being a chord-arc curve on a certain scale, and we discuss the weak chord-arc condition. The proofs of Theorem 1.2 and Theorem 1.3 are given in Section 4. Finally, two examples, based on homogeneous snowflakes, illustrating the weak chord-arc condition are presented in Section 5.

# 2 Preliminaries

## 2.1 Definitions and notation

An embedding $f$ of a metric space $(X, d_X)$ into a metric space $(Y, d_Y)$ is said to be *$\eta$-quasisymmetric* if there exists a homeomorphism $\eta \colon [0, \infty) \to [0, \infty)$ such that for all $x, a, b \in X$ and $t > 0$ with $d_X(x, a) \le t d_X(x, b)$,

$$d_Y(f(x), f(a)) \le \eta(t) d_Y(f(x), f(b)).$$

A metric $n$-sphere $\mathbb{S}$ that is quasisymmetrically homeomorphic to $\mathbb{S}^n$ is called a *quasisymmetric sphere* when $n \geq 2$, and a *quasisymmetric circle* when $n = 1$.

A doubling metric circle $\gamma$ is called a *C-quasicircle* for some $C > 1$ if it satisfies the *2-point condition*:

$$\text{for all } x, y \in \gamma, \ \operatorname{diam} \gamma(x, y) \leq C|x - y|, \tag{2.1}$$

where $\gamma(x, y)$ denotes the subarc of $\gamma$ connecting $x$ and $y$ of smaller diameter, or either subarc when both have the same diameter. Tukia and Väisälä [9] proved that quasisymmetric circles are exactly the quasicircles. Ahlfors [1] showed that a planar Jordan curve $\gamma \subset \mathbb{R}^2$ is a $C$-quasicircle if and only if, there exists an $\eta$-quasisymmetric homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $\mathbb{S}^1$ onto $\gamma$.

A rectifiable metric circle or arc $\gamma$ is called a *c-chord-arc curve* for some $c > 1$ if

$$\text{for all } x, y \in \gamma, \ \ell(\gamma'(x, y)) \leq c|x - y|, \tag{2.2}$$

where $\gamma'(x, y)$ is the shortest component of $\gamma \setminus \{x, y\}$ and $\ell(\cdot)$ denotes length. It is easy to see that a rectifiable metric circle or arc $\gamma$ is a chord-arc curve if and only if $\ell(\gamma(x, y)) \leq c'|x - y|$ for some $c' > 1$; here constants $c$ and $c'$ are quantitatively related.

A Jordan arc is any proper subarc of a Jordan curve and a quasiarc is any proper subarc of a quasicircle. For the rest, all quasicircles, quasiarcs and chord-arc curves are assumed to be subsets of $\mathbb{R}^2$.

The notion of linear local connectivity generalizes the 2-point condition (2.1) on curves to general spaces. A metric space $X$ is *$\lambda$-linearly locally connected* (or $\lambda$–LLC) for $\lambda \geq 1$ if the following two conditions are satisfied.

1. ($\lambda$–LLC$_1$) If $x \in X$, $r > 0$ and $y_1, y_2 \in B(x, r) \cap X$, then there exists a continuum $E \subset B(x, \lambda r) \cap X$ containing $y_1, y_2$.
2. ($\lambda$–LLC$_2$) If $x \in X$, $r > 0$ and $y_1, y_2 \in X \setminus B(x, r)$, then there exists a continuum $E \subset X \setminus B(x, r/\lambda)$ containing $y_1, y_2$.

A metric space $X$ is said to be *Ahlfors Q-regular* if there is a constant $C > 1$ such that the $Q$-dimensional Hausdorff measure $\mathcal{H}^Q$ of every open ball $B(a, r)$ in $X$ satisfies

$$C^{-1} r^Q \leq \mathcal{H}^Q(B(a, r)) \leq C r^Q, \tag{2.3}$$

when $0 < r \leq 2 \operatorname{diam} X$.

Bonk and Kleiner found in [4] an intrinsic characterization of quasisymmetric 2-spheres and then derived a readily applicable sufficient condition.

**Theorem 2.1** ([4, Theorem 1.1, Lemma 2.5]). *Let X be an Ahlfors* 2*-regular metric space homeomorphic to* $\mathbb{S}^2$. *Then X is quasisymmetric to* $\mathbb{S}^2$ *if and only if X is LLC.*

For $x \in \mathbb{R}^n$ and $r > 0$, define $B^n(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$. For any $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, denote by

$$\pi(a) = (a_1, a_2, 0)$$

the projection of $a$ on the plane $\mathbb{R}^2 \times \{0\}$. For $x, y \in \mathbb{R}^2$, denote by $[x, y]$ the line segment having $x, y$ as its end points.

In the following, we write $u \lesssim v$ (resp. $u \simeq v$) when the ratio $u/v$ is bounded above (resp. bounded above and below) by positive constants. These constants may vary, but are described in each occurrence.

## 2.2 Geometry of level sets

For a planar Jordan domain $\Omega$ and some $\epsilon > 0$ define the $\epsilon$-level set

$$\gamma_\epsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) = \epsilon\}.$$

In general the sets $\gamma_\epsilon$ need not be connected and if connected need not be curves; see [11, Figure 1]. We say that $\Omega$ has the *level quasicircle property* (or LQC *property*), if there exist $\epsilon_0 > 0$ and $C \geq 1$ such that the level set $\gamma_\epsilon$ is a $C$-quasicircle for every $0 \leq \epsilon \leq \epsilon_0$. Sufficient conditions for a domain $\Omega$ to satisfy the LQC property have been given in [11] in terms of the *chordal flatness* of $\partial\Omega$, a scale-invariant parameter measuring the local deviation of subarcs from their chords.

We list some properties of the level sets from [11] which are used in the proof of Theorem 1.2.

**Lemma 2.2** ([11, Lemma 4.1]). *Suppose that $\Omega$ is a Jordan domain and for some $\epsilon > 0$ the set $\gamma_\epsilon$ is a Jordan curve. If $\sigma$ is a closed subarc of $\partial\Omega$ then the set $\sigma' = \{z \in \Omega\colon \mathit{dist}(z, \partial\Omega) = \mathit{dist}(z, \sigma) = \epsilon\}$ is either empty or a subarc of $\gamma_\epsilon$.*

**Lemma 2.3** ([11, Lemma 6.1]). *There is a universal constant $c_0 > 1$ such that if $\sigma$ is a Jordan arc and $\epsilon > 3\operatorname{diam}\sigma$ then the set $\sigma' = \{x \in \mathbb{R}^2\colon \mathit{dist}(x, \sigma) = \epsilon\}$ is a $c_0$-chord-arc curve.*

**Lemma 2.4** ([11, Theorem 1.3]). *Let $\Omega$ be a Jordan domain that has the LQC property. If $\partial\Omega$ is a c-chord-arc curve then there exist $\epsilon_0 > 0$ and $c' > 1$ such that $\gamma_\epsilon$ is a $c'$-chord-arc curve for all $\epsilon \in [0, \epsilon_0]$.*

# 3 A weak-chord arc property

## 3.1 A chord-arc index

Let $\Gamma$ be a Jordan curve or arc. A *partition* $\mathcal{P}$ of $\Gamma$ is a finite set of mutually disjoint (except for their endpoints) subarcs of $\Gamma$ whose union is all of $\Gamma$. We denote with $|\mathcal{P}|$ the number of elements a partition $\mathcal{P}$ has. A *$\delta$-partition* of $\Gamma$, for $\delta \in (0, 1)$, is a partition $\mathcal{P}$ of $\Gamma$ such that for each $\Gamma' \in \mathcal{P}$

$$\frac{\delta}{2}\operatorname{diam}\Gamma \leq \operatorname{diam}\Gamma' \leq \delta\operatorname{diam}\Gamma.$$

Standard compactness arguments show that every Jordan curve or arc has $\delta$-partitions for all $\delta \in (0, 1)$. Moreover, if $\mathcal{P}$ is a $\delta$-partition of $\Gamma$ then $|\mathcal{P}| \geq 1/\delta$. In the next lemma, we show that if $\Gamma$ is a quasiarc and $\mathcal{P}$ is a $\delta$-partition then, $|\mathcal{P}| \lesssim \delta^{-2}$.

**Lemma 3.1.** *For each $C > 1$ there exists a number $C' > 1$ depending only on $C$ such that, if $\Gamma$ is a $C$-quasiarc and $\lambda \in (0, 1]$ then, each partition $\{\Gamma_1, \dots, \Gamma_N\}$ with* $\operatorname{diam}\Gamma_i \geq \lambda\operatorname{diam}\Gamma$ *satisfies $N \leq C'/\lambda^2$.*

*Proof.* We may assume that $\operatorname{diam}\Gamma = 1$. Fix $\lambda \in (0, 1)$ and let $\{\Gamma_1, \dots, \Gamma_N\}$ be a partition of $\Gamma$ with $\operatorname{diam}\Gamma_i \geq \lambda$. Let $A = \{x_0, x_1, \dots, x_N\}$ be the set of endpoints of $\Gamma_1, \dots, \Gamma_N$. The 2-point condition of $\Gamma$ implies that $|x - y| \geq \lambda/C$ for all $x, y \in A$. Since all points of $A$ lie in $\mathbb{R}^2$ which is a doubling, space, there is a universal constant $C_0 > 1$ such that $N \leq C_0(\lambda/C)^{-2} = C_0C^2/\lambda^2$. □

For a partition $\mathcal{P}$ of $\Gamma$ define

$$M(\Gamma, \mathcal{P}) = \frac{1}{\operatorname{diam}\Gamma}\sum_{\Gamma' \in \mathcal{P}}\operatorname{diam}\Gamma'.$$

The finite number $M(\Gamma, \mathcal{P})$ measures, on the scale of $\mathcal{P}$, the deviation of $\Gamma$ from being a chord-arc curve. When $\ell(\Gamma)$ is finite, the number $M(\Gamma, \mathcal{P})$ is an estimation of $\ell(\Gamma)/\operatorname{diam}\Gamma$: for any $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $\Gamma$ such that $|M(\Gamma, \mathcal{P}) - \ell(\Gamma)/\operatorname{diam}\Gamma| < \epsilon$. On the other hand, if $\ell(\Gamma)$ is infinite then for all $M > 0$ there exists a partition $\mathcal{P}$ of $\Gamma$ such that $M(\Gamma, \mathcal{P}) > M$. The properties of the gauge $M(\Gamma, \mathcal{P})$ are summarized in the following lemma.

