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STABILITY OF DEGENERATE PARABOLIC CAUCHY PROBLEMS

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Abstract. We prove that solutions to Cauchy problems related to the p-parabolic equations are stable with respect to the nonlinearity exponent p. More specifically, solutions with a fixed initial trace converge in an L^q-space to a solution of the limit problem as $p > 2$ varies.

1. Introduction

We study the stability of solutions to the degenerate parabolic equation

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \tag{1.1}$$

with respect to perturbations in the nonlinearity power p. The main issue we address is whether solutions converge in some sense to the solution of the limit problem as the nonlinearity power p varies. In applications, parameters like p are often only known approximately, for instance from experiments. Thus it is natural to ask whether solutions are sensitive to small variations in such parameters or not.

The stability of Dirichlet problems on bounded domains for (1.1) is detailed in [KP10]. See also [Lind87, LM98, Lind93] for elliptic equations similar to the p-Laplacian, [Luk] for the porous medium and fast diffusion equations, and [FHKM] for parabolic quasiminimizers. Here we focus on the Cauchy problem

$$\begin{cases} \partial_t u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0, & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = \nu \end{cases}$$

with $p > 2$. The initial trace ν can be a positive measure with compact support and finite total mass, and the interpretation of the initial condition is that the initial values is attained in the sense of distributions. See Section 2 below for the details. For the existence theory of this initial value problem, we refer to [DH89].

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An important example of solutions to the Cauchy problems is given by the celebrated *Barenblatt solutions*. These functions are the nonlinear counterparts of the fundamental solution, and they are defined for positive times $t > 0$ by the formula

$$\mathcal{B}_p(x,t) = t^{-n/\lambda} \left(C - \frac{p-2}{2} \lambda^{\frac{1}{1-p}} \left(\frac{|x|}{t^{1/\lambda}} \right)^{\frac{p}{p-1}} \right)_+^{\frac{p-1}{p-2}},$$

where $\lambda = n(p-2) + p$. The constant C is at our disposal. We choose the value for which

$$\int_{\mathbb{R}^n} \mathcal{B}_p(x,t)\, dx = 1 \quad \text{for all} \quad t > 0.$$

With this normalization, one easily checks that

$$\lim_{t \to 0} \mathcal{B}_p(\cdot, t) = \delta_0,$$

in the sense of distributions, where δ_0 is Dirac's delta at the origin of $\mathbb{R}^n$. Further, the function $\mathcal{B}_p$ is the unique solution to the Cauchy problem with this initial trace, see [KV88].

Our main result states that solutions and their gradients converge to a limit function in an L^q-space, locally in space and up to the initial time, as p varies. The limit function turns out to be a weak solution to the equation involving the limit exponent, and it also has the correct initial trace. In particular, since the Barenblatt solutions are the unique solutions to the respective Cauchy problems, our stability theorem contains the fact that

$$\mathcal{B}_{\widetilde{p}} \to \mathcal{B}_p \quad \text{as} \quad \widetilde{p} \to p \quad \text{for} \quad p > 2 \tag{1.2}$$

in the above sense. The admissible exponents in the norms are sharp in the sense that for larger exponents, the norms of the Barenblatt solution and its gradient are no longer finite. For general initial measures, uniqueness for the Cauchy problem remains open, and we only get a subsequence converging to *some* solution of the limit problem. If the initial trace is an L^1 function, then the solution to the limit problem is unique, and the whole sequence converges.

The proof of our stability result is based on deriving suitable estimates for solutions and their gradients. We use the truncation device commonly employed in measure data problems for this purpose, see e.g. [BDGO97]. This works well, since the mass of the initial data has the same role in the estimates as the mass of a right hand side source term. Convergence of the solutions is established by employing the ideas of [KP10] and a localization argument. Here we encounter a difficulty: changing the growth exponent p changes the space to which solutions belong. To deal with this, we need to use a local Reverse Hölder inequality for the gradient. Such a result can be found in [KL00].

The paper is organized as follows. In Section 2, we present the necessary preparatory material, including the parabolic Sobolev spaces to which weak solutions belong, and a rigorous description of what is meant by the Cauchy problem in this context. In Section 3, we prove a result concerning the propagation properties of solutions. These properties are then employed in Section 4 in proving the estimates necessary for our stability result. Finally, in Section 5 we prove the main theorem about stability.

2. Preliminaries

We denote by

$$L^p(t_1,t_2;W^{1,p}(\Omega)),$$

where $t_1 < t_2$, the parabolic Sobolev space. This space consists of measurable functions u such that for almost every t, $t_1 < t < t_2$, the function $x \mapsto u(x,t)$ belongs to $W^{1,p}(\Omega)$ and the norm

$$\left(\int_{t_1}^{t_2}\int_\Omega |u(x,t)|^p + |\nabla u(x,t)|^p \, dx\, dt\right)^{1/p}$$

is finite. The definitions for $L^p_{loc}(t_1,t_2;W^{1,p}_{loc}(\Omega))$ and $L^p(t_1,t_2;W^{1,p}_0(\Omega))$ are analogous. The space $C((t_1,t_2);L^q(\Omega))$, $q = 1,2$, comprises of all the functions u such that for $t_1 < s, t < t_2$, we have

$$\int_\Omega |u(x,t) - u(x,s)|^q \, dx \to 0$$

as $s \to t$. The notation $U \Subset \Omega$ means that U is a bounded subset of Ω and the closure of U belongs to Ω. We also denote $U_\varepsilon = U \times (\varepsilon, \infty)$. We study the Cauchy problem

$$\begin{cases} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) & = \frac{\partial u}{\partial t}, \qquad \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(\cdot,0) & = \nu, \end{cases} \tag{2.1}$$

where ν is a compactly supported positive Radon measure. For simplicity, this measure is fixed throughout the paper. To be more precise, the solution $u \geq 0$ satisfies the initial condition in the sense of distributions, meaning that

$$\lim_{t\to 0}\int_{\mathbb{R}^n} u(x,t)\varphi(x)\, dx = \int_{\mathbb{R}^n}\varphi(x)\, d\nu(x)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. Further, u is a weak solution for positive times in the sense of the following definition.

Definition 2.2. *A function*

$$u \in L^p_{loc}(0,\infty\,;W^{1,p}_{loc}(\mathbb{R}^n))$$

is a weak solution in $\mathbb{R}^n \times (0,\infty)$ *if it satisfies*

$$\int_0^\infty\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\, dx\, dt - \int_0^\infty\int_\Omega u\frac{\partial\phi}{\partial t}\, dx\, dt = 0 \tag{2.3}$$

for every test function $\phi \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$.

