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L_2-variation of Lévy driven BSDEs with non-smooth terminal conditions

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We consider the L_2-regularity of solutions to backward stochastic differential equations (BSDEs) with Lipschitz generators driven by a Brownian motion and a Poisson random measure associated with a Lévy process $(X_t)_{t\in[0,T]}$. The terminal condition may be a Borel function of finitely many increments of the Lévy process which is not necessarily Lipschitz but only satisfies a fractional smoothness condition. The results are obtained by investigating how the special structure appearing in the chaos expansion of the terminal condition is inherited by the solution to the BSDE.

Keywords: backward stochastic differential equations; Chaos expansion; L_2-regularity; Lévy processes; Malliavin calculus

1. Introduction

The main objective of this paper consists in studying the relation between fractional smoothness of the terminal condition of a BSDE and the L_2-variation of its according solution.

A motivation to investigate this relation arises from the fact that the convergence rate of the discretization error of BSDEs with Lipschitz generator is determined by the convergence of the discretized terminal condition to its limit and by the L_2-variation properties of the exact solution (Y, Z).

In the Brownian scenario, the discretization of BSDEs has been studied by many authors, see, for example, [4,10,11,14,27,28,36]. If the BSDE is given by

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s, \qquad 0 \le t \le T,$$

we define the L_p-variation

$$\operatorname{var}_p(\xi, F, \tau) := \sup_{1\le i\le n}\ \sup_{t_{i-1}<s\le t_i} \|Y_s - Y_{t_{i-1}}\|_p + \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|Z_t - \hat{Z}_{t_{i-1}}\|_p^2\,\mathrm{d}t\right)^{1/2},$$

where $\tau = (t_i)_{i=0}^n$ is a deterministic time-net $0 = t_0 < \cdots < t_n = T$ and

$$\hat{Z}_{t_{i-1}} := \frac{1}{t_i - t_{i-1}} \mathbb{E}\left[\int_{t_{i-1}}^{t_i} Z_s\,\mathrm{d}s \,|\, \mathcal{F}_{t_{i-1}}\right].$$

Gobet and Makhlouf [21] considered L_2-regularity of (Y, Z) for a terminal condition given by $\xi = g(X_T)$ where g does not need to be Lipschitz and X denotes the forward process. In [14], the L_p-regularity of (Y, Z) for $p \geq 2$ was shown if the terminal condition depends on the forward process at finitely many time points, $\xi = g(X_{r_1}, \dots, X_{r_m})$, and satisfies a path-dependent Malliavin fractional smoothness condition which is weaker than the Lipschitz condition on g. Using these results and following the ideas of [11], one can show that the convergence rate of the error between the discretizations (Y^τ, Z^τ) and the solution (Y, Z) is of order $\frac{1}{2}$, that is

$$\mathrm{Err}_{\tau,2}(Y, Z) := \left\{ \sup_{0<t\leq T} \mathbb{E}\left|Y_t - Y_t^\tau\right|^2 + \int_0^T \mathbb{E}\left|Z_t - Z_t^\tau\right|^2 \mathrm{d}t \right\}^{1/2} \leq c|\tau|^{1/2}$$

provided that the time grid for the discretization is chosen in an appropriate way (like in [14]), and the discretized terminal condition converges in this order. Without any assumptions on the dependence of the terminal condition ξ on a forward process X, Hu, Nualart and Song [22] have shown the convergence rate $\frac{1}{2}$ supposing Malliavin differentiability properties of ξ (and of the generator).

For a BSDE driven by a Poisson random measure, Bouchard and Elie [9] proved that the convergence rate is of order $\frac{1}{2}$ (in the non-degenerate case) if the terminal condition is given by $\xi = g(X_T)$ where g is Lipschitz.

Here we study whether the L_2-variation would allow to achieve the convergence rate $\frac{1}{2}$ with a terminal condition $\xi = g(X_{r_1}, \dots, X_{r_m})$ and whether the Lipschitz condition on g can be weakened to a Malliavin fractional smoothness condition. The method we use allows to answer this question in the case where X is the Lévy process itself.

In the Brownian setting, in case of a zero generator it is stated in [20], relation (8), that the rate $\frac{1}{2}$ is the best possible as long as ξ can not be represented as a linear function of W_T. Moreover, in [20], Theorem 3.5, it is shown that for equidistant grids there is a direct connection between the convergence rate and the index of fractional smoothness of the terminal condition. In [15], Theorems 5 and 6, the same results are recovered for W replaced by a square integrable Lévy process X, even if the Lévy process does not have a Brownian part.

The paper is organised as follows. In Section 2, we describe the setting and recall some needed facts. In Section 3 we recall some basic facts about Malliavin calculus in the Lévy setting. Furthermore, we state a result about Malliavin differentiability of the solution (Y, Z) to a BSDE. Our method to show the L_2-regularity of solutions to BSDEs exploits the fact that their Malliavin derivative is piece-wise constant in time. This is ensured by selecting a terminal condition which has this property. For this purpose, we introduce a space of suitable terminal conditions and investigate the chaos expansion of the according solution in Section 4. Section 5 contains our main result, equivalences and implications concerning the L_2-regularity of (Y, Z). An important fact, which will be considered in Section 6, is a sufficient condition for the L_2-regularity of the solution: a certain Malliavin fractional smoothness of the terminal condition (in addition to our standing assumption that the generator is Lipschitz).

2. Setting

Let $X = (X_t)_{t\in[0,T]}$ be an L_2-Lévy-process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Lévy-measure ν. We will denote the augmented natural filtration of X by $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ and let $L_2 := L_2(\Omega, \mathcal{F}_T, \mathbb{P})$.

The Lévy–Itô decomposition of an L_2-Lévy-process X can be written as

$$X_t = \gamma t + \sigma W_t + \int_{]0,t]\times(\mathbb{R}\setminus\{0\})} x\tilde{N}(\mathrm{d}s, \mathrm{d}x), \tag{1}$$

where $\sigma \geq 0$, W is a Brownian motion and $\tilde{N}$ is the compensated Poisson random measure corresponding to X.

We define the random measure M by

$$M(\mathrm{d}t, \mathrm{d}x) := \sigma\,\mathrm{d}W_t\delta_0(\mathrm{d}x) + x\tilde{N}(\mathrm{d}t, \mathrm{d}x). \tag{2}$$

Then $\mathbb{E}M(B)^2 = \int_B \mathbb{m}(\mathrm{d}t, \mathrm{d}x)$ for $B \in \mathcal{B}([0,T]\times\mathbb{R})$ where

$$\mathbb{m}(\mathrm{d}t, \mathrm{d}x) = (\lambda \otimes \mu)(\mathrm{d}t, \mathrm{d}x)$$

with

$$\mu(\mathrm{d}x) = \sigma^2\delta_0(\mathrm{d}x) + x^2\nu(\mathrm{d}x)$$

and λ denotes the Lebesgue measure. For $0 \leq t \leq T$, we consider the BSDE

$$Y_t = \xi + \int_t^T F(s, Y_s, \bar{Z}_s)\,\mathrm{d}s - \int_{]t,T]\times\mathbb{R}} Z_{s,x} M(\mathrm{d}s, \mathrm{d}x), \tag{3}$$

where

$$\bar{Z}_s = \int_{\mathbb{R}} Z_{s,x}\kappa(\mathrm{d}x)$$

and $\kappa(\mathrm{d}x) := \kappa'(x)\mu(\mathrm{d}x)$ such that $\kappa' \in L_2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$. We will use the following assumptions:

(A_ξ) $\xi \in L_2$,

(A_F) $F : \Omega \times [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is jointly measurable, adapted to $(\mathcal{F}_t)_{t\in[0,T]}$, Lipschitz-continuous in the last two variables, uniformly in ω and t, that is,

$$\left|F(t, y_1, z_1) - F(t, y_2, z_2)\right| \leq L_f\left(|y_1 - y_2| + |z_1 - z_2|\right), \tag{4}$$

and satisfies

$$\mathbb{E}\int_0^T \left|F(t, 0, 0)\right|^2 \mathrm{d}t < \infty.$$

For later use, we introduce spaces of stochastic processes.

Definition 2.1. 1. *Let S denote the space of all $(\mathcal{F}_t)$-adapted and càdlàg processes $(y_t)_{0\le t\le T}$ such that*

$$\|y\|_S^2 := \mathbb{E} \sup_{0\le t\le T} |y_t|^2 < \infty.$$

2. *We define H as the space of all random fields $z : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ which are measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ (where $\mathcal{P}$ denotes the predictable σ-algebra on $\Omega \times [0,T]$ generated by the left-continuous $\mathbb{F}$-adapted processes) such that*

$$\|z\|_H^2 := \mathbb{E} \int_{[0,T]\times\mathbb{R}} |z_{s,x}|^2 \mathbb{m}(\mathrm{d}s, \mathrm{d}x) < \infty.$$

The space $S \times H$ is equipped with the norm $\|(y,z)\|_{S\times H} := (\|y\|_S^2 + \|z\|_H^2)^{1/2}$.

A pair $(Y,Z) \in S \times H$ which satisfies (3) is called a solution to the BSDE (3).

Theorem 2.2. *Assume (ξ, F) satisfies (A_ξ) and (A_F). Then the BSDE (3) has a unique solution $(Y,Z) \in S \times H$.*

This assertion can be found in Tang and Li [35], Lemma 2.4 and in Barles, Buckdahn and Pardoux [5], Theorem 2.1. We next cite the stability result of [5] comparing the distance between solutions to the BSDE (3) with different terminal conditions and generators.

Theorem 2.3 ([5], Proposition 2.2). *Let (ξ, F) and (ξ', F') satisfy (A_ξ) and (A_F). For the corresponding solutions (Y,Z) and (Y',Z') to (3), it holds*

$$\begin{aligned}&\|Y - Y'\|_S^2 + \|Z - Z'\|_H^2 \\ &\quad \le C\left(\|\xi - \xi'\|_{L_2}^2 + \int_0^T \|F(s, Y_s, \bar{Z}_s) - F'(s, Y_s, \bar{Z}_s)\|_{L_2}^2 \mathrm{d}s\right),\end{aligned}$$

where $C = C(T, L_{F'}, \mu)$.

Remark 2.4. According to [34], Proposition 3 (see also [29], Proposition 3 or [2], Lemma 2.2) for any $z \in L_2(\mathbb{P} \otimes \mathbb{m})$ there exists a process

$$^p z \in L_2\big(\Omega \times [0,T] \times \mathbb{R}, \mathcal{P} \otimes \mathcal{B}(\mathbb{R}), \mathbb{P} \otimes \mathbb{m}\big)$$

such that for any fixed $x \in \mathbb{R}$ the function $(^p z)_{\cdot,x}$ is a version of the predictable projection (in the classical sense) of $z_{\cdot,x}$. In the following, we will always use this result to get predictable projections which are measurable w.r.t. a parameter.

3. Malliavin differentiability of (Y, Z)

We will use the Malliavin derivative which is defined via chaos expansions, that is series of multiple stochastic integrals. Following Itô [23], for $n \ge 1$ we define elementary functions of the type

$$f_n\big((t_1,x_1),\dots,(t_n,x_n)\big)=\sum_{k=1}^m a_k\prod_{i=1}^n \mathbb{1}_{B_i^k}(t_i,x_i),$$

where $a_k\in\mathbb{R}$, and for all k the sets $B_i^k\in\mathcal{B}([0,T]\times\mathbb{R}), i=1,\dots,n$ are disjoint and satisfy $\mathbb{m}(B_i^k)<\infty$. The multiple stochastic integral I_n of order n of the elementary function f_n with respect to the random measure M is defined by

$$I_n(f_n):=\sum_{k=1}^m a_k\prod_{i=1}^n M\big(B_i^k\big).$$

Since the elementary functions given above are dense in $L_2^n:=L_2(([0,T]\times\mathbb{R})^n,\mathbb{m}^{\otimes n})$, by linearity and continuity of I_n its domain extends to L_2^n. For $n=0$, we set $L_2^n:=\mathbb{R}$ and $I_0(f_0):=f_0$ for $f_0\in\mathbb{R}$. The properties of I_n are very similar to those in the Brownian case, especially it holds

$$I_n(f_n)=I_n(\tilde{f}_n),$$

where $\tilde{f}_n$ denotes the symmetrization of f with respect to the n pairs $(t_1,x_1),\dots,(t_n,x_n)$. Moreover,

$$\mathbb{E}I_n(f_n)I_m(g_m)=\begin{cases}\langle\tilde{f}_n,\tilde{g}_n\rangle_{L_2^n}, & n=m,\\ 0, & n\neq m.\end{cases}$$

Any $G\in L_2$ has a chaos expansion

$$G=\sum_{n=0}^\infty I_n(f_n)$$

which is unique if symmetric $f_n\in L_2^n$ are used (which we will be our standing assumption from now on), and it holds

$$\|G\|^2:=\|G\|_{L_2}^2=\sum_{n=0}^\infty n!\|f_n\|_{L_2^n}^2.$$

For example, for X_s from (1) we have

$$X_s=I_0(\gamma s)+I_1(\mathbb{1}_{[0,s]\times\mathbb{R}})=\gamma s+M\big([0,s]\times\mathbb{R}\big). \tag{5}$$

A straightforward generalisation of [30], Lemma 1.2.5, implies ($\mathbb{E}_t$ stands for the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t]$)

$$\mathbb{E}_tG=\sum_{n=0}^\infty I_n(f_n\mathbb{1}_{[0,t]^n}).$$

The space $\mathbb{D}_{1,2}$ consists of all random variables $G \in L_2$ such that

$$\|G\|_{\mathbb{D}_{1,2}}^2 := \sum_{n=0}^{\infty} (n+1)! \|f_n\|_{L_2^n}^2 < \infty.$$

For $G \in \mathbb{D}_{1,2}$ the Malliavin derivative

$$\mathcal{D}G \in L_2(\mathbb{P} \otimes \mathbb{m}) := L_2\big(\Omega \times [0,T] \times \mathbb{R}, \mathcal{F}_T \otimes \mathcal{B}\big([0,T] \times \mathbb{R}\big), \mathbb{P} \otimes \mathbb{m}\big)$$

is given by

$$\mathcal{D}_{t,x} G(\omega) := \sum_{n=1}^{\infty} n I_{n-1}\big(f_n\big((t,x),\cdot\big)\big)(\omega),$$

for $\mathbb{P} \otimes \mathbb{m}$-a.e. $(\omega, t, x) \in \Omega \times [0,T] \times \mathbb{R}$. For example, for X_s from (1) representation (5) implies

$$\mathcal{D}_{t,x} X_s = \mathcal{D}_{t,x} I_1(\mathbb{1}_{[0,s]\times\mathbb{R}}) = \mathbb{1}_{[t,T]}(s) \qquad \text{for } \mathbb{P} \otimes \mathbb{m}\text{-a.e. } (\omega,t,x) \in \Omega \times [0,T] \times \mathbb{R}. \tag{6}$$

For random variables in $\mathbb{D}_{1,2}$ there is an explicit expression for the integrand in the formulation of the predictable representation property (for an introduction to stochastic integration w.r.t. random measures see, e.g., [3]).

