Affine Decomposition of Isometries in Nilpotent Lie Groups

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Tässä työssä esitetään uusi tulos koskien isometrioiden säännöllisyyttä nilpotenttien yhtenäisten metristen Lien ryhmien välillä. Termillä metrinen Lien ryhmä tarkoitamme Lien ryhmää, joka on varustettu etäisyysfunktiolla siten, että ryhmän (vasen) siirtokuvaus on isometria, ja etäisyysfunktio indusoi topologian, joka Lien ryhmällä on monistona alun perin olemassa. Todistamme, että isometriat tässä tilanteessa ovat välttämättä affiinikuvauksia: jokainen isometria voidaan esittää yhdistettynä kuvauksena siirrosta ja isomorfismista. Tämän seurauksena kaksi isometrista ryhmää ovat välttämättä isomorfiset.

Klassisesti isometrioiden lineaariaffiinisuus on tunnettu Euklidisessa avaruudessa, mutta myöhemmin vastaava yleistetty tulos on todistettu reaalisissa normiavaruuksissa (Mazur–Ulam lause) ja nilpotenteissa yhtenäisissä Riemannilaisissa Lien ryhmissä (E.N. Wilson). Viime vuosina tulos on onnistuttu todistamaan myös sub-Riemannilaisissa ja subFinsleriläisissä Carnot'n ryhmissä. Metrinen Lien ryhmä on näitä kaikkia yleisempi avaruus, lukuun ottamatta ääretönulotteisia normiavaruuksia.

Todistus perustuu Montgomery–Zippinin lokaalisti kompaktien ryhmien teoriasta johdettaviin isometrioiden säännöllisyysominaisuuksiin sekä mainitun Wilsonin tuloksen käyttöön.

Toteamme lopuksi, että niin yhtenäisyys kuin nilpotenttiuskin ovat välttämättömiä oletuksia siinä mielessä, että voimme esittää vastaesimerkit kummasta tahansa näistä oletuksista luovuttaessa.

Abstract We show that any isometry between two connected nilpotent metric Lie groups can be expressed as a composition of a translation and an isomorphism, i.e. isometries have an affine decomposition. By the term metric Lie group we mean a Lie group with a left-invariant distance that induces the topology of the manifold. It also follows that two isometric groups are isomorphic in this setting. Classically isometries are known to have the affine decomposition in the setting of Euclidean space and more generally in normed vector spaces over $\mathbb{R}$ [MU32], nilpotent connected Riemannian Lie groups [Wil82], and subRiemannian (even subFinsler) Carnot groups [Ham90], [Kis03], [LDO14]. Metric Lie groups are more general spaces than these, excluding the infinite dimensional normed spaces. Our proof is based on the theory of locally compact groups of Montgomery–Zippin and on the usage of the above mentioned result by Wilson [Wil82]. In a sense our result is a maximal generalization: After proving the result we construct counterexamples to the result in the cases where either nilpotency or connectedness is dropped from the list of assumptions.

Contents

1 Introduction

In this section I will explain our result together with its Euclidean motivation or analogue first without any mathematics and then with some simple mathematical concepts. I will also show the reader an explicit proof of our result in the simple Euclidean setting.

With exception of this introduction section, which should be accessible to any undergraduate student of mathematics, there are some prerequisities to be able to completely understand this thesis. The reader is assumed to have the knowledge of Lie group theory at the level of some basic course treating the manifold-viewpoint and thus naturally also basics of differentiable manifolds. A reference covering such topics could be the book [War13]. Basic definitions and facts are still recalled here in Section 2 to fix notations and for a reminder. Any advanced techniques of algebra or Lie theory are introduced in preliminaries carefully, as well as anything that is not appropriate to assume for an undergraduate student to know.

An experienced reader can start directly from the Section 3 where we really state the theorem and its proof rigorously and with all details.

1.1 Some words without any mathematics

Think about the familiar 3-dimensional space we see around us. In how many ways you can think of moving a body atom by atom from one place to another? To be more precise we don't want to break the structure of the body. So let's make a restriction: We move the body such a way that distances between any pair of atoms is preserved, i.e. distances are the same finally as they were initially. So how many ways there are? Well, we can just translate the body. We can also rotate it. Or do any combination of translations and rotations. Oh, and there is one thing more: We can make a reflected copy of the body[1]. It seems that this is everything you can do, but how can we be sure?

Later in this introduction, we shall for a warm-up prove mathematically that there are no more ways in non-curved n-dimensional space obeying the familiar geometric laws we are used to (Euclidean space). This thesis is devoted to studying an analogue of this elementary problem in its most generality. We move to the spaces that not only are curved, but where possibly the curvature can't even be defined naturally, and still some geometry is present. Still we can study the generalizations of translations, rotations and reflections. We will precisely identify which extra assumptions then are necessary for the analogous result to hold, namely that there are no more ways of moving bodies atom by atom preserving mutual distances of atoms, than combinations of translations, rotations and reflections.

[1]In practice this may cause damage to the body: You can't easily make a left-handed ice-hockey stick from a right-handed one. But in principle the distance of pairs of atoms would be preserved.

1.2 Explaining the result

Let's move some steps towards mathematics. The result we will prove is stated in its fully formal form as follows

Theorem 1.1. *Let* (N_1, d_1) *and* (N_2, d_2) *be two connected nilpotent metric Lie groups. Then any isometry* $F\colon N_1 \to N_2$ *is affine.*

In the preliminaries section we are going to go through the necessary definitions precisely, but let's for now have some idea what is this result about.

Isometry Isometry is a map between metric spaces that preserves distances, i.e. any two points have the same separation from each other as their respective image points under the map.

Affine map In Euclidean space the map $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is called *linear affine* if it has the decomposition

$$F(x) = Ax + c\,,$$

where $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear map and $c \in \mathbb{R}^n$ is the translation part of the map.

More generally in a space equipped with some "multiplication", an operation similar to the vector sum of Euclidean space, an affine map is a map which breaks nicely to the translation part and the origin fixing part. Nicely means here that the origin fixing part preserves the multiplication of the space just like a linear map in Euclidean space preserves the sum: $L(x + y) = L(x) + L(y)$.

The space equipped with a properly behaving multiplication is a *group.* In a group G the translation by an element g is the map $L_g\colon G \to G$, $L_g(h) \mapsto gh$ so called *left-translation*[2]. In a group the map which presevers the multiplication, the analogue of a linear map, is a *homomorphism*, i.e. a map $\psi\colon G \to G$ for which $\psi(gh) = \psi(g)\psi(h)$ for every $g, h \in G$. In a group G we therefore call a map $F\colon G \to G$ *affine* if it has the decomposition[3]

$$F(x) = (L_g \circ \psi)(x)$$

with $g \in G$ and ψ a homomorphism.

Metric Lie group Lie groups are groups with a delicate topological structure, namely that of a differentiable manifold[4]. With the term *metric Lie group* we refer to a Lie group that is also a metric space in such a way that group structure, manifold structure, and metric structure are combatible with each other. This compatibility

[2] We could as well use right-translation $h \mapsto hg$, but let's stick with the left one since we must choose one and be consistent. Note that we will always denote the product in a group just by writing the elements side by side, i.e. $gh := g * h$.

[3] Actually we studied the problem in more generality where F can be an isometry between two *different* groups. But let me forget this for a moment because thinking about one group is simpler, and serves well for this introduction.

[4] Besides the topology, the structure of a differentiable manifold also provides us the notion of differentiability for maps defined on the manifold.

means that the distance function induces the manifold topology and that the distance is invariant under the left-translations. Notice that this really is the most general setting that makes sense: It would be weird to study the properties of a space with a topology, a distance and a group structure if the distance does not respect the group structure nor the topology.

Nilpotency We shall later introduce nilpotency very precisely, but let's say for now that it is an algebraic constraint on the group related to commutativity of group elements. All commutative (a.k.a. Abelian) groups are nilpotent, but there are many more nilpotent groups. For a non-Abelian example consider the algebra of linear operators on some vector space. One defines *commutator* of operators A and B to be $[A, B] := AB - BA$. For some restricted subgroup $\mathcal{F}$ of all linear operators (that is closed under the commutator) it can be the situation that there are some non-zero commutators but that all second order commutators vanish, i.e. $[A, [B, C]] = 0$ for all operators $A, B, C \in \mathcal{F}$. This is an example of a nilpotent but non-Abelian Lie algebra[5]. In a general nilpotent group there exist an analogue for the commutator above and nilpotency then means exactly that commutators of some order must vanish.

1.3 Warm-up: Proof in Euclidean space

In the Euclidean setting the statement of our result is the following:

Proposition 1.2. *Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be an isometry, i.e. a map for which $d(x, y) = d(F(x), F(y))$ for all $x, y \in \mathbb{R}^n$, where d denotes the usual Euclidean distance*

$$d(x, y) := \sqrt{\sum (x_i - y_i)^2}\,.$$

Then there exists a linear map $A\colon \mathbb{R}^n \to \mathbb{R}^n$ and a vector $c \in \mathbb{R}^n$ such that

$$F(x) = Ax + c \qquad \forall x \in \mathbb{R}^n\,.$$

Notice that Euclidean space with the usual vector addition is a commutative group. Commutative groups are nilpotent so this fact can be deduced from our theorem. But in any case this is classical and well known fact, although it is not immediately clear why it holds. We shall next construct a proof for it. It is just for doing some warm-up gymnastics with a simple case, we don't need this result for anything.

Proof. Recall that the Euclidean distance function d actually comes from the Euclidean scalar product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i\,.$$

[5]This is exactly the situation of the Heisenberg algebra in the quantum mechanics. In the quantum mechanics the fundamental operators of position and momentum don't commute, but their commutators commute with the original operators.

As any scalar product, this induces a norm on $\mathbb{R}^n$ defined by $\|x\| = \sqrt{\langle x, x\rangle}$ and a norm always induces a distance function by the relation $d(x,y) = \|x-y\|$. This is exactly how the Euclidean distance, norm, and scalar product are connected to each other. Notice the formula (to prove this, consider the scalar product $\langle x-y, x-y\rangle$)

$$\langle x, y\rangle = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x-y\|^2)\,.$$

This formula in hand, we observe that any isometry in Euclidean space preserves the scalar product if it fixes the origin:

$$\begin{aligned}\langle x, y\rangle &= \frac{1}{2}(d(x,0)^2 + d(y,0)^2 - d(x,y)^2)\\ &= \frac{1}{2}(d(F(x),F(0))^2 + d(F(y),F(0))^2 - d(F(x),F(y))^2)\\ &= \frac{1}{2}(d(F(x),0)^2 + d(F(y),0)^2 - d(F(x),F(y))^2) = \langle F(x), F(y)\rangle\,.\end{aligned}$$

Actually it is enough to show that an isometry fixing the origin is linear. Namely, if G is an isometry that does not fix the origin, then the map $\tilde{G}(x) := G(x) - G(0)$ fixes and thus is linear. Therefore the original map G has the affine decomposition asked for in the claim:

$$G(x) = \tilde{G}(x) + G(0)\,.$$

Let's check that an isometry $F\colon \mathbb{R}^n \to \mathbb{R}^n$ fixing the origin is linear. In Euclidean space length minimizing curves are lines. Because an isometry preserves distance, it must send lines to lines: If $A, B \in \mathbb{R}^n$, then the point C found from the line joining A to B must be mapped to the line joining $F(A)$ to $F(B)$ because otherwise

$$d(F(A),F(C)) + d(F(C),F(B)) > d(F(A),F(B)) = d(A,B) = d(A,C) + d(C,B)$$

which is a contradiction because F is isometry.

Therefore

$$\forall \mathbf{a},\mathbf{b} \in \mathbb{R}^n \quad \exists \mathbf{x},\mathbf{y} \in \mathbb{R}^n \quad \text{such that} \quad F(\mathbf{a} + t\mathbf{b}) = \mathbf{x} + f(t)\mathbf{y} \quad \forall t \in \mathbb{R} \tag{1}$$

for some $f\colon \mathbb{R} \to \mathbb{R}$. The idea of the proof is to work towards the linearity condition

$$F(\mathbf{a} + t\mathbf{b}) = F(\mathbf{a}) + tF(\mathbf{b})\,.$$

To have that, we should argue we can replace $\mathbf{x} = F(\mathbf{a})$, $f(t) \equiv t$ and $\mathbf{y} = F(\mathbf{b})$ in the above. This turns out to be possible because expressing lines with two vectors $\mathbf{a}, \mathbf{b}$ as $\mathbf{a} + t\mathbf{b}$ contains a lot of useless information, e.g. one can choose $\|\mathbf{b}\| = 1$ without any change to the actual line.

Set $t = 0$ to get $F(\mathbf{a}) = \mathbf{x} + f(0)\mathbf{y}$. Therefore

$$\begin{aligned}F(\mathbf{a} + t\mathbf{b}) &= \mathbf{x} + f(t)\mathbf{y} + F(\mathbf{a}) - F(\mathbf{a})\\ &= F(\mathbf{a}) + (f(t) - f(0))\mathbf{y}\,,\end{aligned}$$

and so we can express our condition (1) above in the form

$$\forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \quad \exists \mathbf{y} \in \mathbb{R}^n \quad \text{such that} \quad F(\mathbf{a} + t\mathbf{b}) = F(\mathbf{a}) + g(t)\mathbf{y} \quad \forall t \in \mathbb{R}$$

for some $g \colon \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$.

At this point it can be in principle that in the above expression $\mathbf{y}$ depends on both vectors $\mathbf{a}, \mathbf{b}$, but we shall argue that it can't depend on $\mathbf{a}$. Indeed, the lines with different $\mathbf{a}$ but same $\mathbf{b}$ are parallel. If $\mathbf{y}$ chances to $\mathbf{y}'$ when changing $\mathbf{a}$ to $\mathbf{a}'$ then the image lines $F(\mathbf{a} + t\mathbf{b})$ and $F(\mathbf{a}' + t\mathbf{b})$ would intersect. This is against injectivity[6] of F, since the lines in the domain were parallel and thus disjoint.

Because we now know that $\mathbf{y}$ can't depend on $\mathbf{a}$, we can extract information on g and $\mathbf{y}$ by setting $\mathbf{a} = 0$. We get

$$F(t\mathbf{b}) = F(0) + g(t)\mathbf{y} = g(t)\mathbf{y} \, .$$

We can assume that $\|\mathbf{y}\| = 1$, otherwise we just scale the function g. Choosing also $\|\mathbf{b}\| = 1$ and using that F preserves the norm we get

$$t = \|t\mathbf{b}\| = \|F(t\mathbf{b})\| = \|g(t)\mathbf{y}\| = |g(t)| \qquad \forall t \, .$$

We may assume $t = g(t)$ by possibly changing $\mathbf{y}$ to its opposite vector, as g must clearly be continuous. We have now the condition (1) in the form

$$\forall \mathbf{a} \in \mathbb{R}^n,\ \forall \mathbf{b} \in \mathbb{S}^{n-1} \quad \exists \mathbf{y} \in \mathbb{S}^{n-1} \quad \text{such that} \quad F(\mathbf{a} + t\mathbf{b}) = F(\mathbf{a}) + t\mathbf{y} \quad \forall t \in \mathbb{R} \, ,$$

where $\mathbf{y}$ does not depend on $\mathbf{a}$. Setting $\mathbf{a} = 0$ and $t = 1$ gives $F(\mathbf{b}) = \mathbf{y}$. We got finally

$$\forall \mathbf{a} \in \mathbb{R}^n,\ \forall \mathbf{b} \in \mathbb{S}^{n-1} \quad \text{it holds} \quad F(\mathbf{a} + t\mathbf{b}) = F(\mathbf{a}) + tF(\mathbf{b}) \quad \forall t \in \mathbb{R} \, .$$

This proves linearity. □

1.4 The general case

As we discussed, to generalize the Euclidean result above we need the generalized concept of affine map, for which the group structure is necessary. In order to talk about isometries on the other hand, the space in question must be a metric space. Seen in this way it actually is not immediate that one should require the manifold structure, i.e. require the space to be a Lie group. The minimal setting where the result would make sense is a group with a distance function that is invariant under group multiplication from the left (or right). Still, Lie groups appear in numerous applications and come with a rich theory to attack the problem. The tools of this theory are heavily used in our proof. Also the setting of Lie groups is already more general than any previous result in this area. Thus we will not discuss more about dropping the manifold structure.

[6] Indeed, an isometry is necessarily injective.

The result of this thesis is new as far as we know. The case of Euclidean space that we just considered has been known since centuries perhaps, if not by the ancient Greeks. There are of course more serious results, which are more like the real motivation for this study. The Euclidean case is just a convenient and understandable example. We shall discuss those more advanced results in the Section 5. But to summarize a little, Theorem 1.1 was known due to Wilson in the setting of (nilpotent and connected) *Riemannian* Lie groups [Wil82], i.e. Lie groups where the distance comes from a Riemannian metric tensor. The work of Hamenstädt [Ham90] and Kishimoto [Kis03] on the other hand gives the result in the setting of so called *subRiemannian Carnot groups* (see the complete proof in [LDO14]). Carnot groups have more assumptions on the algebraic structure of the space, but Riemannian Lie groups have more assumptions on the metric structure than subRiemannian Carnot groups. Our setting of metric Lie groups is a generalization of the both. But to stress the logic, our proof of the general result actually uses the result of Wilson, thus it does not make much sense to deduce the result of Wilson from our result, altough formally that would be correct.