Definition of a solution can be also written in the form

$$\begin{aligned} \int_{\tau_1}^{\tau_2} \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx \, dt - \int_{\tau_1}^{\tau_2} \int_\Omega u \frac{\partial \phi}{\partial t} \, dx \, dt \\ + \int_\Omega u(x,\tau_2)\phi(x,\tau_2) \, dx - \int_\Omega u(x,\tau_1)\phi(x,\tau_1) \, dx \; = \; 0 \end{aligned} \tag{2.4}$$

for a test function $\phi \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$ and for almost every $0 < \tau_1 < \tau_2 < \infty$.

Our assumptions imply that the Cauchy problem has a solution, see [DH89] and [KP09]. These solutions are constructed by approximating the initial trace by regular functions, proving uniform estimates, and then passing to the limit by compactness arguments.

Remark 2.5. By a solution to the Cauchy problem (2.1) we mean any solution obtained as a limit of approximating the initial trace ν via standard mollifiers. The reason for this is that we want to choose the approximations so that their support does not expand much.

The above definition does not include a time derivative of u in any sense. Nevertheless, we would like to employ test functions depending on u, and thus the time derivative of u inevitably appears. We deal with this defect by using the convolution

$$u_\sigma(x,t) = \frac{1}{\sigma} \int_\varepsilon^t e^{(s-t)/\sigma} u(x,s) \, ds, \tag{2.6}$$

defined for $t \geq \varepsilon > 0$. This is expedient for our purpose; see for example [Nau84], [BDGO97], and [KL06]. In the above, $\varepsilon > 0$ is a Lebesgue instant of u. The reason for using a positive number ε instead of zero is that solutions are not a priori assumed to be integrable up to the initial time.

The convolution (2.6) has the following properties.

Lemma 2.7. *(i) If* $u \in L^p(\Omega_\varepsilon)$, *then*

$$||u_\sigma||_{L^p(\Omega_\varepsilon)} \leq ||u||_{L^p(\Omega_\varepsilon)},$$

$$\frac{\partial u_\sigma}{\partial t} = \frac{u - u_\sigma}{\sigma} \in L^p(\Omega_\varepsilon),$$

and

$$u_\sigma \to u \quad in \quad L^p(\Omega_\varepsilon) \quad as \quad \sigma \to 0.$$

(ii) If $\nabla u \in L^p(\Omega_\varepsilon)$, *then* $\nabla u_\sigma = (\nabla u)_\sigma$ *componentwise,*

$$||\nabla u_\sigma||_{L^p(\Omega_\varepsilon)} \leq ||\nabla u||_{L^p(\Omega_\varepsilon)},$$

and

$$\nabla u_\sigma \to \nabla u \quad in \quad L^p(\Omega_\varepsilon) \quad as \quad \sigma \to 0.$$

(iii) *Furthermore, if* $u^k \to u$ *in* $L^p(\Omega_\varepsilon)$, *then also*

$$u^k_\sigma \to u_\sigma \quad \text{and} \quad \frac{\partial u^k_\sigma}{\partial t} \to \frac{\partial u_\sigma}{\partial t}$$

in $L^p(\Omega_\varepsilon)$.

(iv) *If* $\nabla u^k \to \nabla u$ *in* $L^p(\Omega_\varepsilon)$, *then* $\nabla u^k_\sigma \to \nabla u_\sigma$ *in* $L^p(\Omega_\varepsilon)$.

(v) *Analogous results hold for the weak convergence in* $L^p(\Omega_\varepsilon)$.

(vi) *Finally, if* $\varphi \in C(\overline{\Omega}_\varepsilon)$, *then*

$$\varphi_\sigma(x,t) + e^{-\frac{t-\varepsilon}{\sigma}}\varphi(x,\varepsilon) \to \varphi(x,t)$$

uniformly *in* Ω_ε *as* $\sigma \to 0$.

To derive the necessary estimates, we need the equation satisfied by the mollified solution u_σ. By using (2.4), we may write

$$\begin{aligned}
\int_s^{T-\varepsilon} \int_{\mathbb{R}^n} &|\nabla u(x,t+\varepsilon-s)|^{p-2}\,\nabla u(x,t+\varepsilon-s)\cdot\nabla\phi(x,t+\varepsilon)\,dx\,dt \\
&- \int_s^{T-\varepsilon}\int_{\mathbb{R}^n} u(x,t+\varepsilon-s)\frac{\partial\phi(x,t+\varepsilon)}{\partial t}\,dx\,dt \\
&+ \int_{\mathbb{R}^n} u(x,T-s)\phi(x,T)\,dx \\
&= \int_{\mathbb{R}^n} u(x,\varepsilon)\phi(x,s+\varepsilon)\,dx
\end{aligned}$$

when $0 \le s \le T-\varepsilon$. Notice that $t+\varepsilon-s \in [\varepsilon, T-s]$. Then we multiply the above inequality by $e^{-s/\sigma}/\sigma$, integrate over $[0,T-\varepsilon]$ with respect to s, change the order of integration using $\int_0^{T-\varepsilon}\int_s^{T-\varepsilon}\dots\,dt\,ds = \int_0^{T-\varepsilon}\int_0^t\dots\,ds\,dt$, and finally perform a change of variables $s_{\text{new}} = s+\varepsilon, t_{\text{new}} = t+\varepsilon$, and for simplicity denote the new variables also with s and t . We end up with

$$\begin{aligned}
\int_\varepsilon^T \int_{\mathbb{R}^n} &(|\nabla u(x,t)|^{p-2}\,\nabla u(x,t))_\sigma\cdot\nabla\phi(x,t)\,dx\,dt \\
&- \int_\varepsilon^T\int_{\mathbb{R}^n} u(x,t)_\sigma\frac{\partial\phi(x,t)}{\partial t}\,dx\,dt + \int_{\mathbb{R}^n} u(x,T)_\sigma\phi(x,T)\,dx \\
&= \int_{\mathbb{R}^n} u(x,\varepsilon)\int_\varepsilon^T \phi(x,s)e^{-(s-\varepsilon)/\sigma}/\sigma\;ds\;dx.
\end{aligned} \tag{2.8}$$

It is often convenient to write the last two terms on the right as

$$\begin{aligned}
-\int_\varepsilon^T\int_{\mathbb{R}^n} u(x,t)_\sigma\frac{\partial\phi(x,t)}{\partial t}\,dx\,dt &+ \int_{\mathbb{R}^n} u(x,T)_\sigma\phi(x,T)\,dx \\
&= \int_\varepsilon^T\int_{\mathbb{R}^n}\frac{\partial u_\sigma(x,t)}{\partial t}\phi(x,t)\,dx\,dt,
\end{aligned}$$

where we integrated by parts in time and used the fact that $u_\sigma(x,\varepsilon)$ vanishes by the properties of the mollifier.

