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Assouad Dimension, Nagata Dimension, and Uniformly Close Metric Tangents

ENRICO LE DONNE & TAPIO RAJALA

ABSTRACT. We study the Assouad dimension and the Nagata dimension of metric spaces. As a general result, we prove that the Nagata dimension of a metric space is always bounded from above by the Assouad dimension. Most of the paper is devoted to the study of when these metric dimensions of a metric space are locally given by the dimensions of its metric tangents. Having uniformly close tangents is not sufficient. What is needed, in addition, is either that the tangents have dimension with uniform constants independent from the point and the tangent, or that the tangents are unique. We will apply our results to equiregular sub-Riemannian manifolds and show that, locally, their Nagata dimension equals the topological dimension.

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1. INTRODUCTION

Assouad dimension, Nagata dimension, and metric tangents are relevant in the program of doing analysis in the metric space setting. The Assouad dimension is a quantification of the doubling property. It plays a role, for example, in the

study of spaces that are quasisymmetrically embeddable in Euclidean spaces, in the study of fractal sets, and in the study of boundaries of groups [Ass79, Ass83, Luu98, KL04, Mac11]. The Nagata dimension, introduced in [Nag58, Ass82], is a local-and-global metric version of the topological dimension. The bounded-scale version is called *linearly controlled dimension* and the large-scale version is called *asymptotic Assouad-Nagata dimension*. Such dimensions are relevant for embeddings, in particular in Geometric Group Theory [BDS07]. Nagata dimension links with quasisymmetric embedding into metric trees, and also with Lipschitz extension properties [LS05, WY10].

In [LS05], it has been shown that doubling metric spaces have finite Nagata dimension. In this paper, we prove the sharp bound.

Theorem 1.1. *For all metric spaces X , the Nagata dimension of X is less than or equal to the Assouad dimension of X .*

Metric tangents provide a way of studying the infinitesimal properties of metric spaces. Limits of metric spaces were introduced by Gromov in the setting of Geometric Group Theory to study asymptotic cones of groups of polynomial growth [Gro81]. However, many of the results hold in the context of metric spaces with finite Assouad dimension, and have applications in different areas of mathematics, for instance, in the study of limits of Riemannian manifolds with curvature bounds and Reifenberg-flat metric spaces (see [CC97], [DT99], and subsequent work). Additional results regarding tangents of general metric spaces can be found in [HH00, LD11, Her11].

In this paper we study how and when one can deduce the Nagata dimension or the Assouad dimension of a space, knowing the respective dimensions of its tangents. A concrete application of our results is given by the Lipschitz extension problem for sub-Riemannian manifolds. Lang and Schlichenmaier gave a connection of the Lipschitz extension problem with the Nagata dimension and the Lipschitz connection property. Lipschitz connectivity and Lipschitz homotopy groups have recently been studied in [DHLT11, WY10, HS13]. By the results in this paper, knowing now the Nagata dimension of equiregular sub-Riemannian manifolds, one can deduce that, for example, a partially defined Lipschitz map $f: A \rightarrow Y$ from a subset A of an equiregular sub-Riemannian manifold M can be Lipschitz extended on compact sets of M if Y is Lipschitz m -connected for all m strictly smaller than the topological dimension of M . (For such applications, see [LS05, Theorem 1.5 and Theorem 1.6].) Another property one deduces for a space (X, d) with Nagata dimension at most some number $n \in \mathbb{N}$ is that, for sufficiently small $p \in (0, 1)$, there exists a bi-Lipschitz embedding of (X, d^p) into the product of $n + 1$ metric trees (see [LS05, Theorem 1.3]).

Let us now present in detail our results on tangent spaces. Let X be a metric space. For each $x \in X$, let $\text{Tan}(X, x)$ be the collection of all the metric spaces tangent to X at x , in the pointed Gromov-Hausdorff sense. We say that X has *uniformly close tangents* if, for each $x \in X$, the convergence of the dilated spaces of (X, x) toward $\text{Tan}(X, x)$ is uniform. In other words and more generally, we

say that, on a subset $K \subset X$, the *convergence to tangents is uniform* if, for all $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ such that, for all $k \in K$ and all $\lambda > \lambda_\varepsilon$, there exists a tangent Y of X at k with

$$\text{Dist}_{\text{GH}}((\lambda X, k), Y) < \varepsilon.$$

Here, Dist_{GH} is a specific distance that we fix in Section 2.3 in order to induce the Gromov-Hausdorff topology on pointed metric spaces, and λX is the metric space $(X, \lambda d_X)$. The condition of uniform convergence to tangents is motivated by the fact that this is what happens on equiregular sub-Riemannian manifolds (see Theorem 4.1).