**Lemma 3.2.** *Suppose that $\Gamma$ is a Jordan curve or arc.*

1. *If $\mathcal{P}$ is a $\delta$-partition then $\frac{1}{2}|\mathcal{P}|\delta \leq M(\Gamma, \mathcal{P}) \leq |\mathcal{P}|\delta$.*

2. *Suppose that* $\mathcal{P} = \{\Gamma_1, \dots, \Gamma_N\}$ *is a partition of* $\Gamma$ *and for each* $i = 1, \dots, N$, $\mathcal{P}_i$ *is a partition of* $\Gamma_i$. *Then,*
$$M(\Gamma, \bigcup_{i=1}^{N} \mathcal{P}_i) \geq M(\Gamma, \mathcal{P}).$$
3. *Assume that* $\Gamma$ *is a* $C$*-quasiarc and* $\delta \in (0,1)$. *There exists* $M_1 > 1$ *depending on* $C$ *such that* $M(\Gamma, \mathcal{P})/M(\Gamma, \mathcal{P}') \leq M_1$ *for any two* $\delta$*-partitions* $\mathcal{P}, \mathcal{P}'$ *of* $\Gamma$.

*Proof.* Property *(1)* is an immediate consequence of the definition. For property *(2)* simply note that $\operatorname{diam} \Gamma_i \leq \sum_{\Gamma' \in \mathcal{P}_i} \Gamma'$. Finally, to show *(3)*, let $\mathcal{P} = \{\Gamma_1, \dots, \Gamma_N\}$ and $\mathcal{P}' = \{\Gamma'_1, \dots, \Gamma'_{N'}\}$ be two $\delta$-partitions of $\Gamma$. By Lemma 3.1 there exists $N_0 > 0$ depending on $C$ such that each $\Gamma_i$ can contain at most $N_0$ subarcs $\Gamma'_j$. Thus, $N' \leq N_0 N$ and *(3)* follows from *(1)*. □

In the next lemma we show that if a quasiarc resembles a chord-arc curve for partitions of small scale then it should also do that for partitions of larger scale.

**Lemma 3.3.** *Suppose that* $\Gamma$ *is a* $C$*-quasiarc,* $N_0 > 0$ *and* $\mathcal{P}$ *is a* $\delta$*-partition of* $\Gamma$ *with* $\delta \in (0,1)$ *and* $|\mathcal{P}| \leq N_0/\delta$. *There exists* $N_1$ *depending on* $C, N_0$ *such that for all* $\delta' \in [\delta, 1)$ *and all* $\delta'$*-partitions* $\mathcal{P}'$ *of* $\Gamma$ *we have* $|\mathcal{P}'| \leq N_1/\delta'$.

*Proof.* We prove the lemma for $N_1 = 5M_1N_0$ where $M_1$ is as in the third part of Lemma 3.2. Contrary to the claim, suppose that there exist a number $\delta' \in [\delta, 1)$ and a $\delta'$-partition $\mathcal{P}' = \{\Gamma_1, \dots, \Gamma_N\}$ such that $N > N_1/\delta'$. For each $i = 1, \dots, N$ consider a $\frac{\delta \operatorname{diam} \Gamma}{\operatorname{diam} \Gamma_i}$-partition $\mathcal{P}_i$ of $\Gamma_i$; for those $i$ that $\operatorname{diam} \Gamma_i \leq \delta \operatorname{diam} \Gamma$ set $\mathcal{P}_i = \{\Gamma'_i\}$. Note that $|\mathcal{P}_i| \geq \frac{\operatorname{diam} \Gamma_i}{\delta \operatorname{diam} \Gamma}$ and that $\mathcal{P}^\star = \bigcup_{i=1}^N \mathcal{P}_i$ is a $\delta$-partition of $\Gamma$. By Lemma 3.2,
$$M(\Gamma, \mathcal{P}^\star) \geq (|\mathcal{P}_1| + \cdots + |\mathcal{P}_N|)\frac{\delta}{2} > M_1 N_0.$$
But then $M(\Gamma, \mathcal{P}^\star)/M(\Gamma, \mathcal{P}) > M_1$ which is a contradiction. □

The following lemma is used in the proof of Theorem 1.3.

**Lemma 3.4.** *Let* $\Gamma$ *be a Jordan arc and* $0 < \delta < \delta' < 1$. *Suppose that each* $\delta$*-partition of* $\Gamma$ *has at least* $N$ *elements for some* $N > 1$. *Then, there exists a subarc* $\Gamma' \subset \Gamma$ *and a* $\delta'$*-partition* $\mathcal{P}'$ *of* $\Gamma'$ *with* $|\mathcal{P}'| \geq N^{\frac{\log \delta' + 1}{\log(\delta\delta'/2)}}$.

*Proof.* Contrary to the lemma, suppose that for each subarc $\sigma \subset \Gamma$, every $\delta'$-partition $\mathcal{P}$ of $\sigma$ has $|\mathcal{P}| \leq N^{\frac{\log \delta' + 1}{\log(\delta\delta'/2)}}$. We construct a partition of $\Gamma$ as follows.

Let $k_0 \in \mathbb{N}$ be the smallest integer that is greater than $\frac{\log(\delta/2)}{\log \delta'}$. Let $\mathcal{P}_0 = \{\Gamma_1, \dots, \Gamma_{N_0}\}$ be a $\delta'$-partition of $\Gamma$. For each $i = 1, \dots, N_0$ let $\mathcal{P}_i = \{\Gamma_{i1}, \dots, \Gamma_{iN_i}\}$ be a $\delta'$-partition of $\Gamma_i$. Inductively, suppose that $\Gamma_w$ has been defined where $w = i_1 i_2 \dots i_k$, $i_j \in \mathbb{N}$ and $k < k_0$. Then let $\mathcal{P}_w = \{\Gamma_{w1}, \dots, \Gamma_{wN_w}\}$ be a $\delta'$-partition of $\Gamma_w$.

Define $\mathcal{P}^\star = \bigcup_w \mathcal{P}_w$ where the union is taken over all words $w = i_1 \dots i_{k_0}$. Note that for each $\Gamma_w \in \mathcal{P}^\star$,
$$\left(\frac{\delta'}{2}\right)^{k_0} \leq \frac{\operatorname{diam} \Gamma_w}{\operatorname{diam} \Gamma} \leq (\delta')^{k_0} < \frac{\delta}{2}.$$
Moreover, by our assumptions, for all partitions $\mathcal{P}_w$ defined,
$$|\mathcal{P}_w| \leq N^{\frac{\log \delta' + 1}{\log(\delta\delta'/2)}}.$$
Consequently, since $k_0 \leq \frac{\log(\delta/2)}{\log \delta'} + 1$,
$$|\mathcal{P}^\star| \leq \left(N^{\frac{\log \delta' + 1}{\log(\delta\delta'/2)}}\right)^{k_0} < \left(N^{\frac{\log \delta'}{\log(\delta\delta'/2)}}\right)^{k_0} \leq N.$$

If $\mathcal{P}$ is a $\delta$-partition of $\Gamma$ then, for each $\Gamma' \in \mathcal{P}$ and each $\Gamma_w \in \mathcal{P}^\star$, $\operatorname{diam} \Gamma_w < \operatorname{diam} \Gamma'$. Thus, $|\mathcal{P}| \leq |\mathcal{P}^\star| < N$ which is a contradiction. □

### 3.2 A weak chord-arc property

Note that if a curve $\Gamma$ is a chord-arc curve then there exists $M_0 > 1$ such that for all subarcs $\Gamma' \subset \Gamma$, all $\delta > 0$ and all $\delta$-partitions $\mathcal{P}$ of $\Gamma'$, we have $M(\Gamma', \mathcal{P}) < M_0$.

*A curve or arc $\Gamma$ is said to have the weak chord-arc property if there exists $M_0 > 0$ such that for all subarcs $\Gamma' \subset \Gamma$ with $\operatorname{diam} \Gamma' < 1$, there exists a $\operatorname{diam} \Gamma'$-partition $\mathcal{P}$ of $\Gamma'$ satisfying $M(\Gamma', \mathcal{P}) \leq M_0$.*

In other words, a curve $\Gamma$ is a weak chord-arc curve if there exists $M_0 \geq 1$ such that any subarc $\Gamma'$ of $\Gamma$ can be partitioned to at most $2M_0/\operatorname{diam} \Gamma'$ subarcs $\Gamma_1, \dots, \Gamma_N$ of diameters comparable to $(\operatorname{diam} \Gamma')^2$. It is clear that a chord-arc curve is a weak chord-arc curve but the converse fails; see Section 5.1. The third claim of Lemma 3.2 implies that, for quasicircles, the weak chord-arc property is equivalent to a stronger condition.