3. Propagation properties

In this section, we prove a finite propagation result for the Cauchy problem. More specifically, if the initial trace has compact support, then this property is preserved in the time evolution, meaning that $x \mapsto u(x,t)$ also has compact support for positive times t. This is in principle well known, but it seems hard to find a convenient reference covering the case when the initial trace is a measure. For stronger initial conditions, see for example Theorem 3.1 in [KP09].

Here we use a comparison with an explicit solution and adapt the arguments for the porous medium equation in [Vaz06], see in particular Lemma 14.5 and Proposition 14.24. We start by extracting a family of barrier functions from the Barenblatt solution

$$\mathcal{B}_p(x,t) = t^{-n/\lambda}\left(C - \frac{p-2}{2}\lambda^{1/(1-p)}\left(\frac{|x|}{t^{1/\lambda}}\right)^{p/(p-1)}\right)_+^{(p-1)/(p-2)},$$

where $\lambda = n(p-2)+p$. The Barenblatt solution is of the form

$$\mathcal{B}_p(x,t) = F(t, |x|^{p/(p-1)})$$

for a suitable function F. From this, we see that the function

$$v(x,t) = F(T-t, -|x-x_1|^{p/(p-1)})$$

is also a weak solution, since both the terms in the p-parabolic equation change sign. With the choice $C = 0$ and some simplifications, we get a weak solution

$$v(x,t) = c(n,p)(T-t)^{-\frac{1}{p-2}}\,|x-x_1|^{\frac{p}{p-2}}\,. \tag{3.1}$$

The first result concerns finite propagation for bounded initial data.

Theorem 3.2. *Let u be the solution to the Cauchy problem*

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\,\nabla u) = 0, \\ u(x,0) = u_0(x) \end{cases}$$

where

$$u_0 \geq 0 \quad \textit{and} \quad ||u_0||_\infty = H < \infty.$$

Assume that there is a point $x_0 \in \mathbb{R}^n$ and a radius $R > 0$ such that

$$u_0(x) = 0 \quad \textit{for all} \quad x \in B(x_0, R).$$

Then there exists a nonnegative function $R(t)$ such that

$$R(t) \geq R - c(n,p)H^{\frac{p-2}{p}}t^{1/p},$$

and

$$u(x,t) = 0 \quad \textit{for all} \quad x \in B(x_0, R(t)).$$

Remark 3.3. It can be shown that the function $R(t)$ is nonincreasing, but we do not need this fact here. In Theorems 3.2 and 3.5, it can happen that $R(t) = 0$ after some time instant.

Proof. By the comparison principle, we have $0 \le u \le H$. Pick a point $x_1 \in B(x_0, R)$ and denote

$$d(x_1) = \operatorname{dist}(x_1, \partial B(x_0, R)) = R - |x_1 - x_0|\,.$$

We derive an estimate for the time T up to which $u(x_1, T) = 0$ by comparing u with the barrier function v given by (3.1) in $B(x_0, R) \times (0, T)$. At the initial time zero, $u \le v$ since u vanishes. On the lateral boundary $\partial B(x_0, R) \times (0, T)$ we have $u \le v$ if

$$H \le cT^{-\frac{1}{p-2}} d(x_1)^{\frac{p}{p-2}}. \tag{3.4}$$

Choose now T small enough so that equality holds in (3.4). Then

$$d(x_1) = cH^{\frac{p-2}{p}} T^{1/p},$$

from which we get

$$|x_1 - x_0| = R - cH^{\frac{p-2}{p}} T^{1/p}.$$

The choice of T made above is admissible in (3.4) for all $\tilde{x}$ such that $d(\tilde{x}) \ge d(x_1)$. Thus $u(\tilde{x}, t) = 0$ for all $0 \le t \le T$ and $\tilde{x} \in B(x_0, R(T))$ where

$$R(T) \ge R - cH^{\frac{p-2}{p}} T^{1/p},$$

as desired. □

The next step is to generalize the above result to the case of initial data with finite mass. For this purpose, we need the so-called $L^1 - L^\infty$ regularizing effect, i.e. the estimate

$$||u(\cdot, t)||_\infty \le c(n, p)\nu(\mathbb{R}^n)^\sigma t^{-\alpha}.$$

Here $\lambda = n(p-2) + p$, $\alpha = n/\lambda$, and $\sigma = p/\lambda$, see Theorem 2.1 in [DiB93] on page 318. The proof is an iteration of the estimate in Theorem 3.2 with the help of the regularizing effect.

Theorem 3.5. *Assume that ν is a positive measure with $\nu(\mathbb{R}^n) < \infty$, and that there is a point $x_0 \in \mathbb{R}^n$, a radius $R > 0$, and a number $\varepsilon > 0$ such that*

$$\nu(B(x_0, R + \varepsilon)) = 0.$$

Then there is a solution u to the Cauchy problem

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0, \\ u(x, 0) = \nu \end{cases}$$

Then there exists a nonnegative function $R(t)$ such that

$$R(t) \ge R - c(n, p, \nu)t^\gamma,$$

and

$$u(x, t) = 0 \quad \textit{for all} \quad x \in B(x_0, R(t))$$

where

$$\gamma = \frac{1}{p}\left(1 - \frac{n(p-2)}{n(p-2)+p}\right) \quad \textit{and} \quad c(n,p,\nu) = c(n,p)\nu(\mathbb{R}^n)^{\sigma(p-2)/p}.$$

Proof. We derive the estimate for the case where the initial trace is a bounded function u_0 such that

$$u_0(x) = 0 \quad \text{for all} \quad x \in B(x_0, R).$$

The general case then follows by mollifying the initial trace ν so that the above assumption holds for the approximations, and passing to the limit. Indeed, the estimate depends on the initial trace only via its total mass, so the result holds also for the limit function.

Fix a small $t > 0$ and let $t_k = 2^{-k}t$ for $k = 0, 1, 2, \ldots$. Then $t_{k-1} = 2t_k$ and $t_{k-1} - t_k = t_k$. Denote $H_k = ||u(\cdot, t_k)||_\infty$; then

$$H_k^{(p-2)/p} \leq c\nu(\mathbb{R}^n)^{\sigma(p-2)/p} t^{-\alpha(p-2)/p} 2^{k\alpha(p-2)/p}$$

by the regularizing effect. Further, we have

$$t_k^{1/p} = c2^{-k/p}t^{1/p}.$$

We combine the estimate of Theorem 3.2 over the time interval (t_{k+1}, t_k) and the last two display formulas to get

$$R(t_k) \geq R(t_{k+1}) - c\nu(\mathbb{R}^n)^{\sigma(p-2)/p} t^\gamma 2^{-k\gamma}$$

where we used the fact that

$$\frac{1}{p} - \alpha\frac{p-2}{p} = \frac{1}{p}\left(1 - \frac{n(p-2)}{n(p-2)+p}\right) = \gamma > 0.$$

Iteration of the above estimate leads to

$$R(t) = R(t_0) \geq R(0) - c(n,p,\nu)t^\gamma \sum_{k=1}^{\infty} 2^{-k\gamma}.$$

The series is convergent since γ is positive, and the proof is complete. □

Remark 3.6. We will use Theorem 3.5 in the following way: given a time T and an initial trace ν with compact support, we can find a bounded open set U such that the support of $u(\cdot, t)$ is contained in U for all $0 < t \leq T$. To see this, note that we may choose the point x_0 in Theorem 3.5 to be arbitrarily far away from the support of ν.