Assuming uniform convergence towards unique tangents, our first result for the Nagata dimension, which we denote by \dim_{N} , is the following.

Theorem 1.2. *Let X be a metric space that at every point admits a single tangent space. Let $Y \subseteq X$ be a relatively compact set with $\dim_{\text{N}} Y < \infty$. Assume that the convergence toward the tangents is uniform on the closure of Y (as just defined). Then, we have*

$$\sup_{x \in \text{int } Y} \dim_{\text{N}} T_x X \leq \dim_{\text{N}} Y \leq \sup_{x \in \text{cl } Y} \dim_{\text{N}} T_x X.$$

Here, $\text{int } Y$ and $\text{cl } Y$ denote the interior and the closure of the set Y , respectively. From Example 3.3, we see that the assumption on uniqueness of tangents is necessary in Theorem 1.2. Relative compactness of Y is needed already to handle the large scales, and the necessity of the interior of Y is seen by taking $Y = \{0\}$ and $X = \mathbb{R}$. We shall prove Theorem 1.2 at the end of Section 3.1 as a consequence of Theorem 3.4, with assumption (ii). An application of Theorem 1.2 is the following result, which was actually our initial goal (the notion of equiregular sub-Riemannian manifold is recalled in Section 4).

Corollary 1.3. *Let (M, d_{cc}) be an equiregular sub-Riemannian manifold. Then, the Nagata dimension of any open bounded nonempty subset of M equals the topological dimension of the manifold.*

Corollary 1.3 relies also on a result by Urs Lang and the first-named author which states that the Nagata dimension of a Carnot group equals its topological dimension. For completeness, we include a short proof of this fact in Section 4.

We prove the following analog of Theorem 1.2 for the Assouad dimension, which we denote by \dim_{A} .

Theorem 1.4. *Let X be a metric space that at every point admits a single tangent space. Let $Y \subseteq X$ be a relatively compact set. Assume that the convergence toward the tangents is uniform on Y (as defined above). Then, we have*

$$\sup_{x \in \text{int } Y} \dim_{\text{A}} T_x X \leq \dim_{\text{A}} Y \leq \sup_{x \in \text{cl } Y} \dim_{\text{A}} T_x X.$$

Theorem 1.4 is a consequence of Theorem 3.9. These results will be proved in Section 3.2. Theorem 1.4 gives an alternative proof of a fact that is essentially

proven in [NSW85]: namely, that the Assouad dimension of an equiregular sub-Riemannian manifold equals the Assouad dimension of its tangents.

One direction of research where the above results can be used is the study of generalizations of Reifenberg vanishing-flat metric spaces where the model space \mathbb{R}^n is replaced by any fixed doubling metric space. Namely, we say that a metric space X is *vanishing-flat modeled on a metric space Y* if Y is the tangent space at any point x of X , and if the convergence toward the tangents is uniform. The case when Y is an Euclidean space is called Reifenberg vanishing-flat metric spaces, and it has been mainly considered in [CC97, DT99]. It is a natural problem to study what properties of X can then be deduced from the ones of Y .

We shall also provide results when the tangents are not assumed to be single spaces. However, without such an assumption, it is necessary to require uniformity in values of the constants appearing in the definition of the dimension of the tangents. Such uniformity of constants is true for, as an example, many self-similar spaces, and these spaces usually have more than one tangent at every point. Let us briefly recall that the linearly controlled dimension is the bounded-scale version of the Nagata dimension.

Theorem 1.5. *Let X be a metric space of finite linearly controlled dimension. Assume that X has uniformly close tangents (as defined above). Then, the linearly controlled dimension of X equals the infimum of all integers n for which, for all $x \in X$, the linearly controlled dimension of any $Y \in \text{Tan}(X, x)$ is at most n with constants independent from Y and x .*

See Theorem 3.4, with assumption (i), for a more explanatory statement of the upper bound. The lower bound follows from Corollary 2.19.

We also have the analogue of Theorem 1.5 for the Assouad dimension.

Theorem 1.6. *Let X be a metric space. Assume that X has uniformly close tangents (as defined above). Then, the Assouad dimension of X equals the infimum of all $\alpha \geq 0$ for which, for all $x \in X$, the Assouad dimension of any $Y \in \text{Tan}(X, x)$ is at most α with constants independent from Y and x .*

A more explanatory statement of the upper bound is given in Theorem 3.7, which will be an immediate consequence of Proposition 3.8. The lower bound is given by Corollary 2.17.