**Lemma 3.5.** *Suppose that $\Gamma$ is a $C$-quasicircle that satisfies the weak chord-arc property with constant $M_0 > 1$. For each $\alpha \in (0,1)$ there exists $M_\alpha > 1$ depending only on $C, M_0, \alpha$ such that for all subarcs $\Gamma' \subset \Gamma$ of diameter less than 1 and all $(\operatorname{diam} \Gamma')^{\frac{1}{\alpha}-1}$-partitions $\mathcal{P}$ we have $M(\Gamma', \mathcal{P}) \leq M_\alpha$.*

*Proof.* If $\alpha \in [\frac{1}{2}, 1)$ then the claim follows immediately from Lemma 3.3 with $M_\alpha = 10M_0M_1$ where $M_1$ is as in Lemma 3.2.

Suppose now that $\alpha \in (0, \frac{1}{2})$ and let $k_0$ be the smallest integer with $2^{-k_0} \leq \alpha$. Fix a subarc $\Gamma_0 \subset \Gamma$ with $\operatorname{diam} \Gamma_0 < 1$ and a $\operatorname{diam} \Gamma_0$-partition $\mathcal{P}_0 = \{\Gamma_1, \dots, \Gamma_{N_0}\}$ of $\Gamma_0$ with $|\mathcal{P}_0| \leq 2M_0$. For each $i = 1, \dots, N_0$ let $\mathcal{P}_i$ be a $\frac{(\operatorname{diam} \Gamma_0)^4}{\operatorname{diam} \Gamma_i}$-partition of $\Gamma_i$. The weak chord-arc property of $\Gamma_i$ and Lemma 3.3 imply $|\mathcal{P}_i| \leq 10M_0M_1(\operatorname{diam} \Gamma_0)^{-2}$. Inductively, suppose that $\Gamma_w$ has been defined where $w = i_1 \cdots i_k$, $i_j \in \mathbb{N}$ and $k < k_0$. Then, let $\mathcal{P}_w = \{\Gamma_{w1}, \dots, \Gamma_{wN_w}\}$ be a $\frac{(\operatorname{diam} \Gamma_0)^{2^{k+1}}}{\operatorname{diam} \Gamma_w}$-partition of $\Gamma_w$. Again, note that $|\mathcal{P}_w| \leq 10M_0M_1(\operatorname{diam} \Gamma_0)^{-2^k}$.

Define $\mathcal{P}^\star = \bigcup_w \mathcal{P}_w$ where the union is taken over all words $w = i_1 \cdots i_{k_0}$ constructed as above. Note that $\mathcal{P}^\star$ is a $(\operatorname{diam} \Gamma_0)^{2^{k_0}-1}$-partition of $\Gamma_0$ with

$$M(\Gamma_0, \mathcal{P}^\star) \leq \frac{(\operatorname{diam} \Gamma_0)^{2^{k_0}}}{\operatorname{diam} \Gamma_0} \sum_{w=i_1\cdots i_{k_0}} |\mathcal{P}_w| \leq (10M_0M_1)^{k_0}.$$

Since $(\operatorname{diam} \Gamma_0)^{1/\alpha} \geq (\operatorname{diam} \Gamma_0)^{2^{k_0}} \geq \operatorname{diam} \Gamma_w$ for all $\Gamma_w \in \mathcal{P}^\star$, it follows from Lemma 3.3 that every $(\operatorname{diam} \Gamma')^{\frac{1}{\alpha}-1}$-partition $\mathcal{P}$ of $\Gamma_0$ satisfies $M(\Gamma_0, \mathcal{P}) \leq M_\alpha$ with $M_\alpha = (10M_0M_1)^{k_0+1}$. □

By interchanging the roles of $\frac{1}{2}$ and $\alpha$, and applying the arguments in the proof of Lemma 3.5, the following converse can be obtained.

**Remark 3.6.** *Suppose that $\Gamma$ is a $C$-quasicircle and that there exist $\alpha \in (0,1)$ and $M_\alpha > 1$ such that every subarc $\Gamma' \subset \Gamma$ with $\operatorname{diam} \Gamma' < 1$ has a $(\operatorname{diam} \Gamma')^{\frac{1}{\alpha}-1}$-partition $\mathcal{P}$ satisfying $M(\Gamma', \mathcal{P}) \leq M_\alpha$. Then $\Gamma$ has the weak chord-arc property.*

## 4 Proofs of the main results

The proof of Theorem 1.2 is given in Section 4.1 and Section 4.2. By the theorem of Bonk and Kleiner, it is clear that *(2)* implies *(1)*.

To show that *(3)* implies *(2)* we first show in Lemma 4.6 that if $\Omega$ satisfies the LQC property then $\Sigma(\Omega, t^\alpha)$ is LLC for all $\alpha \in (0,1)$. Then, in Proposition 4.1 we prove that if $\Omega$ has the LQC property and $\partial\Omega$ is a weak chord-arc curve then $\Sigma(\Omega, t^\alpha)$ is 2-regular.

To prove that *(1)* implies *(3)*, we show in Lemma 4.6 that if $\Sigma(\Omega, t^\alpha)$ is LLC for some $\alpha \in (0,1)$ then $\Omega$ satisfies the LQC property. Then, in Proposition 4.7 we show that if $\Omega$ has the LQC property and $\Sigma(\Omega, t^\alpha)$ is quasisymmetric to $\mathbb{S}^2$ then $\partial\Omega$ is a weak chord-arc curve.

The proof of Theorem 1.3 is given in Section 4.3.

## 4.1 Ahlfors 2-regularity

The following proposition connects the weak chord-arc property of $\partial\Omega$ with the 2-regularity of $\Sigma(\Omega, t^\alpha)$.

**Proposition 4.1.** *Suppose that $\Omega$ has the LQC property, $\partial\Omega$ has the weak chord-arc property and $\alpha \in (0, 1)$. Then, $\Sigma(\Omega, t^\alpha)$ is 2-regular.*

For the rest of Section 4.1 we assume that there exist $C_0 > 1$ and $\epsilon_0 \in (0, \frac{1}{8C_0})$ such that for any $\epsilon \in [0, \epsilon_0]$, the level set $\gamma_\epsilon$ is a $C_0$-quasicircle. To show (2.3) we first apply some reductions on $a \in \Sigma(\Omega, t^\alpha)$ and $r > 0$.

*Reduction 1.* Let $\Sigma(\Omega, t^\alpha)^+ = \Sigma(\Omega, t^\alpha) \cap \{x \in \mathbb{R}^3 : x_3 \geq 0\}$. If $a \in \Sigma(\Omega, t^\alpha)^+$ and $r > 0$ then by symmetry of $\Sigma(\Omega, t^\alpha)$ with respect to $\mathbb{R}^2 \times \{0\}$ we have

$$\mathcal{H}^2(B^3(a, r) \cap \Sigma(\Omega, t^\alpha)^+) \leq \mathcal{H}^2(B^3(a, r) \cap \Sigma(\Omega, t^\alpha)) \leq 2\mathcal{H}^2(B^3(a, r) \cap \Sigma(\Omega, t^\alpha)^+).$$

Therefore, it is enough to verify

$$C^{-1}r^2 \leq \mathcal{H}^2(B^3(a, r) \cap \Sigma(\Omega, t^\alpha)^+) \leq Cr^2, \tag{4.1}$$

for some $C > 1$ and for all $a \in \Sigma(\Omega, t^\alpha)^+$ and $r \leq 2 \operatorname{diam} \Sigma(\Omega, t^\alpha)$.

*Reduction 2.* Let $r_0 = \min\{\epsilon_0/3, \frac{1}{180C_0^2} \operatorname{diam} \Omega\}$. By the doubling property of $\mathbb{R}^3$, there exists $c_1 > 1$ depending only on $\epsilon_0$, $\operatorname{diam} \Omega$ and $\alpha$ such that each ball $B^3(a, r) \cap \Sigma(\Omega, t^\alpha)^+$ with $r \geq r_0$ can be covered by at most $c_1$ balls $B^3(a_i, r_i) \cap \Sigma(\Omega, t^\alpha)^+$ with $a_i \in \Sigma(\Omega, t^\alpha)^+$ and $r_i < r_0$. Hence, it suffices to show (4.1) for $r \leq r_0$.

*Reduction 3.* We claim that it is enough to verify (4.1) only for those points $a \in \Sigma(\Omega, t^\alpha)^+$ whose projection satisfies $\operatorname{dist}(\pi(a), \partial\Omega) \leq \epsilon_0$. Indeed, following the notation in [12], let $\Delta_{\epsilon_0/3}$ be the set of all points in $\Omega$ whose distance from $\partial\Omega$ is greater than $\epsilon_0/3$ and $\Delta^+_{\epsilon_0/3}$ be the subset of $\Sigma(\Omega, t^\alpha)^+$ whose projection on $\mathbb{R}^2 \times \{0\}$ is $\Delta_{\epsilon_0/3}$. Since $\partial\Delta_{\epsilon_0/3} = \gamma_{\epsilon_0/3}$ is a quasicircle, the domain $\Delta_{\epsilon_0/3}$ is 2-regular. Therefore, the surface $\Delta^+_{\epsilon_0/3}$, which is the graph of a Lipschitz function on $\Delta_{\epsilon_0/3}$, is 2-regular as well. Thus, if $a \in \Sigma(\Omega, t^\alpha)^+$ satisfies $\operatorname{dist}(\pi(a), \partial\Omega) > \epsilon_0$ and $r \in (0, \epsilon_0/3)$ then $B^3(a, r) \cap \Sigma(\Omega, t^\alpha)^+ \subset \Delta^+_{\epsilon_0/3}$ and $\mathcal{H}^2(B^3(a, r) \cap \Sigma(\Omega, t^\alpha)^+) \simeq r^2$.