4. A priori estimates

In this section, we derive some estimates we will employ for our stability result. More specifically, we estimate certain L^q-norms of solutions and their gradients locally in space, up to the initial time. We begin with a simple lemma for passing up to the initial time in our estimates.

Lemma 4.1. *Let u be a solution to the Cauchy problem with initial trace ν, and let $U \Subset \mathbb{R}^n$ be an open set. Then*

$$\limsup_{\varepsilon \to 0} \int_U u(x,\varepsilon)\, dx \le \nu(\overline{U}).$$

Proof. Since u is a local weak solution for positive times, it is also continuous in time with values in $L^2_{loc}(\mathbb{R}^n)$ on $(0,\infty)$. Thus $x \mapsto u(x,\varepsilon)$ is a locally integrable function for *every* $\varepsilon > 0$, and we may identify it with the measure defined by

$$\nu_\varepsilon(E) = \int_E u(x,\varepsilon)\, dx.$$

Observe that we need the continuity in time only for strictly positive times. Now the fact that ν is the initial trace of u means exactly that $\nu_\varepsilon \to \nu$ in the sense of weak convergence of measures. Thus

$$\limsup_{\varepsilon \to 0} \int_U u(x,\varepsilon)\, dx \le \limsup_{\varepsilon \to 0} \int_{\overline{U}} u(x,\varepsilon)\, dx \le \nu(\overline{U})$$

by the properties of weak convergence of measures. □

Lemma 4.2. *Let u be a solution to* (2.1). *Then there is a constant $c = c(n,p)$ such that*

$$\sup_{0<t<T} \int_U \min(u,j)^2\, dx + \int_{U_T} |\nabla \min(u,j)|^p \, dx\, dt \le cj\nu(\mathbb{R}^n).$$

Remark 4.3. We can replace the left hand side by

$$\sup_{\varepsilon<t<T} \int_U \min(u,j)^2\, dx + \int_{U_{\varepsilon,T}} |\nabla \min(u,j)|^p \, dx\, dt$$

by a trivial estimate.

Proof. We test the regularized equation with $\varphi = u_j = \min(u,j)$. This is an admissible test function, since u has a 'finite speed of propagation' i.e. for a given time T, there is U such that $u(\cdot,t)$ is supported on U, see Remark 3.6. We get

$$\begin{aligned}\int_{U_{\varepsilon,T}} &\frac{\partial u_\sigma}{\partial t} u_j + (|\nabla u|^{p-2} \nabla u)_\sigma \cdot \nabla u_j \, dx\, dt \\ &= \int_U u(x,\varepsilon) \left(\frac{1}{\sigma} \int_\varepsilon^T e^{-(s-\varepsilon)/\sigma} u_j(x,s)\, ds \right) dx.\end{aligned}$$

We need to eliminate the time derivative in the first term on the left. To accomplish this, we note that

$$\begin{aligned}\frac{\partial u_\sigma}{\partial t} u_j =& \frac{\partial u_\sigma}{\partial t} \min(u_\sigma, j) + \frac{\partial u_\sigma}{\partial t}(\min(u,j) - \min(u_\sigma, j)) \\ \ge& \frac{\partial u_\sigma}{\partial t} \min(u_\sigma, j),\end{aligned}$$

so that this term is estimated from below by

$$\begin{aligned}\int_{U_{\varepsilon,T}} \frac{\partial u_\sigma}{\partial t} u_j \, dx \, dt &= \int_{U_{\varepsilon,T}} \frac{\partial}{\partial t} \int_0^{u_\sigma(x,t)} \min(s,j) \, ds \; dt \, dx \\ &= \int_U F(u_\sigma(x,T)) \, dx - \int_U F(u_\sigma(x,\varepsilon)) \, dx\end{aligned}$$

where

$$F(t) = \int_0^t \min(s,j) \, ds = \begin{cases} \frac{1}{2} t^2, & t \leq j, \\ \frac{1}{2} j^2 + jt - j^2 \geq \frac{1}{2} j^2, & t > j. \end{cases}$$

In particular, $F(t) \geq \frac{1}{2} \min(t,j)^2$. Since $u_\sigma(x,\varepsilon) = 0$, we get

$$\int_{U_{\varepsilon,T}} \frac{\partial u_\sigma}{\partial t} u_j \, dx \, dt \geq \int_U \frac{1}{2} \min(u_\sigma, j)^2(x,T) \, dx.$$

The time derivative has been eliminated, so we may pass to the limit $\sigma \to 0$. We take this limit and get

$$\frac{1}{2} \int_U u_j^2(x,T) \, dx + \int_{U_{\varepsilon,T}} |\nabla u_j|^p \; dx \, dt \leq \int_U u(x,\varepsilon) u_j(x,\varepsilon) \, dx.$$

We then use the fact that $u_j \leq j$

$$\int_U u_j^2(x,T) \, dx + \int_{U_{\varepsilon,T}} |\nabla u_j|^p \; dx \, dt \leq cj \int_U u(x,\varepsilon) \, dx.$$

Then pass to the limit $\varepsilon \to 0$; the right hand side is bounded by $cj\nu(\overline{U})$ by Lemma 4.1, so we get the desired quantity to the right hand side by a trivial estimate.

The proof is then completed by replacing T in the above by τ chosen so that

$$\int_U u_j^2(x,\tau) \, dx \geq \frac{1}{2} \operatorname*{ess\,sup}_{0<t<T} \int_U u_j^2(x,t) \, dx.$$

This leads to an estimate for the supremum in terms of the right hand side in the claim. □

We need the following well known Sobolev type inequality.

Lemma 4.4. *Let $u \in L^p(\varepsilon, T; W_0^{1,p}(U))$. Then*

$$\int_{U_{\varepsilon,T}} |u|^{\kappa p} \; dx \, dt \leq c \int_{U_{\varepsilon,T}} |\nabla u|^p \; dx \, dt \left(\operatorname*{ess\,sup}_{\varepsilon<t<T} \int_U u^2(x,t) \, dx \right)^{p/n},$$

where

$$\kappa = \frac{n+2}{n}.$$

Next we show that solutions to the Cauchy problem, as well as their gradients, are integrable to certain powers. The proof is based on estimating the decay of certain level sets by applying the previous two lemmas. This is optimal as can be seen from the Barenblatt solution.