It has been also widely known that there are counterexamples to Theorem 1.1 if we do not assume anything else than a metric Lie group. We will thus assume nilpotency from the group structure and connectedness from the topological structure. By providing counterexamples in the Section 4, we will also prove that one can't remove either of requirements of nilpotency and connectedness. Namely, we shall present 1) a connected non-nilpotent and 2) a non-connected nilpotent metric Lie group, where there exists isometries that do not have a decomposition to a homomorphism and a translation. In some sense one can think of this fact as maximality of our result. But notice still: It can well be the case that it is possible to relax a little bit for example the assumption of nilpotency. If some property is weaker than nilpotency and is not satisfied by our non-nilpotent counterexample, then perhaps one can establish Theorem 1.1 in connected metric Lie groups with that property.

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2 Preliminaries

In this section we shall go through some theory and techiniques on which the proof is based. We shall recall some important definitions and facts even if some of them are basically assumed as a prerequisite so that it is easy for the reader to check the exact forms as these facts and notions appear along the proof. For elementary facts we don't give references, but in general a good reference for the basic Lie theory is the book [War13] and for the topology of the homeomorphism group the books [McC88] and [Kel12]. For some results with a very short proof we will give the proof.

After discussing these topics, we shall present in more detail some more advanced concepts: Haar measures, nilpotent groups, semidirect product of groups and group actions.

2.1 The basics of Lie groups

Definition. *A Lie group* is a differentiable manifold G together with the group operations of multiplication $\mathrm{Mult}_G : G \times G \to G$, $(p, q) \mapsto p \cdot g$ and inversion $\mathrm{inv}_G : G \to G$ such that these are smooth maps with respect to the differentiable structure of the manifold. When we call a map *smooth*, it will always mean that it is of the class C^∞. The *left-translations* of the group, i.e. the maps $p \mapsto gp$, are denoted by L_g.

Definition. *A Lie algebra* is a vector space $\mathfrak{g}$ equipped with bilinear operator $[\cdot, \cdot]$ called *bracket* that satisfies

i) $[X, Y] = -[Y, X]$ (antisymmetry) and

ii) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity)

for all $X, Y, Z \in \mathfrak{g}$.

Let G be a Lie group. The *Lie algebra of* G, denoted by $\mathrm{Lie}(G)$, is the set of left-invariant vector fields on G equipped with the commutator of vector fields. The Lie algebra of a Lie group G is canonically isomorphic to $(T_eG, [\cdot, \cdot])$ where the bracket is calculated in the tangent space by extending vector fields left-invariantly and taking the usual commutator of vector fields calculated at the identity, i.e. for $v, w \in T_eG$ we set

$$[v, w]_{T_eG} := [\tilde{v}, \tilde{w}]_{\mathrm{Lie}(G)}(e) \in T_eG \, ,$$

where $\tilde{v}$ and $\tilde{w}$ are left-invariant extensions of vectors $v, w \in T_eG$. The left-invariant extension is constructed by the formula $\tilde{v}_g = (L_g)_* v$. From now on we do not usually bother to make the distinction between $\mathrm{Lie}(G)$ and T_eG.

Morphisms on Lie groups Recall that a *morphism of Lie groups* is a smooth algebraic homomorphism between two Lie groups[7]. A morphism is a map preserving the structure and Lie groups have the structures of a differentiable manifold and a

[7]Sometimes the term *Lie group homomorphism* is used, but we shall preserve the word "homomorphism" for morphisms of groups, i.e. requiring just the algebraic structure to be preserved.

group. Instead, an *isomorphism of Lie groups* is a map that is bijective and morphism to both directions: It is a diffeomorphism and a group isomorphism. A *morphism of Lie algebras* is a linear map that preserves the bracket, i.e. $\psi([X,Y]) = [\psi(X), \psi(Y)]$.

Fact 2.1. *Let $\varphi\colon G \to H$ be a morphism (resp. isomorphism) of Lie groups. It induces a morphism (resp. isomorphism) of Lie algebras $\varphi_*\colon \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ defined by $\varphi_* v = (\mathrm{d}\varphi)_e v_e$ where $v_e \in T_e G$ is the left-invariant vector field v calculated at the identity.*

Fact 2.2. *Let G, H be Lie groups, where G is simply connected. Then for all morphisms of Lie algebras $\psi\colon \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ there exists a unique morphism of Lie groups $\varphi\colon G \to H$ with $\varphi_* = \psi$.*

Fact 2.3. *Let G, H be Lie groups. If $\varphi\colon G \to H$ is a continuous homomorphism, then it is smooth, i.e. a morphism of Lie groups.*

The exponential map In a Lie group G the *exponential map* $\exp\colon \mathrm{Lie}(G) \to G$ is the map which associates to a vector $X \in T_e G$ the point where the integral curve of its left-invariant extension gets at time $t = 1$. Formally we write this as $\exp(X) = \Phi_X^1(e)$, where $\Phi_X^t(e)$ denotes the flow of X starting from e calculated at time t. The exponential map is globally smooth and a local diffeomorphism.

Fact 2.4. *Let $\varphi\colon G \to H$ be a morphism of Lie groups and $\varphi_*\colon \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ the induced morphism of Lie algebras. Then $\varphi \circ \exp = \exp \circ \varphi_*$.*

Lie subgroups For a subset of a Lie group G to be a "Lie subgroup" we should not only require it is algebraicly a subgroup but that it is also a submanifold of G. Different authors have different definitions of what actually is a submanifold which reflect to the different notions of a Lie subgroup. We shall use the following definition:

Definition. Let G, H be Lie groups such that $H \leq G$ as groups[8]. If

i) the inclusion map $H \hookrightarrow G$ is smooth[9] and has injective differential (i.e. is an immersion), we say that H is a *Lie subgroup* of G.

ii) the inclusion map is in addition a homeomorphism into its image (i.e. is an embedding), we say that H is a *regular Lie subgroup* of G.

An example where nontrivial things happen is the torus $\mathbb{S}^1 \times \mathbb{S}^1$. There the lines $\ell(x) = ax$, $a \in \mathbb{R}$ are one-dimensional subgroups, but they are not embedded submanifolds if the slope a is irrational.

Fact 2.5. *If H is a Lie subgroup of G, then* $\mathrm{Lie}(H)$ *is a subalgebra of* $\mathrm{Lie}(G)$ *(up to canonical isomorphism).*

[8] By the notation $H \leq G$ we always mean that H is a subgroup of G.

[9] Notice that smoothness is only a statement about differentiable structure in H. Also notice that the inclusion map is automatically a homomorphism and an injective map.

Proof. By Fact 2.1, for the inclusion map there exists an associated morphism of Lie algebras $\iota_*\colon \mathrm{Lie}(H) \to \mathrm{Lie}(G)$ which is injective by the definition of Lie subgroup. Therefore, if we restrict ι_* to its image, it is invertible. Thus ι_* makes $\mathrm{Lie}(H)$ isomorphic (as a Lie algebra) to a subset (and thus a subalgebra) of $\mathrm{Lie}(G)$. □

Fact 2.6. *Let G be a Lie group and $H \leq G$. If H is a closed set in G, then H is a regular Lie subgroup of G.*

Fact 2.7. *Let G be a Lie group. For every subalgebra $\mathfrak{h} \subset \mathrm{Lie}(G)$ there is a unique connected Lie subgroup $H \leq G$ with $\mathfrak{h}$ as its Lie algebra. In fact H is the group generated by $\exp(\mathfrak{h})$.*

2.2 The topology of the homeomorphism group

It is clear what is the topology of a Lie group: it is the manifold topology. Considering the metric Lie groups, the topology induced by the metric also agrees the manifold topology. Manifold topology has almost all the nice properties one could require for a topology because a manifold is locally just a Euclidean space. But in this thesis there appears one class of objects which we should treat carefully from topological viewpoint. We should make a sense of the topology of the isometry group of a metric Lie group. In general, if X and Y are topological spaces, there is one good topology in the set $C(X, Y)$, the space of continuous maps $X \to Y$. That topology is the following:

Definition. Let X and Y be topological spaces. For any $K \subset X$ compact and $U \subset Y$ open, denote

$$[K; U] = \{f \in C(X, Y) \mid f(K) \subset U\}\,.$$

The *compact-open topology* on the set $C(X, Y)$ is the topology τ_{co} that has the sets $[K; U]$ as its subbase, i.e. τ_{co} contains all unions and finite intersections of the sets of the form $[K; U]$ with K compact and U open.

In metric spaces the convergence of a sequence of maps corresponds to the usual uniform convergence in compact sets:

Fact 2.8. *Let X and Y be metric spaces, $(f_i) \subset C(X, Y)$ a sequence of continuous maps and $f \in C(X, Y)$. Then $f_n \to f$ in the compact-open topology if and only if $(f_n) \to f$ uniformly on compact sets, i.e.*

$$\forall K \subset X \text{ compact} \quad \forall \varepsilon > 0 \quad \exists m \in \mathbb{N} \text{ such that } \sup_{a \in K}(d(f_n(a), f(a)) < \varepsilon \quad \forall n \geq m\,.$$

The following is from [Are46b, Thm 5.]:

Fact 2.9. *Let A and B be second countable Hausdorff spaces with A locally compact. Then the space $(C(A, B), \tau_{co})$ is second countable.*

Every second countable space is a *sequential space*, which means that its topology is specified completely in terms of sequences. In particular we shall use the following:

Fact 2.10. *Let A and B be second countable Hausdorff spaces with A locally compact. Then a set $K \subset (C(A, B), \tau_{co})$ is*

i) compact if and only if it is sequentially compact, i.e. every sequence has a subsequence that converges in K.

ii) closed if and only if it is sequentially closed, i.e. every sequence of K that converges in $C(A, B)$ converges in K.

The compact-open topology makes the homeomorphism group a "topological group". This is a more general concept than a Lie group:

Definition. *A topological group* is a topological space (X, τ) together with the group operations of multiplication $\mathrm{Mult}_X : X \times X \to X$, $(p, q) \mapsto p \cdot g$ and inversion $\mathrm{inv}_X : X \to X$ such that these are continuous maps.

Fact 2.11. *Let X be a locally compact and locally connected Hausdorff-space. Then the set of homeomorphisms on X, denoted by* $\mathrm{Homeo}(X)$*, with the compact-open topology and the composition of mappings as the group operation is a topological group.*

One requires an argument to prove the continuity of the multiplication, let alone the continuity of the inversion. This was first proven by R. Arens in [Are46a, Thm 4.]. In this thesis our topological space X is a metric space and a manifold so the assumptions of Fact 2.11 are fulfilled. In particular, *the isometry group of a metric Lie group is a topological group when equipped with the compact-open topology.* For the rest of this thesis, we will always consider the homeomorphism/isometry group to be equipped with the compact-open topology.

Later in the Section 2.7 we shall study the concept of group action. In the language of actions the following fact means that the action $\mathrm{Homeo}(X) \curvearrowright X$ is continuous:

Fact 2.12. *If X is a locally compact Hausdorff space, then the map $\Phi\colon$ $\mathrm{Homeo}(X) \times X \to X$, $\Phi(F)(x) = F(x)$ is continuous.*

With the compact-open topology many natural continuity results, like the two following facts, hold:

Fact 2.13. *Let X, Y, Z be topological spaces with Y locally compact Hausdorff space. Then the map*

$$C(Y, Z) \times C(X, Y) \to C(X, Z) \qquad (f, g) \mapsto f \circ g$$

is continuous when the compact open topology is considered in the spaces of continuous maps.

Fact 2.14. *If X, Y, Z are topological spaces and $\phi\colon X \times Y \to Z$ is continuous then the corresponding map $\tilde{\phi}\colon X \to (C(Y, Z), \tau_{co})$ with $\tilde{\phi}(x) = \phi(x, \cdot)$ is continuous.*

These two facts gives us two corollaries that we need in our proof:

Fact 2.15. *Let X, Y be locally compact Hausdorff spaces and $F\colon X \to Y$ a homeomorphism. Then the map*

$$\hat{F}\colon \mathrm{Homeo}(X) \to \mathrm{Homeo}(Y) \qquad I \mapsto F \circ I \circ F^{-1}$$

is a homeomorphism.

Proof. For the continuity of $\hat{F}$, apply Fact 2.13 to the maps $I \mapsto F \circ I$ and $I \mapsto I \circ F^{-1}$. Because F^{-1} is also a homeomorphism, the inverse map $I \mapsto F^{-1} \circ I \circ F$ is continuous by the first part of the proof. □

Fact 2.16. *Let G be a topological group. Then the map $G \to \mathrm{Homeo}(G)$ for which $h \mapsto L_h$ is continuous.*

Proof. Apply Fact 2.14 fact to the case $\phi = \mathrm{Mult}_G$. □

In addition to the facts related to the homeomorphism group we will need the following result of the general theory of topological groups:

Fact 2.17. *Let G be a topological group and $K_1, K_2 \subset G$ compact sets. Then $K_1 \cdot K_2 := \{k_1 k_2 \mid k_1 \in K_1,\ k_2 \in K_2\}$ is compact.*

2.3 Basics of algebraic topology

Definition. Let X and $\tilde{X}$ be topological spaces. A continuous surjective map $\pi\colon \tilde{X} \to X$ is a *covering map* if any point $x \in X$ has a neighbourhood $U \subset X$ in such a way that $\pi^{-1}(U)$ is a disjoint union of open sets $V_\alpha \subset \tilde{X}$ and $\pi|_{V_\alpha}\colon V_\alpha \to U$ is a homeomorphism for all indexes α. In the case when such a map π exists, the space $\tilde{X}$ is called a *covering space* of X.

Definition. Let X be topological space with a covering space $\tilde{X}$ and a covering map $\pi\colon \tilde{X} \to X$. If $\tilde{X}$ is simply connected (a path connected space for which the fundamental group is trivial), then $\tilde{X}$ is called a *universal covering space.*

For a connected manifold, a universal covering space always exists. For Lie groups we have everything we can ask for (see [War13, p. 100]):

Fact 2.18. *Let G be a connected Lie group. Then there exists a Lie group $\tilde{G}$ and a map $\pi\colon \tilde{G} \to G$ in such a way that π is a morphism of Lie groups and a covering map, and $\tilde{G}$ is a universal covering space of G.*

Quotient groups If G is a topological group with a subgroup $\Gamma < G$, we can form the quotient space G/Γ whose elements are the equivalence classes $[g]$ with $g \in G$ and $g \sim g'$ if there exists $k \in \Gamma$ in such a way that $g' g^{-1} = k$. This set is a topological space when endowed with the usual quotient topology, i.e. the largest topology that makes the projection map

$$p\colon G \to G/\Gamma \qquad p(g) = [g]$$

continuous.

Let G be a topological group that has the topological group $\tilde{G}$ as its covering space with $\pi\colon \tilde{G} \to G$ the covering map. Then $\pi^{-1}(e_G) =: \Gamma$ is a subgroup of $\tilde{G}$ and we may consider the quotient space $\tilde{G}/\Gamma$. There exist now two different kind of "projection maps" as the diagram below illustrates

$$\begin{array}{ccc} \tilde{G} & & \\ {\scriptstyle\pi}\downarrow & \searrow^{p} & \\ G & & \tilde{G}/\Gamma \end{array} \; .$$

Actually, the spaces G and $\tilde{G}/\Gamma$ are homeomorphic. From the diagram above one can read the obvious candidate for this homeomorphism: $\Psi([g]) = \pi(g)$ where $[g]$ denotes the equivalence class in the space $\tilde{G}/\Gamma$. This is well defined: if $g' \sim g$, i.e.

$$g'g^{-1} = k \in \Gamma = \pi^{-1}(e) \,,$$

then

$$e = \pi(g'g^{-1}) = \pi(g')(\pi(g))^{-1}$$

so $\pi(g') = \pi(g)$. This reasoning also holds backwards, meaning that $[g] = [g']$ if and only if $\pi(g) = \pi(g')$. Therefore the map Ψ is bijective. We also know that both maps p and π are open: The covering map is always open, as well as the projection to a quotient group. Therefore the map Ψ is continuous and open and hence a homeomorphism.

The following fact is related to the general quotient spaces:

Fact 2.19. *Let X and Y be topological spaces with some partitions $X = \bigsqcup X_i$ and $Y = \bigsqcup Y_i$ inducing equivalence relations $\sim_X$ and $\sim_Y$. If $f\colon X \to Y$ is a homeomorphism for which $f(X_i) = Y_i$ for all i, then f induces a homeomorphism $F\colon X/{\sim_X} \to Y/{\sim_Y}$ by the formula $F([x]) = [f(x)]$.*

2.4 Haar measures

Let X be a topological space. Recall that a *Borel measure* is a measure $\mu\colon \mathcal{M}_X \to [0,\infty]$ for which any open set is measurable. Here $\mathcal{M}_X$ denotes the σ-algebra of measurable sets of X. A *Radon measure* has in addition the properties

i) $\mu(K) < \infty$ for any $K \subset X$ compact,

ii) for all $B \in \mathcal{B}_X$ holds $\mu(B) = \inf\{\mu(V) \mid B \subset V,\ V \text{ open}\}$ and

iii) for all open sets U holds $\mu(U) = \sup\{\mu(K) \mid K \subset U,\ K \text{ compact}\}$.

A nonzero Radon measure μ on a Lie group G is called *left-Haar measure* (respectively *right-Haar measure*) if it is invariant by left-translations, i.e. if $\mu(A) = \mu(L_g(A))$ (respectively if $\mu(A) = \mu(R_g(A))$) for any measurable set A and for any $g \in G$. A measure is called a *Haar-measure* if it is both left- and right-invariant. In compact groups such a Haar measure always exists [Kna13, p. 239]:

Fact 2.20. *i) In a Lie group there always exists a left-Haar-measure.*

ii) In a compact Lie group any left-Haar measure is also a right-Haar measure.