*Reduction 4.* We show in Lemma 4.2 that it is enough to check the Hausdorff 2-measure of certain subsets on $\Sigma(\Omega, t^\alpha)$ defined as follows. Suppose that $x_1, y_1, x_2, y_2$ are points in $\Sigma(\Omega, t^\alpha)^+$ satisfying the following properties:

(i) $\pi(x_1), \pi(y_1) \in \gamma_{t_1}$ and $\pi(x_2), \pi(y_2) \in \gamma_{t_2}$ for some $0 \leq t_2 < t_1 \leq \epsilon_0$,
(ii) $|\pi(x_1) - \pi(x_2)| = \operatorname{dist}(\pi(x_1), \gamma_{t_2}) = |\pi(y_1) - \pi(y_2)| = \operatorname{dist}(\pi(y_1), \gamma_{t_2}) = t_1 - t_2$,
(iii) $\frac{1}{20C_0}|x_1 - y_1| \leq t_1 - t_2 + t_1^\alpha - t_2^\alpha \leq \frac{1}{3}|x_1 - y_1| \leq \frac{1}{10C_0} \operatorname{diam} \Omega$.

Property (i) implies that for each $i = 1, 2$, $x_i$ is on the same horizontal plane as $y_i$. Property (ii) implies that $\pi(x_2), \pi(y_2)$ are the points of $\gamma_{t_2}$ which are closest to $\pi(x_1), \pi(y_1)$ respectively. Property (iii) implies that $|x_i - y_j|$, $|x_1 - x_2|$ and $|y_1 - y_2|$ are all comparable and sufficiently small. If $x_0, y_0 \in \partial\Omega$ are the closest points to $\pi(x_1), \pi(y_1)$ respectively, then $|\pi(x_2) - x_0| = |\pi(y_2) - y_0| = t_2$. Therefore, $\pi(x_1), \pi(x_2), x_0$ are colinear and the line segment $[\pi(x_1), \pi(x_2)]$ joining the points $\pi(x_1)$ and $\pi(x_2)$ is entirely in $\Omega$. Similarly, $[\pi(y_1), \pi(y_2)] \subset \Omega$.

Denote with $D = D(x_1, y_1, x_2, y_2)$ the subset on $\Sigma(\Omega, t^\alpha)^+$ whose projection on $\mathbb{R}^2$ is the quadrilateral bounded by $[\pi(x_1), \pi(x_2)]$, $\gamma_{t_1}(\pi(x_1), \pi(y_1))$, $[\pi(y_1), \pi(y_2)]$, $\gamma_{t_2}(\pi(x_2), \pi(y_2))$. We call $D$ a *square piece* on $\Sigma(\Omega, t^\alpha)^+$.

The following lemma is a corollary of [12, (6.4)] and [12, Remark 6.1].

**Lemma 4.2.** *There exist $C_1, C_2 > 1$ depending on $C_0, \epsilon_0, \operatorname{diam} \Omega$ such that*

1. $\operatorname{diam} D \leq C_1|x_1 - y_1|$ *for every square piece* $D = D(x_1, y_1, x_2, y_2)$,
2. *for all $a \in \Sigma(\Omega, t^\alpha)^+$ and $r > 0$ as above, there exist square pieces $D_1$ and $D_2$ such that* $1 < \operatorname{diam} D_2 / \operatorname{diam} D_1 \leq C_2$ *and*

$$D_1 \subset B^3(a, r) \cap \Sigma(\Omega, t^\alpha)^+ \subset D_2.$$

Thus, by (iii), choosing $\epsilon_0$ small enough, we may assume from now on that $\operatorname{diam} D \le (4C_0)^{-1}$ for all square pieces $D$. By Lemma 4.2 and the discussion above, Proposition 4.1 is now equivalent to the following lemma.

**Lemma 4.3.** *There exists $C > 1$ depending on $C_0$ and the weak chord-arc constant $M_0$ such that*

$$C^{-1}(\operatorname{diam} D)^2 \le \mathcal{H}^2(D) \le C(\operatorname{diam} D)^2 \tag{4.2}$$

*for all square pieces $D$ on $\Sigma(\Omega, t^\alpha)^+$ defined as above.*

The lower bound in (4.2) was shown in [12, Section 6.2.1]. For the upper bound, the following lemma is used; we only give a sketch of its proof since it is similar to the discussion in [12, Section 6.2.1].

**Lemma 4.4.** *Let $S$ be a closed subset of $\Sigma(\Omega, t^\alpha)^+$ and $0 \le t_2 \le t_1$ be such that $\pi(S)$ intersects with $\gamma_t$ if and only if $t \in [t_2, t_1]$. Suppose that, for all $t, t' \in [t_2, t_1]$, the Hausdorff distance between $\pi(S) \cap \gamma_t$ and $\pi(S) \cap \gamma_{t'}$ is less than $c_1|t - t'|$ for some $c_1 > 1$, and $\pi(S) \cap \gamma_t$ is a $c_2$-chord-arc curve for some $c_2 > 1$. Then $\mathcal{H}^2(S) \le C(\operatorname{diam} S)^2$ for some $C$ depending on $c_1, c_2$.*

*Proof.* Fix $\epsilon > 0$. Let $t_2 = \tau_1 < \cdots < \tau_N = t_1$ be such that the sets $S_i = S \cap (\gamma_{\tau_i} \times \{\tau_i^\alpha\})$ satisfy $\epsilon/4 \le \operatorname{dist}(S_i, S_{i+1}) \le \epsilon/2$. By the first assumption of the lemma, it is straightforward to check that $N \le N_1 \operatorname{diam} S/\epsilon$ for some $N_1$ depending on $c_1$. The second assumption implies that each $S_i$ contains points $x_{i,1}, \dots, x_{i,n_i}$ satisfying $|x_{i,j} - x_{i,j+1}| \le \epsilon/2$ and $n_i \le N_2 \operatorname{diam} S_i/\epsilon \le N_2 \operatorname{diam} S/\epsilon$ for some $N_2$ depending on $c_2$. Thus, $S$ can be covered by at most $N_1 N_2 (\operatorname{diam} S)^2/\epsilon^2$ balls of radius $\epsilon$ and the lemma follows. □

For the upper bound of (4.2) we consider two cases. The first case is an application of Lemma 4.4 while in the second case we use the weak chord-arc condition to subdivide $D$ into smaller pieces on which the first case applies.

*Proof of Lemma 4.3.* As mentioned already, the lower bound of (4.2) follows from the discussion in [12, Section 6.2.1] even without the weak chord-arc assumption of $\partial\Omega$. It remains to show the upper bound. By Lemma 3.3, there exists $M_\alpha > 1$ such that for any subarc $\Gamma \subset \partial\Omega$ with $\operatorname{diam} \Gamma < 1$ and for any $(\operatorname{diam} \Gamma)^{\frac{1}{\alpha}-1}$-partition $\mathcal{P}$ of $\Gamma$ we have $M(\Gamma, \mathcal{P}) \le M_\alpha$.

Fix a square piece $D = D(x_1, y_1, x_2, y_2)$ where $x_1, y_1, x_2, y_2$ satisfy equations (i)-(iii). Let $x_0, y_0 \in \partial\Omega$ be such that

$$|x_0 - \pi(x_1)| = |y_0 - \pi(y_1)| = t_1 \text{ and } |x_0 - \pi(x_2)| = |y_0 - \pi(y_2)| = t_2$$

and set $\Gamma_0$ to be the subarc of $\partial\Omega$, of smaller diameter, with endpoints $x_0, y_0$. The choice of $\epsilon_0$ implies $\operatorname{diam} \Gamma_0 \le C_0|x_0 - y_0| \le C_0(\operatorname{diam} D + 2t_1) \le C_0(\operatorname{diam} D + 2\epsilon_0) < 1/2$.

*Case 1.* Suppose that $(\operatorname{diam} \Gamma_0)^{\frac{1}{\alpha}} \le t_2/10$. We claim that there is $C > 1$ depending on $C_0, M_\alpha$ such that for each $\epsilon \in [t_2, t_1]$, the arc $\gamma_\epsilon \cap \pi(D)$ is a $C$-chord-arc curve. Assuming the claim, the upper bound of (4.2) follows from Lemma 4.4. To prove the claim, it suffices to show that $\gamma_{t_2} \cap \pi(D)$ is a $C$-chord-arc curve. Then, using the fact that for each $\epsilon > t_2$, the set $\gamma_\epsilon$ is the $(\epsilon - t_2)$-level set of $\gamma_{t_2}$, the claim follows from Lemma 2.4. Let $\sigma$ be a subarc of $\gamma_{t_2} \cap D$ and $x, y$ be the endpoints of $\sigma$. Since $\gamma_{t_2}$ is a quasicircle, it suffices to show that there exists $C > 1$ such that $\ell(\sigma) \le C \operatorname{diam} \sigma$. Let $x', y' \in \Gamma_0$ be such that $|x - x'| = |y - y'| = t_2$ and set $\sigma' = \Gamma_0(x', y')$.