The paper is organized as follows. Section 2 is devoted to preliminaries. We recall the definitions and basic properties of Assouad dimension, Nagata dimension, and locally controlled dimension. In Section 2.3, we give the definition of Gromov-Hausdorff distance for pointed metric spaces, and we define the set of tangents. We provide some remarks about the lower semicontinuity of Assouad dimension and Nagata dimension. In particular, Corollary 2.17 (respectively, Corollary 2.19) gives the lower bound for Theorem 1.4 and Theorem 1.6 (respectively, Theorem 1.2 and Theorem 1.5). In Example 2.20, we show that in general, even for compact subsets of \mathbb{R} , the dimension of all the tangents could be strictly smaller than the dimension of the set. However, in Proposition 2.21,

we prove that, if a doubling space has Nagata dimension equal to one, it has some weak tangent with Nagata dimension one.

In Section 3, we study metric spaces with uniformly close tangents. In Example 3.3 and Example 3.10, we show that having uniformly close tangents does not imply an upper bound for the dimension of the space in terms of the dimensions of the tangents. In Theorem 3.4, we provide such a bound with the additional assumption that either the tangents have linearly controlled dimension less than n with respect to a uniform constant c , or the tangents are unique. Such a theorem gives the missing upper bound for Theorem 1.5 and Theorem 1.2. In Section 3.2, we consider Assouad dimension. We prove Theorem 3.7 (respectively, Theorem 3.9), giving the upper bound needed for concluding the proof of Theorem 1.6 (respectively, Theorem 1.4).

In Section 4, we apply Theorem 1.2 to prove Corollary 1.3, after recalling some results on sub-Riemannian geometry and Carnot groups. In Section 5, we prove Theorem 1.1 and Theorem 5.1, which is a bounded-scale version of Theorem 1.1.

2. PRELIMINARIES

2.1. Assouad dimension and Nagata dimension. In this paper, we consider the notion of Assouad dimension. This is also known by other names, such as metric covering dimension, uniform metric dimension, or doubling dimension. We recall here the definition from [Hei01, p. 81]. The *Assouad dimension* of a metric space X is denoted by $\dim_A X$, and is defined as the infimum of all numbers $\beta > 0$ with the property that there exists some $C > 1$ such that, for every $\varepsilon > 0$, every set of diameter D can be covered by using no more than $C\varepsilon^{-\beta}$ sets of diameter at most εD . In this case, we say that the Assouad dimension is less than or equal to β with constant C .

We will need a quantified and local version of the definition that makes explicit the constants involved.

Definition 2.1 (Assouad dimension up to a scale). Let $\bar{R} > 0$ and $C > 1$. We say that a metric space X has *Assouad dimension at most β up to scale \bar{R} with constant C* if, for all $0 < r < R < \bar{R}$, any ball of radius R in X can be covered with $C(R/r)^\beta$ or less balls of radius r in X . In this case, we write $\dim_A(X, C, \bar{R}) \leq \beta$.

Metric spaces with finite Assouad dimension are precisely the doubling metric spaces. We recall that a metric space is *doubling with constant L* , for some $L > 0$, if, for every $s > 0$, every subset of the metric space with diameter at most $2s$ can be covered by L or fewer sets of diameter at most s .

Another notion of metric dimension we consider is the Nagata dimension. Before giving the definition, let us recall some basic terminology, following [LS05]. Two subsets A, B of a metric space are *s-separated*, for some constant $s \geq 0$, if $\text{dist}(A, B) := \inf\{d(a, b) \mid a \in A, b \in B\} \geq s$. A family of subsets is called *s-separated* if each distinct pair of element in it is *s-separated*. Let \mathcal{B} be a cover of a metric space X . Then, for $s > 0$, the *s-multiplicity* of \mathcal{B} is the infimum of all n

such that every subset of X with diameter at most s meets at most n members of the family \mathcal{B} . Furthermore, \mathcal{B} is called *D-bounded*, for some constant $D \geq 0$, if $\text{diam } B := \sup\{d(x, x') \mid x, x' \in B\} \leq D$, for all $B \in \mathcal{B}$.