Lemma 3.1 (Clark–Ocone–Haussmann formula [33], Theorem 10). *Assume* $G \in \mathbb{D}_{1,2}$. *Then*

$$G = \mathbb{E}G + \int_{[0,T]\times\mathbb{R}} {}^{p}(\mathcal{D}G)_{t,x} M(\mathrm{d}t, \mathrm{d}x). \tag{7}$$

The Malliavin derivative $\mathcal{D}_{\cdot,0}$ can be interpreted as a Malliavin derivative in the Brownian setting with values in the L_2-space of random variables depending on the jump part of the Lévy process (see [1,32]). On the other hand, for $x \neq 0$, the Malliavin derivative $\mathcal{D}_{\cdot,x}$ behaves like a difference quotient (see [1,32]). This is also illustrated by the next lemma which contains formulae for the Malliavin derivative of differentiable Lipschitz functions depending on random variables in $\mathbb{D}_{1,2}$.

Lemma 3.2. *Let* $f \in C^1(\mathbb{R}^n; \mathbb{R})$ *with bounded partial derivatives. If* $G_1, \dots, G_n \in \mathbb{D}_{1,2}$ *then* $f(G_1, \dots, G_n) \in \mathbb{D}_{1,2}$ *and*

(i) *for* $x = 0$ *it holds*

$$\mathcal{D}_{t,0} f(G_1, \dots, G_n) = \sum_{i=1}^{n} (\partial_i f)(G_1, \dots, G_n) \mathcal{D}_{t,0} G_i,$$

for $\mathbb{P} \otimes \lambda$-*a.e.* (ω, t);

(ii) *for* $x \neq 0$ *we get the difference quotient*

$$\mathcal{D}_{t,x} f(G_1, \dots, G_n) = \frac{f(G_1 + x\mathcal{D}_{t,x}G_1, \dots, G_n + x\mathcal{D}_{t,x}G_n) - f(G_1, \dots, G_n)}{x},$$

for $\mathbb{P} \otimes \mathbb{m}$-*a.e.* (ω, t, x).

Proof. Assertion (i) follows immediately from [32], Proposition 3.5, combined with [32], Proposition 3.3 and [30], Proposition 1.2.3. Assertion (ii) is a straightforward extension of [16], Lemma 5.1. □

We will make use of the following properties for the Malliavin derivative [13], Lemmas 3.1–3.3.

Lemma 3.3. (i) *Let $G \in \mathbb{D}_{1,2}$. Then for $0 \le s \le T$, $\mathbb{E}_s G \in \mathbb{D}_{1,2}$ and*

$$\mathcal{D}_{t,x}\mathbb{E}_s G = \mathbb{E}_s(\mathcal{D}_{t,x} G)\mathbb{1}_{[0,s]}(t), \qquad \mathbb{P} \otimes \mathbb{m}\text{-}a.e.$$

(ii) *Let $F : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a product measurable and adapted process, ρ a finite measure on $([0, T] \times \mathbb{R}, \mathcal{B}([0, T] \times \mathbb{R}))$ such that the conditions*

(a) $\mathbb{E}\int_{[0,T]\times\mathbb{R}} |F(s, y)|^2 \rho(\mathrm{d}s, \mathrm{d}y) < \infty$,
(b) $F(s, y) \in \mathbb{D}_{1,2}$ *for ρ-a.e.* $(s, y) \in [0, T] \times \mathbb{R}$,
(c) $\mathbb{E}\int_{[0,T]\times\mathbb{R}}\int_{[0,T]\times\mathbb{R}} |\mathcal{D}_{t,x}F(s, y)|^2 \rho(\mathrm{d}s, \mathrm{d}y)\mathbb{m}(\mathrm{d}t, \mathrm{d}x) < \infty$

are satisfied. Then

$$\int_{[0,T]\times\mathbb{R}} F(s, y)\rho(\mathrm{d}s, \mathrm{d}y) \in \mathbb{D}_{1,2},$$

and the differentiation rule

$$\mathcal{D}_{t,x}\int_{[0,T]\times\mathbb{R}} F(s, y)\rho(\mathrm{d}s, \mathrm{d}y) = \int_{[0,T]\times\mathbb{R}} \mathcal{D}_{t,x}F(s, y)\rho(\mathrm{d}s, \mathrm{d}y)$$

holds for $\mathbb{P} \otimes \mathbb{m}$-a.e. $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}$.

(iii) *Let $F : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a predictable process satisfying*

$$\mathbb{E}\int_{[0,T]\times\mathbb{R}} \big|F(s, y)\big|^2 \mathbb{m}(\mathrm{d}s, \mathrm{d}y) < \infty.$$

Then conditions (a)–(c) *of* (ii) *are satisfied for $\rho = \mathbb{m}$ if and only if*

$$\int_{[0,T]\times\mathbb{R}} F(s, y)M(\mathrm{d}s, \mathrm{d}y) \in \mathbb{D}_{1,2}.$$

In this case, the formula

$$\mathcal{D}_{t,x}\int_{[0,T]\times\mathbb{R}} F(s, y)M(\mathrm{d}s, \mathrm{d}y) = F(t, x) + \int_{[0,T]\times\mathbb{R}} \mathcal{D}_{t,x}F(s, y)M(\mathrm{d}s, \mathrm{d}y)$$

holds $\mathbb{P} \otimes \mathbb{m}$-a.e.

The following theorem is concerned with Malliavin differentiability of the solution to a BSDE of the form

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, \bar{Z}_s)\,\mathrm{d}s - \int_{]t,T]\times\mathbb{R}} Z_{s,x} M(\mathrm{d}s, \mathrm{d}x), \qquad 0 \le t \le T, \tag{8}$$

where we will assume

(A_f) $f \in \mathcal{C}([0, T] \times \mathbb{R}^3)$ is Lipschitz-continuous in (x, y, z), uniformly in t, that is,

$$\bigl|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\bigr| \le L_f\bigl(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\bigr). \tag{9}$$

(A_f1) f satisfies (A_f) and $f \in \mathcal{C}^{0,1,1,1}([0, T] \times \mathbb{R}^3)$.

Note that (8) is a special form of (3), and $F(\omega, t, y, z) := f(t, X_t(\omega), y, z)$ satisfies (A_F) if f does satisfy (A_f).

Theorem 3.4. *Let $\xi \in \mathbb{D}_{1,2}$ and assume (A_f1). Then the following assertions hold.*

(i) *For $\mathbb{m}$-a.e. $(r, v) \in [0, T] \times \mathbb{R}$, there exists a unique solution $(U^{r,v}, V^{r,v}) \in S \times H$ to the BSDE*

$$\begin{aligned} U_t^{r,v} &= \mathcal{D}_{r,v}\xi + \int_t^T F_{r,v}\bigl(s, U_s^{r,v}, \bar{V}_s^{r,v}\bigr)\,\mathrm{d}s - \int_{]t,T]\times\mathbb{R}} V_{s,x}^{r,v} M(\mathrm{d}s, \mathrm{d}x), \qquad t \in [r, T], \\ U_t^{r,v} &= V_{s,x}^{r,v} = 0, \qquad t \in [0, r), \end{aligned} \tag{10}$$

where $\bar{V}_s^{r,v} := \int_{\mathbb{R}} V_{s,x}^{r,v} \kappa(\mathrm{d}x)$,

$$F_{r,0}\bigl(s, U_s^{r,0}, \bar{V}_s^{r,0}\bigr) := \bigl\langle \nabla f(s, X_s, Y_s, \bar{Z}_s), \bigl(\mathbb{1}_{[r,T]}(s), U_s^{r,0}, \bar{V}_s^{r,0}\bigr)\bigr\rangle,$$

with $\nabla = (\partial_x, \partial_y, \partial_z)$, *and*

$$F_{r,v}\bigl(s, U_s^{r,v}, \bar{V}_s^{r,v}\bigr) := \frac{1}{v}\bigl[f\bigl(s, X_s + v\mathbb{1}_{[r,T]}(s), Y_s + vU_s^{r,v}, \bar{Z}_s + v\bar{V}_s^{r,v}\bigr) - f(s, X_s, Y_s, \bar{Z}_s)\bigr]$$

for $v \neq 0$.

(ii) *For the solution (Y, Z) of (8) it holds*

$$Y \in L_2\bigl([0, T]; \mathbb{D}_{1,2}\bigr), \qquad Z \in L_2\bigl([0, T] \times \mathbb{R}; \mathbb{D}_{1,2}\bigr), \tag{11}$$

and $\mathcal{D}_{r,v}Y$ admits a càdlàg version for $\mathbb{m}$-a.e. $(r, v) \in [0, T] \times \mathbb{R}$.

(iii) *$(\mathcal{D}Y, \mathcal{D}Z)$ is a version of (U, V), that is, for $\mathbb{m}$-a.e. (r, v) it solves*

$$\begin{aligned} \mathcal{D}_{r,v}Y_t = \mathcal{D}_{r,v}\xi &+ \int_t^T F_{r,v}\left(s, \mathcal{D}_{r,v}Y_s, \int_{\mathbb{R}} \mathcal{D}_{r,v}Z_{s,x}\kappa(\mathrm{d}x)\right)\mathrm{d}s \\ &- \int_{]t,T]\times\mathbb{R}} \mathcal{D}_{r,v}Z_{s,x} M(\mathrm{d}s, \mathrm{d}x), \qquad t \in [r, T]. \end{aligned} \tag{12}$$

(iv) *The process* ${}^p((\mathcal{D}_{r,v}Y_r)_{r\in[0,T],v\in\mathbb{R}})$ *is a version of* Z *where we set*

$$\mathcal{D}_{r,v}Y_r(\omega) := \lim_{t\searrow r}\mathcal{D}_{r,v}Y_t(\omega)$$

for all (r, v, ω) *for which* $\mathcal{D}_{r,v}Y$ *is càdlàg and* $\mathcal{D}_{r,v}Y_r(\omega) := 0$ *otherwise.*

In the setting of time, delayed BSDEs a similar result was proved by Delong and Imkeller [13] assuming that the time horizon T or the Lipschitz constant L_f of the generator are sufficiently small. For the convenience of the reader a proof of Theorem 3.4 is contained in the preprint version [17].

4. Chaotic representation of (Y, Z)

The goal of this section is to investigate properties of the chaos expansions of the solution (Y, Z) to (8) with terminal values ξ of the following form:

Let $r_0 = 0 < r_1 < \cdots < r_m = T$ be a partition of $[0, T]$. Define $\Lambda_k :=]r_{k-1}, r_k]$ for $k = 1, \dots, m$ and $V_m^n := \{1, \dots, m\}^n$. The set of cuboids

$$\{\Lambda_\alpha := \Lambda_{\alpha_1} \times \cdots \times \Lambda_{\alpha_n} \colon \alpha = (\alpha_1, \dots, \alpha_n) \in V_m^n\}$$

forms a partition of $]0, T]^n$. Furthermore, we let

$$\mathbb{H} := \Bigg\{\xi = \sum_{n=0}^{\infty} I_n(f_n) \in L_2\colon \exists g_n^\alpha \in L_2\big(\mathbb{R}^n, \mu^{\otimes n}\big) \text{ such that}$$

$$f_n\big((t_1, x_1), \dots, (t_n, x_n)\big) = \sum_{\alpha\in V_m^n} g_n^\alpha(x_1, \dots, x_n)\mathbb{1}_{\Lambda_\alpha}(t_1, \dots, t_n)\Bigg\}.$$

Hence, on each cuboid Λ_α the function f_n is constant in $(t_1, \dots, t_n)$. In particular, this space contains random variables of the form

$$g(X_{r_m} - X_{r_{m-1}}, \dots, X_{r_1} - X_0) \in L_2,$$

where g is a Borel function (see [6]).

The benefit to consider terminal conditions from $\mathbb{H}$ lies in the fact that $t \mapsto \mathcal{D}_{tx}\xi$ is a.e. constant as long as t is within an interval Λ_k. This property will be used several times below, especially in the proofs of Lemmas 5.3–5.5.