If $\operatorname{diam} \sigma' \le t_2/10$ then, by Lemma 2.3, $\sigma$ is a $c_0$-chord-arc curve for some universal $c_0 > 1$.

If $\operatorname{diam} \sigma' \in [t_2/10, 10t_2]$ then apply Lemma 3.1 to get a $10^{-2}$-partition $\mathcal{P} = \sigma'_1, \dots, \sigma'_N$ of $\sigma'$ with $N$ bounded above by a positive constant depending on $C_0$. For each $n = 1, \dots, N$ let $\sigma_n$ be the set of all points $z$ in $\sigma$ that satisfy $\operatorname{dist}(z, \sigma'_n) = t_2$. By Lemma 2.2, $\{\sigma_n\}$ are subarcs of $\sigma$, perhaps not mutually disjoint. Thus, $\sigma$ is a union of $N$ $c_0$-chord-arc curves, hence $\ell(\sigma) \le c_1 \operatorname{diam} \sigma$ for some $c_1 > 1$ depending on $C_0$.

Finally, if $\operatorname{diam} \sigma' > 10t_2$ note that $\operatorname{diam} \sigma \simeq \operatorname{diam} \sigma'$. Since $(\operatorname{diam} \sigma')^{\frac{1}{\alpha}} \le (\operatorname{diam} \Gamma_0)^{\frac{1}{\alpha}} \le t_2/10$, Lemma 3.3 and the weak chord-arc property of $\sigma'$ imply that there exists a $(\frac{t_2}{10 \operatorname{diam} \sigma'})$-partition $\mathcal{P} = \{\sigma'_1, \dots, \sigma'_N\}$ of $\sigma'$ with $N \le N_0 \operatorname{diam} \sigma'/t_2$ and $N_0$ depending on $C_0, M_\alpha$. Define $\sigma_1, \dots, \sigma_N$ as above and note that each $\sigma_n$ is a

$c_0$-chord-arc curve and satisfies $\operatorname{diam}\sigma_n \le \operatorname{diam}\sigma'_n + t_2 \lesssim t_2$. Hence,

$$\ell(\sigma) \le \sum_{n=1}^{N} \ell(\sigma_n) \le \sum_{n=1}^{N} c_1 \operatorname{diam}\sigma_n \lesssim Nt_2 \lesssim \operatorname{diam}\sigma' \simeq \operatorname{diam}\sigma.$$

*Case 2.* Suppose that $(\operatorname{diam}\Gamma_0)^{\frac{1}{\alpha}} > t_2/10$. Let $H = 10^{2/\alpha}$. For a subarc $\Gamma' \subset \Gamma_0$ define $D(\Gamma')$ to be the set of all points $(z, (\operatorname{dist}(z,\partial\Omega))^{\alpha})$ such that $z \in \overline{\Omega}$, $\operatorname{dist}(z,\partial\Omega) = \operatorname{dist}(z,\Gamma')$ and

$$0 \le \operatorname{dist}(z,\partial\Omega) \le H(\operatorname{diam}\Gamma')^{1/\alpha}.$$

Note that $\operatorname{diam} D(\Gamma') \le 2(H^{\alpha}\operatorname{diam}\Gamma' + H(\operatorname{diam}\Gamma')^{1/\alpha} + \operatorname{diam}\gamma_{t_1}(x_1,y_1)) \lesssim \operatorname{diam}\Gamma'$. From the middle inequality of (iii), it is easy to see that $t_1 < H(\operatorname{diam}\Gamma_0)^{1/\alpha}$ and thus $D \subset D(\Gamma_0)$. Let

$$E_0 = \{x \in D(\Gamma_0) \colon \operatorname{dist}(\pi(x),\partial\Omega) \ge 10(\operatorname{diam}\Gamma_0)^{2/\alpha}\}$$

and note that $\operatorname{diam} E_0 \le \operatorname{diam} D(\Gamma_0) \lesssim \operatorname{diam}\Gamma_0$.

We claim that $\mathcal{H}^2(E_0) \le C(\operatorname{diam}\Gamma_0)^2$ for some $C > 1$ depending on $C_0, M_\alpha$. As in *Case 1*, it is enough to show that $\gamma = \gamma_t \cap \pi(E_0)$ is a chord-arc curve when $t = 10(\operatorname{diam}\Gamma_0)^{\frac{2}{\alpha}}$. The claim then follows from Lemma 2.4 and Lemma 4.4. Let $\sigma \subset \gamma$ and define $\sigma' \subset \Gamma_0$ as in *Case 1*. If $\operatorname{diam}\sigma' \le 10t$ then the claim follows from Lemma 2.3 and Lemma 3.3 as in *Case 1*. If $\operatorname{diam}\sigma' \ge 10t$ then let $\{\sigma'_1,\dots,\sigma'_m\}$ be a $(\operatorname{diam}\sigma')^{\frac{1}{\alpha}-1}$-partition of $\sigma'$ with $m \le 2M_\alpha$. For each $i = 1,\dots,m$ let $\{\sigma'_{i1},\dots,\sigma'_{iN_i}\}$ be a $(t(\operatorname{diam}\sigma')^{-\frac{1}{\alpha}})$-partition of $\sigma'_i$. Define subarcs $\{\sigma_{ij}\}$ of $\sigma$ as in *Case 1*. Since $\operatorname{diam}\sigma'_{ij} \le t < 10t$, each $\sigma_{ij}$ is a $c_1$-chord-arc curve for some $c_1 > 1$ depending on $C_0$. Moreover, by the weak chord-arc property and Lemma 3.5, $N_1 + \dots + N_m \le 10M_1M_\alpha^2 \operatorname{diam}\sigma'/t$ where $M_1$ is as in Lemma 3.2. Since $\operatorname{diam}\sigma' \simeq \operatorname{diam}\sigma$,

$$\ell(\sigma) \le \sum_{i,j} \ell(\sigma_{ij}) \le c_1 \sum_{i,j} \operatorname{diam}\sigma_{ij} \lesssim t\sum_{i=1}^{m} N_i \lesssim \operatorname{diam}\sigma' \lesssim \operatorname{diam}\sigma.$$

and the claim follows.

Let $\mathcal{P}_1 = \{\Gamma_1,\dots,\Gamma_N\}$ be a $(\operatorname{diam}\Gamma_0)^{\frac{1}{\alpha}-1}$-partition of $\Gamma_0$ with $N \le 2M_\alpha(\operatorname{diam}\Gamma_0)^{1-\frac{1}{\alpha}}$. Define $D(\Gamma_i)$, $E_i$ as above and note that the choice of $H$ yields $D(\Gamma_0) \subset E_0 \cup \bigcup_{i=1}^{N} D(\Gamma_i)$. Similarly as above, $\mathcal{H}^2(E_i) \le C(\operatorname{diam}\Gamma_0)^{\frac{2}{\alpha}}$.

For each $i = 1,\dots,N$ let $\{\Gamma_{i1},\dots,\Gamma_{iN_i}\}$ be a $\frac{(\operatorname{diam}\Gamma_0)^{1/\alpha^2}}{\operatorname{diam}\Gamma_i}$-partition of $\Gamma_i$ and set $\mathcal{P}_2 = \{\Gamma_{ij}\}$. The weak chord-arc condition and Lemma 3.1 imply that $N_i \le 10M_\alpha M_1(\operatorname{diam}\Gamma_i)^{1-\frac{1}{\alpha}} \le 10M_\alpha M_1 2^{\frac{1}{\alpha}-1}(\operatorname{diam}\Gamma_0)^{\frac{1}{\alpha}-\frac{1}{\alpha^2}}$ where $M_1$ is as in Lemma 3.2. Thus, $|\mathcal{P}_2| \le (m_0 2^{\frac{1}{\alpha}-1})^2(\operatorname{diam}\Gamma_0)^{1-\frac{1}{\alpha^2}}$ with $m_0 = 10M_\alpha M_1$. Again, the choice of $H$ yields

$$D(\Gamma_0) \subset E_0 \cup \bigcup_{i=1}^{N} E_i \cup \bigcup_{i=1}^{N}\bigcup_{j=1}^{N_i} D(\Gamma_{ij}).$$

Inductively, we obtain partitions $\mathcal{P}_k = \{\Gamma_{i_1\cdots i_k}\}$ with

$$|\mathcal{P}_k| \le (m_0 2^{\frac{1}{\alpha}-1})^k(\operatorname{diam}\Gamma_0)^{1-\frac{1}{\alpha^k}}$$

and square-like pieces $E_{i_1\cdots i_k}$ with

$$D(\Gamma_0) \subset E_0 \cup \bigcup_{i} E_i \cup \bigcup_{i_1,i_2} E_{i_1,i_2} \cup \dots \cup \bigcup_{\Gamma_{i_1\cdots i_k}\in\mathcal{P}_k} D(\Gamma_{i_1\cdots i_k})$$

and $\mathcal{H}^2(E_{i_1\cdots i_k}) \le C(\operatorname{diam}\Gamma_0)^{\frac{2}{\alpha^k}}$. Therefore, since $D \subset D(\Gamma_0)$,

$$\begin{aligned}\mathcal{H}^2(D) &\le \mathcal{H}^2(E_0) + \sum_{i}\mathcal{H}^2(E_i) + \sum_{i_1,i_2}\mathcal{H}^2(E_{i_1,i_2}) + \dots\\ &\lesssim (\operatorname{diam}\Gamma_0)^2\sum_{k=0}^{\infty}(m_0 2^{\frac{1}{\alpha}-1})^k(\operatorname{diam}\Gamma_0)^{\frac{1}{\alpha^k}-1}\\ &\le (\operatorname{diam}\Gamma_0)^2\sum_{k=0}^{\infty}\frac{(m_0 2^{\frac{1}{\alpha}-1})^k}{2^{\frac{1}{\alpha^k}-1}}.\end{aligned}$$

It is easy to see that the latter series converges. Since $\operatorname{diam}\Gamma_0 \le \operatorname{diam} D$, we conclude that $\mathcal{H}^2(D) \lesssim (\operatorname{diam} D)^2$ and the proof is complete. □

## 4.2 The LQC property and Väisälä's method

The connection between the LQC property of $\Omega$ and the LLC property of $\Sigma(\Omega, t^\alpha)$ is established in the following proposition from [12].