Definition 2.2 (Nagata dimension). Let X be a metric space. The *Nagata dimension* (or Assouad-Nagata dimension) of X is denoted by $\dim_{\text{N}} X$, and is defined as the infimum of all integers n with the following property: there exists a constant $c > 0$ such that, for all $s > 0$, X admits a cs -bounded cover with s -multiplicity at most $n + 1$.

As shown in [LS05, Proposition 2.5], the Nagata dimension can be defined equivalently as the infimum of all integers n with the following property:

$$(2.1) \quad \left\| \begin{array}{l} \text{There exists a constant } c > 0 \text{ such that, for all } s > 0, \\ \text{the metric space admits an } s\text{-bounded cover of the} \\ \text{form } \mathcal{B} = \bigcup_{k=0}^n \mathcal{B}_k \text{ where each } \mathcal{B}_k \text{ is } cs\text{-separated.} \end{array} \right.$$

The constant c in (2.1) can be different from the constant c in Definition 2.2.

We should notice that the notion of Nagata dimension is global. Moreover, it is both for small and large scales. We will need the bounded-scale version. For a better formulation of our statements, we give a definition that points out both the scale and the constant.

Definition 2.3 (Nagata dimension at a scale). We say that a metric space X has Nagata dimension bounded from above by $n \in \mathbb{N} := \{0, 1, 2, \dots\}$ with constant $c > 0$ at scale $s > 0$, and write

$$\dim_{\text{N}}(X, c, s) \leq n$$

if the metric space admits an s -bounded cover of the form $\mathcal{B} = \bigcup_{k=0}^n \mathcal{B}_k$ where each \mathcal{B}_k is cs -separated.

Definition 2.4 (Linearly controlled dimension). We say that a metric space X has *linearly controlled dimension* bounded from above by n , written as

$$\dim_{\text{LC}} X \leq n,$$

provided that there exists $c > 0$ and $s_0 > 0$ such that $\dim_{\text{N}}(X, c, s) \leq n$, for all $0 < s < s_0$.

The notation ℓ -dim is also used for the linearly controlled dimension (see [BS07, Section 9.1.4]).

Notice that the space X has Nagata dimension bounded from above by n exactly when we can take $s_0 = \infty$, in other words, if there exists $c > 0$ such that $\dim_{\text{N}}(X, c, s) \leq n$ for all $s > 0$.

Since the restriction of any cover for a space X to a subset $Y \subset X$ gives a cover for Y , we have the easy inequality $\dim_{\text{N}} Y \leq \dim_{\text{N}} X$, for all $Y \subset X$. Moreover, if $Y \subset X$, then

$$\dim_{\text{N}}(X, c, s) \leq n \implies \dim_{\text{N}}(Y, c, s) \leq n.$$

2.2. Basic facts about the dimensions. Here is a first lemma showing that, in the local version of the Assouad dimension, we have no problem if the radius a priori depends on the exponent.

Lemma 2.5. *Let X be a metric space and $\alpha \geq 0$. Assume that, for all $\beta > \alpha$, there exist two constants $C_\beta > 1$ and $R_\beta > 0$ such that $\dim_A(X, C_\beta, R_\beta) \leq \beta$. Then, there exists $R' > 0$ such that, for all $\beta > \alpha$, there exists a constant $C'_\beta > 1$ such that $\dim_A(X, C'_\beta, R') \leq \beta$.*

Proof. Fix $\eta > \alpha$. We can take $R' := R_\eta$. Indeed, pick any $\beta > \alpha$. Fix $0 < r < R < R_\eta$ and $x \in X$. If $R < R_\beta$, we are done. Thus, we assume $R \geq R_\beta$. Cover $B(x, R_\eta)$ with $C_\eta(R_\eta/R_\beta)^\eta$ or less balls of radius R_β . Cover each of these balls with at most $C_\beta(R_\beta/r)^\beta$ balls of radius r . Hence, the ball $B(x, R)$, which is inside $B(x, R_\eta)$, needs no more than $C_\eta(R_\eta/R_\beta)^\eta C_\beta(R_\beta/r)^\beta$ balls of radius r to cover it. The proof is concluded by putting $C'_\beta := C_\eta C_\beta (R_\eta/R_\beta)^\eta$, since then $C'_\beta (R_\beta/r)^\beta < C'_\beta (R/r)^\beta$. \square

Regarding the Nagata dimension, we shall study what happens when we consider spaces that are the union of spaces whose Nagata dimension we know. We start with an easy statement, whose proof is straightforward.