Let's study a little about integrating with respect to a Haar measure. Let $K \leq G$ be a compact subgroup of a Lie group G. The group K is a Lie group by itself (because it is a closed subgroup), so it has a left-Haar measure μ and this measure is also right-invariant by the previous fact. The total measure of K is finite so given $k \in K$ and $f\colon G \to [0,\infty[$ we can form the integral

$$\int_{h\in K} f(hk)\,\mathrm{d}\mu \equiv \int_K f(hk)\,\mathrm{d}\mu(h) \;=\; \int_K f(R_k(h))\,\mathrm{d}\mu(h) \ .$$

Let's make a "change of variables" to get rid of the translation R_k. How does it work when integrating with respect to a general measure? We will show the correct formula for the change of variables to be

$$\int_K (f\circ F)(h)\,\mathrm{d}\mu(h) \;=\; \int_K f(h)\,\mathrm{d}F_*\mu(h)$$

for any homeomorphism $F\colon K \to K$. Here $F_*\mu$ is the pushed forward measure defined by $F_*\mu(A) = \mu(F^{-1}(A))$.

Proof of the formula. Let $F\colon X \to Y$ be a homeomorphism, μ a measure in X and $A \subset Y$ a measurable set. For any set B we denote by $\mathbb{I}_B$ the characteristic function of that set. Recall that the measure of B with respect to some measure is then given by integrating the characteristic function of B with that measure. We calculate

$$\int_Y \mathbb{I}_A\,\mathrm{d}F_*\mu = (F_*\mu)(A) = \mu(F^{-1}(A)) = \int_X \mathbb{I}_{F^{-1}(A)}\,\mathrm{d}\mu \;=\; \int_X (\mathbb{I}_A \circ F)\,\mathrm{d}\mu \ .$$

The integration is defined via approximating the integrand function by so called *simple functions* whose image is a finite set. A simple function is therefore a finite linear combination of characteristic functions of some sets. If $\psi\colon Y \to \mathbb{R}$ is an arbitrary (measurable) function, and $\psi_0 = \sum_{i=1}^n c_i\,\mathbb{I}_{B_i}$ is a simple function approximating it, then by the above formula

$$\int_Y \psi_0\,\mathrm{d}F_*\mu = \int_Y \sum_{i=1}^n c_i\,\mathbb{I}_{B_i}\,\mathrm{d}F_*\mu = \sum_{i=1}^n c_i \int_Y (\mathbb{I}_{B_i}\circ F)\,\mathrm{d}\mu = \int_X \psi_0 \circ F\,\mathrm{d}\mu \ .$$

This holds for the function ψ itself also, since we can make better approximations. □

Using the change of variables formula to $F = R_k$ (it is important that $k \in K$ so that $R_k(K) \subset K$) we get for a Haar measure μ the result

$$\int_K f(hk)\,\mathrm{d}\mu(h) \;=\; \int_K f(h)\,\mathrm{d}(R_k)_*\mu(h) \;=\; \int_K f(h)\,\mathrm{d}\mu(h) \ . \qquad (2)$$

To summarize, integrating in a compact Lie group with respect to its Haar measure one can just drop any right-translation[10]. We shall refer a couple of times to this formula in our proof.

[10] This holds equally well for a left-translation, but for us only the right-translation appear in these kind of situations.

2.5 Nilpotency

This section is devoted to study the notion of a nilpotent group. Actually, for a Lie group, there are two seemingly different definitions that in the end coincide.

The group commutator If R is an (algebraic) group, we can define a *commutator* of two elements $x, y \in R$ to be

$$[x, y] \coloneqq x^{-1}y^{-1}xy \in R .$$

The commutator is thus the unique element with the property

$$xy = yx[x, y] .$$

A Lie group has in this way a commutator on the group level and another one on its Lie algebra.

Let $\mathfrak{r}$ be a Lie algebra with $\mathfrak{h}, \mathfrak{p} \leq \mathfrak{r}$ subalgebras and let R be a group with $H, P \subset R$ subgroups. We denote

$$[\mathfrak{h}, \mathfrak{p}] \coloneqq \operatorname{span}_{\mathbb{R}}\{[X, Y] \mid X \in \mathfrak{h},\ Y \in \mathfrak{p}\}$$
$$\text{and} \qquad [H, P] \coloneqq \langle\{h^{-1}p^{-1}hp \mid h \in H,\ p \in P\}\rangle ,$$

where $\langle X\rangle$ stands for the subgroup generated by the set X. We had to take the generated group (respectively the span) as otherwise the set is not necessarily closed under multiplication (respectively sum).

The *center* of a Lie algebra (or respectively a group) $\mathfrak{g}$ is the set of elements that commute with the whole Lie algebra (respectively the group):

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = 0\} .$$

The two definitions of nilpotency A Lie algebra is nilpotent if there are no arbitrarily long commutators:

Definition. Let $\mathfrak{g}$ be a Lie algebra. The *lower central series* of $\mathfrak{g}$ is formed by the Lie subalgebras

$$\mathfrak{g}_1 = \mathfrak{g} \qquad \text{and} \qquad \mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i] \quad \text{for } i \in \mathbb{N} .$$

If for some $s \in \mathbb{N}$ we have $\mathfrak{g}_{s+1} = \{0\}$, but $\mathfrak{g}_s \neq \{0\}$, then the Lie algebra $\mathfrak{g}$ is said to be *nilpotent of step s*.

The definition for groups is very similar:

Definition. Let R be a group. The *lower central series* of R is formed by the subgroups

$$R_1 = R \qquad \text{and} \qquad R_{i+1} = [R, R_i] \quad \text{for } i \in \mathbb{N} .$$

If for some $s \in \mathbb{N}$ we have $R_{s+1} = \{e_R\}$, but $R_s \neq \{e_R\}$, then the group R is said to be *nilpotent of step s*.

In a connected Lie group there is indeed a correspondence between these notions (see [Rag72, p. 2]):

Fact 2.21. *Let G be a connected Lie group. Then its Lie algebra is nilpotent if and only if G is nilpotent.*

Consequently, if a Lie group is non-connected and nilpotent, then the connected component of the identity is a nilpotent subgroup and therefore the Lie algebra of the group is nilpotent.

If G is a nilpotent group with more than one element, it has necessarily non-trivial center. Indeed, if G is nilpotent of step s, then $G_s \subset Z(G)$.

In the nilpotent groups, there is a little more we can say about the exponential map:

Fact 2.22. *For a connected nilpotent Lie group the exponential map is surjective.*

Fact 2.23. *For a simply connected nilpotent Lie group the exponential map is a diffeomorphism.*

Nilpotency is a property that transfers to the universal covering space:

Fact 2.24. *If N is a connected nilpotent Lie group and the Lie group $\tilde{N}$ is its universal covering space, then $\tilde{N}$ is nilpotent.*

Proof. The projection map $\pi\colon \tilde{N} \to N$ is locally a Lie group isomorphism. Thus the Lie algebra of the universal covering space is isomorphic to the Lie algebra of N. Therefore $\mathrm{Lie}(N)$ is nilpotent if and only if $\mathrm{Lie}(\tilde{N})$ is nilpotent. Fact 2.21 completes the proof. □

Relation to nilpotent matrices To give some intuition to the concept of nilpotency in perhaps more familiar terms, consider a vector space V. A linear map $M\colon V \to V$ is said to be *nilpotent* if for some $k \in \mathbb{N}$ it holds $M^k = 0$.

The relation to the nilpotent Lie algebras is the following (see [Kna13, p. 2]):

Fact 2.25. *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if for all $X \in \mathfrak{g}$ the linear map $\mathrm{ad}_X = [X, \cdot\,] \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent.*

Examples of nilpotent groups Abelian groups are nilpotent as we already mentioned. This also holds for the non-connected ones, which is important when we discuss the maximality of our result in the Section 4.

Here are some non-Abelian examples of nilpotent Lie algebras and groups:

i) The *Heisenberg algebra* is the 2-step nilpotent 3-dimensional Lie algebra spanned by the vectors $\{X, Y, Z\}$ with the only non-trivial bracket relation

$$[X, Y] = Z\ .$$

More generally, the $2n + 1$-dimensional Heisenberg algebra for $n \in \mathbb{Z}_+$ is the Lie algebra spanned by the vectors $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ with the only non-trivial bracket relations

$$[X_i, Y_j] = \delta_{ij} Z\ .$$

ii) A Lie algebra $\mathfrak{g}$ is *stratifiable* if there is a direct sum decomposition $\mathfrak{g} = \bigoplus_{i=1}^{\infty} V_i$ with only finitely many V_i different than $\{0\}$ and with the property $[V_1, V_i] = V_{i+1}$. Stratifiable Lie algebras are nilpotent, but nilpotency is a strictly weaker assumption as demonstrated by the following example: A non-stratifiable nilpotent Lie algebra is the 7-dimensional Lie algebra spanned by the vectors $\{X_1, \ldots, X_7\}$ with the only non-trivial bracket relations

$$[X_i, X_j] = (j - i)X_{i+j} \,.$$

iii) The group of upper tringular matrices (with ones in the diagonal) is nilpotent in any dimension. The (3-dimensional) Heisenberg group, a Lie group whose Lie algebra is the (3-dimensional) Heisenberg algebra, is isomorphic to the group of 3×3 upper triangular matrices.

To give an example of a *non-nilpotent* Lie group and to see how one can observe the non-nilpotency, let's study the group $SU(2)$. One can use the so called Pauli spin matrices

$$\sigma^1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma^2 := \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \qquad \sigma^3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

to span the Lie algebra of $SU(2)$. They obey the commutation relation

$$[\sigma^i, \sigma^j] = \mathrm{i}\epsilon_{ijk}\sigma^k \,. \tag{3}$$

If the Lie algebra were to be nilpotent, then by Fact 2.25 every ad_{σ^i} should form a nilpotent matrix. But using (3) one sees that

$$\mathrm{ad}_{\sigma^1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{bmatrix}$$

clearly is not nilpotent as $(\mathrm{ad}_{\sigma^1})^3 = \mathrm{ad}_{\sigma^1}$.

As remarked above, all stratifiable Lie groups are nilpotent. The following definition gives an important family of examples of nilpotent groups:

Definition. Let G be a simply connected Lie group with a stratifiable Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_n$. Fix a norm $\|\cdot\|_e$ in V_1. We form a subbundle of the tangent bundle by vector spaces $\Delta_g := (L_g)_* V_1$. Any Δ_g is equipped with the norm $\|v\|_g := \|(L_g)^* v\|_e$. The distance function in G is now defined by

$$d_{CC}(p, q) = \inf\Big\{ \int \|\dot\sigma(t)\| \,\mathrm{d}t \mid \sigma \text{ AC curve from } p \text{ to } q \text{ with } \dot\sigma(t) \in \Delta_{\sigma(t)} \text{ for a.e. } t \Big\}$$

Here AC stands for absolutely continuous. The metric Lie group (G, d_{CC}) is then called a *subFinsler Carnot group.*

The result we shall prove was already known in Carnot groups. We shall discuss the results that were known before in the Section 5.

2.6 Semidirect product of groups

If H and N are two (algebraic) groups, then we can form a new group as a *direct product* of H and N. This is achieved by equipping the set $H \times N$ with the product

$$(h_1, n_1) \cdot (h_2, n_2) = (h_1 h_2, n_1 n_2) .$$

This group has the subgroups $H \times \{e_N\}$ and $\{e_H\} \times N$ which are isomorphic to H and N respectively. Both of these subgroups are normal in $H \times N$. Recall that a subgroup $H \leq G$ is said to be *normal* in G, we write $H \lhd G$, if $gHg^{-1} \in H$ for all $g \in G$.

Let's study a more general construction. Still, let H and N be two groups and let $\pi \colon H \to \mathrm{Aut}(N)$ be a homomorphism[11]. The *semidirect product* of H and N with respect to π, denoted by $H \ltimes_\pi N$, is setwise $H \times N$ and has the product

$$(h_1, n_1) * (h_2, n_2) = (h_1 h_2, n_1 \pi_{h_1}(n_2)) . \tag{4}$$

Again $H \times \{e_N\}$ and $\{e_H\} \times N$ are both subgroups of $H \ltimes_\pi N$, and $\{e_H\} \times N$ is even a normal subgroup:

$$\begin{aligned}
(h_1, n_1) * (e, n) * (h_1, n_1)^{-1} &= (h_1, n_1) * (e, n) * (h_1^{-1}, \pi_{h_1}^{-1}(n_1^{-1})) \\
&= (h_1, n_1) * (h_1^{-1}, n\pi_e(\pi_{h_1}^{-1}(n_1^{-1}))) \\
&= (h_1, n_1) * (h_1^{-1}, n\pi_{h_1}^{-1}(n_1^{-1})) \\
&= (h_1 h_1^{-1}, n_1 \pi_{h_1}(n\pi_{h_1}^{-1}(n_1^{-1}))) \\
&= (e, n_1 \pi_{h_1}(n) n_1^{-1}) \in \{e_H\} \times N .
\end{aligned}$$

Notice that in the non-symmetric notation $H \ltimes_\pi N$ the normal subgroup is the one on the right hand side. Notice also that the direct product is recoverd by choosing $\pi(h) = \mathrm{Id}_N$ for all $h \in H$.

In the above, the groups H and N were totally unrelated. What happens in the situation where we have a group G and there are two subgroups H and N of it? Can we in some case recover (an isomorphic copy of) G by taking the semidirect product of H and N with respect to some π? Usually this does not happen. It is not even enough if $N \lhd G$. Still we have the following:

Fact 2.26. *Let G be a group with subgroups $H \leq G$ and $N \lhd G$ in such a way that $H \cap N = \{e_G\}$ and $G = N \cdot H$. Then by setting $\psi_h(n) = hnh^{-1}$ we get a homomorphism $\psi \colon H \to \mathrm{Aut}(N)$ for which $H \ltimes_\psi N$ is isomorphic to G.*

2.7 Group actions

The *group action* is a general concept which we define probably best by giving a particular example first. Let G be a group and X a topological space. A group homomorphism $\Phi \colon G \to \mathrm{Homeo}(X)$ is called an *action* of G in X. If this action is known implicitly, we say that G *acts* on X and denote $G \curvearrowright X$.

[11]Here $\mathrm{Aut}(N)$ refers to the group of automorphisms of N, i.e. isomorphisms $N \to N$.

A priori the group G and the space X are totally non-related. The intuition behind this definition is that an action Φ makes it possible to identify G with some group of symmetries of X.

The generalization is that one can replace the assumption of X being topological space by some other space with a mathematical structure and the set of homeomorphisms by the group of isomorphisms of that structure. For example one could choose X to be a differentiable manifold and replace $\mathrm{Homeo}(X)$ by $\mathrm{Diffeo}(X)$. If one wants to make explicit what kind of structure is $\Phi(g)$ (where $g \in G$) preserving, one can say for example $G \curvearrowright X$ *by isometries* in the case X is a metric space.

An equivalent definition would be to say that an action is a mapping $\Psi\colon G \times X \to X$ with properties $\Psi(e,p) = p$ and $\Psi(st,p) = \Psi(s, \Psi(t,p))$ requiring in addition that mappings $\Psi(t,\cdot)\colon X \to X$ are isomorphisms of the structure on X for all $t \in G$.

If G is a group and M is a differentiable manifold, we say $G \curvearrowright M$ *smoothly* or that *the action is smooth* if the mapping Ψ is a smooth mapping between manifolds. Notice that this is a stronger condition than $\Phi(G) \leq \mathrm{Diffeo}(M)$. Indeed, $\Phi(G) \leq \mathrm{Diffeo}(M)$ means just that $G \curvearrowright M$ by diffeomorphisms, i.e. the restrictions $\{g\} \times M$ are smooth. Similarly, we say $G \curvearrowright M$ *continuously* or *the action is continuous* if the mapping $\Psi\colon G \times X \to X$ is continuous.

Definition. Let Φ be an action for which $G \curvearrowright X$. We say that G acts

i) *transitively* if for all $x, y \in X$ there exists $g \in G$ with help of which x can be mapped to y, i.e. $\Phi(g)(x) = y$. If the element g is furthermore unique, we say that G acts *simply transitively*.

ii) *effectively* if
$$\Phi(g) = \mathrm{id}_X \quad \Leftrightarrow \quad g = e_G \, .$$

iii) *freely* if already the existence of a fixed point, i.e. an element $x \in X$ for which $\Phi(g)(x) = x$, implies $g = e_G$.

For one example, any Lie group G acts on itself simply transitively, effectively and freely by the action $g \mapsto L_g$. Moreover, the action is smooth. Let's go this through: For transitivity, let $x, y \in G$. Then for $g := yx^{-1}$ it holds uniquely
$$\Phi(g)(x) = L_g(x) = L_{yx^{-1}}(x) = y \, .$$
The action is effective because L_g and L_h are not the same mappings if $h \neq g$. It is also free: If $h \neq e$, then the map L_h does not have fixed points. The action is smooth by the definition of a Lie group.

Definition. Let G be a group acting on a space X with the action map $\Phi\colon G \times X \to X$. The *stabilizer* (or the *isotropy group*) of a point $x \in X$ is the set
$$\mathrm{Stab}(x) := \{g \in G \mid \Phi(g)(x) = x\} \, .$$

3 Our result and its proof

3.1 Stating the result and its corollaries

Let's first state our result in a form that is possible to understand without the extra terminology introduced in the earlier sections:

Theorem 3.1 (Main theorem, explicit form)**.** *Let (N_1, d_1) and (N_2, d_2) be two connected nilpotent Lie groups endowed with left-invariant distances that induce the manifold topologies. Then for any distance preserving bijection $F\colon N_1 \to N_2$ there exists a left-translation $\tau\colon N_2 \to N_2$ and a group isomorphism $\Phi\colon N_1 \to N_2$ such that $F = \tau \circ \Phi$.*

Next we will restate here the definitions already mentioned at the introduction because this section should be a stand-alone rigorous treatment. Then we restate our theorem with this new terminology.