**Proposition 4.5** ([12, Proposition 5.1, Lemma 5.6]). *Suppose that $\alpha \in (0, 1)$ and $\Omega$ is a Jordan domain whose boundary $\partial\Omega$ is a quasicircle. Then $\Sigma(\Omega, t^\alpha)$ is LLC if and only if $\Omega$ has the LQC property.*

It turns out that the quasicircle assumption of $\partial\Omega$ can be dropped in Proposition 4.5.

**Lemma 4.6.** *Suppose that $\Omega$ is a Jordan domain and $\alpha \in (0, 1)$. Then, $\Sigma(\Omega, t^\alpha)$ is LLC if and only if $\Omega$ has the LQC property.*

*Proof.* Suppose that $\Sigma = \Sigma(\Omega, t^\alpha)$ is $\lambda$-LLC for some $\lambda > 1$. We show that $\partial\Omega$ is a $(4\lambda^2)$-quasicircle and the proof follows then from Proposition 4.5.

Let $x, y \in \partial\Omega$ and $\gamma, \gamma'$ be the two components of $\partial\Omega \setminus \{x, y\}$. By the $\lambda - \mathrm{LLC}_1$ property, the points $x, y$ are contained in a continuum $E$ in $B^3(x, 2\lambda|x-y|) \cap \Sigma$. Then the projection $\pi(E)$ is a continuum in $\overline{\Omega}$ containing $x, y$. Suppose that there exist points $z, z'$ in $\gamma, \gamma'$ respectively, which lie outside of $B^3(x, 2\lambda^2|x-y|)$. The $\lambda - \mathrm{LLC}_2$ property implies that there exists a continuum $E' \subset \Sigma \setminus B^3(x, 2\lambda|x-y|)$ that contains $z, z'$. But then, the projection $\pi(E')$ is a continuum in $\overline{\Omega}$ containing $z, z'$. It follows that $\pi(E) \cap \pi(E') \neq \emptyset$ and, since $\Sigma$ is symmetric with respect to $\mathbb{R}^2 \times \{0\}$, $E'$ intersects $B^3(x, 2\lambda|x-y|)$ which is a contradiction. Therefore, at least one of $\gamma, \gamma'$ lies in $\overline{B}^3(x, 2\lambda^2|x-y|)$ and $\min\{\operatorname{diam}\gamma, \operatorname{diam}\gamma'\} \leq 4\lambda^2|x-y|$. □

We now show that the weak chord-arc condition of $\partial\Omega$ is necessary for $\Sigma(\Omega, t^\alpha)$ to be quasisymmetric to $\mathbb{S}^2$. This concludes the proof of Theorem 1.2. The proof follows closely that of [12, Proposition 6.2]. The main idea used is due to Väisälä from [10].

**Proposition 4.7.** *Suppose that $\Omega$ has the LQC property and $\alpha \in (0, 1)$. If $\Sigma(\Omega, t^\alpha)$ is quasisymmetric to $\mathbb{S}^2$ then $\partial\Omega$ is a weak chord-arc curve.*

*Proof.* By our assumptions, there exist $\epsilon_0 > 0$ and $C > 1$ such that, for all $\epsilon \in [0, \epsilon_0]$, the set $\gamma_\epsilon$ satisfies (2.1) with constant $C$. Set $\partial\Omega = \Gamma$.

Suppose that the claim is false. Then, by Remark 3.6, for each $n \in \mathbb{N}$, there exists a subarc $\Gamma_n \subset \Gamma$, of diameter less than 1, and a $(\operatorname{diam}\Gamma_n)^{\frac{1}{\alpha}-1}$-partition $\mathcal{P}_n = \{\Gamma_{n,1}, \dots, \Gamma_{n,N_n}\}$ with $M(\Gamma_n, \mathcal{P}_n) > 4Cn$. By Lemma 3.2, the latter implies that

$$N_n > 4Cn(\operatorname{diam}\Gamma_n)^{1-\frac{1}{\alpha}}.$$

Let $\{x_{n,0}, x_{n,1}, \dots, x_{n,N_n}\}$ be the endpoints of the arcs $\Gamma_{n,1}, \dots, \Gamma_{n,N_n}$, ordered consecutively according to the orientation in $\Gamma_n$ with $x_{n,0}, x_{n,N_n}$ being the endpoints of $\Gamma_n$. For these points, (2.1) implies that

$$\frac{1}{2C}(\operatorname{diam}\Gamma_n)^{\frac{1}{\alpha}} \leq |x_{n,i} - x_{n,i-1}| \leq (\operatorname{diam}\Gamma_n)^{\frac{1}{\alpha}}.$$

By adding more points from each subarc $\Gamma_{n,i}$ in this collection, if necessary, we may further assume that

$$\frac{1}{4C}(\operatorname{diam}\Gamma_n)^{\frac{1}{\alpha}} \leq |x_{n,i} - x_{n,i-1}| \leq \frac{1}{2C}(\operatorname{diam}\Gamma_n)^{\frac{1}{\alpha}}.$$

The existence of these new points follows easily from the doubling property of $\mathbb{R}^2$ and the 2-point condition of $\Gamma$. As we may have increased the number of points, it follows that

$$\sum_{i=1}^{N_n} |x_{n,i-1} - x_{n,i}| \geq N_n \frac{1}{4C}(\operatorname{diam}\Gamma_n)^{\frac{1}{\alpha}} > n \operatorname{diam}\Gamma_n.$$

Set $d_n = (10C^2)^{-1} \min_{1 \leq i \leq N_n} |x_{n,i} - x_{n,i-1}|$ and note that $d_n^\alpha \simeq \operatorname{diam}\Gamma_n$.

The rest of the proof is identical to that of [12, Proposition 5.1] by setting $\varphi(t) = t^\alpha$ therein. We sketch the remaining steps for the sake of completeness.

Fix $n \in \mathbb{N}$ and write $N_n = N$ and $x_{n,i} = x_i$. Assume that there exists an $\eta$-quasisymmetric map $F$ from $\Sigma(\Omega, t^\alpha)$ onto $\mathbb{S}^2$. Composing $F$ with a suitable Möbius map we may assume that $F(\Sigma(\Omega, t^\alpha)^+)$ is contained in the unit disc $\mathbb{B}^2$. Since $\partial\Omega$ is a quasicircle, we can find points $w_0, \dots, w_N$ on $\Gamma$ and points $w'_0, \dots, w'_N$ on $\gamma_d = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) = d\}$ which follow the orientation of $\{x_0, \dots, x_N\}$ such that $|w_i - x_i| \in [d, 3C_0 d]$ and $|w_i - w'_i| = d$. Hence, $|w_i - w_{i+1}| \simeq d$. Let $D$ be the square-like piece on $\Sigma(\Omega, t^\alpha)^+$ whose projection on $\mathbb{R}^2 \times \{0\}$ is the Jordan domain bounded by $\Gamma(w_0, w_N)$, $\gamma_d(w'_0, w'_N)$, $[w_0, w'_0]$, $[w_N, w'_N]$.

The partition of $\Gamma(w_0, w_N)$ into the subarcs $\Gamma(w_{i-1}, w_i)$, $i = 1, \dots, N$ induces a partition of $D$ into $N$ tall and narrow strips $D_i$ with height in the magnitude $d^\alpha$ and width in the magnitude $d$. Each $D_i$ is further partitioned by planes parallel to $\mathbb{R}^2 \times \{0\}$ into $k$ square-like pieces $D_{ij}$ with $k \simeq d^{\alpha-1}$. The quasisymmetry of $F$ implies that $(\operatorname{diam} F(D_{ij}))^2 \leq c_1 \mathcal{H}^2(D_{ij})$ with $c_1 > 1$ depending on $\eta, C$. Summing first over $j$ and then over $i$, and applying Hölder's inequality twice, we obtain $(\operatorname{diam} F(D))^2 \leq Nkc_1\mathcal{H}^2(D) \leq c_2 n \mathcal{H}^2(D)$ with $c_2 > 1$ depending on $\eta, C$. On the other hand, the quasisymmetry of $F$ on $D$ implies that $\mathcal{H}^2(F(D)) \leq c_3 (\operatorname{diam} F(D))^2$ with $c_3 > 1$ depending on $\eta, C$. Since $\operatorname{diam} F(D) \neq 0$, letting $n \to \infty$, we obtain a contradiction. □

## 4.3 Assouad dimension

The Assouad dimension of a metric space $(X, d)$, introduced in [2], is the infimum of all $s > 0$ that satisfy the following property: there exists $C > 1$ such that for any $Y \subset X$ and $\delta \in (0, 1)$, the set $Y$ can be covered by at most $C\delta^{-s}$ subsets of diameter at most $\delta \operatorname{diam} Y$. In a sense, the main difference between Hausdorff and Assouad dimension of a space $X$ is that that the former is related to the average small scale structure of $X$, while the latter measures the size of $X$ in all scales. The doubling metric spaces are exactly the metric spaces of finite Assouad dimension. See [6] for a detailed survey of the concept.