The essential point is that the constant on the right can be chosen to be independent of u.

Theorem 4.5. *Let u be a solution to* (2.1). *Then*

$$\int_0^T \int_{\mathbb{R}^n} u^q \, dx \, dt \le C(\nu, p, q) < \infty \quad \textit{whenever} \quad q < p - 1 + \frac{p}{n}, \ \textit{and}$$
$$\int_0^T \int_{\mathbb{R}^n} |\nabla u|^q \, dx \, dt \le C(\nu, p, q) < \infty \quad \textit{whenever} \quad q < p - 1 + \frac{1}{n+1}.$$

Above the constant remains bounded when $p \to p_0 > 2$.

Proof. Since T and ν are given, we can choose open $U \Subset \mathbb{R}^n$ such that $u(\cdot, t)$ is compactly supported in U for each $t \in [0, T]$, see Remark 3.6. Let us define the sets

$$E_j^\varepsilon = \{(x,t) \in U_{\varepsilon,T} : j \le u(x,t) < 2j\}.$$

The reason for introducing a positive $\varepsilon > 0$ here is that we only know the integrability of the solution u and its gradient for strictly positive times. The aim is to derive estimates independent of ε on the time interval (ε, T), and then pass to the limit $\varepsilon \to 0$. This is possible since the right hand side in the estimate of Lemma 4.2 is independent of ε.

Recall the notation $u_{2j} = \min(u, 2j)$. We have

$$\begin{aligned}
j^{\kappa p} \left| E_j^\varepsilon \right| &\le \int_{E_j^\varepsilon} u_{2j}^{\kappa p} \, dx \, dt \\
&\le \int_{U_{\varepsilon,T}} u_{2j}^{\kappa p} \, dx \, dt \\
&\le c \int_{U_{\varepsilon,T}} |\nabla u_{2j}|^p \, dx \, dt \left(\operatorname*{ess\,sup}_{\varepsilon<t<T} \int_U u_{2j}^2 \, dx \right)^{p/n} \\
&\le c \nu(\mathbb{R}^n) j^{1+p/n}
\end{aligned}$$

by applications of the parabolic Sobolev inequality and Lemma 4.2; see also Remark 4.3. It follows that

$$\left| E_j^\varepsilon \right| \le c j^{1-p-p/n} \tag{4.6}$$

with a constant independent of ε.

With (4.6), the estimates follow by essentially the same computations as in the proof of Lemma 3.14 in [KL05]. By (4.6), we get

$$\begin{aligned}\int_\varepsilon^T \int_{\mathbb{R}^n} u^q \, dx \, dt &\le \int_\varepsilon^T \int_{\{u<1\}} u^{q-1} u \, dx \, dt + \sum_{j=1}^\infty \int_{E^\varepsilon_{2^{j-1}}} u^q \, dx \, dt \\ &\le \nu(\mathbb{R}^n) T + \sum_{j=1}^\infty 2^{jq} \left| E^\varepsilon_{2^{j-1}} \right| \\ &\le \nu(\mathbb{R}^n) T + 2^{p+p/n-1} \sum_{j=1}^\infty 2^{j(q+1-p-p/n)} < \infty.\end{aligned}$$

The sum on the last line is finite by the choice of q. We let $\varepsilon \to 0$ to conclude the desired estimate for u.

Next we estimate the gradient. We use Hölder's inequality, (4.6) and Lemma 4.2 to obtain

$$\begin{aligned}&\int_\varepsilon^T \int_U |\nabla u|^q \, dx \, dt \\ &\quad= \int_{\{u<1\}} |\nabla u_1|^q \, dx \, dt + \sum_{j=1}^\infty \int_{E^\varepsilon_{2^{j-1}}} |\nabla u|^q \, dx \, dt \\ &\quad\le \nu(\mathbb{R}^n)^{q/p} (T\,|U|)^{1-q/p} + \sum_{j=1}^\infty \Big(\int_{E^\varepsilon_{2^{j-1}}} |\nabla u|^p \, dx \, dt \Big)^{q/p} \left| E^\varepsilon_{2^{j-1}} \right|^{1-q/p} \\ &\quad\le \nu(\mathbb{R}^n)^{q/p} (T\,|U|)^{1-q/p} \\ &\qquad + \sum_{j=1}^\infty 2^{(j-1)(1-q/p)(1-p-p/n)} \Big(\int_{E^\varepsilon_{2^{j-1}}} |\nabla u_{2^{j-1}}|^p \, dx \, dt \Big)^{q/p}.\end{aligned}$$

We apply Lemma 4.2 once more to estimate the last integral by $C2^{j-1}$. This gives the bound

$$C \sum_{j=1}^\infty 2^{(j-1)(1-q/p)(1-p-p/n)} 2^{(j-1)q/p} = C \sum_{j=1}^\infty 2^{(j-1)(1-p-p/n+q+q/n)}$$

for the second term. Here the sum converges since $1-p-p/n+q+q/n < 0$. Letting $\varepsilon \to 0$ completes the proof. $\square$

5. Stability

Our stability result now follows from the estimates of the previous section by arguments similar to those in [KP10].

We work with a sequence of exponents (p_i) such that

$$\lim_{i\to\infty} p_i = p > 2.$$

By the convergence, we are free to assume that $p_i > 2$, and that all the exponents belong to a compact subinterval $[p^-, p^+]$of $(2, \infty)$.

Theorem 5.1. *Let (p_i) be a sequence such that $p_i \to p > 2$, (u_i) the corresponding solutions to* (2.1) *(see Remark 2.5), $T > 0$ and $U \Subset \mathbb{R}^n$. Then there exists a subsequence still denoted by (u_i) and a function $u \in L^q(0,T;W^{1,q}(U))$, $q < p - 1 + 1/(n+1)$, such that*

$$u_i \to u \quad in \quad L^q(U_T)$$

and

$$\nabla u_i \to \nabla u \quad weakly\ in \quad L^q(U_T),$$

as $i \to \infty$.