Definition. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A map $F\colon M_1 \to M_2$ is said to be an *isometry* if it is bijective and it holds $d_1(p, q) = d_2(F(p), F(q))$ for all $p, q \in M_1$.

Notice that we do not require an isometry to be a smooth map. We only require isometries to preserve distances and be surjective[12]. Injectivity and continuity we get for free, but smoothness shall only be a consequence of the main theorem. Along the way for the proof we actually have to use advanced techniques like the results of Montgomery–Zippin even to get the self-isometries of a Lie group[13] smooth. It remains an open problem if isometries between two general metric Lie groups are always smooth.

Definition. Let G_1 and G_2 be two Lie groups. A map $F\colon G_1 \to G_2$ is said to be *affine* if there exists a left-translation τ of G_2 and a Lie group isomorphism $\Phi\colon G_1 \to G_2$ such that $F = \tau \circ \Phi$.

Notice that since left-translations commute with a homomorphism, then affine maps have equivalently the composition $F = \Phi \circ \tau'$, where the translation $\tau'\colon G_1 \to G_1$ is related to $\tau = L_g$ by $\tau' = L_{\Phi^{-1}(g)}$. We will stick in our notation to the translations that come last in the composition just as in Euclidean space it is more natural to write $F = Ax + v$ instead of $F = A(x + v')$.

Definition. Let G be a Lie group and d a distance function on G. If d induces the manifold topology of G, and d is moreover left-invariant, i.e. $d(p, q) = d(gp, gq)$ for all $p, q, g \in G$, we call the pair (G, d) a *metric Lie group.*

[12]The surjectivity/bijectivity is important in the idea that an isometry should be the "isomorphism of metric spaces". We want to say that two metric spaces are the same if they are isometric. A non-surjective version of an isometry is called *isometric embedding* and our result does not apply to such a map.

[13]This means isometries $G \to G$.

Motivation for the notion of affine is discussed in the introduction and it is general terminology. Instead, the notion of metric Lie group is, as far as we know, only presented here. The reason to use such term is that the setting of metric Lie group has only the most general assumptions that still make sense. If one endows a Lie group with distance function not satisfying these assumptions, the distance is not respecting the Lie group structure[14].

Theorem 3.1 (Main theorem, convenient form). *Let (N_1, d_1) and (N_2, d_2) be two connected nilpotent metric Lie groups. Then any isometry $F\colon N_1 \to N_2$ is affine.*

For now on we will be using the introduced extra terminology stating the corollaries and going through proofs.

Corollary 3.2. *Let (N_1, d_1) and (N_2, d_2) be two connected nilpotent metric Lie groups. Any isometry $F\colon N_1 \to N_2$ fixing the identity element is a group isomorphim. In particular, if the groups are isometric, they are isomorphic.*

Notice that this corollary is actually equivalent to the main theorem: If identity fixing isometries are isomorphisms, then all the rest are necessarily affine maps.

Corollary 3.3. *Let (N_1, d_1) and (N_2, d_2) be two connected nilpotent metric Lie groups. Then the isometries $F\colon N_1 \to N_2$ are C^∞ maps.*

Proof. Any isometry F has the affine decomposition $F = \tau \circ \Phi$, where Φ is a homomorphism. The map $\tau^{-1} \circ F = \Phi$ is still an isometry and thus continuous. Continuous homomorphisms between Lie groups are necessarily smooth (Fact 2.3). Thus also F is C^∞. □

Corollary 3.4. *Let (N, d) be a connected nilpotent metric Lie group. Then its isometry group has a decomposition[15] as a semidirect product of the stabilizer of the identity and the group of left-translations:* $\mathrm{Isome}(N, d) = \mathrm{Stab}(e) \ltimes N^L$.

In the corollary above (its proof shall be discussed in the Section 3.6) a new notation was introduced: by N^L we mean the group of left-translations of a Lie group N. Actually we could also omit such extra notation and just write N instead of N^L as the group of left-translations $N^L := \{L_g \mid g \in N\}$ is naturally isomorphic to N via the map $g \mapsto L_g$. Still, I think keeping this notation explicit can help avoiding confusion, and I will stick with it. Notice how in the case of a metric Lie group this isomorphism allows one to see the group as a subgroup of its own isometry group[16]: $N \cong N^L < \mathrm{Isome}(N, d)$.

[14]Notice also that the statements like "an isometry is always continuous" is already a statement about topology: this happens in every metric space with a topology induced by the metric but not necessarily otherwise. If the manifold topology would be different, we don't even know which "continuity" we should refer to.

[15]To be very precise, we of course mean they are isomorphic.

[16]We will use occasionally the notation $G \cong G'$ to mean that G is isomorphic to G'.

3.2 The strategy of the proof

Let's present a roadmap for proving the main theorem. In the parenthesis we refer to the explicit and exact forms of these statements. The assumption of connectedness is required in almost every step, but we do not write it in this list. First we study the group of self-isometries:

a. The stabilizers of the action $\mathrm{Isome}(G, d) \curvearrowright G$ are compact for (G, d) a metric Lie group by the Ascoli–Arzelà theorem (Proposition 3.7).

b. The isometry group of a metric Lie group is a locally compact topological group (Proposition 3.8).

c. The isometry group of a metric Lie group is itself a Lie group by the theory of Montgomery–Zippin (Corollary 3.9).

d. The self-isometries of a Lie group are smooth (Proposition 3.10).

e. For a metric Lie group (G, d) we can find a Riemannian metric g such that $\mathrm{Isome}(G, d) \leq \mathrm{Isome}(G, g)$ (Theorem 3.13).

Then we continue by operating between two different groups:

f. An isometry between metric spaces induces an isomorphism between their respective isometry groups (Lemma 3.15).

g. If the induced isomorphism of the step f maps the left-translations to left-translations, then the corresponding isometry is affine (Lemma 3.16).

h. If the nilradical condition[17] holds for groups N_1 and N_2, then the induced isomorphism of the step f maps the left-translations to left-translations (Lemma 3.17).

Finally, we can study more the self-isometries and find out that with the help of the result by Wilson we can deduce:

i. The nilradical condition holds in nilpotent metric Lie groups (Proposition 3.19).

The logical dependence of these steps are described in Figure 1. Notice how the assumption of nilpotency only appears in the step i. All the other results are independent of nilpotency and thus the techniques based on nilpotency shall only appear towards the end of the proof. To reduce the level of confusion, I have made the most important steps in the roadmap Figure 1 to appear in boldface. The others can be considered more like "intermediate steps".

[17]See the Definition 3.11.

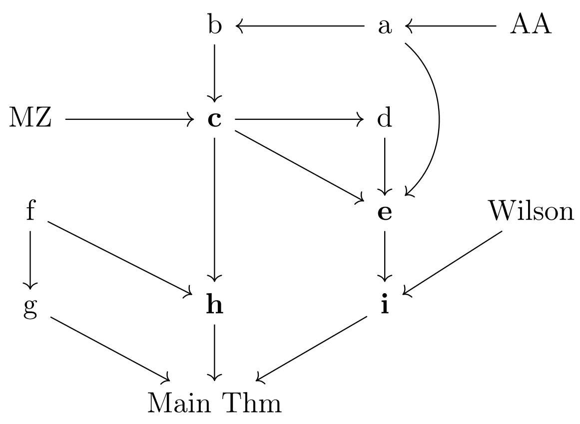

Figure 1: Logical dependence of the steps a–i of the proof. An arrow $x \to y$ means that in proving y we need to use x. "AA" stands for Ascoli–Arzelà theorem, "MZ" for Montgomery–Zippin theorem and "Wilson" for Theorem 3.12. The more important steps are in boldface.

3.3 The group of self-isometries

Let (G, d) be a metric Lie group. We will study the structure of its isometry group $\mathrm{Isome}(G, d)$. This is a group with the composition of mappings as the multiplication. Also, it is a topological group when endowed with the compact-open topology (Fact 2.11).

We would want to have a Lie group structure on $\mathrm{Isome}(G, d)$. This structure can be deduced from the following "big theorem", originally presented in [MZ55] (with a contribution of A. M. Gleason's paper [Gle52]). This modified version is from [LDO14, p. 7]:

Theorem 3.5 (Montgomery–Zippin)**.** *Let H be a second countable, locally compact topological group and X a locally compact, locally connected metric space with finite topological dimension*[18]*. Assume $H \curvearrowright X$ by isometries continuously, effectively and transitively. Then H is a Lie group and X is a differentiable manifold.*

Now we just have to carefully go through the assumptions and see that they are fullfilled. In the next subsection we prove that the isometry group is locally compact. This essentially follows from the Ascoli–Arzelà theorem. Actually, it is more convient to first prove that the stabilizer of the action $\mathrm{Isome}(G, d) \curvearrowright G$ is compact, as this result shall also be used later when constructing a metric tensor on G to accomplish the step e of the road map of the proof. Then using the compactness of the stabilizers we get the local compactness of the isometry group.

[18] Topological dimension (also called the Lebesque covering dimension) of a topological space is less or equal to n, if for every open cover one can find a finite refinement cover (the new sets are the subsets of the old ones) such that any point belongs to at most $n + 1$ sets. For us these details does not matter because topological dimension of a manifold is that of $\mathbb{R}^n$, which is n.

3.3.1 Local compactness of the isometry group

The Ascoli–Arzelà theorem can be found from the literature rephrased in uncountably many ways. Here is a version we shall use (for a more general discussion check for example the book by Kelley [Kel12]):

Theorem 3.6 (Ascoli–Arzelà)**.** *Let (A, d_A) and (B, d_B) metric spaces with A compact. Suppose a set $\mathcal{F} \subset C(A, B)$ is equi-uniformly continuous and pointwise precompact, i.e. suppose*

i) for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d_B(f(x), f(y)) < \varepsilon$ whenever $d_A(x, y) < \delta$ and $f \in \mathcal{F}$.

ii) for all $x \in A$ the subset $\{f(x) \mid f \in \mathcal{F}\} \subset B$ is precompact, i.e. its closure is compact.

Then every sequence $(f_n) \subset \mathcal{F}$ has a subsequence (f_{n_k}) that converges to some $f \in C(A, B)$ uniformly.

Proposition 3.7. *Let (G, d) be a connected metric Lie group and let $p \in G$. Then the stabilizer of p under the action by isometries, i.e. the set*

$$\mathrm{Stab}(p) := \mathrm{Isome}_p(G, d) := \{f \in \mathrm{Isome}(G, d) \mid f(p) = p\}$$

is compact with respect to the compact-open topology.

Proof. It is enough to prove that the stabilizer is sequentially compact (see Fact 2.10). Fix an arbitrary sequence (g_n) of isometries fixing the point p. For the claim we want that it has a subsequence that converges to some map $g \in \mathrm{Isome}_p(G, d)$. We have the convergence $g_n \to g$ in the compact-open topology if and only if (Fact 2.8) in all compact sets the convergence is uniform. Our strategy is to apply Ascoli–Arzelà to the situation where we restrict the maps g_n to an increasing sequence of compact sets.

Observe that a Lie group G has a global frame for its tangent bundle TG, consisting of a basis of left-invariant vector-fields. Thus one can define a Riemannian metric on G by declaring this frame orthonormal. Resulting metric tensor η is then automatically left-invariant, i.e. G-invariant: $\eta_h(v, w) = \eta_{h'h}((L_{h'})_* v, (L_{h'})_* w)$. Denote by ρ the distance function induced by the Riemannian metric tensor η. Now there are two distances ρ and d on G, and we have to be careful: Both distances are left-invariant and induce the same topology but the maps g_n are only isometries with respect to d.

Let $r > 0$ be arbitrary and denote $K := \overline{B_\rho(p, r)}$. We know that every complete Riemannian manifold is boundedly compact[19], i.e. the closed balls are compact, thus K is a compact set. The set $\mathcal{F}_K := \{g_n|_K\}$ is equi-uniformly continuous because the maps are isometries (one can choose uniformly $\delta = \varepsilon$). We will next prove that the

[19] By Hopf–Rinow theorem, a Riemannian manifold is boundedly compact if and only if it is a complete metric space. A Lie group G endowed with the Riemannian metric by declaring some frame of TG orthonormal is always a complete metric space [Zhe05, p. 196]

set $\mathcal{F}_q := \{f(q) \mid f \in \mathcal{F}_K\}$ is precompact for every $q \in K$. It is enough to prove that $\mathcal{F}_q$ is bounded in ρ, again because (G, ρ) is boundedly compact space[20].

Let $s > 0$ be such that $B_d(a, s) \subset B_\rho(a, 1)$ for all $a \in G$. Such an s exists pointwise because d and ρ induce the same topology, but we claim that s is actually independent of a. If $b \in G$ is any other point, then $L_{ba^{-1}}(B_d(a, s)) = B_d(b, s)$ by left-invariance of the metric d. Similar relation holds for the left-invariant Riemannian metric ρ, making s a global parameter.

Fix $q \in K$. Because K is compact there exist $N \in \mathbb{N}$ and points $q_1, \ldots, q_N \in K$ such that

$$K \subset \bigcup_{i=1}^{N} B_d(q_i, s/2) .$$

It is enough to show that for any $g \in \mathcal{F}_K$ we have $\rho(p, g(q)) \leq N$ (it is allowed here for N to depend on q, although this does not happen in our case). Because K is path connected[21], we can take a curve $\gamma\colon [0, 1] \to K$ from p to q. We may assume by reordering points q_i that γ starts from the ball $B_1 := B_d(q_1, s/2)$. If we also have $q \in B_1$, then further reordering is not necessary and we stop. If this is not the case, then γ leaves the ball B_1 last time at some $t_1 \in \,]0, 1[$ (it may leave the ball many times but after t_1 it does not come back anymore). By above we mean that $t_1 = \sup\{t \mid \gamma(t) \in B_1\}$. Because the sets cover, by reordering we may assume that $\gamma(t_1)$ is in the ball $B_2 := B_d(q_2, s/2)$. If $q \in B_2$, we stop, otherwise we continue to define times t_i with this system. Because $q \in B_d(q_k, s/2)$ for some k, this process eventually stops and we have times $t_1, \ldots, t_{k-1} \in \,]0, 1[$, where $k \leq N$. Define in addition $t_0 = 0$ and $t_k = 1$. By construction

$$d(\gamma(t_i), \gamma(t_{i+1})) \leq s \qquad \forall i \in \{0, \ldots, k-1\} .$$

This implies that for arbitrary $g \in \mathcal{F}_K$ we have

$$d(g(\gamma(t_i)), g(\gamma(t_{i+1}))) \leq s \qquad \text{and so} \qquad \rho(g(\gamma(t_i)), g(\gamma(t_{i+1}))) \leq 1 .$$

Therefore

$$\rho(p, g(q)) \leq \sum_{i=0}^{k-1} \rho(g(\gamma(t_i)), g(\gamma(t_{i+1}))) \leq k \leq N$$

as was claimed.

By Ascoli–Arzelà some subsequence of $(g_n|_K)$ has a limit $g\colon K \to G$. Notice that Ascoli–Arzéla is not telling us that $g \in \mathrm{Isome}_p(G, d)$ but only that $g \in C(K, G)$. From the fact that the distance function d is continuous, it follows immediately that also the map g preserves distances, but it is not immediate that g is bijective[22].

---

[20] If (M, d) is a boundedly compact space and a subset $X \subset M$ is bounded, i.e. there exists $x \in M$ and $r > 0$ such that $X \subset B_d(x, r)$, then X is precompact because $\bar{X}$ is closed: it is a closed subset of a compact set $\overline{B_d(x, r)}$.

[21] A connected complete Riemannian manifold has path-connected balls because any point of a ball is connected to the center of the ball with the geodesic (Hopf–Rinow theorem).

[22] In general it is not true that the set of distance preserving bijections is closed under pointwise convergence: Consider the sequence $g_n\colon \mathbb{N} \to \mathbb{N}$ with $g_n(0) = 0$, $g_n(k) = k + 1$ for $k < n$, $g_n(n) = 1$ and $g_n(k) = k$ for $k > n$ and equip $\mathbb{N}$ with discrete topology. The key point is that $\mathbb{N}$ is not connected.

Another thing to notice: The above constructed subsequence, as well as its limit, can depend on the compact set K chosen. Next we should find a subsequence of (g_n) that is uniformly convergent in any compact set.

Take as a sequence of compact sets the incresing ρ-balls $K_n := \overline{B_\rho(p,n)}$. For K_1 the argument above holds and we have some subsequence and some limit $f_1 \colon K_1 \to G$. For K_2, using Ascoli–Arzelà again we can choose a subsequence of the preceding subsequence, which was for K_1, and again we have some limit map $f_2 \colon K_2 \to G$. Necessarily it holds $f_2|_{K_1} = f_1$ and in this sense the limit is the same[23]. Proceeding inductively we get infinitely many nested subsequencies in such a way that the nth subsequence is converging uniformly in K_n to the limit function g.

If $(g_n) \subset \mathrm{Isome}_p(G,d)$ is a sequence of isometries, is there now a subsequence (g_{n_k}) converging uniformly in any compact set, i.e. a subsequence that converges in the compact-open topology? Yes: it can be "read from the diagonal". Choose as the kth map in the sequence the kth map of the subsequence corresponding to the set K_k. If $\hat{K}$ is now any compact set, it is ρ-bounded and thus subset of some K_n. The subsequence picked as above is converging uniformly in K_n because it is a subsequence of the subsequence that was constructed for K_n. Thus it is converging in $\hat{K}$ also.

Denote by (g_{n_k}) the subsequence described above. We know $g_{n_k} \to g \in C(G)$ in compact open topology and g preserves distances necessarily. But why should it be bijective? Consider the sequence $(g_{n_k}^{-1})$ constructed out of the inverse maps of the above sequence. This is also a sequence of isometries in $\mathrm{Isome}_p(G,d)$, and therefore anything argued above can be applied it: This sequence has a subsequence $(g_{n_{k_m}}^{-1})$ that converges uniformly on compact sets to some map $h \in C(G)$. Denote the corresponding subsequence of the original maps g_n by $(g_m) := (g_{n_{k_m}})$ for simplicity. We complete the proof by showing that $h(g(x)) = x = g(h(x))$ for all $x \in G$.