**Remark 4.8.** *If $X$ is a quasicircle, then all sets in the definition of the Assouad dimension can be replaced by subarcs of $X$.*

The claim of the remark becomes evident after noticing that for all subsets $Y$ of a $C$-quasicircle $\Gamma$ there exists a subarc $\Gamma' \subset \Gamma$ containing $Y$, such that $\operatorname{diam} \Gamma' \leq C \operatorname{diam} Y$.

We now turn to the proof of Theorem 1.3.

*Proof of Theorem 1.3.* Suppose that $\Gamma$ is a $C$-quasicircle with Assouad dimension greater than 1; in particular, greater than $1 + \epsilon$ for some fixed $\epsilon \in (0, 1)$. We claim that $\Gamma$ does not have the weak chord-arc property.

Contrary to the claim, assume that $\Gamma$ satisfies the weak chord-arc condition for some $M_0 > 1$. By Lemma 3.3 there exists $N_0 > 1$ depending on $M_0$, $C$ such that for all $\Gamma' \subset \Gamma$ with $\operatorname{diam} \Gamma' < 1$ and all $\operatorname{diam} \Gamma'$-partitions $\mathcal{P}$ of $\Gamma'$ we have $|\mathcal{P}| < N_0$. By our assumption on the Assouad dimension of $\Gamma$, there exists a subarc $\Gamma' \subset \Gamma$ and a number $\delta \in (0, 1)$ such that all $\delta$-partitions $\mathcal{P}$ of $\Gamma'$ satisfy $|\mathcal{P}| \geq M\delta^{-1-\epsilon}$ where $M = 10N_0M_1M_\alpha$, $M_1 > 1$ is as in the third claim of Lemma 3.2, $M_\alpha > 1$ is the number in Lemma 3.5 associated to $\alpha = \frac{1-\sqrt{\beta}}{1+\sqrt{\beta}}$ and $\beta = \frac{1+\epsilon/2}{1+\epsilon}$. The subarc $\Gamma'$ can be chosen small enough so that $\operatorname{diam} \Gamma' < \min\{(2M_0M_1)^{-2/\epsilon}, e^{-1/(1-\sqrt{\beta})}\}$.

*Case 1.* Suppose that $\operatorname{diam} \Gamma' \leq 2\delta^{(1-\sqrt{\beta})/\sqrt{\beta}}$. Assume first that $\delta > \operatorname{diam} \Gamma'$. Let $\mathcal{P}$ be a $\delta$-partition of $\Gamma'$. The weak chord-arc property of $\Gamma'$ and Lemma 3.3 yield $|\mathcal{P}| \leq N_0\delta^{-1} < M\delta^{-1-\epsilon}$ which is false.

Assume now that $\delta \leq \operatorname{diam} \Gamma'$. Let $\alpha' \in (0, 1)$ be such that $(\operatorname{diam} \Gamma')^{\frac{1}{\alpha'}-1} = \delta$. The assumption on $\operatorname{diam} \Gamma'$, $\delta$ and the fact that $\operatorname{diam} \Gamma' < 1/2$ yield $\alpha' \geq \frac{1-\sqrt{\beta}}{1+\sqrt{\beta}} = \alpha$. By Lemma 3.3 and Lemma 3.5, every $\delta$-partition $\mathcal{P}$ of $\Gamma'$ satisfies $|\mathcal{P}| \leq 10M_\alpha M_1\delta^{-1} < M\delta^{-1-\epsilon}$ which is also false.

*Case 2.* Suppose that $\operatorname{diam}\Gamma' > 2\delta^{(1-\sqrt{\beta})/\sqrt{\beta}}$. By our assumptions on $\delta$ and $\operatorname{diam}\Gamma'$ we have

$$\begin{aligned}\frac{\log(\operatorname{diam}\Gamma')+1}{\log(\frac{\delta}{2}\operatorname{diam}\Gamma')} &= \frac{\log(\operatorname{diam}\Gamma')+1}{\log(\operatorname{diam}\Gamma')}\,\frac{\log\delta}{\log(\frac{\delta}{2}\operatorname{diam}\Gamma')}\,\frac{\log(\operatorname{diam}\Gamma')}{\log\delta}\\ &\geq \beta\frac{\log(\operatorname{diam}\Gamma')}{\log\delta}.\end{aligned}$$

Apply Lemma 3.4 for $\Gamma'$ with $\delta' = \operatorname{diam}\Gamma$ and $N = \delta^{-1-\epsilon}$. There exists a subarc $\Gamma'' \subset \Gamma'$ and a $\operatorname{diam}\Gamma'$-partition $\mathcal{P}'$ such that

$$|\mathcal{P}'| \geq \delta^{(-1-\epsilon)\beta\frac{\log(\operatorname{diam}\Gamma')}{\log\delta}} \geq (\operatorname{diam}\Gamma')^{-(1+\epsilon/2)}.$$

We create a $\operatorname{diam}\Gamma''$-partition of $\Gamma''$ as follows. For each $\sigma \in \mathcal{P}'$ let $\mathcal{P}(\sigma)$ be a $\frac{(\operatorname{diam}\Gamma'')^2}{\operatorname{diam}\sigma}$-partition of $\sigma$; for those $\sigma \in \mathcal{P}'$ that satisfy $\operatorname{diam}\sigma < (\operatorname{diam}\Gamma'')^2$ set $\mathcal{P}(\sigma) = \{\sigma\}$. Define $\mathcal{P}''$ to be the union of all partitions $\mathcal{P}(\sigma)$. It is easy to see that $\mathcal{P}''$ is a $\operatorname{diam}\Gamma''$-partition of $\Gamma''$ and Lemma 3.2 gives

$$M(\Gamma'',\mathcal{P}'') \geq M(\Gamma'',\mathcal{P}') \geq \frac{1}{2}(\operatorname{diam}\Gamma')^{-\epsilon/2} > M_0M_1.$$

The latter, however, is false by Lemma 3.2 and the proof is complete. □

# 5 Examples from homogeneous snowflakes

Let $N \geq 4$ be a natural number and $p \in (1/4, 1/2)$. A homogeneous $(N,p)$-snowflake in $\mathbb{R}^2$ is constructed as follows. Let $S_0$ be a regular $N$-gon, of diameter equal to $1/2$. At the $n$-th step, the polygon $S_{n+1}$ is constructed by replacing *all* of the $N4^n$ edges of $S_n$ by the *same* rescaled and rotated copy of one of the two polygonal arcs of Figure 1, in such a way that the polygonal regions are expanding. The curve $\mathcal{S}$ is obtained by taking the limit of $S_n$, just as in the construction of the usual von Koch snowflake. It is easy to verify that every homogeneous snowflake satisfies (2.1) for some $C$ depending on $N, p$, and as a result is a quasicircle. See [8] for relevant results.


**Figure 1**

Let $E$ be an edge of some $S_n$ towards the construction of $\mathcal{S}$. Denote with $\mathcal{S}_E$ the subarc of $\mathcal{S}$, of smaller diameter, having the same endpoints as $E$. The next lemma will ease some of the computations in the rest.

**Lemma 5.1.** *A homogeneous $(N,p)$-snowflake $\mathcal{S}$ is a weak chord-arc curve if and only if there exists $M^* > 1$ such that every subarc $\mathcal{S}_E$ has a* $\operatorname{diam}\mathcal{S}_E$*-partition $\mathcal{P}$ with $M(\mathcal{S}_E,\mathcal{P}) \leq M_0$.*

*Proof.* The necessity is clear. For the sufficiency, fix a subarc $\Gamma \subset \mathcal{S}$ and let $n_0$ be the greatest integer $n$ for which $\Gamma$ is contained in $\mathcal{S}_E$ for some edge $E$ of $S_n$. Assume for now that $n_0 > 0$. Denote by $E_i$, $i = 1,\dots,4$, the oriented four segments constructed after $E$ in the $n_0+1$ step, that is $\mathcal{S}_E = \bigcup_{i=1}^4 \mathcal{S}_{E_i}$. Inductively, if $w = i_1\cdots i_k$ with $i_j \in \{1,\dots,4\}$, let $E_{wi}$, be the oriented four segments constructed after $E_w$ in the $n_0+k+1$ step.

Suppose that $\Gamma$ contains a subarc $\mathcal{S}_{E_i}$, $i = 1,\dots,4$. Then $\operatorname{diam}\Gamma \geq \frac{1}{4}\operatorname{diam}\mathcal{S}_E = \frac{1}{4}\operatorname{diam}E$. Let $\mathcal{P}$ be a $(\frac{1}{4}\operatorname{diam}E)$-partition of $\mathcal{S}_E$ and $\mathcal{P}' = \{\Gamma'\cap\Gamma\colon \Gamma' \in \mathcal{P}\}$. The weak chord-arc property of $\mathcal{S}_E$ and Lemma 3.2 imply that there exists $N_0 > 0$ such that $|\mathcal{P}'| \leq |\mathcal{P}| \leq N_0(\operatorname{diam}\Gamma)^{-1}$. Since $\operatorname{diam}\Gamma' \leq (\operatorname{diam}\Gamma)^2$ for each $\Gamma' \in \mathcal{P}'$ it follows that $\Gamma$ has the weak chord-arc property for some $M^*$ depending on $N_0$.