Remark 4.1. By convolution with mollifiers, we construct for any function $f \in \mathcal{C}([0, T] \times \mathbb{R}^3)$ satisfying (A_f) a sequence $(f_n)_{n\geq 0}$ converging uniformly to f in $\mathcal{C}([0, T] \times \mathbb{R}^3)$, such that for all $n \in \mathbb{N}$ and $t \in [0, T]$ we have $f_n(t, \cdot) \in \mathcal{C}^\infty(\mathbb{R}^3)$, and f_n satisfies the Lipschitz-condition (9) with the same constant L_f for all n.

Let $(\xi_n)_{n\geq 0} \subseteq \mathbb{D}_{1,2}$ be a sequence converging to ξ in L_2. By (Y^n, Z^n), we will denote the solution to (8) with terminal condition ξ_n and generator f_n. Then Theorem 2.3 implies that

$$\big(Y^n, Z^n\big) \to (Y, Z) \qquad \text{if } n \to \infty \text{ in } S \times H. \tag{13}$$

If $\xi \in \mathbb{H}$, then the solution (Y, Z) has a chaos expansion which resembles those of the elements of $\mathbb{H}$.

Theorem 4.2. *Let (A_f) hold. For $\xi \in \mathbb{H}$ the chaos expansion of $(Y, Z) \in S \times H$ has the representation*

$$Y_t = \sum_{n=0}^{\infty} I_n\Bigg(\sum_{\alpha \in V_m^n} \varphi_n^\alpha(t) \mathbb{1}_{\Lambda_\alpha \cap]0,t]^n}\Bigg), \qquad \mathbb{P} \otimes \lambda\text{-}a.e., \tag{14}$$

where $\varphi_n^\alpha : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is jointly measurable, $\varphi_n^\alpha(t) \in L_2(\mathbb{R}^n, \mu^{\otimes n})$ and

$$Z_{t,x} = \sum_{n=0}^{\infty} I_n\Bigg(\sum_{\alpha \in V_m^n} \psi_n^\alpha(t, x) \mathbb{1}_{\Lambda_\alpha \cap]0,t]^n}\Bigg), \qquad \mathbb{P} \otimes \mathbb{m}\text{-}a.e., \tag{15}$$

where $\psi_n^\alpha : [0, T] \times \mathbb{R}^{n+1} \to \mathbb{R}$ is jointly measurable and $\psi_n^\alpha(t) \in L_2(\mathbb{R}^{n+1}, \mu^{\otimes n+1})$.

Proof. We may use Remark 4.1 and approximate $\xi \in \mathbb{H}$ by a sequence $(\xi_n)_n \subseteq \mathbb{H} \cap \mathbb{D}_{1,2}$ and f by $(f_n)_n$ satisfying $(A_f 1)$. Since the convergence in $S \times H$ implies convergence w.r.t. the norm

$$\big\|\big|(y, z)\big|\big\| := \big(\|y\|^2_{L_2(\mathbb{P}\otimes\lambda)} + \|z\|^2_H\big)^{1/2}, \tag{16}$$

and the space of processes (y, z) with representations (14) and (15) is closed with respect to the norm (16) we only need to show that the assertion holds for any solution (Y^n, Z^n) w.r.t. (ξ_n, f_n). Hence we may assume that $\xi \in \mathbb{H} \cap \mathbb{D}_{1,2}$ and $f \in \mathcal{C}^{0,1,1,1}([0, T] \times \mathbb{R}^3)$ such that $\partial_x f$, $\partial_y f$ and $\partial_z f$ are bounded by L_f. According to Theorem 3.4, we can differentiate (8) and obtain for $\mathbb{m}$-a.e. (t, x) and all $s \in [t, T]$ that

$$\mathcal{D}_{t,x} Y_s = \mathcal{D}_{t,x}\xi + \int_s^T \mathcal{D}_{t,x} f(r, X_r, Y_r, \bar{Z}_r)\,\mathrm{d}r - \int_{]s,T]\times\mathbb{R}} \mathcal{D}_{t,x} Z_{r,y} M(\mathrm{d}r, \mathrm{d}y).$$

Theorem 3.4 yields that Z is a version of ${}^p(\mathcal{D}_{t,x} Y_t)$, hence

$$Z_{t,x} = \mathcal{D}_{t,x} Y_t, \qquad \mathbb{P} \otimes \mathbb{m}\text{-a.e.}$$

We define the recursion

$$\begin{aligned} &Y_s^0 := 0, \qquad Z_{s,y}^0 := 0, \\ &\mathcal{Y}_s^{k+1} := \xi + \int_s^T f\big(r, X_r, Y_r^k, \bar{Z}_r^k\big)\,\mathrm{d}r, \qquad Y^{k+1} := {}^o\big(\mathcal{Y}^{k+1}\big), \end{aligned} \tag{17}$$

where ^{o} denotes the optional projection, which is according to [12], Theorem 47 and Remark 50, càdlàg. Since $Y_u^{k+1} = \mathbb{E}_u \mathcal{Y}_u^{k+1}$ $\mathbb{P}$-a.s. one gets by Lemma 3.3

$$\begin{aligned}\mathcal{D}_{s,y} Y_u^{k+1} &= \mathcal{D}_{s,y}\mathbb{E}_u \xi + \mathcal{D}_{s,y}\mathbb{E}_u \int_u^T f\big(r, X_r, Y_r^k, \bar{Z}_r^k\big)\,\mathrm{d}r \\ &= \mathbb{E}_u \mathcal{D}_{s,y}\xi + \mathbb{E}_u \int_u^T \mathcal{D}_{s,y} f\big(r, X_r, Y_r^k, \bar{Z}_r^k\big)\,\mathrm{d}r, \qquad u \in [s, T].\end{aligned} \tag{18}$$

Since $\mathcal{D}_{s,y}\xi + \int_u^T \mathcal{D}_{s,y} f(r, X_r, Y_r^k, \bar{Z}_r^k)\,\mathrm{d}r$, $u \in [s, T]$, has continuous paths for a.e. (s, y) we can again apply [12], Theorem 47 and Remark 50, to get a càdlàg optional projection. Hence, we may define the set

$$A_k := \big\{(s, y) \in [0, T] \times \mathbb{R}\colon \text{ the RHS of (18) is càdlàg on } [s, T]\ \mathbb{P}\text{-a.s.}\big\}$$

and assume a pathwise càdlàg version of $\mathcal{D}_{s,y} Y^{k+1}$ for any $(s, y) \in A_k$ while we let $\mathcal{D}_{s,y} Y^{k+1}$ be zero otherwise. In this sense, we can set

$$\mathcal{Z}_{s,y}^{k+1} := \lim_{t \searrow s} \mathcal{D}_{s,y} Y_t^{k+1}, \qquad Z^{k+1} := {}^p\big(\mathcal{Z}^{k+1}\big) \tag{19}$$

for $k = 0, 1, 2, \ldots$.

The process Y^{k+1} has a càdlàg version, therefore, $(Y^k, Z^k) \in S \times H$ for all $k \in \mathbb{N}$. In the proof of [35], Theorem 2.2, it is shown that (Y^k, Z^k) converges to (Y, Z) with respect to the norm (16).

Consequently, we only need to show that (14) and (15) hold for (Y^k, Z^k).

For fixed $t \in]0, T[$, we describe (14) by introducing the space

$$\mathbb{H}_t := \left\{\sum_{n=0}^{\infty} I_n\left(\sum_{\alpha \in V_m^n} g_n^\alpha \mathbb{1}_{\Lambda_\alpha \cap]0,t]^n}\right) \in L_2\colon\ g_n^\alpha \in L_2\big(\mathbb{R}^n, \mu^{\otimes n}\big)\right\}.$$

From [6], one can conclude the following fact.

Lemma 4.3. *For any Borel function $h : \mathbb{R}^d \to \mathbb{R}$ and $\xi_1, \ldots, \xi_d \in \mathbb{H}_t$ such that $h(\xi_1, \ldots, \xi_d) \in L_2$ it holds $h(\xi_1, \ldots, \xi_d) \in \mathbb{H}_t$.*

Assume now that (14) and (15) hold for Y^k and Z^k, respectively. We have

$$\begin{aligned}\bar{Z}_t^k &= \int_{\mathbb{R}} \sum_{n=0}^{\infty} I_n\left(\sum_{\alpha \in V_m^n} \psi_n^\alpha(t, x) \mathbb{1}_{\Lambda_\alpha \cap [0,t]^{\otimes n}}\right) \kappa(\mathrm{d}x) \\ &= \sum_{n=0}^{\infty} I_n\left(\sum_{\alpha \in V_m^n} \int_{\mathbb{R}} \psi_n^\alpha(t, x) \kappa(\mathrm{d}x) \mathbb{1}_{\Lambda_\alpha \cap [0,t]^{\otimes n}}\right) \\ &= \sum_{n=0}^{\infty} I_n\left(\sum_{\alpha \in V_m^n} \bar{\psi}_n^\alpha(t) \mathbb{1}_{\Lambda_\alpha \cap [0,t]^{\otimes n}}\right) \in \mathbb{H}_t.\end{aligned} \tag{20}$$

From Lemma 4.3, it follows that $f(t, X_t, Y_t^k, \bar{Z}_t^k) \in \mathbb{H}_t$ that is,

$$f\big(t, X_t, Y_t^k, \bar{Z}_t^k\big) = \sum_{n=0}^{\infty} I_n\bigg(\sum_{\alpha \in V_m^n} g_n^{\alpha}(t) \mathbb{1}_{\Lambda_\alpha \cap]0,t]^{\otimes n}}\bigg),$$

with $g_n^{\alpha}(t) \in L_2(\mathbb{R}^n, \mu^{\otimes n})$. Because $f(\cdot, X_\cdot, Y_\cdot^k, \bar{Z}_\cdot^k)$ is square integrable w.r.t. $\mathbb{P} \otimes \lambda$ on $\Omega \times [0, T]$ one can show that the g_n^{α} can be chosen jointly measurable. This implies

$$\begin{aligned}\mathbb{E}_t \int_t^T f\big(r, X_r, Y_r^k, \bar{Z}_r^k\big)\,\mathrm{d}r &= \int_t^T \sum_{n=0}^{\infty} I_n\bigg(\sum_{\alpha \in V_m^n} g_n^{\alpha}(r) \mathbb{1}_{\Lambda_\alpha \cap]0,t]^{\otimes n}}\bigg)\,\mathrm{d}r \\ &= \sum_{n=0}^{\infty} I_n\bigg(\sum_{\alpha \in V_m^n} \int_t^T g_n^{\alpha}(r)\,\mathrm{d}r \mathbb{1}_{\Lambda_\alpha \cap]0,t]^{\otimes n}}\bigg).\end{aligned}$$

From (17), we have that $Y_t^{k+1} = \mathbb{E}_t \mathcal{Y}_t^{k+1}$ $\mathbb{P}$-a.s. and since $\mathbb{E}_t \xi \in \mathbb{H}_t$ we conclude representation (14) for Y_t^{k+1}. To find out the representation of Z^{k+1}, we will use (19). Let $\underline{\alpha} := (\alpha_2, \ldots, \alpha_n)$. Assuming $\xi = \sum_{n=0}^{\infty} I_n(\sum_{\alpha \in V_m^n} \hat{g}_n^{\alpha} \mathbb{1}_{\Lambda_\alpha})$ with symmetric $f_n = \sum_{\alpha \in V_m^n} \hat{g}_n^{\alpha} \mathbb{1}_{\Lambda_\alpha}$ we get by Lemma 3.3 for $\mathbb{P} \otimes \mathfrak{m}$-a.e. $(t, y, \omega) \in]0, s] \times \mathbb{R} \times \Omega$ that

$$\begin{aligned}\mathcal{D}_{t,y} Y_s^{k+1} &= \mathcal{D}_{t,y} \mathbb{E}_s \xi + \mathcal{D}_{t,y} \mathbb{E}_s \int_s^T f\big(r, X_r, Y_r^k, \bar{Z}_r^k\big)\,\mathrm{d}r \\ &= \mathcal{D}_{t,y} \mathbb{E}_s \xi + \mathcal{D}_{t,y} \int_s^T \sum_{n=0}^{\infty} I_n\bigg(\sum_{\alpha \in V_m^n} g_n^{\alpha}(r) \mathbb{1}_{\Lambda_\alpha \cap]0,s]^{\otimes n}}\bigg)\,\mathrm{d}r \\ &= \sum_{n=1}^{\infty} n I_{n-1}\bigg(\sum_{\alpha \in V_m^n} \hat{g}_n^{\alpha}(y, \cdot) \mathbb{1}_{\Lambda_{\alpha_1}}(t) \mathbb{1}_{\Lambda_{\underline{\alpha}} \cap]0,s]^{\otimes (n-1)}}\bigg) \\ &\quad + \int_s^T \sum_{n=1}^{\infty} n I_{n-1}\bigg(\sum_{\alpha \in V_m^n} g_n^{\alpha}(r, y, \cdot) \mathbb{1}_{\Lambda_{\alpha_1}}(t) \mathbb{1}_{\Lambda_{\underline{\alpha}} \cap]0,s]^{\otimes (n-1)}}\bigg)\,\mathrm{d}r,\end{aligned}$$

where we again have chosen symmetric integrands $\sum_{\alpha \in V_m^n} g_n^{\alpha}(r) \mathbb{1}_{\Lambda_\alpha \cap]0,s]^{\otimes n}}$. One easily checks the L_2-convergence

$$\begin{aligned}\lim_{s \searrow t} \mathcal{D}_{t,y} Y_s^{k+1} &= \sum_{n=1}^{\infty} n I_{n-1}\bigg(\sum_{\alpha \in V_m^n} \hat{g}_n^{\alpha}(y, \cdot) \mathbb{1}_{\Lambda_{\alpha_1}}(t) \mathbb{1}_{\Lambda_{\underline{\alpha}} \cap]0,t]^{\otimes (n-1)}}\bigg) \\ &\quad + \int_t^T \sum_{n=1}^{\infty} n I_{n-1}\bigg(\sum_{\alpha \in V_m^n} g_n^{\alpha}(r, y, \cdot) \mathbb{1}_{\Lambda_{\alpha_1}}(t) \mathbb{1}_{\Lambda_{\underline{\alpha}} \cap]0,t]^{\otimes (n-1)}}\bigg)\,\mathrm{d}r.\end{aligned}$$

If we consider the càdlàg version of $\mathcal{D}_{t,y}Y^{k+1}$, we obtain the same expression for the pathwise limit, that is, $\mathbb{P}$-a.s.

$$\begin{aligned} Z_{t,y}^{k+1} &= \lim_{s\searrow t} \mathcal{D}_{t,y} Y_s^{k+1} \\ &= \sum_{n=1}^{\infty} n I_{n-1}\biggl(\sum_{\alpha\in V_m^n}\biggl[\hat{g}_n^{\alpha}(y,\cdot) + \int_t^T g_n^{\alpha}(r,y,\cdot)\,\mathrm{d}r\biggr]\mathbb{1}_{\Lambda_{\alpha_1}}(t)\mathbb{1}_{\Lambda_{\underline{\alpha}}\cap]0,t]^{\otimes(n-1)}}\biggr). \end{aligned}$$

□

5. L_2-variation of (Y, Z)

The next theorem is our main statement, which allows conclusions on the L_2-regularity of the solutions to BSDE (8) by observing regularity properties of Y_{r_k} for fixed time points $r_0 = 0 < r_1 < \cdots < r_m = T$.