Fix $x \in G$. Although not all closed d-balls are compact, small balls are because G is a manifold. If $\varepsilon > 0$ is such that $\overline{B_d(h(x),\varepsilon)} =: Q$ is compact, then after some index we have $g_m^{-1}(x) \in Q$, so we may assume this is always the case. The convergence $g_m \to g$ is uniform in Q, and thus the map $g|_Q$ is continuous. Therefore, since $g_m^{-1}(x) \to h(x)$, there exists an index N such that

$$d(g(g_m^{-1}(x)), g(h(x))) < \varepsilon \qquad \forall m \geq N\,.$$

Let $M \geq N$ be an index after which

$$\sup\{d(g_m(y), g(y)) \mid y \in Q\} < \varepsilon$$

which exists because the convergence is uniform. Then we have for all $m \geq M$

$$\begin{aligned} d(x, g(h(x)) &= d(g_m(g_m^{-1}(x)), g(h(x))) \\ &\leq d(g_m(g_m^{-1}(x)), g(g_m^{-1}(x))) + d(g(g_m^{-1}(x)), g(h(x))) < 2\varepsilon\,. \end{aligned}$$

This proves $g(h(x)) = x$. Similarly one gets $h(g(x)) = x$. □

[23] Consider pointwise sequences: necessarily a subsequence of a converging sequence converges to the same limit.

Notice that compactness of the stabilizers is not true without the assumption of connectedness. From Proposition 3.7 the assumption of connectedness is induced basically to every theorem of this thesis.

Proposition 3.8. *Let (G, d) be a connected metric Lie group. Then its isometry group is locally compact.*

Proof. We need to find a compact neighbourhood for every point in $\text{Isome}(G, d)$. Let's fix $F \in \text{Isome}(G, d)$ and denote $F(e) =: q$. A compact neighbourhood of F shall be the set

$$V_\varepsilon^F := \bigcup_{h \in \overline{B_\varepsilon(q)}} \{L_h \circ J \mid J \in \text{Stab}(e)\} ,$$

where $\varepsilon > 0$ is actually arbitrary. At least $F \in V_\varepsilon^F$ because $F = L_q \circ (L_{q^{-1}} \circ F)$. Moreover, F is an interior point of this set because if (F_i) is a sequence of isometries converging to F in the compact-open topology, then by pointwise convergence $q_i := F_i(p) \to q$. This means that in the expression $F_i = L_{q_i} \circ (L_{q_i^{-1}} \circ F)$ we have sooner or later $q_i \in \overline{B_\varepsilon(q)}$, as is required for the sequence (F_i) to enter the set V_ε^F.

Next we should prove the set V_ε^F compact. Using the group product of $\text{Isome}(G, d)$ we can write

$$V_\varepsilon^F = K \cdot \text{Stab}(e) , \qquad \text{where} \qquad K := \{L_h \mid h \in \overline{B_\varepsilon(q)}\} .$$

We have expressed V_ε^F as product of two compact sets, hence it is compact (Fact 2.17). The set K is compact as the image of the compact set $\overline{B_\varepsilon(q)}$ under the continuous mapping $G \to \text{Homeo}(G, d)$ for which $h \mapsto L_h$ (Fact 2.16). □

3.3.2 The isometry group as a Lie group

Corollary 3.9 (Consequence of MZ)**.** *Let (G, d) be a connected metric Lie group. Then its isometry group* $\text{Isome}(G, d)$ *is a (finite-dimensional) Lie group.*

Proof. Using the terminology of MZ theorem, as stated in Theorem 3.5, we have a metric Lie group (G, d) as our metric space X. It is locally compact, locally connected and finite dimensional as a manifold because the local properties of a manifold are the same as those of $\mathbb{R}^n$.

The topological group H is the isometry group endowed with the compact-open topology, and H is locally compact by Proposition 3.8. The compact-open topology of H is second countable (Fact 2.9) as isometries are continuous.

Because the distance d is left-invariant, the action $\text{Isome}(G, d) \curvearrowright G$ is transitive, and effective it is trivially. Note that a priori the action by isometres could be discontinuous because acting continuously requires the map $H \times X \to X$ to be continuous. Still, with compact-open topology in the isometry group, the action $\text{Isome}(G, d) \curvearrowright G$ is continuous (Fact 2.12). □

3.3.3 Smoothness of the self-isometries

In this section we will prove that self-isometries of a metric Lie group are smooth. Notice that we don't claim such to hold for isometries between two different metric Lie groups. Between two different groups we work instead with the techniques based on the induced isomorphism of the isometry groups, which we will introduce in Lemma 3.15 and use in Lemma 3.16.

Proposition 3.10. *If G is a metric Lie group and $F\colon G \to G$ is an isometry, then F is smooth.*

Proof. Let $g \in G$ be a point where we want to check smoothness. Notice that the map $L_{F(g)}^{-1} \circ F \circ L_g =: F'$ is smooth at the identity $e \in G$ if and only if F is smooth at g. In addition F' fixes the identity, so if G° denotes the connected component of G containing e, then $F'(G^\circ) \subset G^\circ$ as F' is continuous (as an isometry). This means $F' \in \mathrm{Isome}(G^\circ, d)$. Hence, without loss of generality, we can assume G is connected.

Let's prove a bit stronger result as it comes with the same effort: the action $\mathrm{Isome}(G) \curvearrowright G$ is smooth. The proof is based on the following facts:

(i) If H is a Lie group and $S \leq H$ a regular Lie subgroup, then the quotient space H/S (with the quotient topology) has a unique differentiable structure $\omega_{H/S}$ for which the left-action[24] $H \curvearrowright H/S$ is smooth [Hel01, p. 123].

(ii) The quotient space $\mathrm{Isome}(G)/\mathrm{Isome}_e(G)$ is homeomorphic to G via the map $[F] \mapsto F(e)$. [Hel01, p. 121].

We know that $\mathrm{Isome}(G) =: H$ is a Lie group and $\mathrm{Isome}_e(G) =: S$ is a compact subgroup, in particular closed. Topologically closed subgroups of a Lie group are regular Lie subgroups (Fact 2.6).

Thus by (i) there is a differentiable structure on $\mathrm{Isome}(G)/\mathrm{Isome}_e(G) = H/S$ making the action $H \curvearrowright H/S$ smooth. Denote this action by $\xi\colon H \times H/S \to H/S$. We can construct a differentiable structure to G by declaring the homeomorphism in (ii), which we denote by ψ, to be smooth. Let's denote that structure of G by ω. Then the action $H \curvearrowright (G, \omega)$ is smooth as is evident from the following diagram:

$$\begin{array}{ccc} H \times H/S & \xrightarrow{\ \xi\ } & H/S \\ \big\downarrow{\scriptstyle (\mathrm{Id},\psi)} & & \big\downarrow{\scriptstyle \psi} \\ H \times (G,\omega) & \longrightarrow & (G,\omega) \end{array} .$$

We still did not prove that isometries act smoothly because there was already a differentiable structure ω_0 on Lie group G, and although structures ω and ω_0 are by our construction topologically similar, we still want $\omega = \omega_0$.

Because the action $H \curvearrowright (G, \omega)$ is smooth, in particular the action by left-translations $G \curvearrowright (G, \omega)$ is smooth. To be precise, we mean the action $(G^L, \omega_L) \curvearrowright$

[24] The left action is given by the map $H \times H/S \to H/S$, $(h, h'S) \mapsto hh'S$ so the left coset of element h' is sent to the left coset of element hh'.

(G, ω) where the set of left-translations G^L is equipped with the induced differentiable structure ω_L which it gets as a submanifold of Isome(G, d).

We can apply again (i) to the case $H = G^L$ and $S = \{\mathrm{Id}\}$ in which case $H/S = G$ (they are homeomorphic) and (i) gives that there exists a unique differentiable structure $\tilde{\omega}$ in G such that the action $(G^L, \omega_L) \curvearrowright (G, \tilde{\omega})$ is smooth. We had exactly this condition for $\tilde{\omega} = \omega$. For $\tilde{\omega} = \omega_0$ it holds because (G, ω_0) is a Lie group, so we conclude $\omega = \omega_0$. □

3.3.4 A little stop for the ideas

Until now we have gone through the steps a, b, c and d in the roadmap of the proof: Figure 1. This was more or less just developing the tools that were necessary: Because of the smoothness of the self-isometries we can now use their differentials, the compactness of the stabilizers lets us integrate over that set, and we can invoke the Lie group structure of the isometry group to use the Lie theory techniques there.

As Figure 1 tells, the logic is now going to branch into two. The branch of steps h, f and g is less complicated using mainly algebraic observations and some basic Lie group theory. One can think of the harder points of our proof lying on the right branch of the Figure 1 instead.

A key idea for how to prove that isometries are affine was to consider the techniques of Wilson in [Wil82], where he proved that the self-isometries of nilpotent connected *Riemannian* Lie groups are affine. His theorem stood on the observation that in this setting the group of left-translation is actually a normal subgroup in the isometry group. Actually, the normality of this subgroup is equivalent to the self-isometries being affine, see Section 3.6.

Definition 3.11. Let $\mathfrak{g}$ be a Lie algebra. The *nilradical* of $\mathfrak{g}$, denoted by $\mathrm{nil}(\mathfrak{g})$, is the sum of all nilpotent ideals of $\mathfrak{g}$. We say that a metric Lie group (G, d) satisfies *the nilradical condition*, if it holds $\mathrm{Lie}(G^L) = \mathrm{nil}(\mathrm{Lie}(\mathrm{Isome}(G, d)))$.

Recall that a subalgebra $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is an ideal, if $[\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a}$. The nilradical of a Lie algebra $\mathfrak{g}$ is itself a nilpotent ideal of $\mathfrak{g}$, because a sum of any two nilpotent ideals is a nilpotent ideal [HN11, p. 93] (and we only consider finite dimensional Lie algebras). Thus the nilradical is the unique maximal nilpotent ideal.

Wilson observed that in the level of Lie algebras the key result to prove that the isometries are affine was the following theorem [Wil82, p. 342 theorem 2]:

Theorem 3.12 (Wilson)**.** *Let N be a nilpotent connected Lie group with a left-invariant Riemannian metric tensor g. Then* $\mathrm{Lie}(N^L) = \mathrm{nil}(\mathrm{Lie}(\mathrm{Isome}(N, g)))$.

We oberved that in our case this kind of result would also be the key. Indeed, the left branch of the Figure 1 finishes to the step h saying exactly that it would be enough to prove that the nilradical condition holds in our setting. So far I have been telling that Wilson's result 3.12 gives the idea how to proceed, but indeed we shall also use it explicitly to give the nilradical condition in our more general setting as shown in Figure 1.

How shall we then get the nilradical condition? We observe that in a general (connected) metric Lie group (G, d), it is possible to construct a Riemannian metric tensor g in such a way that the old isometries are now Riemannian isometries, i.e. $\mathrm{Isome}(G, d) \leq \mathrm{Isome}(G, g)$. Combining with Wilson we have then immediately the following result:

> *In a connected nilpotent metric Lie group (N, d) there exists a Riemannian metric tensor g such that* $\mathrm{Lie}(N^L) = \mathrm{nil}(\mathrm{Lie}(\mathrm{Isome}(N, g)))$.

But what about the original distance d of the metric Lie group (N, d)? The condition $\mathrm{Lie}(N^L) = \mathrm{nil}(\mathrm{Lie}(\mathrm{Isome}(N, g)))$ means that $\mathrm{Lie}(N^L)$ is a nilpotent ideal of $\mathrm{Lie}(\mathrm{Isome}(N, g))$ and there are no bigger nilpotent ideals. When we know that $\mathrm{Isome}(N, d) \leq \mathrm{Isome}(N, g)$, then $\mathrm{Lie}(N^L)$ is also a nilpotent ideal[25] of the Lie algebra $\mathrm{Lie}(\mathrm{Isome}(N, d))$, but a priori it can be that $\mathrm{Lie}(N^L)$ is no more the biggest as the notion of ideal is less strict in the smaller space $\mathrm{Lie}(\mathrm{Isome}(N, d))$. This issue shall be completed in the Proposition 3.19.

3.3.5 A Riemannian metric preserving the old isometries

We ask if for an arbitrary metric Lie group (G, d) there exists a Riemannian metric tensor such that the old isometries are still isometries with respect to this Riemannian metric. Put in other words, if $I := \mathrm{Isome}(G, d)$ we want to construct a metric tensor g on G for which $I \curvearrowright (G, g)$ by isometries[26]. Recall that isometries are defined in the Riemannian setting as those diffeomorphisms F that leave the metric tensor invariant (when pulled back with F). So we ask for a metric tensor g such that $F^* g = g$ for all "old" isometries $F \in \mathrm{Isome}(G, d)$. Notice how at this point it is important to know the isometries of (G, d) to be smooth: We need to differentiate them.

Theorem 3.13. *Let (G, d) be a connected*[27] *metric Lie group. Then there exists a Riemannian metric tensor g such that* $\mathrm{Isome}(G, d) \leq \mathrm{Isome}(G, g)$.

Approach the idea of the proof by first examining the following problem:

> *On an arbitrary Lie group G construct a Riemannian metric that is G-invariant.*

Recall that we already solved this problem during the proof of Proposition 3.7: Such a Riemannian metric tensor can be defined by declaring a global frame of TG

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[25]That it is an ideal actually needs some reasoning because we have to go from group level to the algebra level, but we will do it when it is the time for the rigorous proof. This is just giving the idea.

[26]This does not mean that the metric tensor gives exactly the original distance. For example, if g gives the distance d_g, then $2g$ gives the distance $2d_g$, but the isometries of these distances are the same. Moreover, a priori it might be the case that there are really more of these Riemannian isometries.

[27]The connectedness assumption may actually be removed by restricting the problem to the identity component G°, as in the proof of Proposition 3.10. However, we don't wish to go through these technicalities for such an intermediate step.

orthonormal. A philosophical note is that in this exercise the construction gives us a left-invariant metric tensor because declaring some set of left-invariant vectors orthogonal is exactly what is asked for G-invariance of the metric tensor. Why this case was so easy is that the action $G \curvearrowright G$ has trivial stabilizers, i.e. $\mathrm{Stab}(p) = \{L_e\}$ for any $p \in G$. More general actions $I \curvearrowright G$ have complicated stabilizers, and we have to care about the stabilizer.

The strategy to prove Theorem 3.13 is the following: Lemma 3.14 shall show first how to define a stabilizer-invariant scalar product on T_eG by averaging over the stabilizer. Then we move that scalar product to every point of the Lie group using left-translations. To make the mentioned averaging procedure possible we need the compactness of stabilizers given by Proposition 3.7. Recall that the isometry group is a Lie group and this makes it possible to average using Haar measures. These observations explain the arrows pointing to e in Figure 1.

Lemma 3.14. *Let I_e be a compact Lie group that acts continuously on a Lie group G. Assume that the action $I_e \curvearrowright G$ is via diffeomorphisms and fixes the identity of G. Then there exists a I_e-invariant scalar product on T_eG.*

Proof. Let $\langle \cdot, \cdot \rangle$ be any scalar product on T_eG. We denote by $\Phi \colon I_e \to \mathrm{Diffeo}(G)$ the action map. Because for every $f \in I_e$ we have $\Phi(f)(e) = e$, then each $f \in I_e$ gives birth to a new scalar product on T_eG, namely $\Phi(f)^*\langle \cdot, \cdot \rangle$. Recall that if $F \colon M \to N$ is a smooth mapping between manifolds, we can pull back a scalar product η defined at a point $F(p) \in F(M)$ to a scalar product at p by the formula $(F^*\eta)(X_p, Y_p) = \eta(F_*X_p, F_*Y_p)$ where the vectors at p are pushed forward by the differential of F, i.e. $F_*X_p \equiv \mathrm{d}F_p X_p$.

By Fact 2.20, we have a Haar measure μ on the Lie group I_e such that the total measure is finite. Therefore we can define

$$\langle\langle \cdot, \cdot \rangle\rangle := \int_{I_e} \Phi(f)^*\langle \cdot, \cdot \rangle \,\mathrm{d}\mu(f)$$

since this an integral of a continuous function over a compact set, being then well defined and finite (this is just the usual integration of real valued function over a measure space when the inputs are fed in). A more concrete formula is

$$\langle\langle X, Y \rangle\rangle = \int_{I_e} \Phi(f)^*\langle X, Y \rangle \,\mathrm{d}\mu(f) = \int_{I_e} \langle \Phi(f)_*X, \Phi(f)_*Y \rangle \,\mathrm{d}\mu(f)\,.$$

We claim that this is the desired I_e-invariant scalar product. Indeed, for any $h \in I_e$ we have

$$\begin{aligned}
\Phi(h)^*\langle\langle X, Y\rangle\rangle &= \langle\langle \Phi(h)_*X, \Phi(h)_*Y\rangle\rangle \\
&= \int_{I_e} \langle \Phi(f)_*\Phi(h)_*X, \Phi(f)_*\Phi(h)_*Y\rangle \, \mathrm{d}\mu(f) \\
&= \int_{I_e} \langle (\Phi(f)\circ\Phi(h))_*X, (\Phi(f)\circ\Phi(h))_*Y\rangle \, \mathrm{d}\mu(f) \\
&= \int_{I_e} \langle \Phi(fh)_*X, \Phi(fh)_*Y\rangle \, \mathrm{d}\mu(f) \\
&\overset{(2)}{=} \int_{I_e} \langle \Phi(f)_*X, \Phi(f)_*Y\rangle \, \mathrm{d}\mu(f) \\
&= \langle\langle X, Y\rangle\rangle \, .
\end{aligned}$$

Moreover, $\langle\langle\cdot,\cdot\rangle\rangle$ is really a scalar product: it is positive definite, symmetric and linear in both variables as immediate consequence of $\langle\cdot,\cdot\rangle$ having those properties. □

Proof of Theorem 3.13. We must now construct an I-invariant Riemannian metric on the Lie group G (remember, $I = \mathrm{Isome}(G,d)$). As the stabilizers are compact, the above lemma gives us a scalar product $\langle\langle\cdot,\cdot\rangle\rangle =: \eta_e$ on T_eG that is invariant under $\mathrm{Stab}(e) \subset I$. The idea is to use the left-translations to move this scalar product to everywhere on the manifold, thus defining a Riemannian metric[28]. More precisely, if $p \in G$, then we define $\eta_p := (L_{p^{-1}})^*\eta_e$ as cotensors must be pulled back. There are two things to check:

a) η varies smoothly on the manifold and thus defines really a metric tensor.

b) metric tensor η is I-invariant.