Suppose now that $\Gamma$ contains none of the $\mathcal{S}_{E_i}$, $i = 1, \dots, 4$. Then, there exist $i \in \{1, 2, 3\}$ and maximal integers $r, q \geq 0$ such that $\Gamma \subset \mathcal{S}_{E_{i4^q}} \cup \mathcal{S}_{E_{(i+1)1^r}}$. In this case, apply the arguments above for $\Gamma \cap \mathcal{S}_{E_{i4^q}}$ and $\Gamma \cap \mathcal{S}_{E_{(i+1)1^r}}$.

If $n_0 = 0$ then apply the arguments above for $\Gamma \cap \mathcal{S}_{E_1}, \dots, \Gamma \cap \mathcal{S}_{E_N}$ where $E_1, \dots, E_N$ are the edges of $S_0$. □

## 5.1 A non-rectifiable Jordan curve that satisfies the weak chord-arc property

Let $\mathcal{S}$ be the homogeneous $(N, p)$-snowflake where the first polygonal arc in Figure 1 is used only at the $10^n$-th steps, $n \in \mathbb{N}$. We also require that $\operatorname{diam} \mathcal{S} < (4p)^{-1}$. Note that at the $10^k$ step of construction, the length of the polygonal curve $S_{10^k}$ is equal to $(4p)^k$. Thus, $\mathcal{S}$ is not rectifiable. We claim that $\mathcal{S}$ has the weak chord-arc property.

By Lemma 5.1, it suffices to check that all subarcs $\mathcal{S}_E$ are weak chord-arc curves. Fix an edge $E$ built at step $n$. Then $\operatorname{diam} E \geq 4^{-n}$. Let $k_0$ be the smallest $k \in \mathbb{N}$ such that $\operatorname{diam} E' \leq (\operatorname{diam} E)^2$ for all $E' \in S_{n+k}$. We claim that $k_0 \leq 9n$. Indeed, the construction of $\mathcal{S}$ implies that at step $10n$, each edge has diameter equal to $(4p)4^{-9n} \operatorname{diam} E \leq (4p)(\operatorname{diam} E)^{10}$ since the first polygonal arc in Figure 1 has been used only once. The claim follows from our assumption that $\operatorname{diam} E \leq \operatorname{diam} \mathcal{S} < (4p)^{-1}$.

Let $\mathcal{P}$ be the set of all subarcs $\mathcal{S}_{E'}$ where $E'$ are constructed at step $n + k_0$ and have $E$ as their common parent. Then, $\operatorname{diam} E' = 4^{-k_0} C \operatorname{diam} E$ with $C = 4p$ if the first polygonal arc has been used in the $k_0$ steps or $C = 1$ otherwise. Since $k_0$ is minimal,

$$\frac{1}{4}(\operatorname{diam} \mathcal{S}_E)^2 \leq \operatorname{diam} \mathcal{S}_{E'} \leq (\operatorname{diam} \mathcal{S}_E)^2.$$

Therefore, $|\mathcal{P}| = 4^{k_0} \leq 4C(\operatorname{diam} E)^{-1} = 4C(\operatorname{diam} \mathcal{S}_E)^{-1}$. By Lemma 3.1, there exists a $\operatorname{diam} \mathcal{S}_E$-partition of $\mathcal{S}_E$ that has at most $C'(\operatorname{diam} \mathcal{S}_E)^{-1}$ elements, for some $C' > 1$ depending on $N, p$. Hence, $\mathcal{S}_E$ has the weak chord-arc property.

**Corollary 5.2.** *There exists a Jordan domain $\Omega$ with nonrectifiable boundary such that $\Sigma(\Omega, t^\alpha)$ is a quasisymmetric sphere for all $\alpha \in (0, 1]$.*

*Proof.* It follows from the discussion in Section 7 of [11] that there exists $p_0 \in (\frac{1}{4}, \frac{1}{2})$ and an integer $N_0 \geq 4$ such that every homogeneous $(N, p)$-snowflake with $p \leq p_0$, $N \geq N_0$ bounds a domain that satisfies the LQC property. Let $\Omega$ be the domain bounded by the snowflake constructed above with $p \leq p_0$, $N \geq N_0$. Since $\mathcal{S}$ is a quasicircle, $\Sigma(\Omega, t)$ is a quasisymmetric sphere. Moreover, the weak chord-arc property of $\mathcal{S}$ and Theorem 1.2 imply that $\Sigma(\Omega, t^\alpha)$ is a quasisymmetric sphere for all $\alpha \in (0, 1)$. □

## 5.2 A quasicircle of Assouad dimension 1 that does not satisfy the weak chord-arc property

Let $\mathcal{S}$ be the homogeneous $(N, p)$-snowflake where the first polygonal arc in Figure 1 is used only at the $n^2$-th steps for $n \in \mathbb{N}$. For convenience we also assume that each edge of $S_0$ has length equal to 1. Then, if $E$ is an edge of $S_n$, $\operatorname{diam} E = 4^{-n}(4p)^{\lfloor\sqrt{n}\rfloor}$ where, $\lfloor x \rfloor$ denotes the greatest integer which is smaller than $x$.

We show that $\mathcal{S}$ has Assouad dimension equal to 1 but does not have the weak chord-arc property.

Fix $\epsilon > 0$; we claim that $\mathcal{S}$ has Assouad dimension less than $1+\epsilon$. Similarly to Lemma 5.1, it is easy to show that it is enough to verify the Assouad condition only for the subarcs $\mathcal{S}_E$. Take $\delta \in (0, 1)$ and an edge $E$ of the $n$-th step polygon $S_n$, for some $n \in \mathbb{N}$. Let $m$ be the largest integer such that $\operatorname{diam} E' \geq \delta \operatorname{diam} E$ for all edges $E'$ of $S_m$ and $\mathcal{P}$ be the set of all subarcs $\mathcal{S}_{E'} \subset \mathcal{S}_E$ where $E'$ is an edge of $S_m$. Then, $4^{n-m}(4p)^{\lfloor\sqrt{m}\rfloor - \lfloor\sqrt{n}\rfloor} \geq \delta$ and

$$(m - n) \log 4 - (\sqrt{m} - \sqrt{n}) \log 4p \leq -\log \delta + \log 4p \tag{5.1}$$

By elementary calculus, there exists $M > 0$ depending on $\epsilon$, $p$ such that $\sqrt{x}-\sqrt{y} \leq \frac{\epsilon}{1+\epsilon}\frac{\log 4}{\log 4p}(x-y)$ for all $x > M$ and $0 < y < x$. If $m < M$ then clearly $|\mathcal{P}| = 4^{m-n} \leq 4^M < 4^M\delta^{-1-\epsilon}$. If $m \geq M$ then $(m-n)\log 4-(\sqrt{m}-\sqrt{n})\log 4p \geq \frac{\log 4}{1+\epsilon}(m-n)$ and by (5.1)

$$(m-n)\log 4 \leq \log \delta^{-1-\epsilon} + 2\log 4p.$$

Therefore, $|\mathcal{P}| = 4^{m-n} \leq 4^M(4p)^2\delta^{-1-\epsilon}$ and the claim follows. Since $\epsilon$ was chosen arbitrarily, $\mathcal{S}$ has Assouad dimension equal to 1.

We show now that $\mathcal{S}$ does not have the weak chord-arc property. Let $n \in \mathbb{N}$, $E$ be an edge of the $n$-th step polygon $S_n$ and $m \geq n$ be the greatest integer such that $\operatorname{diam} E' \geq (\operatorname{diam} E)^2$ for each edge $E'$ of $S_m$. Let $\mathcal{P}$ be the set of all subarcs $\mathcal{S}_{E'} \subset \mathcal{S}_E$ where $E'$ are edges of $S_m$. Then,

$$\left(4^{-n}(4p)^{\sqrt{n}-1}\right)^2 \leq (\operatorname{diam} E)^2 \leq \operatorname{diam} E' \leq 4^{-m}(4p)^{\sqrt{m}}$$

which yields

$$\frac{1}{2}-\frac{n}{m} \leq \frac{\log(4p)}{\sqrt{m}\log 4}\left(\frac{1}{2}-\sqrt{\frac{n}{m}}\right)+\frac{\log 4p}{m\log 4}.$$

Note that as $n$ goes to infinity, $m$ goes to infinity and $n/m$ goes arbitrarily close to $1/2$. Hence, $m > \frac{25}{16}n$ for all sufficiently large $n$. Therefore,

$$M(\mathcal{S}_E,\mathcal{P}) \simeq |\mathcal{P}|\frac{\operatorname{diam}\mathcal{S}_{E'}}{\operatorname{diam}\mathcal{S}_E} \geq (4p)^{\sqrt{m}-\sqrt{n}-1} \gtrsim (4p)^{\sqrt{m}-\sqrt{n}} \geq (4p)^{\sqrt{n}/4}$$

which goes to infinity as $n$ goes to infinity. Thus $\mathcal{S}$ is not a weak chord-arc curve.

**Acknowledgement:** The author is grateful to Pekka Pankka, Kai Rajala and his adviser Jang-Mei Wu for stimulating discussions and to the anonymous referee for valuable comments on the manuscript. The research work was supported by the Academy of Finland project 257482.

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