Theorem 5.1. *Assume that (A_f) is satisfied and $\xi \in \mathbb{H}$. Let $k \in \{1,\dots,m\}$ and $\theta_k \in]0,1]$. For the solution (Y, Z) of (8), consider the following assertions*:

(i) *There is some $c_1 > 0$ such that for all $s \in [r_{k-1}, r_k]$,*

$$\|Y_{r_k} - \mathbb{E}_s Y_{r_k}\|^2 \le c_1(r_k - s)^{\theta_k}.$$

(ii) *There is some $c_2 > 0$ such that for all $r_{k-1} \le s < t \le r_k$,*

$$\|Y_t - Y_s\|^2 \le c_2 \int_s^t (r_k - r)^{\theta_k - 1}\,\mathrm{d}r.$$

(iii) *There is some $c_3 > 0$ such that for λ-a.e. $s \in [r_{k-1}, r_k[$,*

$$\|Z_{s,\cdot}\|^2_{L_2(\mathbb{P}\otimes\mu)} \le c_3(r_k - s)^{\theta_k - 1}.$$

(iv) *There is some $c_4 > 0$ and a Borel set N_k with $\lambda(N_k) = 0$ such that for all $s, t \in [r_{k-1}, r_k[\backslash N_k$ with $s < t$ and for all $h \in L_2(\mu)$ it holds*

$$\biggl\|\int_{\mathbb{R}} (Z_{t,x} - Z_{s,x})h(x)\mu(\mathrm{d}x)\biggr\|^2 \le \|h\|^2_{L_2(\mu)} c_4 \int_s^t (r_k - r)^{\theta_k - 2}\,\mathrm{d}r.$$

Then it holds that

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Rightarrow \text{(iv)}.$$

Remark 5.2. (i) Analogously to [14], Definition 1, we may introduce the concept of *path-dependent fractional smoothness*: fix $\Theta = (\theta_1,\dots,\theta_m) \in]0,1[^m$. If (Y, Z) is the solution to BSDE (8) with generator f and terminal condition $\xi \in \mathbb{H}$, we let

$$(\xi, f) \in B^{\Theta}_{2,\infty}(X)$$

provided that there is some $c > 0$ such that

$$\|Y_{r_k} - \mathbb{E}_s Y_{r_k}\|^2 \le c(r_k - s)^{\theta_k}, \qquad r_{k-1} \le s < r_k, k = 1, \dots, m.$$

If $f = 0$ we simply write $\xi \in B^{\Theta}_{2,\infty}(X)$. If, moreover, $T = 1$ and $m = 1$ then the space $B^{\Theta}_{2,\infty}(X)$ can be understood as the real interpolation space $(L_2, \mathbb{D}_{1,2})_{\theta_1,\infty}$ which describes *fractional smoothness*. For $\xi = \sum_{n=0}^{\infty} I_n(f_n) \in \mathbb{H}$ set $T_\xi(t) := \|\mathbb{E}_t \xi\|^2 = \sum_{n=0}^{\infty} \|I_n(f_n)\|^2 t^n$, and using the ideas of [19], Proposition 3.2 and [20], formula (13), one can conclude that

$$\begin{aligned}\|\xi\|_{(L_2,\mathbb{D}_{1,2})_{\theta_1,\infty}} &\sim_c \|\xi\| + \sup_{0\le t<1} (1-t)^{-\theta_1/2}\sqrt{T_\xi(1) - T_\xi(t)} \\ &= \|\xi\| + \sup_{0\le t<1} (1-t)^{-\theta_1/2}\|\xi - \mathbb{E}_t\xi\|.\end{aligned}$$

By assumption, we have $\|\xi - \mathbb{E}_t\xi\|^2 \le c(1-t)^{\theta_1}$ hence the RHS is finite. From the lexicographical scale of the real interpolation spaces (see [8] or [7]), it follows

$$(L_2, \mathbb{D}_{1,2})_{\theta_1',2} \supseteq (L_2, \mathbb{D}_{1,2})_{\theta_1,\infty} \qquad \text{for all } \theta_1' \in\,]0, \theta_1[.$$

Especially, $\|\xi - \mathbb{E}_t\xi\|^2 \le c(1-t)^{\theta_1}$ implies that $\sum_{n=0}^{\infty} n^{\theta_1'}\|I_n(f_n)\|^2 < \infty$ for all $\theta_1' \in\,]0, \theta_1[$ (see [20], Remark A.1).

(ii) In general (iv) $\not\Rightarrow$ (iii): let $(p_n)_{n=1}^{\infty}$ be an ONB in $L_2(\mu)$. For simplicity, assume $T = 1, m = 1, f \equiv 0$ and $\xi = \sum_{n=0}^{\infty} I_n(g_n \mathbb{1}^{\otimes n}_{]0,T]})$ so that

$$Z_{s,x} = \sum_{n=1}^{\infty} n I_{n-1}\big(g_n(x,\cdot)\mathbb{1}^{\otimes(n-1)}_{]0,s]}\big).$$

Setting $g_n := \beta_n (n!)^{-1/2} p_n^{\otimes n}$ we have

$$\|Z_{s,\cdot}\|^2_{L_2(\mathbb{P}\otimes\mu)} = \sum_{n=1}^{\infty} n\beta_n^2 s^{n-1}.$$

For a sequence (β_n) such that $\beta_1^2 := 1$, $\beta_2^2 := 0$, $\beta_n^2 := \frac{1}{n(\log(n-1))^2}$, $n \ge 3$, we use Lemma A.1 of [31] which states that

$$1 + \sum_{n=2}^{\infty} (\log n)^{-2} s^n \sim_c \frac{1}{(1-s)(1-\log(1-s))^2}$$

(where for some $c \ge 1$ and $A, B > 0$ the expression $A \sim_c B$ is a short notation for $c^{-1}A \le B \le cA$). Hence

$$\|Z_{s,\cdot}\|^2_{L_2(\mathbb{P}\otimes\mu)} \sim_c \frac{1}{(1-\log(1-s))^2}(1-s)^{-1},$$

so that there does not exist any $\theta \in\,]0, 1]$ for which property (iii) holds.

But for any $h = \sum_{n=1}^{\infty} \alpha_n p_n$ such that $\|h\|^2_{L_2(\mu)} = \sum_{n=1}^{\infty} \alpha_n^2 = 1$ we have

$$\left\| \int_{\mathbb{R}} (Z_{t,x} - Z_{s,x}) h(x) \mu(\mathrm{d}x) \right\|^2 = \sum_{n=3}^{\infty} \alpha_n^2 \frac{1}{(\log(n-1))^2} \left(t^{n-1} - s^{n-1}\right) \le \frac{1}{(\log 2)^2},$$

which means that (iv) is fulfilled for any $\theta \in]0,1]$.

We prepare some lemmas to prove Theorem 5.1.

Lemma 5.3. *Let $\eta \in \mathbb{H} \cap \mathbb{D}_{1,2}$ and $k \in \{1,\dots,m\}$. Then for λ-a.e. $s,t \in]r_{k-1}, r_k[$ with $s < t$ it holds*

$$\|\mathbb{E}_t \mathcal{D}_{t,\cdot}\eta - \mathbb{E}_s \mathcal{D}_{s,\cdot}\eta\|^2_{L_2(\mathbb{P}\otimes\mu)} \le 4 \int_s^t \frac{\|\mathbb{E}_{r_k}\eta - \mathbb{E}_r \eta\|^2}{(r_k - r)^2} \mathrm{d}r.$$

Proof. Let $\eta \in \mathbb{H} \cap \mathbb{D}_{1,2}$ be given by $\eta = \sum_{n=0}^{\infty} I_n(\sum_{\alpha \in V_m^n} g_n^\alpha \mathbb{1}_{\Lambda_\alpha})$ where we assume that the functions $f_n((t_1,x_1),\dots,(t_n,x_n))$ are symmetric. In the following, we use again the notation $\underline{\alpha} := (\alpha_2,\dots,\alpha_n)$. Since

$$\mathcal{D}_{t,x}\eta = \sum_{n=1}^{\infty} n I_{n-1}\left(\sum_{\alpha \in V_m^n} g_n^\alpha(x,\cdot) \mathbb{1}_{\Lambda_{\alpha_1}}(t) \mathbb{1}_{\Lambda_{\underline{\alpha}}} \right)$$

and since there exists a version of $\mathcal{D}\eta$ which is constant on $]r_{k-1}, r_k[$ we get for $s,t \in]r_{k-1}, r_k[$ that

$$\begin{aligned} &\|\mathbb{E}_t \mathcal{D}_{t,\cdot}\eta - \mathbb{E}_s \mathcal{D}_{s,\cdot}\eta\|^2_{L_2(\mathbb{P}\otimes\mu)} \\ &\quad = \left\| \sum_{n=1}^{\infty} n I_{n-1}\left(\sum_{\alpha \in V_m^n} g_n^\alpha \mathbb{1}_{\Lambda_{\alpha_1}}(t) \mathbb{1}_{\Lambda_{\underline{\alpha}}} \left(\mathbb{1}_{]0,t]}^{\otimes(n-1)} - \mathbb{1}_{]0,s]}^{\otimes(n-1)} \right) \right) \right\|^2_{L_2(\mathbb{P}\otimes\mu)} \qquad (21) \\ &\quad = \sum_{n=2}^{\infty} n n! \sum_{\substack{\alpha \in V_m^n \\ \alpha_1 = k}} \|g_n^\alpha\|^2_{L_2(\mu^{\otimes n})} \lambda^{\otimes(n-1)}\left(\Lambda_{\underline{\alpha}} \cap \left( ]0,t]^{n-1} \setminus]0,s]^{n-1} \right) \right). \end{aligned}$$

For $\beta \in V_m^n$ and $1 \le l \le m$, we define

$$\gamma_l(\beta) := \#\{i \mid \beta_i = l,\ i = 1,\dots,n\}.$$

Notice that the intersection $\Lambda_{\underline{\alpha}} \cap (]0,t]^{n-1} \setminus]0,s]^{n-1})$ is empty if $\sum_{d=k+1}^{m} \gamma_d(\underline{\alpha}) > 0$. Therefore, setting

$$\delta_{\underline{\alpha}} := \mathbb{1}_{\{0\}}\left(\sum_{d=k+1}^{m} \gamma_d(\underline{\alpha}) \right)$$

we have

$$\lambda^{\otimes(n-1)}\big(\Lambda_{\underline{\alpha}} \cap \big(]0,t]^{n-1} \setminus]0,s]^{n-1}\big)\big)$$
$$= \big((t-r_{k-1})^{\gamma_k(\underline{\alpha})} - (s-r_{k-1})^{\gamma_k(\underline{\alpha})}\big) \prod_{1\le l<k} (r_l - r_{l-1})^{\gamma_l(\underline{\alpha})} \delta_{\underline{\alpha}}.$$

Using the symmetry of the functions in the chaos decomposition, we get that

$$g_n^{\alpha}(x_1,\dots,x_n) = g_n^{\pi(\alpha)}(x_{\pi(1)},\dots,x_{\pi(n)})$$

and hence $\|g_n^{\alpha}\|^2_{L_2(\mu^{\otimes n})} = \|g_n^{\pi(\alpha)}\|^2_{L_2(\mu^{\otimes n})}$ for all $\pi \in \mathrm{S}(n)$ where we used the notation $\pi(\alpha) := (\alpha_{\pi(1)},\dots,\alpha_{\pi(n)})$. Applying this fact, we reduce our summation over $\alpha \in V_m^n$ to a summation over equivalence classes $[\alpha] \in V_m^n/\sim$ where

$$\alpha \sim \beta \quad \Leftrightarrow \quad \exists \pi \in \mathrm{S}(n)\colon \alpha = \pi(\beta).$$