Problem a: Recall that a tensorfield η is smooth if and only if the $\mathbb{R}$-valued map on the manifold $\eta(X,Y)$ is smooth for any fixed smooth vectorfields X, Y. Our formula is

$$\eta_p(X_p, Y_p) = (L_{p^{-1}})^*\eta_e(X_p, Y_p) = \langle\langle (L_{p^{-1}})_*X_p, (L_{p^{-1}})_*Y_p\rangle\rangle = \langle\langle \mathrm{d}L_{p^{-1}}X_p, \mathrm{d}L_{p^{-1}}Y_p\rangle\rangle$$

Here $\mathrm{d}L_{p^{-1}}X_p \in T_eG$ for all p and as $\langle\langle\cdot,\cdot\rangle\rangle$ is just a scalar product on T_eG we can check very concretely in coordinates that the map $M \to \mathbb{R}$ for which $p \mapsto \langle\langle \mathrm{d}L_{p^{-1}}X_p, \mathrm{d}L_{p^{-1}}Y_p\rangle\rangle$ is smooth: Expressions $v(p) := \mathrm{d}L_{p^{-1}}X_p$ and $w(p) := \mathrm{d}L_{p^{-1}}Y_p$ define vectors v and w always at T_eG that vary smoothly with p, i.e. the coordinate functions

$$v_i(p) := \mathrm{d}\varphi^i(v(p)) = \frac{\partial(\varphi^i \circ L_{p^{-1}} \circ \varphi^{-1})}{\partial x^j}\Big|_{\varphi(p)} X_p^j$$

[28]With help of left-translations every point can be reached in unique way. In fancy words, the action is simply transitive.

are smooth functions $M \to \mathbb{R}$ in any frame, when φ is the local chart. We conclude by noticing that the scalar product has the simple expression $\langle\langle v, w\rangle\rangle = \sum v_i w_i$ in its associated orthonormal basis at p.

Problem b: Let $F \in I$ and denote $F(e) =: p$. We want to prove $F^*\eta_q = \eta_{F^{-1}(q)}$ at every point q. First of all

$$F^*\eta_q = F^*(L_{q^{-1}})^*\eta_e = (L_{q^{-1}} \circ F)^*\eta_e \ .$$

Notice that this is a tensor calculated at $F^{-1}(q) = (L_{q^{-1}} \circ F)^{-1}(e)$, but the scalar product was brought to this point via translation, i.e. $\eta_{F^{-1}(q)} = (L_{(F^{-1}(q))^{-1}})^*\eta_e$. We realize that it did not matter that we chose exactly the left-translation to bring scalar product from e to other points: any isometry does the same thing. To make this argument simple, choose two maps $H, \hat{H} \in I$ such that $H^{-1}(e) = \hat{H}^{-1}(e)$. We claim that $H^*\eta_e = \hat{H}^*\eta_e$. Remember that η_e is stabilizer-invariant and $\hat{H} \circ H^{-1}$ fixes the identity. Observe that

$$(\hat{H} \circ H^{-1})^*\eta_e = \eta_e \qquad \Rightarrow \qquad (H^{-1})^*\hat{H}^*\eta_e = \eta_e \qquad \Rightarrow \qquad \hat{H}^*\eta_e = H^*\eta_e \ ,$$

so the claim follows. We proved that the pull back of η does not depend on the isometry making the pull back as long as the points are the same. Therefore

$$F^*\eta_q = (L_{q^{-1}} \circ F)^*\eta_e = (L_{(F^{-1}(q))^{-1}})^*\eta_e = \eta_{F^{-1}(q)} \ ,$$

which completes the proof of I-invariance of the metric tensor. □

3.4 Operating between two different groups

So far we have studied the self-isometries of a metric Lie group. Now it is time to focus on isometries $F\colon N_1 \to N_2$ between two possibly different Lie groups. This means going through the left branch of the roadmap of the proof in Figure 1.

Lemma 3.15. *If two metric spaces (M_1, d_1) and (M_2, d_2) are isometric via $F\colon M_1 \to M_2$, their groups of isometries are isomorphic as topological groups via the map*

$$\hat{F}\colon \mathrm{Isome}(M_1, d_1) \to \mathrm{Isome}(M_2, d_2) \qquad I \mapsto F \circ I \circ F^{-1} \ .$$

Proof. First of all, the map $\hat{F}$ has the correct codomain because for any isometry I of (M_1, d_1) its image $\hat{F}(I)$ is a mapping $M_2 \to M_2$ and has the property

$$\begin{aligned} d_2\big(\hat{F}(I)(p), \hat{F}(I)(q)\big) &= d_2\big(F(I(F^{-1}(p))), F(I(F^{-1}(q)))\big) \\ &= d_1\big(I(F^{-1}(p)), I(F^{-1}(q))\big) \\ &= d_1\big(F^{-1}(p), F^{-1}(q)\big) = d_2(p, q) \qquad \forall p, q \in M_1 \ , \end{aligned}$$

where we just used the fact that all these maps are isometries. Moreover, $\hat{F}$ is invertible: its inverse is $(\hat{F})^{-1} = \widehat{F^{-1}}$ as is obvious. The homorphism property is proved as follows:

$$\hat{F}(I \circ H) = F \circ (I \circ H) \circ F^{-1} = (F \circ I \circ F^{-1}) \circ (F \circ H \circ F^{-1}) = \hat{F}(I) \circ \hat{F}(H)$$

for any $I, H \in \mathrm{Isome}(M_1, d_1)$.

The fact that $\hat{F}$ is a homeomorphism follows from Fact 2.15. □

The main theorem is basically about proving isometric groups isomorphic. The above lemma applied to two metric Lie groups gives that for isometric groups the groups of isometries are isomorphic via the map $\hat{F}$ that is induced by the isometry F. But we want the isomorphism-property to occur at the group level. Could it happen through the map F? This is the case if the induced map $\hat{F}$ works correctly with translations:

Lemma 3.16. *Let (G_1, d_1) and (G_2, d_2) be metric Lie groups. Suppose $F\colon G_1 \to G_2$ is an isometry fixing the identity for which $\hat{F}(G_1^L) = G_2^L$. Then G_1 is isomorphic to G_2 via the map F.*

This result makes much sense: if one thinks "$G = G^L$", which is quite legal as they are isomorphic, then the assumption reads "$\hat{F}(G_1) = G_2$" and thus it guarantees with Lemma 3.15 that there is a bijective homomorphism between the groups G_1 and G_2.

We can make this point precise immediately, as G_i is isomorphic to G_i^L and thus $G_1 \cong G_1^L \cong G_2^L \cong G_2$ using that $\hat{F}$ is an isomorphism. The only thing we did not get by this reasoning is that the groups are isomorphic via the map F. With these ideas, let's write the proof:

Proof. An idea to make the above reasoning precise is that it didn't use the fact that F fixes the identity. Let's show first that at least an isomorphism is induced between the groups G_i, even if it is not the map F. Indeed, given $p_1 \in G_1$, we have $\hat{F}(L_{p_1}) = L_{p_2}$ for some $p_2 \in G_2$. This defines one-to-one correspondence between elements of G_1 and G_2. Call this bijection $\tilde{F}$. Then by the definition $\hat{F}(L_{p_1}) = L_{\tilde{F}(p_1)}$. We want to show that $\tilde{F}$ is a homomorphism, and therefore indeed the desired isomorphism $G_1 \to G_2$.

Fix $p_1, q_1 \in G_1$ and let $h_2 := \tilde{F}(p_1 q_1)$, $p_2 := \tilde{F}(p_1)$ and $q_2 := \tilde{F}(q_1)$. Then we have by the homomorphism property of $\hat{F}$ that

$$L_{h_2} = \hat{F}(L_{p_1 q_1}) = \hat{F}(L_{p_1} L_{q_1}) = \hat{F}(L_{p_1}) \circ \hat{F}(L_{q_1}) = L_{p_2} \circ L_{q_2} = L_{p_2 q_2} \, .$$

This means that $h_2 = p_2 q_2$ which is exactly the required homorphism property

$$\tilde{F}(p_1 q_1) = \tilde{F}(p_1) \tilde{F}(q_1) \, .$$

Moreover, if we use the assumption that the isometry F fixes the identity, i.e. $F(e_{G_1}) = e_{G_2}$, then the isomorphism $\tilde{F}$ is really the map F. Indeed, for arbitrary $p \in G_1$ we have

$$\tilde{F}(p) = L_{\tilde{F}(p)}(e_{G_2}) = \hat{F}(L_p)(e_{G_2}) = (F \circ L_p \circ F^{-1})(e_{G_2}) = (F \circ L_p)(e_{G_1}) = F(p)$$

so $\tilde{F} \equiv F$. □

We observe next, that the condition that left-translations map to left-translations actually follows from the nilradical condition. Therefore the nilradical condition actually implies that isometries are affine. Later we go from this result to the Main Theorem by proving that the nilradical condition is automatic in nilpotent Lie groups. It is immediate that the nilradical condition implies the group to be nilpotent.

Lemma 3.17. *Let (N_1, d_1) and (N_2, d_2) be two connected nilpotent metric Lie groups. Assume* $\mathrm{Lie}(N_i^L) = \mathrm{nil}(\mathrm{Lie}(\mathrm{Isome}(N_i, d_i)))$ *for both $i \in \{1, 2\}$. Then for any isometry $F : N_1 \to N_2$ fixing the identity it holds $\hat{F}(N_1^L) = N_2^L$.*

Proof. First of all, isometry groups are Lie groups by Proposition 3.9 and thus it makes sense to talk about their Lie algebras. The map $\hat{F}$ is a group isomorphism between two Lie groups. Actually, the map $\hat{F}$ is even an isomorphism of Lie groups because it is continuous and thus smooth (Fact 2.3).

Corresponding to the isomorphism $\hat{F}$ there exists (Fact 2.1) the Lie algebra isomorphism $\hat{F}_* : \mathrm{Lie}(\mathrm{Isome}(N_1, d_1)) \to \mathrm{Lie}(\mathrm{Isome}(N_2, d_2))$. Because isomorphisms preserve all algebraic properties they preserve the nilradical:

$$\hat{F}_*(\mathrm{nil}(\mathrm{Lie}(\mathrm{Isome}(N_1, d_1)))) = \mathrm{nil}(\mathrm{Lie}(\mathrm{Isome}(N_2, d_2)))\ .$$

Using now the nilradical condition, we have

$$\begin{aligned} & \hat{F}_*(\mathrm{Lie}(N_1^L)) = \mathrm{Lie}(N_2^L) \\ \Rightarrow\quad & \exp(\hat{F}_*(\mathrm{Lie}(N_1^L))) = \exp(\mathrm{Lie}(N_2^L)) \\ \Rightarrow\quad & \hat{F}(\exp(\mathrm{Lie}(N_1^L))) = \exp(\mathrm{Lie}(N_2^L))\ , \end{aligned}$$

where in the last line we used Fact 2.4. Both sides of the last equation are sets with a multiplication but not necessarily groups. Taking the group generated we get $\langle \exp(\mathrm{Lie}(N_i^L)) \rangle = N_i^L$ as the groups N_i are connected (Fact 2.7). This means

$$\begin{aligned} & \langle \hat{F}(\exp(\mathrm{Lie}(N_1^L))) \rangle = N_2^L \\ \Rightarrow\quad & \hat{F}(\langle \exp(\mathrm{Lie}(N_1^L)) \rangle) = N_2^L \\ \Rightarrow\quad & \hat{F}(N_1^L) = N_2^L\ , \end{aligned}$$

where we also observed that a homomorphism goes through the group generating operation $\langle \cdot \rangle$, more precisely $\langle F(S) \rangle = F(\langle S \rangle)$, as is clear (generated group contains only finite products and inverses of the original set S). □

To stress this point, if for the groups (N_i, d_i) the nilradical condition holds and $F : N_1 \to N_2$ is an isometry, then F is necessarily affine. Indeed, from Lemmas 3.16 and 3.17 together one deduces, that the map $\Phi := L_{p^{-1}} \circ F$, where $p := F(e_{N_1})$, which is an identity fixing isometry, is an isomorphism $N_1 \to N_2$. Therefore $F = L_p \circ \Phi$ is the affine decomposition of F.

3.5 The nilradical condition

The next Proposition 3.19, combined with Lemma 3.17 and Lemma 3.16, gives the main theorem. The proof relies on Theorem 3.13. Or to be precise, it relies on the following corollary of that theorem:

Corollary 3.18. *Let (G, d) be a connected metric Lie group. Then there exists a Riemannian metric g such that* $\mathrm{Lie}(\mathrm{Isome}(G, d)) \leq \mathrm{Lie}(\mathrm{Isome}(G, g))$.

Proof. From Theorem 3.13 we have a metric tensor g in G in such a way that $\mathrm{Isome}(G, d) \leq \mathrm{Isome}(G, g)$. By Montgomery–Zippin theorem we know (see Corollary 3.9) that the groups $\mathrm{Isome}(G, d)$ and $\mathrm{Isome}(G, g)$ are Lie groups. The set $\mathrm{Isome}(G, d)$ is closed in $\mathrm{Isome}(G, g)$ as it is sequentially closed (see Fact 2.10). Therefore $\mathrm{Isome}(G, d)$ is a regular Lie subgroup of $\mathrm{Isome}(G, g)$, which is even more than we would need to deduce the claim, a Lie subgroup would be enough (Fact 2.5). □

Proposition 3.19. *For a nilpotent connected metric Lie group* (N, d) *it holds*

$$\mathrm{Lie}(N^L) = \mathrm{nil}(\mathrm{Lie}(\mathrm{Isome}(N, d))) .$$

Proof. We shall show the two inclusions. First we treat the easy claim

$$\mathrm{Lie}(N^L) \subset \mathrm{nil}(\mathrm{Lie}(\mathrm{Isome}(N, d))) =: \mathfrak{h} .$$

By the theorem of Wilson (Theorem 3.12) we know that $\mathrm{Lie}(N^L)$ is a nilpotent ideal of the Lie algebra $\mathrm{Lie}(\mathrm{Isome}(N, g))$. As we have Corollary 3.18, then $\mathrm{Lie}(N^L)$ is of course also an ideal in the smaller Lie algebra $\mathrm{Lie}(\mathrm{Isome}(N, d))$ as the notion of ideal is less restrictive in smaller set.

Now we must show $\mathrm{Lie}(N^L) \supset \mathfrak{h}$. Denote $H := \langle \exp(\mathfrak{h}) \rangle$ and pick any $\phi \in H$. It is enough to show that ϕ is a translation, i.e. that the map $f := L_{\phi(e)^{-1}} \circ \phi$ is the identity map. Indeed, then $H \subset N^L$ and hence their Lie algebras are necessarily in the same order (Fact 2.5).

Notice first that $f \in \mathrm{Stab}(e)$. Moreover, because we already proved the other direction of the inclusion, taking the exponentials and the groups generated we know that $N^L \subset H$ which means that translations are in H. Because H is stable by multiplication of its elements as a generated group, we conclude $f \in \mathrm{Stab}(e) \cap H$.

The Lie group H is nilpotent because by Fact 2.7 it has $\mathfrak{h}$ as its Lie algebra, which is nilpotent. The same fact gives the connectedness, so we deduce from Lemma 3.20 below that its topological closure $\mathrm{Cl}(H)$ is also a nilpotent and connected group. As a consequence of Lemma 3.21 below all compact subgroups of a nilpotent and connected group are central in that group. In particular, since the intersection of two groups is always a group and the intersection of a closed set and a compact set is a compact set, we have $*$ in

$$\mathrm{Stab}(e) \cap H \subset \mathrm{Stab}(e) \cap \mathrm{Cl}(H) \overset{*}{\subset} Z(\mathrm{Cl}(H)) \subset Z(N^L) ,$$

where the fact $N^L \subset H$ was again used. We got $f \in \mathrm{Stab}(e) \cap Z(N^L)$. But the only map that commutes with all the translations and fixes the identity is the identity map. Indeed for such a map it necessarily holds

$$f(q) = f(L_q(e)) = L_q(f(e)) = L_q(e) = q \qquad \forall q \in N .$$

Therefore $f = L_{\phi(e)^{-1}} \circ \phi = \mathrm{id}$ and the proof is finished. □

Lemma 3.20. *Let* G *be a topological group and* $H \leq G$ *a nilpotent and connected subgroup. Then its closure* $\mathrm{Cl}(H)$ *in* G *is a nilpotent and connected subgroup of* G*.*

Proof. The closure of a connected set is connected. First we must prove that the closure is a group and then prove nilpotency. If we close H, we possibly get some additional elements of G to $\mathrm{Cl}(H)$, call this additional part $\mathcal{A} \coloneqq \mathrm{Cl}(H)\backslash H$. First of all, any $a \in \mathcal{A}$ has inverse $a^{-1} \in G$. If such a point a^{-1} had a neighbourhood U not intersecting H, then U^{-1} contains a and does not intersect H either (H is stable under inversion), which is impossible. Thus $a^{-1} \in \mathrm{Cl}(H)$.