Proof. Since $q > 1$, weak convergence follows from the reflexivity of L^q and the uniform L^q-bounds of Theorem 4.5. For the norm convergence of the solutions, we estimate the time derivative of u and appeal to the parabolic version of Rellich's theorem by Simon [Sim87], see also page 106 of Showalter's monograph [Sho97]. Denote $1/q' + 1/q = 1$. For all $\varphi \in C_0^\infty(U_T)$, we have

$$\begin{aligned} |\langle \partial_t u_i, \varphi \rangle| &= \left| \int_{U_T} |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla \varphi \, dx \, dt \right| \\ &\le C \Big(\int_{U_T} |\nabla u_i|^{(p-1)q'} \, dx \, dt \Big)^{1/q'} ||\nabla \varphi||_{L^q(U_T)} \end{aligned}$$

for any $\phi \in C_0^\infty(U_T) \cap L^q(0,T;W_0^{1,q}(U))$. Since $(p-1)q' < q$, it follows that $\Big(\int_{U_T} |\nabla u_i|^{(p-1)q'} \, dx \, dt \Big)^{1/q'} < C$. Thus, by the density of smooth functions in $L^q(0,T;W_0^{1,q}(U))$, the time derivative of u_i is bounded in the dual space of $L^q(t_1,t_2;W_0^{1,q}(\Omega'))$. Now the theorem mentioned above implies the claim. □

The previous theorem is not sufficient for our stability result, since the weak convergence of the gradients is not enough to identify the weak limit of the nonlinear quantity $|\nabla u_i|^{p_i-2} \nabla u_i$. This is rectified in the following theorem by proving the pointwise convergence of the gradients.

Theorem 5.2. *Let (p_i), (u_i), q, U be as in Theorem 5.1. There exists a subsequence (u_i) and a function $u \in L^q(0,T;W^{1,q}(U))$ such that*

$$u_i \to u \quad in \quad L^q(0,T;W^{1,q}(U)),$$

as $i \to \infty$.

Proof. By Remark 3.6, we can again assume that $\operatorname{spt} u(\cdot,t) \subset U$. We can focus our attention to the convergence of the gradients in $L^q(U_T)$. To establish this, let u_j and u_k be two solutions in the sequence. Since both u_j and u_k satisfy the mollified equation (2.4), by subtracting, we obtain

$$\begin{aligned}\int_{U_{\varepsilon,T}} &\frac{\partial}{\partial t}(u_j-u_k)_\sigma \phi \,dx\,dt\\ &+\int_{U_{\varepsilon,T}} (|\nabla u_j|^{p_j-2}\nabla u_j - |\nabla u_k|^{p_k-2}\nabla u_k)_\sigma\cdot\nabla\phi\,dx\,dt \\ &= \int_U (u_j(x,\varepsilon)-u_k(x,\varepsilon))\frac{1}{\sigma}\int_\varepsilon^T e^{-(s-\varepsilon)/\sigma}\phi(x,s)\,ds\,dx.\end{aligned} \tag{5.3}$$

We use the test function

$$\phi(x,t)=u_j(x,t)-u_k(x,t).$$

Also observe that p_j, p_k are close enough to p, both the functions actually belong to a higher parabolic Sobolev space with power $p+\delta$ with some $\delta>0$ by the higher integrability result in [KL00].

We estimate the first term on the left hand side of (5.3). A substitution of the test function and the properties of the convolution imply

$$\begin{aligned}\int_{U_{\varepsilon,T}} &\frac{\partial}{\partial t}(u_j-u_k)_\sigma\phi\,dx\,dt\\ &=\int_{U_{\varepsilon,T}} \frac{\partial}{\partial t}(u_j-u_k)_\sigma(u_j-u_k)\,dx\,dt\\ &=\int_{U_{\varepsilon,T}} \Big((u_j-u_k)-(u_j-u_k)_\sigma\Big)^2\,dx\,dt\\ &\quad+\int_{U_{\varepsilon,T}} \frac{\partial}{\partial t}(u_j-u_k)_\sigma(u_j-u_k)_\sigma\,dx\,dt\\ &\geq \frac12\int_U (u_j-u_k)_\sigma^2(x,T)\,dx-\frac12\int_U(u_j-u_k)_\sigma^2(x,\varepsilon)\,dx.\end{aligned}$$

This estimate is free of the time derivatives of the functions u_j and u_k, which is essential for us in the passage to the limit with σ. Now, letting $\sigma\to 0$, we conclude that

$$\begin{aligned}\int_{U_{\varepsilon,T}} &(|\nabla u_j|^{p_j-2}\nabla u_j-|\nabla u_k|^{p_k-2}\nabla u_k)\cdot(\nabla u_j-\nabla u_k)\,dx\,dt\\ &\leq -\int_U (u_j-u_k)^2(x,T)\,dx+\int_U(u_j-u_k)^2(x,\varepsilon)\,dx.\end{aligned}$$

Observe that the first term on the right hand side is nonpositive. Thus

$$\begin{aligned}\int_{U_{\varepsilon,T}} &(|\nabla u_j|^{p_j-2}\nabla u_j-|\nabla u_k|^{p_k-2}\nabla u_k)\cdot(\nabla u_j-\nabla u_k)\eta\,dx\,dt\\ &\leq \int_U (u_j-u_k)^2(x,\varepsilon)\eta(x)\,dx.\end{aligned}\tag{5.4}$$

We divide the left hand side in three parts as

$$
\begin{aligned}
\int_{U_{\varepsilon,T}} & (|\nabla u_j|^{p_j-2}\,\nabla u_j - |\nabla u_k|^{p_k-2}\,\nabla u_k)\cdot(\nabla u_j - \nabla u_k)\,dx\,dt \\
&= \int_{U_{\varepsilon,T}} (|\nabla u_j|^{p-2}\,\nabla u_j - |\nabla u_k|^{p-2}\,\nabla u_k)\cdot(\nabla u_j - \nabla u_k)\,dx\,dt \\
&\quad + \int_{U_{\varepsilon,T}} (|\nabla u_j|^{p_j-2} - |\nabla u_j|^{p-2})\nabla u_j\cdot(\nabla u_j - \nabla u_k)\,dx\,dt \\
&\quad + \int_{U_{\varepsilon,T}} (|\nabla u_k|^{p-2} - |\nabla u_k|^{p_k-2})\nabla u_k\cdot(\nabla u_j - \nabla u_k)\,dx\,dt \\
&= I_1 + I_2 + I_3.
\end{aligned}
\tag{5.5}
$$

First, we concentrate on I_2 and I_3. A straightforward calculation shows that

$$
\begin{aligned}
\left|\,|\zeta|^a - |\zeta|^b\,\right| &= |\exp(a\log|\zeta|) - \exp(b\log(|\zeta|)| \\
&\le \max_{s\in[a,b]}\left|\frac{\partial\exp(s\log|\zeta|)}{\partial s}\right| |a-b| \\
&\le |\log|\zeta||\,(|\zeta|^a + |\zeta|^b)\,|a-b|\,,
\end{aligned}
$$

where $\zeta\in\mathbb{R}^n$ and $a,\, b\ge 0$. If $|\zeta|\ge 1$, then

$$
|\log|\zeta||\,(|\zeta|^a + |\zeta|^b) \le \frac{1}{\gamma}\,|\zeta|^{\max(a,b)+\gamma}\,,
$$

and if $|\zeta|\le 1$, then

$$
|\log|\zeta||\,(|\zeta|^a + |\zeta|^b) \le \frac{1}{e}\left(\frac{1}{a}+\frac{1}{b}\right).
$$

This leads to

$$
||\zeta|^a - |\zeta|^b| \le \left(\frac{1}{\gamma}\,|\zeta|^{\max(a,b)+\gamma} + \frac{1}{e}\left(\frac{1}{a}+\frac{1}{b}\right)\right)|a-b| \tag{5.6}
$$

for every $\zeta\in\mathbb{R}^n$ and $a, \ge 0$.