Thus, since in (21) we fixed α_1, by taking equivalence classes for V_m^{n-1} we obtain

$$\begin{aligned}
&\|\mathbb{E}_t \mathcal{D}_{t,\cdot}\eta - \mathbb{E}_s \mathcal{D}_{s,\cdot}\eta\|^2_{L_2(\mathbb{P}\otimes\mu)} \\
&\quad= \sum_{n=2}^{\infty} n n! \sum_{[\underline{\alpha}]\in V_m^{n-1}/\sim} \frac{(n-1)!}{\gamma_1(\underline{\alpha})!\cdots\gamma_k(\underline{\alpha})!} \big\| g_n^{(k,\underline{\alpha})}\big\|^2_{L_2(\mu^{\otimes n})} \qquad (22)\\
&\qquad\times \big((t-r_{k-1})^{\gamma_k(\underline{\alpha})} - (s-r_{k-1})^{\gamma_k(\underline{\alpha})}\big) \prod_{1\le l<k} (r_l - r_{l-1})^{\gamma_l(\underline{\alpha})} \delta_{\underline{\alpha}},
\end{aligned}$$

because the cardinality of the equivalence class $[\underline{\alpha}]$ is $\frac{(n-1)!}{\gamma_1(\underline{\alpha})!\cdots\gamma_k(\underline{\alpha})!}$ and $\gamma_l(\underline{\alpha})$ is invariant of permutations of $\underline{\alpha}$. For $\gamma \ge 1$, we estimate

$$(t-r_{k-1})^{\gamma} - (s-r_{k-1})^{\gamma} = \int_s^t \gamma (r-r_{k-1})^{\gamma-1}\,\mathrm{d}r$$

using for the integrand on the right-hand side the inequality

$$(r-r_{k-1})^{\gamma-1} \le \frac{1}{(r_k-u)(u-r)} \int_u^{r_k}\int_r^{\rho} (v-r_{k-1})^{\gamma-1}\,\mathrm{d}v\,\mathrm{d}\rho, \qquad r_{k-1} \le r < u < r_k.$$

For $u = \frac{r_k+r}{2}$ this leads to

$$(t-r_{k-1})^{\gamma} - (s-r_{k-1})^{\gamma} \le \frac{4}{(\gamma+1)} \int_s^t \frac{(r_k-r_{k-1})^{\gamma+1} - (r-r_{k-1})^{\gamma+1}}{(r_k-r)^2}\,\mathrm{d}r.$$

This yields

$$\|\mathbb{E}_t\mathcal{D}_{t,\cdot}\eta - \mathbb{E}_s\mathcal{D}_{s,\cdot}\eta\|^2_{L_2(\mathbb{P}\otimes\mu)}$$
$$\leq 4\int_s^t \sum_{n=1}^{\infty} n! \sum_{[\underline{\alpha}]\in V_m^{n-1}/\sim} \frac{n!}{\gamma_1(\underline{\alpha})!\cdots\gamma_{k-1}(\underline{\alpha})!(\gamma_k(\underline{\alpha})+1)!}\|g_n^{(k,\underline{\alpha})}\|^2_{L_2(\mu^{\otimes n})}$$
$$\times \frac{(r_k - r_{k-1})^{\gamma_k(\underline{\alpha})+1} - (r - r_{k-1})^{\gamma_k(\underline{\alpha})+1}}{(r_k - r)^2} \prod_{1\leq l<k}(r_l - r_{l-1})^{\gamma_l(\underline{\alpha})}\delta_{\underline{\alpha}}\,\mathrm{d}r,$$

where for $\gamma_k(\underline{\alpha}) = 0$ we have used

$$0 = (t - r_{k-1})^{\gamma_k(\underline{\alpha})} - (s - r_{k-1})^{\gamma_k(\underline{\alpha})} \leq \int_s^t \frac{(r_k - r_{k-1}) - (r - r_{k-1})}{(r_k - r)^2}\,\mathrm{d}r.$$

Because of

$$\gamma_l(\alpha) = \gamma_l(\underline{\alpha}), \qquad 0 < l < k \quad \text{and} \quad \gamma_k(\alpha) = \gamma_k(\underline{\alpha}) + 1$$

if $\alpha = (k, \underline{\alpha})$ we finally get

$$\|\mathbb{E}_t\mathcal{D}_{t,\cdot}\eta - \mathbb{E}_s\mathcal{D}_{s,\cdot}\eta\|^2_{L_2(\mathbb{P}\otimes\mu)}$$
$$\leq 4\int_s^t \sum_{n=1}^{\infty} n! \sum_{[\alpha]\in V_m^{n}/\sim} \frac{n!}{\gamma_1(\alpha)!\cdots\gamma_k(\alpha)!}\|g_n^{\alpha}\|^2_{L_2(\mu^{\otimes n})}$$
$$\times \frac{(r_k - r_{k-1})^{\gamma_k(\alpha)} - (r - r_{k-1})^{\gamma_k(\alpha)}}{(r_k - r)^2} \prod_{l<k}(r_l - r_{l-1})^{\gamma_l(\alpha)}\delta_{\alpha}\,\mathrm{d}r$$
$$= 4\int_s^t \sum_{n=1}^{\infty} n! \sum_{\alpha\in V_m^n} \|g_n^{\alpha}\|^2_{L_2(\mu^{\otimes n})} \frac{\lambda^{\otimes n}(\Lambda_\alpha \cap (]0, r_k]^n \setminus]0, r]^n))}{(r_k - r)^2}\,\mathrm{d}r$$
$$= 4\int_s^t \frac{\|\mathbb{E}_{r_k}\eta - \mathbb{E}_r\eta\|^2}{(r_k - r)^2}\,\mathrm{d}r. \qquad \square$$

Lemma 5.4. *If $\eta \in \mathbb{H}\cap\mathbb{D}_{1,2}$ and $k \in \{1, \ldots, m\}$ then for λ-a.e. $t \in]r_{k-1}, r_k[$*

$$\|\mathbb{E}_t\mathcal{D}_{t,\cdot}\eta\|^2_{L_2(\mathbb{P}\otimes\mu)} \leq \frac{\|\mathbb{E}_{r_k}\eta - \mathbb{E}_t\eta\|^2}{r_k - t}.$$

Proof. Similar to the proof of the previous lemma, we get (using the same notation)

$$\|\mathbb{E}_t\mathcal{D}_{t,\cdot}\eta\|^2_{L_2(\mathbb{P}\otimes\mu)}$$
$$= \left\|\sum_{n=1}^{\infty} n I_{n-1}\left(\sum_{\alpha\in V_m^n} g_n^{\alpha}\mathbb{1}_{\Lambda_{\alpha_1}}(t)\mathbb{1}_{\Lambda_{\underline{\alpha}}}\mathbb{1}_{]0,t]}^{\otimes(n-1)}\right)\right\|^2_{L_2(\mathbb{P}\otimes\mu)}$$

$$= \sum_{n=1}^{\infty} n n! \sum_{\substack{\alpha \in V_m^n \\ \alpha_1 = k}} \|g_n^\alpha\|^2_{L_2(\mu^{\otimes n})} (t - r_{k-1})^{\gamma_k(\underline{\alpha})} \prod_{l<k} (r_l - r_{l-1})^{\gamma_l(\underline{\alpha})} \delta_{\underline{\alpha}}$$

$$\leq \sum_{n=1}^{\infty} n! \sum_{\alpha \in V_m^n} \|g_n^\alpha\|^2_{L_2(\mu^{\otimes n})} \frac{(r_k - r_{k-1})^{\gamma_k(\alpha)} - (t - r_{k-1})^{\gamma_k(\alpha)}}{r_k - t} \prod_{l<k} (r_l - r_{l-1})^{\gamma_l(\alpha)} \delta_\alpha$$

$$= \frac{\|\mathbb{E}_{r_k}\eta - \mathbb{E}_t \eta\|^2}{r_k - t}. \qquad \square$$

Lemma 5.5. *Suppose $u \in]r_{k-1}, T]$, $\eta \in \mathbb{H}_u \cap \mathbb{D}_{1,2}$ and $h \in L_2(\mu)$. Then the equation*

$$\frac{\mathbb{E}_s[\eta I_1(\mathbb{1}_{]s,a]} h)]}{a - s} = \frac{\mathbb{E}_s[\eta \int_{]s,a] \times \mathbb{R}} h(x) M(\mathrm{d}t, \mathrm{d}x)]}{a - s} = \int_{\mathbb{R}} \mathbb{E}_s \mathcal{D}_{s,x} \eta h(x) \mu(\mathrm{d}x)$$

is satisfied $\mathbb{P}$-a.s. for λ-a.e. $r_{k-1} < s < a \leq r_k \wedge u$.

Proof. By the Clark–Ocone–Haussmann formula (7), we express η as

$$\eta = \mathbb{E}\eta + \int_{]0,u] \times \mathbb{R}} {}^p(\mathcal{D}\eta)_{t,x} M(\mathrm{d}t, \mathrm{d}x).$$

Thus we can write

$$\mathbb{E}_s\left[\eta \int_{]s,a] \times \mathbb{R}} h(x) M(\mathrm{d}t, \mathrm{d}x)\right]$$
$$= \mathbb{E}_s\left[\int_{]0,u] \times \mathbb{R}} {}^p(\mathcal{D}\eta)_{t,x} M(\mathrm{d}t, \mathrm{d}x) \int_{]s,a] \times \mathbb{R}} h(x) M(\mathrm{d}t, \mathrm{d}x)\right]$$

(the constant $\mathbb{E}\eta$ multiplied with $\int_{]s,a] \times \mathbb{R}} h(x) M(\mathrm{d}t, \mathrm{d}x)$ gives zero when applying $\mathbb{E}_s$). Using now the conditional Itô-isometry, we arrive at

$$\mathbb{E}_s\left[\int_{]0,u] \times \mathbb{R}} {}^p(\mathcal{D}\eta)_{t,x} M(\mathrm{d}t, \mathrm{d}x) \int_{]s,a] \times \mathbb{R}} h(x) M(\mathrm{d}t, \mathrm{d}x)\right]$$
$$= \mathbb{E}_s \int_{]s,a] \times \mathbb{R}} \mathbb{E}_t \mathcal{D}_{t,x} \eta h(x) \mathbb{m}(\mathrm{d}t, \mathrm{d}x)$$
$$= \int_{]s,a] \times \mathbb{R}} \mathbb{E}_s \mathcal{D}_{t,x} \eta h(x) \mathbb{m}(\mathrm{d}t, \mathrm{d}x)$$
$$= (a - s) \int_{\mathbb{R}} \mathbb{E}_s \mathcal{D}_{s,x} \eta h(x) \mu(\mathrm{d}x)$$

since $\mathcal{D}\eta$ is $\mathbb{P} \otimes \mathbb{m}$-a.e. constant on the interval $]r_{k-1}, r_k \wedge u[$ with respect to the time variable because η is in $\mathbb{H}_u$. $\square$

Proof of Theorem 5.1. In the following, we will indicate the dependency of the constants on certain parameters but nevertheless the constants may vary from line to line.

(iii) $\Rightarrow$ (ii): This step is analogous to the proof of [14], Theorem 1, $(C2_l) \Rightarrow (C3_l)$. It holds

$$\begin{aligned}\|Y_t - Y_s\|^2 &\leq 2(t-s)\int_s^t \big\| f(r, X_r, Y_r, \bar{Z}_r)\big\|^2 \mathrm{d}r + 2\int_s^t \|Z_{r,\cdot}\|^2_{L_2(\mathbb{P}\otimes\mu)} \mathrm{d}r \\ &\leq c\big(L_f, \mu(\mathbb{R}), \kappa'\big)\int_s^t \big(1 + \|Y_r\|^2 + \|Z_{r,\cdot}\|^2_{L_2(\mathbb{P}\otimes\mu)}\big)\,\mathrm{d}r \\ &\leq c\big(L_f, \mu(\mathbb{R}), \kappa', c_3\big)\int_s^t (r_k - r)^{\theta_k - 1}\,\mathrm{d}r\end{aligned}$$

since $\int_0^T \|Y_r\|^2\,\mathrm{d}r < \infty$ and $\|\bar{Z}_r\| \leq \|\kappa'\|_{L_2(\mu)}\|Z_{r,\cdot}\|_{L_2(\mathbb{P}\otimes\mu)}$.