If $a, b \in \mathcal{A}$ and $h \in H$, we should prove that all kinds of multiplications ab, ah and ha stay in $\mathrm{Cl}(H)$. We argue again by contradiction: The group multiplication map $\Psi\colon G \times G \to G$ is continuous, so should there be a neighbourhood $U \ni ab$ not intersecting H, its inverse image would be open and thus contains a neighbourhood W of (a, b) not intersecting $H \times H$. But W is formed by unions and finite intersections of products of open neighbourhoods of a and b, and those all intersect H, a contradiction. Everything said here applies to the situations ah and ha.

To prove the closure nilpotent, we must use the group level definition of nilpotency. Suppose H is nilpotent of step $k-1$. This is equivalent to

$$[g_1, [g_2, [\cdots [g_{k-2}, [g_{k-1}, g_k]] \cdots]]] = e \qquad \forall g_i \in H \, .$$

In other words, the map $\Phi\colon G^k \to G$ (where G^k denotes cartesian product with itself k times) defined as the left hand side of the above equation, is constantly e in the set $H^k \subset G^k$. This map is formed only by multiplications and inverses, so it is continuous in all its variables, and thus continuous in the product topology. Therefore $\Phi \equiv e$ also in $\mathrm{Cl}(H)^k$, so $\mathrm{Cl}(H)$ is nilpotent. □

Lemma 3.21. *Let N be a connected nilpotent Lie group. Then every compact subgroup $K \leq N$ is central in N.*

Proof. Let the Lie group $\tilde{N}$ be the universal covering space of the Lie group N in such a way that the projection map $\pi\colon \tilde{N} \to N$ is a morphism of Lie groups (see Fact 2.18). Because $\tilde{N}$ is simply connected as a universal cover, the exponential map $\exp\colon \mathrm{Lie}(\tilde{N}) \to \tilde{N}$ is a diffeomorphism (see Fact 2.23). Denote $\log \coloneqq \exp^{-1}$ and consider the sets

$$B \coloneqq \log(\pi^{-1}(K)) \qquad \text{and} \qquad A \coloneqq \log(\pi^{-1}(e)) \, .$$

We shall prove that $K \subset Z(N)$ with the following steps:

1. $\pi^{-1}(e) \subset Z(\tilde{N})$.
2. $A \subset Z(\mathrm{Lie}(\tilde{N}))$.
3. B/A is a compact set.
4. B/A_S is a compact set when $A_S \coloneqq \mathrm{span}_{\mathbb{R}}(A)$.
5. if $b \in B$ and $m \in \mathbb{N}$, then $mb \in B$.
6. $B \subset A_S$.

7. $B \subset Z(\mathrm{Lie}(\tilde{N}))$.

8. $\pi^{-1}(K) \subset Z(\tilde{N})$.

9. $K \subset Z(N)$.

To prove step 1, notice that $\pi^{-1}(e)$ is discrete and $\pi^{-1}(e) \lhd \tilde{N}$ because π is a covering map and a group homomorphism. Take points $x \in \pi^{-1}(e)$ and $y \in \tilde{N}$ and a curve $\sigma\colon [0,1] \to \tilde{N}$ from $e_{\tilde{N}}$ to y. Define another curve ζ by $\zeta(t) = \sigma(t)x\sigma(t)^{-1}$. Because $\pi^{-1}(e)$ is a normal subgroup of $\tilde{N}$, then $\zeta(t) \in \pi^{-1}(e)$ for all t. We have $\zeta(0) = x$ and because $\pi^{-1}(e)$ is dicrete, then $\zeta(t) = x$ for all t. This proves $yxy^{-1} = x$, so $\pi^{-1}(e) \subset Z(\tilde{N})$.

To prove the step 2, we can use that for any nilpotent Lie group G it holds (see [CG04, p. 24])

$$\exp(Z(\mathrm{Lie}(G))) = Z(G)\,. \tag{5}$$

Therefore, whenever the logarithm is defined, it holds $Z(\mathrm{Lie}(G)) = \log(Z(G))$. Thus by taking the logarithm of the result of step 1 (the universal covering space $\tilde{N}$ is nilpotent, see Fact 2.24), we get

$$A \subset \log(Z(\tilde{N})) = Z(\mathrm{Lie}(\tilde{N}))\,.$$

In the step 3, we mean by B/A the quotient by a group. The fact that A is a group[29] follows from the fact that by step 2 the set A is commutative and that $\pi^{-1}(e)$ is a subgroup. Indeed, if $a, a' \in A$, then

$$\exp(a + a') = \exp(a)\exp(a') \in \pi^{-1}(e)$$

so $a + a' \in A$. To prove that the quotient space B/A is compact, we notice that since A is contained in the center, the exponential map $\exp|_B\colon B \to \pi^{-1}(K)$ sends each coset of A to a coset of $\exp(A) = \pi^{-1}(e)$, namely

$$\exp(b + A) = \exp(b)\exp(A)\,.$$

By Fact 2.19 the map $\exp|_B$ induces a homeomorphism between the spaces B/A and $\pi^{-1}(K)/\pi^{-1}(e)$. The latter space is homeomorphic to K, so B/A is compact.

To prove the step 4, consider the diagram

$$\begin{array}{ccc} B & & \\ \big\downarrow{\scriptstyle \pi} & \searrow^{\pi_S} & \\ B/A & & B/A_S \end{array}\,.$$

We can define a map $\psi\colon B/A \to B/A_S$ in such a way that $F([b]) = \pi_S(b')$ where $b' \in \pi^{-1}(\{b\})$ is chosen arbitrarily. Because $A \subset A_S$ this is well defined map. The map ψ is also continuous, because both maps π and π_S are continuous and open.

[29] The group operation is the vector sum induced from $\mathrm{Lie}(\tilde{N})$.

Therefore, the claim holds because B/A_S is the image of the compact space B/A under the continuous map ψ.

For step 5, take $b \in B$ which means $\exp(b) \in \pi^{-1}(K)$. The set $\pi^{-1}(K)$ is closed under multiplication being preimage under a homomorphism of the subgroup K. Therefore

$$\exp(mb) = (\exp(b))^m \in \pi^{-1}(K) .$$

To prove the step 6, assume by contradiction that there exists $b \in B \backslash A_S$. By step 5, $[nb] \in B/A_S$ for all n. Because B/A_S is compact, this sequence has a subsequence that converges to some $x \in B/A_S$. Then for every $\varepsilon > 0$ starting from some index n_ε this subsequence is in the set $\pi^{-1}(\pi(B(x,\varepsilon)))$ because this is an open neighbourhood of x. Thus for $n \geq n_\varepsilon$ there exists $x'_n \in B(x,\varepsilon)$ such that $\pi(nb) = \pi(x'_n)$, so $nb - x'_n \in A_S$ and we get $d_E(nb, A_S) = d_E(x'_n, A_S)$. This means that for $n \geq n_\varepsilon$

$$|d_E(nb, A_S) - d_E(x, A_S)| = |d_E(x'_n, A_S) - d_E(x, A_S)| < \varepsilon$$

but on the other hand

$$d_E(nb, A_S) = \inf\{\|nb - a\| \mid a \in A_S\} = n d_E(b, A_S) \xrightarrow{n\to\infty} \infty$$

which is a contradiction.

The step 7 we get immediately from steps 2 and 6 because if $A \subset Z(\mathrm{Lie}(\tilde{N}))$, then $\mathrm{span}_{\mathbb{R}}(A) \subset Z(\mathrm{Lie}(\tilde{N}))$.

The step 8 follows from the step 7 using the result (5).

The step 9 follows from the step 8 because π is a surjective homomorphism. □

3.6 Corollary: The semidirect product decomposition

In this section we shall go through the proof of Corollary 3.4. The semidirect product is straightforward to produce, once we know that the group of left-translations is a normal subgroup of the isometry group. We found out that this is the case and there is actually something more:

Proposition 3.22. *Let (G,d) be a metric Lie group. Then its self-isometries are affine maps if and only if $G^L \lhd \mathrm{Isome}(G,d)$.*

For the proof recall Lemma 3.16 and the induced isomorphism $\hat{F}$ of the isometry groups. If we consider the case of just one metric Lie group (G,d), still any isometry $F\colon G \to G$ induces an isomorphism from the isometry group to itself, i.e. an automorphism. This automorphism is a priori non-trivial if F is not the identity map. The proof of the above proposition follows from the lemmas below together with Lemma 3.16.

Lemma 3.23. *Let (G,d) be a metric Lie group. Suppose $G^L \lhd \mathrm{Isome}(G,d)$. Then for any $F \in \mathrm{Isome}(G,d)$ for which $F(e) = e$ it holds $\hat{F}(G^L) = G^L$.*

Proof. Let $g \in G^L$ and $F \in \mathrm{Isome}(G,d)$ be any isometry that fixes the identity. We have then

$$\hat{F}(g) = F \circ g \circ F^{-1} \in G^L \qquad \text{and} \qquad \widehat{F^{-1}}(g) = F^{-1} \circ g \circ (F^{-1})^{-1} \in G^L$$

by normality. This means

$$\hat{F}(G^L) \subset G^L \qquad \text{and} \qquad (\hat{F})^{-1}(G^L) = \widehat{F^{-1}}(G^L) \subset G^L\,.$$

These give together $\hat{F}(G^L) = G^L$. □

Lemma 3.24. *Let (G,d) be a metric Lie group such that any self-isometry $F\colon G \to G$ is an affine map. Then $G^L \lhd \mathrm{Isome}(G,d)$.*

Proof. We assume that every isometry has the affine decomposition into a translation and a group isomorphism. Let L_p be any translation of the group G and $F\colon G \to G$ any isometry. Let the affine decomposition be $F = \tau \circ \Phi$. Then

$$\begin{aligned} F \circ L_p \circ F^{-1} &= (\tau \circ \Phi) \circ L_p \circ (\tau \circ \Phi)^{-1} \\ &= \tau \circ \Phi \circ L_p \circ \Phi^{-1} \circ \tau^{-1} \\ &= \tau \circ L_{\Phi(p)} \circ \Phi \circ \Phi^{-1} \circ \tau^{-1} \\ &= \tau \circ L_{\Phi(p)} \circ \tau^{-1} \in G^L\,. \end{aligned}$$

□

Proof of Corollary 3.4. Recall Fact 2.26. From the Main Theorem and Proposition 3.22 we get that the group of left-translations is a normal subgroup of the isometry group. The Main Theorem also gives us the condition $\mathrm{Isome}(N,d) = N^L \cdot \mathrm{Stab}(e)$. The condition $N^L \cap \mathrm{Stab}(e) = \{\mathrm{Id}\}$ is trivial. Therefore the conjugation gives us the correct semidirect product decomposition. □

4 Maximality of the result: counterexamples

4.1 When connectedness is dropped

In this subsection we shall show that the condition of connectedness can't be removed. It can't be even relaxed to the case of finitely many connected components.

Consider the discrete group $\mathcal{D} = \{1, \mathrm{i}, -1, -\mathrm{i}\} \subset \mathbb{S}^1$ equipped with the usual multiplication of complex numbers. Equip it with the so called discrete metric, i.e. the distance function for which $d(x, y) \equiv 1$ for all $x, y \in \mathcal{D}$. This is known to be a distance and inducing the discrete topology on $\mathcal{D}$, i.e. the topology for which any set is open. It is a left-invariant distance because actually every bijection $\mathcal{D} \to \mathcal{D}$, in particular every left-translation, is an isometry.

This group has four connected components and it is an abelian group and thus nilpotent. If our result would be generalizable to finitely many connected components, then in particular Corollary 3.2 applied to the case $N_1 = N_2 = \mathcal{D}$ states that isometries fixing the identity are group automorphisms.

But how many automorphisms are there? An automorphism ϕ necessarily satisfies $\phi(e) = e$ and now $e = 1$. Observe also that such a map is injective and satisfies $\phi(x^2) = (\phi(x))^2$. Let's figure out what are the options for the element i to be mapped. By the injectivity an automorphism can't have $\phi(\mathrm{i}) = 1$ and for the other options for i to go we have the following conclusions:

a) if $\phi(\mathrm{i}) = -1$, then $\phi(-1) = \phi(\mathrm{i}^2) = (\phi(\mathrm{i}))^2 = (-1)^2 = 1$, which is impossible by injectivity.

b) if $\phi(\mathrm{i}) = \mathrm{i}$, then $\phi(-1) = -1$. This options recovered the identity map.

c) if $\phi(\mathrm{i}) = -\mathrm{i}$, then $\phi(-1) = -1$. This also completely determines the map since then $\phi(-\mathrm{i}) = \mathrm{i}$. This is a bijection different than the identity map.

So only possiblities for an automorphism are the identity map and the map that reflects with respect to x-axis. This is a contradiction because all the bijections fixing the identity should have been automorphisms, for example the permutation $\mathrm{i} \mapsto -1 \mapsto \mathrm{i}$ keeping the other points fixed.

There is one thing we should consider a little more. Our group $\mathcal{D}$ is definitely a nilpotent group with a metric but is it really a Lie group? A Lie group should have a manifold structure, but $\mathcal{D}$ is intuitively "zero-dimensional" and surely it is not locally diffeomorphic to $\mathbb{R}^n$ for $n \geq 1$. We can of course define discrete spaces to be zero-dimensional manifolds. But if this bothers the reader, a more solid way of arguing is the following: Consider as the counter-example the direct product $\mathcal{D} \times \mathbb{R}$ (with the distance $d((g, x), (h, y)) = \max\{d_{\mathcal{D}}(g, h), d_{\mathbb{R}}(x, y)\}$) which is a one-dimensional manifold. The above argument still applies: If ψ is the permutation $\mathrm{i} \mapsto -1 \mapsto \mathrm{i}$, then (ψ, Id) is an isometry of $\mathcal{D} \times \mathbb{R}$. But if Ψ is any bijection of $\mathcal{D} \times \mathbb{R}$ of the form

$\Psi = (f, \mathrm{Id})$, then Ψ is an automorphism if and only if

$$\begin{aligned} & \Psi((a,x)\cdot(b,y)) = \Psi(a,x)\cdot\Psi(b,y) \\ \Leftrightarrow \quad & \Psi(ab,xy) = \Psi(a,x)\cdot\Psi(b,y) \\ \Leftrightarrow \quad & (f(ab),xy) = (f(a),x)\cdot(f(b),y) = (f(a)f(b),xy)\,. \end{aligned}$$

This means that the component map f should be an automorphism. The map ψ above was not, so the same contradiction arises.

4.2 When nilpotency is dropped

Nilpotent groups are *unimodular*, i.e. they admit Radon measures that are both left- and right-invariant. Also all compact Lie groups are unimodular by Fact 2.20. Our result on the affine decomposition of isometries does not extent to unimodular groups. The easiest example is provided by the sphere $\mathbb{S}^3$ equipped with the product of unit quaternions and a distance d that is induced by the Euclidean distance of $\mathbb{R}^4$. Then the group $(\mathbb{S}^3, d)$ is a connected non-nilpotent unimodular metric Lie group. Indeed, one can check that the induced distance is left-invariant using the quaternion product law.

In this case the group inversion is an isometry because it has the formula

$$(a,b,c,d)^{-1} = (a,-b,-c,-d)$$

and this is clearly an isometry of $\mathbb{R}^4$. Thus the inversion is an isometry fixing the identity but it is not a group isomorphism. Since however such a map is defined in terms of the group structure, in the next section we shall provide a (non-unimodular) group with isometries that have nothing to do with the group structure.

4.2.1 Introducing the rototranslation group

Next we will show that the assumption of nilpotency can't be removed. Concretely we shall present a connected non-nilpotent group that again has more isometries than those of the form a translation composed with an isomorphism.

The group of rigid motions of the plane, also called the *roto-translation group*, is

$$\mathrm{Rot}(\mathbb{R}^2) \coloneqq \{x \overset{\psi}{\mapsto} Ax + b \mid A \in SO(2),\, b \in \mathbb{R}^2\}\,.$$

Notice that such maps are parametrized by 3 numbers: for all $\psi \in \mathrm{Rot}(\mathbb{R}^2)$ there exists $\alpha, \beta, \theta \in \mathbb{R}$ such that

$$\psi(x,y) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

From here the group law can be deduced by first calculating the result of two consequent affine motions:

$$\begin{aligned} \psi_1(\psi_2(x,y)) &= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \left(\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \right) + \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \alpha_2\cos\theta_1 - \beta_2\sin\theta_1 + \alpha_1 \\ \alpha_2\sin\theta_1 + \beta_2\cos\theta_1 + \beta_1 \end{bmatrix} \end{aligned}$$

as is also clear just by geometric intuition.

Now we are ready to precisely define the Lie group $\mathrm{Rot}(\mathbb{R}^2)$. It is the manifold $\mathbb{R}^2 \times \mathbb{S}^1$ endowed with the product[30]

$$(\alpha_1, \beta_1, \theta_1) * (\alpha_2, \beta_2, \theta_2) \\ = (\alpha_2 \cos\theta_1 - \beta_2 \sin\theta_1 + \alpha_1,\ \alpha_2 \sin\theta_1 + \beta_2 \cos\theta_1 + \beta_1,\ \theta_1 + \theta_2)\,. \quad (6)$$

However, this group is not going to be our counterexample, but its universal covering group will be. In this case the universal covering group is nothing more complicated than just $\mathbb{R}^3$ endowed with the product law above. Denote this group by G. The reason to pass to the universal cover is to have the space simply connected. We then get one-to-one corresponce between the group automorphisms and the Lie algebra automorphisms (Fact 2.2).