Lemma 2.6. *Let $X = \bigcup_{i \in I} X_i$ with X_i that are r -separated and such that $\dim_N(X_i, c, s) \leq n$, for all $i \in I$. Then,*

$$\dim_N \left(X, \min \left\{ c, \frac{r}{s} \right\}, s \right) \leq n.$$

Regarding finite unions, we show the following Lemma 2.7, which is essentially the quantified version of [LS05, Theorem 2.7].

Lemma 2.7. *Let X, Y be subsets of a metric space, and $c_1, c_2 \in (0, 1)$ and $0 \leq s_0 < s_1 < \infty$, such that*

$$\dim_N(X, c_1, s), \dim_N(Y, c_2, s) \leq n, \quad \text{for all } s_0 \leq s \leq s_1.$$

Then,

$$\dim_N \left(X \cup Y, \frac{c_1 c_2}{5}, s \right) \leq n, \quad \text{for all } \left(2 + \frac{3}{c_1} \right) s_0 \leq s \leq \left(1 + \frac{2}{3} c_1 \right) s_1.$$

Proof. Let $s \in [s_0, s_1]$ and take an s -bounded cover $\bigcup_{j=0}^n \{U_i^j\}_{i \in I_j}$ of X where $\{U_i^j\}_{i \in I_j}$ is $c_1 s$ -separated, for all j . Assuming $\frac{1}{3} c_1 s \geq s_0$, take also a $\frac{1}{3} c_1 s$ -bounded cover $\bigcup_{j=0}^n \{V_i^j\}_{i \in J_j}$ of Y where, for each j , the family $\{V_i^j\}_{i \in J_j}$ is $\frac{1}{3} c_1 c_2 s$ -separated. For each $j = 0, \dots, n$, define

$$K_j = \left\{ i \in J_j \mid \text{dist} \left(V_i^j, \bigcup_{k \in I_j} U_k^j \right) \geq \frac{1}{3} c_1 s \right\}.$$

Using K_j , write new collections of sets covering $X \cup Y$ as follows. For all $i \in I_j$ and $j = 0, \dots, n$ set

$$W_{(1,i)}^j := U_i^j \cup \bigcup_{\text{dist}(V_k^j, U_i^j) < (1/3)c_1 s} V_k^j,$$

and for all $i \in K_j$, $j = 0, \dots, n$, set $W_{(2,i)}^j := V_i^j$. Now, we shall abbreviate $L_j = (\{1\} \times I_j) \cup (\{2\} \times K_j)$. It is easy to check that $\bigcup_{j=0}^n \{W_i^j\}_{i \in L_j}$ is a $(1 + \frac{2}{3}c_1)s$ -bounded cover of $X \cup Y$ where each $\{W_i^j\}_{i \in L_j}$ is $\frac{1}{3}c_1 c_2 s$ -separated. Now,

$$\frac{\text{dist}(W_i^j, W_k^j)}{\text{diam}(W_i^j)} \geq \frac{(1/3)c_1 c_2 s}{(1 + (2/3)c_1)s} \geq \frac{c_1 c_2}{5}.$$

Therefore,

$$\dim_{\mathbb{N}} \left(X \cup Y, \frac{c_1 c_2}{5}, \left(1 + \frac{2}{3}c_1\right)s \right) \leq n,$$

under the assumptions that we have made for s , namely, $\frac{3s_0}{c_1} \leq s \leq s_1$. The claim then follows. \square

Iterating the above lemma, we get the analogous statement for finite unions.

Corollary 2.8. *Assume that there exist $c \in (0, 1)$, $s_0, s_1 > 0$ and $n, N \in \mathbb{N}$ such that*

$$\dim_{\mathbb{N}}(X_i, c, s) \leq n, \quad \text{for all } s_0 \leq s \leq s_1 \text{ and } i = 1, \dots, N.$$

Then,

$$\dim_{\mathbb{N}} \left(\bigcup_{i=1}^N X_i, \frac{c^N}{5^{N-1}}, s \right) \leq n, \quad \text{for all } \left(2 + \frac{3}{c}\right)^{N-1} s_0 \leq s \leq s_1.$$

By the above results (or just by [LS05, Theorem 2.7]), finite unions of sets with Nagata dimension n have Nagata dimension n . The same is not valid for a countable union. Indeed, the space \mathbb{Z} has dimension one, but it is the countable union of points, which have dimension zero. However, we can conclude that the Nagata dimension of a space is n if we know that each ball $B(x, r)$, at a fixed point $x \in X$, has Nagata dimension equal to n for the same constant c . In the case of separable metric spaces, we have the following more general fact. Suppose that there exists an increasing sequence of subsets $X_1 \subset X_2 \subset \dots \subset X$ with $\bigcup_{i=1}^{\infty} X_i = X$ separable and such that all X_i have Nagata dimension equal to n with the same constant $c > 0$ in (2.1). Then, one can show that $\dim_{\mathbb{N}} X = n$. Without the assumption of separability, we can show the following fact, which is the one we will use to give an upper bound for the Nagata dimension of limits of metric spaces.