(ii) $\Rightarrow$ (i): The argument of [14], Theorem 1, $(C3_l) \Rightarrow (C4_l)$, works here as well so that we have

$$\|Y_{r_k} - \mathbb{E}_s Y_{r_k}\|^2 \leq 4\|Y_{r_k} - Y_s\|^2 \leq \frac{4c_2}{\theta_k}(r_k - s)^{\theta_k}.$$

(i) $\Rightarrow$ (iii): Step 1. We first assume that

$$\xi \in \mathbb{H} \cap \mathbb{D}_{1,2} \text{ and } f \text{ satisfies } (A_f 1). \tag{23}$$

Because of the relation

$$Y_r = \mathbb{E}_r Y_{r_k} + \mathbb{E}_r \int_r^{r_k} f(u, X_u, Y_u, \bar{Z}_u)\,\mathrm{d}u, \qquad r_{k-1} < r < r_k, \tag{24}$$

Lemma 3.3 and Theorem 3.4(iv) we have $\mathbb{P}$-a.s. for $\mathbb{m}$-a.e. $(t,x) \in\,]r_{k-1}, r_k[\, \times \mathbb{R}$ that

$$\begin{aligned}Z_{t,x} &= \lim_{r \searrow t} \mathcal{D}_{t,x} Y_r \\ &= \lim_{r \searrow t} \mathcal{D}_{t,x}\bigg(\mathbb{E}_r Y_{r_k} + \mathbb{E}_r \int_r^{r_k} f(u, X_u, Y_u, \bar{Z}_u)\,\mathrm{d}u\bigg) \\ &= \lim_{r \searrow t}\bigg(\mathbb{E}_r \mathcal{D}_{t,x} Y_{r_k} + \mathbb{E}_r \mathcal{D}_{t,x}\int_r^{r_k} f(u, X_u, Y_u, \bar{Z}_u)\,\mathrm{d}u\bigg) \\ &= \lim_{r \searrow t}\bigg(\mathbb{E}_r \mathcal{D}_{t,x} Y_{r_k} + \mathbb{E}_r \int_r^{r_k} \mathcal{D}_{t,x} f(u, X_u, Y_u, \bar{Z}_u)\,\mathrm{d}u\bigg) \\ &= \mathbb{E}_t \mathcal{D}_{t,x} Y_{r_k} + \mathbb{E}_t \int_t^{r_k} \mathcal{D}_{t,x} f(u, X_u, Y_u, \bar{Z}_u)\,\mathrm{d}u,\end{aligned} \tag{25}$$

where we assumed the right continuous versions of the according expressions: Since $Y_{r_k} \in \mathbb{H} \cap \mathbb{D}_{1,2}$ the expression $\mathcal{D}Y_{r_k}$ can be realized such that it is constant in t on $]r_{k-1}, r_k[$ and

$(\mathbb{E}_s \mathcal{D}_{t,x} Y_{r_k})_{s\in]r_{k-1},r_k[}$ is a martingale (for fixed x). From Lemma 5.4, we conclude that

$$\|Z_{t,\cdot}\|_{L_2(\mathbb{P}\otimes\mu)} \le \frac{\|Y_{r_k} - \mathbb{E}_t Y_{r_k}\|}{\sqrt{r_k - t}} + \int_t^{r_k} \big\|\mathcal{D}_{t,\cdot} f(u, X_u, Y_u, \bar{Z}_u)\big\|_{L_2(\mathbb{P}\otimes\mu)} \,\mathrm{d}u. \tag{26}$$

Since Lemma 3.2, the Lipschitz condition and relation (6) imply

$$\big|\mathcal{D}_{t,y} f(r, X_r, Y_r, \bar{Z}_r)\big| \le L_f\big(\mathbb{1}_{[t,T]}(r) + |\mathcal{D}_{t,y} Y_r| + |\mathcal{D}_{t,y} \bar{Z}_r|\big), \qquad y \neq 0,$$

and

$$\mathcal{D}_{t,0} f(r, X_r, Y_r, \bar{Z}_r) = \big(\mathbb{1}_{[t,T]}(r)\partial_x + \mathcal{D}_{t,0} Y_r \partial_y + \mathcal{D}_{t,0}\bar{Z}_r \partial_z\big) f(r, X_r, Y_r, \bar{Z}_r),$$

we have

$$\big\|\mathcal{D}_{t,\cdot} f(r, X_r, Y_r, \bar{Z}_r)\big\|_{L_2(\mathbb{P}\otimes\mu)} \le L_f\big(\sqrt{\mu(\mathbb{R})} + \|\mathcal{D}_{t,\cdot} Y_r\|_{L_2(\mathbb{P}\otimes\mu)} + \|\mathcal{D}_{t,\cdot}\bar{Z}_r\|_{L_2(\mathbb{P}\otimes\mu)}\big). \tag{27}$$

We take the Malliavin derivative of (24), and by Lemmas 3.3 and 5.4, we get

$$\|\mathcal{D}_{t,\cdot} Y_r\|_{L_2(\mathbb{P}\otimes\mu)} \le \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|}{\sqrt{r_k - r}} + \int_r^{r_k} \big\|\mathcal{D}_{t,\cdot} f(u, X_u, Y_u, \bar{Z}_u)\big\|_{L_2(\mathbb{P}\otimes\mu)} \,\mathrm{d}u. \tag{28}$$

In order to estimate $\|\mathcal{D}_{t,\cdot}\bar{Z}_r\|_{L_2(\mathbb{P}\otimes\mu)}$, we will use the representation

$$\bar{Z}_r = \frac{\mathbb{E}_r[(\mathbb{E}_u Y_{r_k}) I_1(\mathbb{1}_{]r,u]}\kappa')]}{u - r} + \int_r^{r_k} \frac{\mathbb{E}_r[f(a, X_a, Y_a, \bar{Z}_a) I_1(\mathbb{1}_{]r,a]}\kappa')]}{a - r} \,\mathrm{d}a,$$

for λ-a.e. u such that $r_{k-1} < r < u \le r_k$, which is a consequence of equation (25), the fact that $\mathbb{E}_u Y_{r_k} \in \mathbb{H}_u$, $f(a, X_a, Y_a, \bar{Z}_a) \in \mathbb{H}_a$ and Lemma 5.5.

Hence for $r_{k-1} < t < r < u < r_k$ the 'conditional' Hölder inequality implies

$$\begin{aligned}
&\|\mathcal{D}_{t,\cdot}\bar{Z}_r\|_{L_2(\mathbb{P}\otimes\mu)} \\
&\quad\le \left\|\frac{\mathbb{E}_r[(\mathcal{D}_{t,\cdot}(\mathbb{E}_u Y_{r_k})) I_1(\mathbb{1}_{]r,u]}\kappa')]}{u - r}\right\|_{L_2(\mathbb{P}\otimes\mu)} \\
&\qquad + \int_r^{r_k} \left\|\frac{\mathbb{E}_r[(\mathcal{D}_{t,\cdot} f(a, X_a, Y_a, \bar{Z}_a)) I_1(\mathbb{1}_{]r,a]}\kappa')]}{a - r}\right\|_{L_2(\mathbb{P}\otimes\mu)} \mathrm{d}a \\
&\quad\le \frac{\|\mathcal{D}_{t,\cdot}(\mathbb{E}_u Y_{r_k})\|_{L_2(\mathbb{P}\otimes\mu)} \|\kappa'\|_{L_2(\mu)}}{\sqrt{u - r}} \\
&\qquad + c\big(L_f, \mu(\mathbb{R}), \kappa'\big) \int_r^{r_k} \frac{1 + \|\mathcal{D}_{t,\cdot} Y_a\|_{L_2(\mathbb{P}\otimes\mu)} + \|\mathcal{D}_{t,\cdot}\bar{Z}_a\|_{L_2(\mathbb{P}\otimes\mu)}}{\sqrt{a - r}} \,\mathrm{d}a,
\end{aligned}$$

where we used that $(\mathbb{E}_r I_1(\mathbb{1}_{]r,u]}\kappa')^2)^{1/2} \le c(\kappa')\sqrt{u - r}$ a.s., and from (27) one gets the estimate for the integral. Choosing $u = \frac{r_k + r}{2}$, we conclude by Lemma 5.4 the inequality

$$\frac{\|\mathcal{D}_{t,\cdot}(\mathbb{E}_u Y_{r_k})\|_{L_2(\mathbb{P}\otimes\mu)}}{\sqrt{u - r}} \le 2\frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|}{r_k - r}.$$

From the estimate (28) for $\mathcal{D}_{t,\cdot}Y_r$ and the above one for $\mathcal{D}_{t,\cdot}\bar{Z}_r$, we obtain

$$\begin{aligned}
&\|\mathcal{D}_{t,\cdot}Y_r\|_{L_2(\mathbb{P}\otimes\mu)} + \|\mathcal{D}_{t,\cdot}\bar{Z}_r\|_{L_2(\mathbb{P}\otimes\mu)} \\
&\quad \le c(\kappa')\frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|}{r_k - r} \\
&\qquad + c(L_f, \mu(\mathbb{R}), \kappa')\int_r^{r_k} \frac{1 + \|\mathcal{D}_{t,\cdot}Y_a\|_{L_2(\mathbb{P}\otimes\mu)} + \|\mathcal{D}_{t,\cdot}\bar{Z}_a\|_{L_2(\mathbb{P}\otimes\mu)}}{\sqrt{a-r}}\,\mathrm{d}a
\end{aligned}$$

which can be treated using an iteration and Gronwall's lemma (see the proof of Lemma 4 in [14]) in order to get

$$\|\mathcal{D}_{t,\cdot}Y_r\|_{L_2(\mathbb{P}\otimes\mu)} + \|\mathcal{D}_{t,\cdot}\bar{Z}_r\|_{L_2(\mathbb{P}\otimes\mu)} \le c(L_f, \mu(\mathbb{R}), \kappa')\frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|}{r_k - r}. \tag{29}$$

Hence from (26) and (27), it follows

$$\begin{aligned}
\|Z_{t,\cdot}\|_{L_2(\mathbb{P}\otimes\mu)} &\le \frac{\|Y_{r_k} - \mathbb{E}_t Y_{r_k}\|}{\sqrt{r_k - t}} \\
&\quad + c(L_f, \mu(\mathbb{R}), \kappa')\int_t^{r_k}\left(1 + \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|}{r_k - r}\right)\mathrm{d}r.
\end{aligned} \tag{30}$$

Step 2. Here we use Remark 4.1 and approximate $\xi \in \mathbb{H}$ by a sequence $(\xi_n)_n \subseteq \mathbb{H}\cap\mathbb{D}_{1,2}$ and f such that (A_f) is fulfilled by $(f_n)_n$ satisfying (A_f1). The convergence (13) implies that we can find a subsequence (Z^{n_k}) which we will again denote by (Z^n) such that for λ-a.e. $t \in [0, T]$

$$\left\|Z^n_{t,\cdot} - Z_{t,\cdot}\right\|^2_{L_2(\mathbb{P}\otimes\mu)} \to 0 \qquad \text{as } n \to \infty. \tag{31}$$

From the first step, we conclude that (30) holds for Z^n and therefore

$$\begin{aligned}
\|Z_{t,\cdot}\|_{L_2(\mathbb{P}\otimes\mu)} &\le \left\|Z_{t,\cdot} - Z^n_{t,\cdot}\right\|_{L_2(\mathbb{P}\otimes\mu)} + \left\|Z^n_{t,\cdot}\right\|_{L_2(\mathbb{P}\otimes\mu)} \\
&\le \left\|Z_{t,\cdot} - Z^n_{t,\cdot}\right\|_{L_2(\mathbb{P}\otimes\mu)} + \frac{\|Y^n_{r_k} - \mathbb{E}_t Y^n_{r_k}\|}{\sqrt{r_k - t}} \\
&\quad + c(L_f, \mu(\mathbb{R}), \kappa')\int_t^{r_k}\left(1 + \frac{\|Y^n_{r_k} - \mathbb{E}_r Y^n_{r_k}\|}{r_k - r}\right)\mathrm{d}r \\
&\le \frac{\|Y_{r_k} - \mathbb{E}_t Y_{r_k}\|}{\sqrt{r_k - t}} + c(L_f, \mu(\mathbb{R}), \kappa')\int_t^{r_k}\left(1 + \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|}{r_k - r}\right)\mathrm{d}r \\
&\quad + \left\|Z_{t,\cdot} - Z^n_{t,\cdot}\right\|_{L_2(\mathbb{P}\otimes\mu)} + \frac{2\|Y_{r_k} - Y^n_{r_k}\|}{\sqrt{r_k - t}} \\
&\quad + c(L_f, \mu(\mathbb{R}), \kappa')\int_t^{r_k}\frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k} - (Y^n_{r_k} - \mathbb{E}_r Y^n_{r_k})\|}{r_k - r}\,\mathrm{d}r.
\end{aligned}$$

For sufficiently large n the terms in the second last line are arbitrarily small. For the last term, we use the relation (24) and get

$$
\begin{aligned}
&\int_t^{r_k} \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k} - (Y_{r_k}^n - \mathbb{E}_r Y_{r_k}^n)\|}{r_k - r} \mathrm{d}r \\
&\quad \le \int_t^{r_k} \frac{\int_r^{r_k} \|f(u, X_u, Y_u, \bar{Z}_u) - f(u, X_u, Y_u^n, \bar{Z}_u^n)\| \,\mathrm{d}u}{r_k - r} \mathrm{d}r \\
&\quad = \int_t^{r_k} \int_t^u \frac{1}{r_k - r} \mathrm{d}r \big\| f(u, X_u, Y_u, \bar{Z}_u) - f\big(u, X_u, Y_u^n, \bar{Z}_u^n\big)\big\| \,\mathrm{d}u \qquad (32)\\
&\quad \le \left[\int_t^{r_k} \left(\int_t^u \frac{1}{r_k - r} \mathrm{d}r\right)^2 \mathrm{d}u\right]^{1/2} \\
&\qquad \times \left[\int_t^{r_k} \big\| f(u, X_u, Y_u, \bar{Z}_u) - f\big(u, X_u, Y_u^n, \bar{Z}_u^n\big)\big\|^2 \,\mathrm{d}u\right]^{1/2},
\end{aligned}
$$

where the last factor is arbitrarily small for sufficiently large n.