Next we apply (5.6) with $\zeta = \nabla u_j$, $a = p_j - 2$, and $b = p-2$. This implies

$$
|I_2| \le c\,|p_j - p|\int_{U_{\varepsilon,T}} (1 + |\nabla u_j|^{\max(p_j-2,p-2)+\gamma})\,|\nabla u_j|\,|\nabla u_j - \nabla u_k|\,dx\,dt.
$$

The integral on the right hand side is uniformly bounded for small enough $\gamma>0$ by Theorem 4.5. Consequently, $I_2\to 0$, as $j,k\to\infty$. A similar reasoning implies that I_3 tends to zero as $j,k\to\infty$. From the elementary inequality

$$
2^{2-p}\,|a-b|^p \le (|a|^{p-2}\,a - |b|^{p-2}\,b)\cdot(a-b),
$$

(5.4) as well as (5.5) we conclude that

$$\int_{U_{\varepsilon,T}} |\nabla u_j - \nabla u_k|^p \, dx\, dt \le c(|I_2| + |I_3|) + \int_U (u_j - u_k)^2(x,\varepsilon)\eta(x)\, dx.$$

The right hand side can be made arbitrary small by choosing j and k large enough, and a suitable ε since the last integral converges for almost every ε (the choice of ε can be done independent of j,k). This shows that (∇u_i) is a Cauchy sequence in $L^p(U_{\varepsilon,T})$, and thus it converges. We can choose a subsequence such that $\nabla u_i \to \nabla u$ a.e. in $U_{\varepsilon,T}$ as $i \to \infty$, and further by diagonalizing with respect to ε we can pass to subsequence such that $\nabla u_i \to \nabla u$ a.e. in U_T as $i \to \infty$. This and the uniform estimate in Theorem 4.5 implies that

$$\nabla u_i \to \nabla u \qquad \text{in} \quad L^q(U_T)$$

as $i \to \infty$, for $q < p - 1 + 1/(n+1)$. To see this, pick $\tilde{q}$ such that $q < \tilde{q} < p-1+1/(n+1)$. By the pointwise convergence of the gradients established above, we also have convergence in measure. For any $\lambda > 0$, we have

$$\begin{aligned}
&\int_{U_T} |\nabla(u-u_j)|^q \, dz \\
&\le c\int_{\{|\nabla(u-u_j)|<\lambda\}} |\nabla(u-u_j)|^q \, dz + c\int_{\{|\nabla(u-u_j)|\ge\lambda\}} |\nabla(u-u_j)|^q \, dz \\
&\le c\,|U_T|\,\lambda^q + c\,|\{|\nabla(u-u_j)| \ge \lambda\}|^{1-q/\tilde{q}} \left(\int_{U_T} |\nabla(u-u_j)|^{\tilde{q}} \, dz\right)^{q/\tilde{q}}.
\end{aligned}$$

by Hölder's inequality. By the convergence in measure and the bound in $L^{\tilde{q}}$, we get that

$$\limsup_{j\to\infty} ||\nabla(u-u_j)||_{L^q(U_T)} \le c\lambda.$$

Since λ was arbitrary, the claim follows from this. $\square$

Theorem 5.7. *Let (p_i), (u_i), q, U, and u be as in Theorem 5.1. Then there is a subsequence $(u_i)$and a function u such that*

$$u_i \to u \quad \text{in } L^q(0,T;W^{1,q}(U)).$$

The function u is a solution to the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\,\nabla u) = 0, & \text{in } \mathbb{R}^n \times (0,T), \\ u(\cdot,0) = \nu. \end{cases}$$

If the initial trace ν is such that the solution to the limiting Cauchy problem is unique, then the whole sequence converges to the unique solution.

Remark 5.8. The uniqueness of solutions to the Cauchy problem is open for general initial measures ν. However, uniqueness holds if ν is an L^1 function, see [DH89]. Further, uniqueness is known for the special case of Dirac's delta, and then the unique solution is the Barenblatt solution, see [KV88]. Hence (1.2) follows from this theorem.

Proof. As the first step, we show that u is a weak solution in $\mathbb{R}^n \times (0, T)$. Pick a test function $\varphi \in C_0^\infty(\mathbb{R}^n \times (0, T))$, and choose the space-time cylinder U_T so that the support of φ is contained in U_T. The sequence $(|\nabla u_i|^{p_i-2} \nabla u_i)$ is bounded in $L^r(U_T)$ for some $r > 1$, and thus converges weakly in $L^r(U_T)$, up to a subsequence, to some limit. By the pointwise convergence of the gradients established in Theorem 5.2 and the fact that $p_i \to p$, the weak limit must be $|\nabla u|^{p-2} \nabla u$. This follows from an estimate similar to (5.6) in the proof of Theorem 5.2. Hence we have

$$\begin{aligned}\int_{U_T} &- u\frac{\partial \varphi}{\partial t} + |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \, dt \\ &= \lim_{i\to\infty} \left(\int_{U_T} -u_i \frac{\partial \varphi}{\partial t} + |\nabla u_i|^{p_i-2} \nabla u_i \cdot \nabla \varphi \, dx \, dt \right) = 0.\end{aligned}$$

Since φ was arbitrary, u is a weak solution.