We still did not define a distance in the roto-translation group or in G. We will just take the Euclidean distance d_E of $\mathbb{R}^3$. This makes (G, d_E) a metric Lie group, actually even a Riemannian Lie group. The fact that the distance d_E is left-invariant must be checked since our product is not the Euclidean vector sum. The left-invariance can be directly seen from product law (6): The question is if for fixed $\alpha_1, \beta_1, \theta_1$ the map

$$(x, y, z) \mapsto (x \cos\theta_1 + y \sin\theta_1 + \alpha_1,\ x \sin\theta_1 + y \cos\theta_1 + \beta_1,\ \theta_1 + z)$$

is an isometry of (G, d_E), i.e. an isometry of $\mathbb{E}^3$. Evidently it is, as it is a rotation of xy-plane composed with a translation.

Considering the group G instead of $\mathrm{Rot}(\mathbb{R}^2)$ actually makes our counterexample in some sense more powerful. Indeed, we get the existence of a connected and even simply connected metric Lie group (not nilpotent!) for which the affine decomposition of isometries fails. If our counterexample would have been the actual group and not its universal cover, one could wonder if the lack of simply connectedness is spoiling the affine decomposition, but that is not the case.

Before going to study the isometries inside (G, d_E) we can make things a little bit more concrete by working on a matrix group that is isomorphic to G. This is handy as we are going to study the Lie algebra of G, and the considerations of Lie algebra are very easy for matrix groups. We have an isomorphism

$$\Psi\colon G \to GL(3, \mathbb{R}) \qquad \Psi(\alpha, \beta, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & \alpha \\ \sin\theta & \cos\theta & \beta \\ 0 & 0 & 1 \end{bmatrix}. \quad (7)$$

The manifold structure of $GL(3, \mathbb{R})$ is the submanifold structure of $\mathbb{R}^{3^2}$, so this isomorphism is a smooth map indeed and thus a Lie group isomorphism.

Denote by $\mathcal{G}$ the group of matrices of the form (7), i.e. the isomorphic copy of G inside $GL(3, \mathbb{R})$. Next we want to find the Lie algebra of the matrix-group $\mathcal{G}$. This

[30] The product is clearly a smooth map. Checking the smoothness of the inversion map is equally straightforward: just figure out its formula.

consists of searching matrices that generate rotations and translations, and studying their commutator-relations. Those generators are the derivatives calculated at zero[31]

$$X := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Y := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Theta := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

The Lie bracket relations are then calculated to be

$$\begin{aligned} [X,Y] &= XY - YX = 0 - 0 = 0 \\ [\Theta,X] &= \Theta X - X\Theta = Y - 0 = Y \\ [\Theta,Y] &= \Theta Y - Y\Theta = -X - 0 = -X . \end{aligned}$$

4.2.2 Calculating the isometric automorphisms

Notice that as (G, d_E) has the Euclidean metric we know its isometry group already. It is

$$\mathrm{Isome}(G, d_E) = \mathrm{Isome}(\mathbb{E}^3) = O(3) \ltimes \mathbb{R}^3 .$$

Thus, the isometries that fix the identity are exactly the matrices of $O(3)$. We shall show that the space of isometric automorphisms of G is much smaller, and thus not every isometry fixing the identity is an automorphism, which contradicts again Corollary 3.2.

Let $\mathfrak{w}$: Lie(G) $\to$ Lie(G) be an arbitrary Lie algebra automorphism. Our first claim is that it satisfies necessarily $\mathfrak{w}(\mathrm{span}\{X,Y\}) \subset \mathrm{span}\{X,Y\}$. This we prove as follows: Write

$$\mathfrak{w}(X) = aX + bY + \mu\Theta \qquad \text{and} \qquad \mathfrak{w}(Y) = cX + dY + \lambda\Theta . \tag{8}$$

By the automorphism property

$$\begin{aligned} 0 = \mathfrak{w}(0) = \mathfrak{w}([X,Y]) &= [\mathfrak{w}(X), \mathfrak{w}(Y)] \\ &= a\lambda[X,\Theta] + b\lambda[Y,\Theta] + \mu c[\Theta, X] + \mu d[\Theta, Y] \\ &= (b\lambda - \mu d)X + (\mu c - a\lambda)Y . \end{aligned}$$

This implies

$$\begin{cases} b\lambda = \mu d \\ \mu c = a\lambda \end{cases} .$$

We claim that $\lambda = 0 = \mu$. If either one is zero, then is also the other because of the pair of equations above and the fact that $\mathfrak{w}$ can't map X nor Y to the zero-vector. So it is enough to assume by contradiction that $\lambda \neq 0 \neq \mu$. In this case one can divide the pair of equations with μ and substituting those equations back to (8) gives

$$\mathfrak{w}(Y) = \frac{a\lambda}{\mu}X + \frac{b\lambda}{\mu}Y + \lambda\Theta .$$

[31]One can also check that the exponentiation gives of these matrices gives respectively the pure rotation and translation matrices.

This is a contradiction because then $\mathfrak{w}(X) = \frac{\mu}{\lambda}\mathfrak{w}(Y)$, which is an impossible situation for an invertible linear map since X and Y were linearly independent vectors.

Now when $\mathfrak{w}(\mathrm{span}\{X,Y\}) \subset \mathrm{span}\{X,Y\}$ is proved, observe that there actually is an equality $\mathfrak{w}(\mathrm{span}\{X,Y\}) = \mathrm{span}\{X,Y\}$. This is just because $\mathfrak{w}$ is a linear bijection as a Lie algebra automorphism, and thus $\mathfrak{w}$ must preserve the vector space dimension.

Let then $F\colon G \to G$ be an arbitrary self-isometry of G that fixes the identity and that is also a group automorphism. Because the distance is Euclidean, we actually know that F is smooth and thus a Lie group automorphism. Because the Lie group G is topologically $\mathbb{R}^3$, it is simply connected and therefore all Lie group automorphisms are induced by some Lie algebra automorphism (Fact 2.2). So we can assume from now on that $\mathfrak{w}$ is the Lie algebra automorphism corresponding the isometric automorphism F. We want to say next that the vectors X, Y and Θ form an orthogonal set, as well as their images $\mathfrak{w}(X)$, $\mathfrak{w}(Y)$ and $\mathfrak{w}(\Theta)$.

Recall that G is diffeomorphic and isometric to $\mathbb{R}^3$. On the other hand $\mathcal{G}$ is isomorphic as a metric Lie group to G. Thus the obvious identification $\mathcal{I}\colon \mathbb{R}^3 \to \mathcal{G}$ (which has the same formula (7) as Ψ) is an isometric diffeomorphism. In $\mathbb{R}^3$ it is not so easy to make the difference between the tangent space of the identity and the space itself, but we must now do that to make sense about what is the scalar product in $T_e\mathcal{G}$. An isometry f of $\mathbb{R}^3 = \{(\alpha,\beta,\theta)\}$ fixing the identity preserves the scalar products between the tangent vectors of the curves

$$\sigma_1(t) = (t,0,0) \qquad \sigma_2(t) = (0,t,0) \qquad \sigma_3(t) = (0,0,t) \qquad .$$

In the space $\mathcal{G}$ the corresponding curves are $\mathcal{I}\circ\sigma_i$ so the scalar product of $T_e\mathcal{G}$ is the one that makes the vectors $(\mathcal{I}\circ\sigma_i)'(0)$ orthogonal. These tangent vectors are exactly the matrices X, Y, Θ.

The tangent of the space $\mathcal{G}$ is not similar to the space itself. An arbitary isometry of $\mathcal{G}$ is of the form $\mathcal{I}\circ f\circ\mathcal{I}^{-1} =: F$. This F is not the map that should preserve X, Y, Θ but only its differential. Concretely we show this to be the case by calculating

$$\begin{aligned} && \mathrm{d}(F)(\mathcal{I}\circ\sigma_i)'(0) &= (\mathcal{I}\circ f\circ\sigma_i)'(0) = \mathrm{d}(\mathcal{I})(f\circ\sigma_i)'(0) \\ \Rightarrow && \langle \mathrm{d}(F)(\mathcal{I}\circ\sigma_i)'(0), \mathrm{d}(F)(\mathcal{I}\circ\sigma_j)'(0)\rangle &= \langle \mathrm{d}(\mathcal{I})(f\circ\sigma_i)'(0), \mathrm{d}(\mathcal{I})(f\circ\sigma_j)'(0)\rangle \\ &&&= \langle (f\circ\sigma_i)'(0), (f\circ\sigma_j)'(0)\rangle \\ &&&= \langle \sigma_i'(0), \sigma_j'(0)\rangle = \delta_{ij}\ . \end{aligned}$$

Because we proved the sets $\{X,Y,\Theta\}$ and $\{\mathfrak{w}(X),\mathfrak{w}(Y),\mathfrak{w}(\Theta)\}$ to be orthogonal sets, then $\mathfrak{w}(\mathrm{span}\{X,Y\}) = \mathrm{span}\{X,Y\}$ implies $\mathfrak{w}(\Theta) = \pm\Theta$. We can now form the matrix of $\mathfrak{w}$ in the basis X, Y, Θ. It is

$$\mathfrak{w} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}, \qquad \text{where} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in O(2)\,,$$

using ones more that $\mathfrak{w}$ preserves the scalar product. Here is a contradiction since the set of isometric Lie algebra automorphisms is then at most[32] of dimension $\dim(O(2)\times$

[32]The space may be smaller, as in the above we only argued to limit the degrees of freedom in the space of isometric automorphisms.

$\mathbb{Z}_2) = 1$, where $\mathbb{Z}_2$ denotes the two element discrete group $\{1, -1\}$ equipped with the multiplication. But the set of isometric Lie algebra automorphisms should equal in dimension to the set of isometries fixing the identity, which is the group $O(3)$ with $\dim O(3) = 3$.

The result is quite ironic: The space on which isometries are not affine is the space of affine isometries of $\mathbb{R}^2$.

5 Discussion

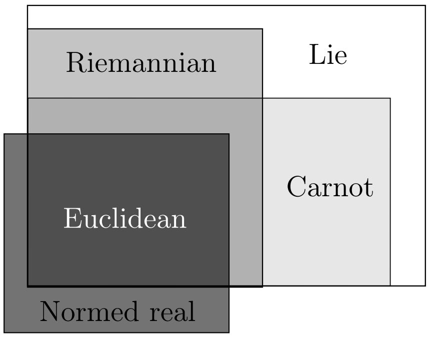

Figure 2: Theorem 3.1 was known already in Carnot groups, normed vectors spaces over $\mathbb{R}$, and connected nilpotent Riemannian Lie groups. In the figure "Lie" stands for general connected nilpotent metric Lie groups.

What was known before The main theorem has been know in several different settings before this study. The situation is described in Figure 2. There were basically three categories of which two were unified by this study.

1. As already mentioned, it is proved by E.N. Wilson in [Wil82] that the self-isometries of a nilpotent connected Lie group equipped with left-invariant *Riemannian* distance[33] are affine. Wilson was only considering the self-isometries, but if one wants to extend Wilson's result for the isometries between two different groups, one can use the steps h, f and g of this thesis and part 2 of his Theorem 2 which is the nilradical condition. Considering the affine decomposition for the self-isometries, Wilson does not state the result in this form outside the introduction, but his Theorem 2 states that the group of left-translations is a normal subgroup of the isometry group in the connected case. In Proposition 3.22 of this thesis we showed using elementary arguments that the normality is equivalent to the fact that the self-isometries are affine. It should also be noted that actually Wilson uses the language of *nilmanifolds*, Riemannian manifolds acted upon isometrically and transitively by a nilpotent Lie group. Still, Wilson himself notes that nilmanifolds are only superficially more general than nilpotent Riemannian Lie groups.

2. By the work of U. Hamenstädt [Ham90] and I. Kishimoto [Kis03] it was known that the isometries between two subRiemannian Carnot groups are affine (see the complete proof in [LDO14]). In the paper [LDO14] this was also generalized to the subFinsler Carnot groups (for the definition, see Section 2.5). A metric

[33] In Riemannian geometry the distance of two points is defined as the length of the shortest curve between them. The length of a curve is defined by integrating the norm of the tangent vector, given by the metric tensor, of the curve. Recall that a Riemannian metric always induces the manifold topology. In the more general setting we were studying in this thesis we had to instead explicitly require that the manifold topology is induced by the distance.

Lie group is more general both as a metric space and as a group. Actually the paper [LDO14] deals with isometries allowed to be defined locally, i.e. from an open set to another. Thus our result does not cover their result completely but only the case of globally defined isometries.

3. The result that the isometries between two normed real vector spaces are affine maps is known as the Mazur–Ulam theorem originally presented in the paper [MU32]. This covers also the infinite dimensional spaces which are not touched by the settings that assume the manifold structure since manifolds are always finite dimensional.

For the completeness of Figure 2 we mention that the Riemannian Heisenberg-group is an example of a Riemannian Carnot-group that is not a vector space since it is non-Abelian. So the intersections of different settings are really meant precisely in Figure 2.

What was *not* known before Every connected subgroup $H < G$ of a Carnot group is an example of nilpotent connected metric Lie group. These subgroups are not necessarily Carnot groups themselves, they can fail to be length spaces. Proper subgroups are never open sets, so even the theory of local isometries in the paper [LDO14] that we discussed in the above list cannot be applied to such sets.

For a concrete example, take the 3-dimensional Heisenberg group seen topologically as $\mathbb{R}^3$. Then the xz-plane is a connected subgroup but not a length space. In a length space (M, d), by the definition, the distance function is such that

$$d(x,y) = \inf\Big\{\sup\{\sum_{i=1}^{N} d(\gamma(t_i),\gamma(t_{i+1})) \mid (t_1,\dots,t_N) \text{ is a partition of } [0,1].\} \\ \mid \gamma\colon [0,1] \to M \text{ is a curve from } x \text{ to } y\Big\}.$$

The term *intrinsic distance* is also used to refer to such a distance function. In words: the distance is the infimum of the lengths of the curves where the lengths are calculated by approximating the curve by a piecewise linear curve. It turns out that in the xz-plane of the Heisenberg group, most curves have infinite length, although the distance function induced from the total space is finite everywhere. From our main theorem we can now deduce that also in such a space the isometries are affine.

A note on Berestovskii theorem In the above we noted that metric Lie groups with non-intrinsic distances are examples of settings that were not covered by the results before ours. Actually, *they are the only examples.* A theorem of V.N. Berestovskii [Ber88, Thm 2] states exactly that if (G, d) is a connected metric Lie group with an intrinsic distance, then the distance is actually a subFinsler distance. Thus such spaces are already covered by the paper [LDO14]. As intrinsic distances are physically natural, one could argue that the generalization we just worked out is non-physical and unimportant. Still, there are non-intrinsic distances appearing in other applications of mathematics.

References

[Are46a] Richard Arens. Topologies for homeomorphism groups. *American Journal of Mathematics*, 68(4):pp. 593–610, 1946.

[Are46b] Richard F. Arens. A topology for spaces of transformations. *Annals of Mathematics*, 47(3):pp. 480–495, 1946.

[Ber88] V.N. Berestovskii. Homogeneous manifolds with intrinsic metric. i. *Siberian Mathematical Journal*, 29(6):887–897, 1988.

[CG04] L. Corwin and F.P. Greenleaf. *Representations of Nilpotent Lie Groups and Their Applications: Volume 1, Part 1, Basic Theory and Examples*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2004.

[Gle52] Andrew M. Gleason. Groups without small subgroups. *Annals of Mathematics*, 56(2):pp. 193–212, 1952.

[Ham90] Ursula Hamenstädt. Some regularity theorems for carnot-carathéodory metrics. *Journal of Differential Geometry*, 32(3):819–850, 1990.

[Hel01] S. Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces*. Crm Proceedings & Lecture Notes. American Mathematical Society, 2001.

[HN11] J. Hilgert and K.H. Neeb. *Structure and Geometry of Lie Groups*. Springer Monographs in Mathematics. Springer New York, 2011.

[Kel12] J.L.R. Kelley. *General Topology*. Literary Licensing, LLC, 2012.

[Kis03] Iwao Kishimoto. Geodesics and isometries of carnot groups. *Journal of Mathematics of Kyoto University*, 43(3):509–522, 2003.

[Kna13] A.W. Knapp. *Lie Groups Beyond an Introduction*. Progress in Mathematics. Birkhäuser Boston, 2013.

[LDO14] Enrico Le Donne and Alessandro Ottazzi. Isometries of carnot groups and sub-finsler homogeneous manifolds. *The Journal of Geometric Analysis*, pages 1–16, 2014.

[McC88] G. McCarty. *Topology: An Introduction with Application to Topological Groups*. Dover Books on Mathematics Series. Dover Publications, 1988.

[MU32] Stanisław Mazur and Stanisław Ulam. Sur les transformations isométriques d'espaces vectoriels normés. *CR Acad. Sci. Paris*, 194(946-948):116, 1932.

[MZ55] Deane Montgomery and Leo Zippin. Topological transformation groups. *Interscience Tracts in pure and applied mathematics*, 1955.

[Rag72] M.S. Raghunathan. *Discrete Subgroups of Lie Groups*. Springer Berlin Heidelberg, 1972.

[War13] F.W. Warner. *Foundations of Differentiable Manifolds and Lie Groups.* Graduate Texts in Mathematics. Springer New York, 2013.

[Wil82] Edward N Wilson. Isometry groups on homogeneous nilmanifolds. *Geometriae Dedicata*, 12(3):337–346, 1982.

[Zhe05] D.P. Zhelobenko. *Principal Structures and Methods of Representation Theory.* Translations of mathematical monographs. American Mathematical Soc., 2005.