(i) $\Rightarrow$ (iv): Step 1. We assume first that (23) holds for (ξ, f). In the following, we use the notation $\mathbf{f}(r) := f(r, X_r, Y_r, \bar{Z}_r)$. Then equation (25) allows us to write

$$
\begin{aligned}
&\left\| \int_{\mathbb{R}} (Z_{t,x} - Z_{s,x}) h(x) \mu(\mathrm{d}x) \right\| \\
&\quad \le \left\| \int_{\mathbb{R}} (\mathbb{E}_t \mathcal{D}_{t,x} Y_{r_k} - \mathbb{E}_s \mathcal{D}_{s,x} Y_{r_k}) h(x) \mu(\mathrm{d}x) \right\| \\
&\qquad + \left\| \int_{\mathbb{R}} \left[\mathbb{E}_t \int_t^{r_k} \mathcal{D}_{t,x} \mathbf{f}(r) \,\mathrm{d}r - \mathbb{E}_s \int_s^{r_k} \mathcal{D}_{s,x} \mathbf{f}(r) \,\mathrm{d}r \right] h(x) \mu(\mathrm{d}x) \right\| \\
&\quad \le \|\mathbb{E}_t \mathcal{D}_{t,\cdot} Y_{r_k} - \mathbb{E}_s \mathcal{D}_{s,\cdot} Y_{r_k}\|_{L_2(\mathbb{P}\otimes\mu)} \|h\|_{L_2(\mu)} \\
&\qquad + \left\| \int_{\mathbb{R}} \left[\mathbb{E}_t \int_t^{r_k} \mathcal{D}_{t,x} \mathbf{f}(r) \,\mathrm{d}r - \mathbb{E}_s \mathbb{E}_t \int_t^{r_k} \mathcal{D}_{t,x} \mathbf{f}(r) \,\mathrm{d}r \right] h(x) \mu(\mathrm{d}x) \right\| \\
&\qquad + \left\| \int_{\mathbb{R}} \mathbb{E}_s \int_s^t \mathcal{D}_{s,x} \mathbf{f}(r) \,\mathrm{d}r h(x) \mu(\mathrm{d}x) \right\|,
\end{aligned}
$$

where we have used that $\mathcal{D}\mathbf{f}(r)$ can be chosen to be constant on $]r_{k-1}, r_k \wedge r[$ that is, we may exchange $\mathcal{D}_{s,x}\mathbf{f}(r)$ by $\mathcal{D}_{t,x}\mathbf{f}(r)$ in the second term.

From Lemma 5.3, we obtain that

$$
\|\mathbb{E}_t \mathcal{D}_{t,\cdot} Y_{r_k} - \mathbb{E}_s \mathcal{D}_{s,\cdot} Y_{r_k}\|_{L_2(\mathbb{P}\otimes\mu)}^2 \le 4 \int_s^t \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|^2}{(r_k - r)^2} \mathrm{d}r.
$$

We combine (27) with (29) to get

$$
\big\| \mathcal{D}_{u,\cdot} \mathbf{f}(r) \big\|_{L_2(\mathbb{P}\otimes\mu)}^2 \le c\big(L_f, \mu(\mathbb{R}), \kappa'\big) \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|^2}{(r_k - r)^2}, \qquad (33)
$$

which is used to estimate

$$\begin{aligned}
&\left\| \int_{\mathbb{R}} \mathbb{E}_s \int_s^t \mathcal{D}_{s,x}\mathbf{f}(r)\,\mathrm{d}r\,h(x)\mu(\mathrm{d}x) \right\| \\
&\quad \le \|h\|_{L_2(\mu)} \int_s^t \big\|\mathcal{D}_{s,\cdot}\mathbf{f}(r)\big\|_{L_2(\mathbb{P}\otimes\mu)}\,\mathrm{d}r \\
&\quad \le \|h\|_{L_2(\mu)} \sqrt{c\big(L_f, \mu(\mathbb{R}), \kappa'\big)} \int_s^t \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|}{r_k - r}\,\mathrm{d}r.
\end{aligned}$$

From Lemma 5.5, we conclude that

$$\int_{\mathbb{R}} \mathbb{E}_t \mathcal{D}_{t,x}\mathbf{f}(r)h(x)\mu(\mathrm{d}x) = \frac{\mathbb{E}_t[I_1(\mathbb{1}_{]t,r]}h)\mathbf{f}(r)]}{r-t}.$$

Applying the Clark–Ocone–Haussmann formula (7) and (33) yields

$$\begin{aligned}
&\left\| \int_{\mathbb{R}} \mathbb{E}_t \mathcal{D}_{t,x}\mathbf{f}(r)h(x)\mu(\mathrm{d}x) - \mathbb{E}_s \int_{\mathbb{R}} \mathbb{E}_t \mathcal{D}_{t,x}\mathbf{f}(r)h(x)\mu(\mathrm{d}x) \right\|^2 \\
&\quad = \frac{1}{(r-t)^2} \left\| \int_{]s,t]\times\mathbb{R}} {}^p\big[\mathcal{D}_{u,y}\mathbb{E}_t\big(I_1(\mathbb{1}_{]t,r]}h)\mathbf{f}(r)\big)\big] M(\mathrm{d}u,\mathrm{d}y) \right\|^2 \\
&\quad \le \frac{1}{(r-t)^2} \int_s^t \int_{\mathbb{R}} \mathbb{E}\big|\mathbb{E}_t\big[I_1(\mathbb{1}_{]t,r]}h)\mathcal{D}_{u,y}\mathbf{f}(r)\big]\big|^2 \mathbb{m}(\mathrm{d}u,\mathrm{d}y) \\
&\quad \le \frac{1}{r-t} \|h\|_{L_2(\mu)}^2 \int_s^t \big\|\mathcal{D}_{u,\cdot}\mathbf{f}(r)\big\|_{L_2(\mathbb{P}\otimes\mu)}^2\,\mathrm{d}u \\
&\quad \le \frac{1}{r-t} \|h\|_{L_2(\mu)}^2 c\big(L_f, \mu(\mathbb{R}), \kappa'\big) \int_s^t \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|^2}{(r_k - r)^2}\,\mathrm{d}u.
\end{aligned}$$

For the first inequality, we have used that for $u < t < r$ it holds $\mathbb{P}\otimes\mathbb{m}$-a.e.

$$\mathcal{D}_{u,y}\big[I_1(\mathbb{1}_{]t,r]}h)\mathbf{f}(r)\big] = I_1(\mathbb{1}_{]t,r]}h)\mathcal{D}_{u,y}\mathbf{f}(r)$$

since $\mathcal{D}_{u,y} I_1(\mathbb{1}_{]t,r]}h) = 0$. This can be proved, for example, applying [16], Corollary 3.1, and approximation. Hence,

$$\begin{aligned}
&\left\| \int_t^{r_k} \left[\int_{\mathbb{R}} \mathbb{E}_t \mathcal{D}_{t,x}\mathbf{f}(r)h(x)\mu(\mathrm{d}x) - \mathbb{E}_s \int_{\mathbb{R}} \mathbb{E}_t \mathcal{D}_{t,x}\mathbf{f}(r)h(x)\mu(\mathrm{d}x) \right] \mathrm{d}r \right\| \\
&\quad \le \|h\|_{L_2(\mu)} \sqrt{c\big(L_f, \mu(\mathbb{R}), \kappa'\big)} \int_t^{r_k} \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|}{(r_k - r)\sqrt{r-t}}\,\mathrm{d}r \sqrt{t-s}.
\end{aligned}$$

Consequently, we infer

$$\begin{aligned}&\left\|\int_{\mathbb{R}}(Z_{t,x}-Z_{s,x})h(x)\mu(\mathrm{d}x)\right\|_{L_2}^2\\&\quad\le\|h\|_{L_2(\mu)}^2c\big(L_f,\mu(\mathbb{R}),\kappa'\big)\left[\int_s^t\frac{\|Y_{r_k}-\mathbb{E}_rY_{r_k}\|^2}{(r_k-r)^2}\,\mathrm{d}r\right.\\&\qquad\left.+\left(\int_t^{r_k}\frac{\|Y_{r_k}-\mathbb{E}_rY_{r_k}\|}{(r_k-r)\sqrt{r-t}}\,\mathrm{d}r\right)^2(t-s)\right].\end{aligned}\tag{34}$$

Obviously (5.1) implies

$$\int_s^t\frac{\|Y_{r_k}-\mathbb{E}_rY_{r_k}\|^2}{(r_k-r)^2}\,\mathrm{d}r\le c_1\int_s^t(r_k-r)^{\theta_k-2}\,\mathrm{d}r$$

and

$$\begin{aligned}\left(\int_t^{r_k}\frac{\|Y_{r_k}-\mathbb{E}_rY_{r_k}\|}{(r_k-r)\sqrt{r-t}}\,\mathrm{d}r\right)^2&\le c_1\left(\int_0^1(1-u)^{(\theta_k/2)-1}u^{-1/2}\,\mathrm{d}u\right)^2(r_k-t)^{\theta_k-1}\\&=c_1B^2\left(\frac{\theta_k}{2},\frac{1}{2}\right)(r_k-t)^{\theta_k-1},\end{aligned}$$

where B denotes the beta function. For $\theta_k<1$ one can see that for all $s,t\in\,]r_{k-1},r_k[$ with $s<t$ it holds

$$\begin{aligned}(r_k-t)^{\theta_k-1}(t-s)&\le\frac{r_k-r_{k-1}}{1-\theta_k}\big((r_k-t)^{\theta_k-1}-(r_k-s)^{\theta_k-1}\big)\\&=(r_k-r_{k-1})\int_s^t(r_k-r)^{\theta_k-2}\,\mathrm{d}r\end{aligned}$$

since this inequality is equivalent to

$$t-s:=\varepsilon(r_k-s)\le\frac{r_k-r_{k-1}}{1-\theta_k}\big[1-(1-\varepsilon)^{1-\theta_k}\big]$$

for $\varepsilon\in\,]0,1[$ and $s\in\,]r_{k-1},r_k[$, and the last inequality can be proved easily. For $\theta_k=1$ we have

$$(r_k-t)^0(t-s)\le\int_s^t\frac{r_k-r_{k-1}}{r_k-r}\,\mathrm{d}r.$$

Summarizing we get the assertion with

$$c_4=c_1c\big(L_f,\mu(\mathbb{R}),\kappa'\big)\left(1+B^2\left(\frac{\theta_k}{2},\frac{1}{2}\right)(r_k-r_{k-1})\right).$$

Step 2. Now we take the sequence $(\xi^n, f^n)_n$ from step 2 of the implication (i) $\Rightarrow$ (iii) and proceed with (34) in the same way as we did with (30). In order to get the analogous estimate, we use the relations

$$\int_s^t \frac{\| \int_r^{r_k} f(u, X_u, Y_u, \bar{Z}_u) - f(u, X_u, Y_u^n, \bar{Z}_u^n)\,\mathrm{d}u \|^2}{(r_k - r)^2}\,\mathrm{d}r \\ \le \int_s^t \frac{1}{r_k - r}\,\mathrm{d}r \int_{r_{k-1}}^{r_k} \big\| f(u, X_u, Y_u, \bar{Z}_u) - f\big(u, X_u, Y_u^n, \bar{Z}_u^n\big)\big\|^2\,\mathrm{d}u$$

which is arbitrarily small for fixed $s, t \in [r_{k-1}, r_k[\backslash N_k$ where $\lambda(N_k) = 0$ and large $n \in \mathbb{N}$, and

$$\int_t^{r_k} \frac{\int_r^{r_k} \| f(u, X_u, Y_u, \bar{Z}_u) - f(u, X_u, Y_u^n, \bar{Z}_u^n)\|\,\mathrm{d}u}{(r_k - r)\sqrt{r - t}}\,\mathrm{d}r \\ \le \frac{2\int_t^{r_k} \| f(u, X_u, Y_u, \bar{Z}_u) - f(u, X_u, Y_u^n, \bar{Z}_u^n)\|\,\mathrm{d}u}{r_k - t} \int_t^{(r_k+t)/2} \frac{1}{\sqrt{r-t}}\,\mathrm{d}r \\ + \frac{\sqrt{2}}{\sqrt{r_k - t}} \int_{(r_k+t)/2}^{r_k} \frac{\int_r^{r_k} \| f(u, X_u, Y_u, \bar{Z}_u) - f(u, X_u, Y_u^n, \bar{Z}_u^n)\|\,\mathrm{d}u}{r_k - r}\,\mathrm{d}r.$$

For the last term, we can apply the estimate (32) to see that the RHS is arbitrarily small for large $n \in \mathbb{N}$. □

6. A sufficient condition on ξ for fractional smoothness

Assume (A_f) is satisfied for (8) and $\xi \in \mathbb{H}$. If $k = m$, condition (i) of Theorem 5.1 means in fact

$$\|\xi - \mathbb{E}_s\xi\|^2 \le c_1(T - s)^{\theta_m}, \qquad s \in\,]r_{m-1}, T].$$

Following the ideas of [14], we will formulate a condition on $\xi \in \mathbb{H}$ which implies that (5.1) of Theorem 5.1 holds for all $k \in \{1, \ldots, m\}$.

Assume that $\check{X}$ and X are processes on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\check{X}$ is an independent copy of the Lévy process X. We define for $0 \le t < r \le T$

$$X_s^{t,r} := \int_0^s \mathbb{1}_{[0,T]\backslash]t,r]}(u)\,\mathrm{d}X_u + \int_0^s \mathbb{1}_{]t,r]}(u)\,\mathrm{d}\check{X}_u, \qquad s \in [0, T], \tag{35}$$

that is, we obtain the process $X^{t,r}$ from X by replacing it on the interval $]t, r]$ by its independent copy. Consequently, for the random measure $M^{t,r}$ w.r.t. $X^{t,r}$ we have the relation

$$M^{t,r}(B) = M\big(B \setminus \big(]t, r] \times \mathbb{R}\big)\big) + \check{M}\big(B \cap \big(]t, r] \times \mathbb{R}\big)\big), \qquad B \in \mathcal{B}\big([0, T] \times \mathbb{R}\big).$$

By $(\hat{\mathcal{F}}_t)_{t\in[0,T]}$ we denote the augmented natural filtration w.r.t. $(X, \check{X})$ and put $L_2 := L_2(\Omega, \hat{\mathcal{F}}_T, \mathbb{P})$ (the notation $(\mathcal{F}_t)_{t\in[0,T]}$ we keep for the augmented natural filtration w.r.t. X).