Next we show that u takes the right initial values in the sense of distributions. Define a linear approximation of the characteristic function of the interval $[t_1, t_2]$ as

$$\chi_{t_1,t_2}^{h,k}(t) = \begin{cases} 0, & t \le t_1 - h \\ (t + h - t_1)/h, & t_1 - h < t < t_1 \\ 1, & t_1 < t < t_2 \\ (t_2 + k - t)/k, & t_2 < t < t_2 + k \\ 0, & t \ge t_2 + k, \end{cases}$$

where $0 \le t_1 - h$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and choose $\varphi(x)\chi_{t_1,t_2}^{h,k}(t)$ as a test function in the weak formulation. We obtain

$$\begin{aligned}&\left| \fint_{t_2}^{t_2+k} \int_{\mathbb{R}^n} u_i \varphi \, dx \, dt - \fint_{t_1-h}^{t_1} \int_{\mathbb{R}^n} u_i \varphi \, dx \, dt \right| \\ &\qquad \le \left| \int_{t_1-h}^{t_2+k} \int_{\mathbb{R}^n} |\nabla u_i|^{p-2} \nabla u_i \chi_{t_1,t_2}^{h,k} \cdot \nabla \varphi \; dx \, dt \right|.\end{aligned} \tag{5.9}$$

Next we pass to limits in a particular order. Since $t \mapsto u_i(\cdot, t)$ is a continuous function having values in L^2, $u_i \in C((0, T), L^2_{\text{loc}}(\mathbb{R}^n))$, see for example page 106 in [Sho97] along with similar estimates as in the proof of Theorem 5.1, it follows that

$$\fint_{t_1-h}^{t_1} \int_{\mathbb{R}^n} u_i \varphi \, dx \, dt \to \int_{\mathbb{R}^n} u_i(x, t_1)\varphi(x) \, dx(x),$$

as $h \to 0$. Furthermore, the initial condition implies

$$\int_{\mathbb{R}^n} u_i(x, t_1)\varphi(x) \, dx(x) \to \int_{\mathbb{R}^n} \varphi(x) \, d\nu(x),$$

as $t_1 \to 0$. As then $i \to \infty$, we obtain

$$\fint_{t_2}^{t_2+k} \int_{\mathbb{R}^n} u_i \varphi \, dx \, dt \to \fint_{t_2}^{t_2+k} \int_{\mathbb{R}^n} u \varphi \, dx \, dt$$

due to Theorem 5.1. Finally, by passing to zero with k, it follows that

$$\fint_{t_2}^{t_2+k} \int_{\mathbb{R}^n} u \varphi \, dx \, dt \to \int_{\mathbb{R}^n} u(x, t_2) \varphi(x) \, dx(x),$$

since the weak solution u belongs to $C((0, T), L^2_{\mathrm{loc}}(\mathbb{R}^n))$. Observe that we only use the continuity on an open interval $(0, T)$.

Consider next the right hand side of (5.9). Let first $h \to 0$ and $t_1 \to 0$ in this order. The uniform integrability estimate in Theorem 4.2 implies

$$\begin{aligned}
\Big| \int_0^{t_2+k} \int_{\mathbb{R}^n} & |\nabla u_i|^{p-2} \nabla u_i \cdot (\chi_{0,t_2}^{0,k} \nabla \varphi) \, dx \, dt \Big| \\
&\leq \int_0^{t_2+k} \int_{\operatorname{spt} \varphi} |\nabla u_i|^{p-1} \, dx \, dt \\
&\leq C(|t_2 + k| \, |U|)^{(q-p+1)/q} \Big(\int_0^{t_2+k} \int_{\operatorname{spt} \varphi} |\nabla u_i|^q \, dx \, dt \Big)^{(p-1)/q}.
\end{aligned}$$

Then we pass to limits with i and k, merge the estimates, and obtain

$$\left| \int_{\mathbb{R}^n} u(x, t_2) \varphi(x) \, dx(x) - \int_{\mathbb{R}^n} \varphi(x) \, d\nu(x) \right| \leq C t_2^{(q-p+1)/q}. \tag{5.10}$$

The right hand side tends to zero, as $t_2 \to 0$, and we have shown that u takes the right initial values.

It remains to prove the claim about the situation when we have uniqueness. We argue by contradiction. Let u be the unique solution to the limit problem. If the whole sequence does not converge to u, there are indices i_k, $k = 1, 2, 3, \ldots$, and a number $\varepsilon > 0$ such that

$$\lim_{k \to \infty} ||u - u_{i_k}||_{L^q(0,T;W^{1,q}(U))} \geq \varepsilon.$$

The first part of the theorem applies to this subsequence, and we get

$$\liminf_{k \to \infty} ||u - u_{i_k}||_{L^q(0,T;W^{1,q}(U))} = 0$$

by uniqueness, which is a contradiction. The proof is complete. $\square$

References

[BDGO97] L. Boccardo, A. Dall'Aglio, T. Gallouët, and L. Orsina. Nonlinear parabolic equations with measure data. *J. Funct. Anal.*, 147(1):237–258, 1997.

[DH89] E. DiBenedetto and M. A. Herrero. On the Cauchy problem and initial traces for a degenerate parabolic equation. *Trans. Amer. Math. Soc.*, 314(1):187–224, 1989.

[DiB93] E. DiBenedetto. *Degenerate parabolic equations.* Universitext. Springer-Verlag, New York, 1993.

[FHKM] Y. Fujishima, J. Habermann, J. Kinnunen and M. Masson Stability for parabolic quasiminimizers. Submitted. Available at `https://www.mittag-leffler.se/preprints/files/IML-1314f-01.pdf`.

[KV88] S. Kamin and J.L. Vázquez. Fundamental solutions and asymptotic behaviour for the p-Laplacian equation *Rev. Mat. Iberoam.*, 4(2):339–354, 1988

[KL00] J. Kinnunen and J. Lewis. Higher integrability for parabolic systems of p-Laplacian type *Duke Math J.*, 102(2):253–272, 2000

[KL05] J. Kinnunen and P. Lindqvist. Summability of semicontinuous supersolutions to a quasilinear parabolic equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 4(1):59–78, 2005.

[KL06] J. Kinnunen and P. Lindqvist. Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation. *Ann. Mat. Pura Appl. (4)*, 185(3):411–435, 2006.

[KP09] T. Kuusi and M. Parviainen. Existence for a degenerate Cauchy problem. *Manuscripta Math.*, 128(2):213–249, 2009.

[KP10] J. Kinnunen and M. Parviainen. Stability for degenerate parabolic equations. *Adv. Calc. Var.*, 3(1):29–48, 2010.

[LM98] G. Li and O. Martio. Stability of solutions of varying degenerate elliptic equations. *Indiana Univ. Math. J.*, 47(3):873–891, 1998.

[Lind87] P. Lindqvist. Stability for the solutions of $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$ with varying p. *J. Math. Anal. Appl.*, 127(1):93–102, 1987.

[Lind93] P. Lindqvist. On nonlinear Rayleigh quotients. *Potential Anal.*, 2(3):199–218, 1993.

[Luk] T. Lukkari. Stability of solutions to nonlinear diffusion equations. Submitted. Available at `http://arxiv.org/abs/1206.2492`.

[Nau84] J. Naumann. *Einführung in die Theorie parabolischer Variationsungleichungen*, volume 64 of *Teubner-Texte zur Mathematik*. BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1984.

[Sho97] R. E. Showalter. *Monotone operators in Banach space and nonlinear partial differential equations*, volume 49 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[Sim87] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[Vaz06] J-L. Vazquez. *The Porous Medium Equation – Mathematical Theory.* Oxford University Press, Oxford, 2007.

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