For symmetric $f_n \in L_2^n$ it holds

$$\left\| I_n^{t,r}(f_n) - I_n(f_n) \right\|^2 = 2n! \left\| f_n (1 - \mathbb{1}_{(([0,T]\setminus]t,r]) \times \mathbb{R})^n}) \right\|_{L_2^n}^2. \tag{36}$$

For any $\eta \in L_2$ given by $\eta = \sum_{n=0}^{\infty} I_n(f_n)$, we define

$$\eta^{t,r} := \sum_{n=1}^{\infty} I_n^{t,r}(f_n).$$

Theorem 6.1. *Assume that $\xi \in \mathbb{H}$ and (A_f) is satisfied for (8). If there exist constants $c > 0$ and $\theta_k \in]0,1]$ such that*

$$\left\| \xi - \xi^{t,r_k} \right\|^2 \le c(r_k - t)^{\theta_k} \qquad \textit{for all } t \in [r_{k-1}, r_k]$$

then

$$\| Y_{r_k} - \mathbb{E}_t Y_{r_k} \|^2 \le C(r_k - t)^{\theta_k} \qquad \textit{for all } t \in [r_{k-1}, r_k].$$

Remark 6.2. (i) For $f = 0$, it follows from Theorem 6.1 the implication

$$\begin{aligned} &\left\| \xi - \xi^{t,r_k} \right\|^2 \le c(r_k - t)^{\theta_k} \qquad \text{for all } t \in [r_{k-1}, r_k] \\ &\quad\Longrightarrow \quad \| \mathbb{E}_{r_k} \xi - \mathbb{E}_t \xi \|^2 \le c(r_k - t)^{\theta_k} \qquad \text{for all } t \in [r_{k-1}, r_k]. \end{aligned} \tag{37}$$

For certain ξ the implication (37) is in fact an equivalence: for example, if $\xi = g(X_{r_m} - X_{r_{m-1}}, \dots, X_{r_1} - X_{r_0}) \in L_2$ such that g is a symmetric function and $r_k = \frac{kT}{m}$, for $k = 0, \dots, m$. A more detailed discussion under which conditions equivalence holds for (37) as well as an example where $\| \mathbb{E}_{r_k} \xi - \mathbb{E}_t \xi \|^2 \le c(r_k - t)^{\theta_k}$, for all $t \in [r_{k-1}, r_k]$ does not imply $\| \xi - \xi^{t,r_k} \|^2 \le c(r_k - t)^{\theta_k}$, for all $t \in [r_{k-1}, r_k]$ can be found in [18].

(ii) If $\xi \in \mathbb{H}$ the case $\Theta = (1, 1, \dots, 1)$ corresponds to Malliavin differentiability:

$$\begin{aligned} \exists c > 0\colon \left\| \xi - \xi^{t,r_k} \right\|^2 \le c(r_k - t) \qquad &\text{for all } t \in [r_{k-1}, r_k], k = 1, \dots, m \\ &\Longleftrightarrow \qquad \xi \in \mathbb{D}_{1,2}. \end{aligned} \tag{38}$$

Indeed, using the notation of the proof of Lemma 5.3 and setting $\binom{n}{\gamma(\alpha)} := \frac{n!}{\gamma_1(\alpha)! \cdots \gamma_m(\alpha)!}$ we have

$$\begin{aligned} \left\| \xi - \xi^{t,r_k} \right\|^2 = 2 \sum_{n=1}^{\infty} n! \sum_{[\alpha] \in V_m^n/\sim} \binom{n}{\gamma(\alpha)} \left\| g_n^{\alpha} \right\|_{L_2(\mu^{\otimes n})}^2 \\ \times \left((r_k - r_{k-1})^{\gamma_k(\alpha)} - (t - r_{k-1})^{\gamma_k(\alpha)} \right) \prod_{\substack{1 \le l \le m \\ l \ne k}} (r_l - r_{l-1})^{\gamma_l(\alpha)}. \end{aligned}$$

This implies for $r := \frac{t-r_{k-1}}{r_k-r_{k-1}}$ and $R := \max_{1\le k\le m} \frac{1}{r_k-r_{k-1}}$ that

$$\begin{aligned}\frac{\|\xi-\xi^{t,r_k}\|^2}{r_k-t} &= \frac{2}{r_k-r_{k-1}}\sum_{n=1}^\infty n! \sum_{[\alpha]\in V_m^n/\sim}\binom{n}{\gamma(\alpha)}\|g_n^\alpha\|^2_{L_2(\mu^{\otimes n})}\lambda^n(\Lambda_\alpha)\\ &\qquad\times \mathbb{1}_{\{\gamma_k(\alpha)\ge 1\}}\big(1+r+\cdots+r^{\gamma_k(\alpha)-1}\big)\\ &\le 2R\sum_{n=1}^\infty n! \sum_{[\alpha]\in V_m^n/\sim}\binom{n}{\gamma(\alpha)}\|g_n^\alpha\|^2_{L_2(\mu^{\otimes n})}\lambda^n(\Lambda_\alpha)\gamma_k(\alpha)\\ &\le 2R\|\mathcal{D}\xi\|^2_{\mathbb{P}\otimes\mathbb{m}}\end{aligned}$$

since $\gamma_k(\alpha)\le n$. On the other hand, we get because of $n=\sum_{k=1}^m \gamma_k(\alpha)$ for $\alpha\in V_m^n$ from the above relation that

$$\begin{aligned}\|\mathcal{D}\xi\|^2_{\mathbb{P}\otimes\mathbb{m}} &= \sum_{k=1}^m\sum_{n=1}^\infty n! \sum_{[\alpha]\in V_m^n/\sim}\binom{n}{\gamma(\alpha)}\|g_n^\alpha\|^2_{L_2(\mu^{\otimes n})}\lambda^n(\Lambda_\alpha)\gamma_k(\alpha)\\ &\le \frac{T}{2}\sup_{1\le k\le m}\sup_{r_{k-1}<t<r_k}\frac{\|\xi-\xi^{t,r_k}\|^2}{r_k-t}.\end{aligned}$$

In [18], there is an example which shows that (38) is not necessarily true without assuming $\xi\in\mathbb{H}$.

Example 6.3. If X is any square integrable Lévy process it holds for $\xi := \mathbb{1}_{]K,\infty[}(X_1)$ with $K\in\mathbb{R}$ that

$$\xi\in\mathbb{D}_{1,2} \quad\Longleftrightarrow\quad \sigma=0 \quad\text{and}\quad \int_{\{|x|\le 1\}}|x|\,\mathrm{d}\nu(x)<\infty$$

(see [26], Example 3.1). If X is a tempered α-stable process given by the Lévy measure

$$\nu_\alpha(\mathrm{d}x)=\frac{d}{|x|^{1+\alpha}}(1+|x|)^{-m}\mathbb{1}_{\{x\neq 0\}}\,\mathrm{d}x,$$

where $d>0$, $\alpha\in]0,2[$ and $m\in]2-\alpha,\infty[$, it follows from [15], Section 4.2, that

$$\xi\in\mathbb{B}^{1/2}_{2,\infty} := (L_2,\mathbb{D}_{1,2})_{1/2,\infty},$$

that is, (see Remarks 5.2(i) and 6.2(i)) there exists a $c>0$:

$$\big\|\xi-\xi^{t,1}\big\|^2\le c(1-t)^{1/2} \qquad \text{for all } t\in[0,1].$$

Consequently, for any $\alpha\in[1,2[$ the above ξ is in $\mathbb{B}^{1/2}_{2,\infty}$ but not in $\mathbb{D}_{1,2}$.

Proof of Theorem 6.1. If (Y,Z) is a solution of (8), then $(Y^{t,r},Z^{t,r})$ solves

$$\mathcal{Y}_u=\xi^{t,r}+\int_u^T f\big(s,X_s^{t,r},\mathcal{Y}_s,\bar{\mathcal{Z}}_s\big)\,\mathrm{d}s-\int_{]u,T]\times\mathbb{R}}\mathcal{Z}_{s,x}M^{t,r}(\mathrm{d}s,\mathrm{d}x).$$

From (36), we conclude that

$$\left\| \mathbb{E}_r I_n^{t,r}(f_n) - \mathbb{E}_r I_n(f_n) \right\|^2 = 2 \left\| \mathbb{E}_t I_n(f_n) - \mathbb{E}_r I_n(f_n) \right\|^2.$$

Since Y_{r_k} is $\mathcal{F}_{r_k}$-measurable this implies for $t \in]r_{k-1}, r_k[$ that

$$2 \| Y_{r_k} - \mathbb{E}_t Y_{r_k} \|^2 = \left\| Y_{r_k} - Y_{r_k}^{t,r_k} \right\|^2.$$

Since M and M^{t,r_k} coincide on $]r_k, T] \times \mathbb{R}$ we have

$$\begin{aligned} Y_{r_k} - Y_{r_k}^{t,r_k} = \xi - \xi^{t,r_k} \\ &+ \int_{r_k}^T f(s, X_s, Y_s, \bar{Z}_s) - f\big(s, X_s^{t,r_k}, Y_s^{r_k}, \bar{Z}_s^{t,r_k}\big)\, \mathrm{d}s \\ &- \int_{]r_k,T]\times\mathbb{R}} \big(Z_{s,x} - Z_{s,x}^{t,r_k}\big) M(\mathrm{d}s, \mathrm{d}x). \end{aligned}$$

By Theorem 2.3, we get

$$\begin{aligned} &\mathbb{E}\big|Y_{r_k} - Y_{r_k}^{t,r_k}\big|^2 + \mathbb{E}\int_{]r_k,T]\times\mathbb{R}} \big|Z_{s,x} - Z_{s,x}^{t,r_k}\big|^2 \mathbb{m}(\mathrm{d}s, \mathrm{d}x) \\ &\quad \le C\left(\mathbb{E}\big|\xi - \xi^{t,r_k}\big|^2 + \mathbb{E}\int_{r_k}^T \big|f(s, X_s, Y_s, \bar{Z}_s) - f\big(s, X_s^{t,r_k}, Y_s, \bar{Z}_s\big)\big|^2\, \mathrm{d}s\right), \end{aligned}$$

which can be reduced by the Lipschitz property of f to

$$\begin{aligned} &\mathbb{E}\big|Y_{r_k} - Y_{r_k}^{t,r_k}\big|^2 + \mathbb{E}\int_{]r_k,T]\times\mathbb{R}} \big|Z_{s,x} - Z_{s,x}^{t,r_k}\big|^2 \mathbb{m}(\mathrm{d}s, \mathrm{d}x) \\ &\quad \le C\left(\mathbb{E}\big|\xi - \xi^{t,r_k}\big|^2 + \mathbb{E}\int_{r_k}^T L_f^2 \big|X_s - X_s^{t,r_k}\big|^2\, \mathrm{d}s\right). \end{aligned}$$

By definition of X^{t,r_k} in (35), we get for $s > r_k$

$$\mathbb{E}\big|X_s - X_s^{t,r_k}\big|^2 = \mathbb{E}\big|X_{r_k} - X_t + (\check{X}_{r_k} - \check{X}_t)\big|^2 = C_1(r_k - t).$$

Thus, there is a constant $\tilde{C}$ such that

$$\begin{aligned} &\mathbb{E}\big|Y_{r_k} - Y_{r_k}^{t,r_k}\big|^2 + \mathbb{E}\int_{]r_k,T]\times\mathbb{R}} \big|Z_{s,x} - Z_{s,x}^{t,r_k}\big|^2 \mathbb{m}(\mathrm{d}s, \mathrm{d}x) \\ &\quad \le C\mathbb{E}\big|\xi - \xi^{t,r_k}\big|^2 + \tilde{C}(r_k - t), \end{aligned}$$

which implies the assertion. □

7. Concluding remarks

1. The assumption that the Lévy process X is square integrable could be avoided by using a more general formulation of the Clark–Ocone–Haussmann formula and modifying the dependency of the generator $f(t, X_t, Y_t, \bar{Z}_t)$ on X_t. (If X is not square integrable, X_t does not belong to $\mathbb{D}_{1,2}$.)
2. A generalization to the setting of a d-dimensional Lévy process seems to be possible as well and similar results might be expected. For example, for a multidimensional Lévy process without Brownian part, a chaos decomposition and a Clark–Ocone–Haussmann formula can be found in [24] and [25]. This could be extended to general Lévy processes.
3. In this paper, the dependency of the driver with respect to the Z process is given by the integral $\int Z_{t,x}\kappa(\mathrm{d}x)$. A generalization to the dependency on finitely many integrals,

$$f\left(s, X_s, Y_s, \int Z_{t,x}\kappa_1(\mathrm{d}x), \dots, \int Z_{t,x}\kappa_n(\mathrm{d}x)\right),$$

where the variables $z_1, \dots, z_n$ in the generator underly the same assumptions as for one z-variable appears to be straightforward. Note that the choice $\kappa = \delta_0$ covers the case for the Z-variable from [5], for instance.
4. The investigation of the case where the terminal condition or the generator depends on paths of a process of a Lévy driven SDE is of major interest for further research, as well as the extension to assumptions beyond the Lipschitz generator setting like quadratic drivers.

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