Lemma 2.9. *Let X be a metric space. Let $n \in \mathbb{N}$, $0 < s_1 < \infty$, $0 < c < 1$, and $x_0 \in X$. Assume there is a sequence of radii $r_m \rightarrow \infty$ such that*

$$\dim_{\mathbb{N}}(B(x_0, r_m), c, s) \leq n, \quad \text{for all } 0 < s \leq s_1 \text{ and all } m = 1, 2, \dots$$

Then,

$$\dim_{\mathbb{N}}\left(X, \frac{c^2}{5}, s\right) \leq n, \quad \text{for all } 0 < s \leq \left(1 + \frac{2}{3}c\right)s_1.$$

Proof. By taking a subsequence if necessary, we may assume $r_{m+1} - r_m > cs_1$. Let

$$A_k := B(x_0, r_{2k}) \setminus B(x_0, r_{2k-1}), \quad Y := \bigcup_{k=1}^{\infty} A_k, \quad \text{and} \quad Z := X \setminus Y.$$

Notice that

$$Z = B(x_0, r_1) \cup \bigcup_{k=1}^{\infty} B(x_0, r_{2k+1}) \setminus B(x_0, r_{2k}).$$

For all $0 < s \leq s_1$, we have $\dim_{\mathbb{N}}(A_k, c, s) \leq \dim_{\mathbb{N}}(B(x_0, r_{2k}), c, s) \leq n$, and the family $\{A_k\}$ is cs_1 -separated. Therefore, by Lemma 2.6,

$$\dim_{\mathbb{N}}(Y, c, s) \leq n, \quad \text{for all } 0 < s \leq s_1.$$

Similarly,

$$\dim_{\mathbb{N}}(Z, c, s) \leq n, \quad \text{for all } 0 < s \leq s_1.$$

Therefore, the claim follows from Lemma 2.7. \square

We observed the next lemma together with Urs Lang. We will use the lemma and the example afterward to show that the Nagata dimension of Carnot groups is equal to the topological one.

Lemma 2.10 (See also [LS05, Proposition 2.8]). *Let X be a metric space. Suppose there exist positive constants r, c and integers L, n such that every ball $B(x, r)$, $x \in X$, is doubling with constant L and has Nagata dimension at most n with constant c . Then, X has linearly controlled dimension at most n .*

Proof. Set $s_0 := r/3$, and choose a set $Z \subseteq X$ that is maximal, with respect to inclusion, subject to the condition that distinct points in Z are at distance more than s_0 from each other. Then, the family of balls $B(z, s_0)$, $z \in Z$, covers X . By the doubling condition, there is an integer N such that every ball $B(z, 3s_0)$ can be covered by N or fewer sets of diameter at most s_0 , and so $Z \cap B(z, 3s_0)$ has cardinality at most N . It follows that there is a coloring of Z by N colors, $k: Z \rightarrow \{1, \dots, N\}$, such that $k(z) \neq k(z')$ whenever $0 < d(z, z') \leq 3s_0$ (see [Ass83, Lemma 2.4]). Let C_k be the union of all balls $B(z, s_0)$ with $k(z) = k$, for

$k = 1, \dots, N$. Clearly, every C_k has linearly controlled dimension at most n . Now, use finite gluing (Corollary 2.8) to show that X has linearly controlled dimension at most n . \square

Example 2.11. Assume G is a Lie group equipped with a left-invariant Riemannian distance. If n is the topological dimension of G , then G has linearly controlled dimension n . Indeed, if we fix $r > 0$ small enough, all balls $B(x, r)$ for $x \in G$ are isometric to each other and are bi-Lipschitz to an open set of the n -dimensional Euclidean space. In particular, such balls are doubling and have linearly controlled dimension at most n , with uniform constants. Hence, Lemma 2.10 gives one of the bounds. Recall that it is a general fact that the topological dimension does not exceed the linearly controlled dimension (see [LS05, Theorem 2.2]).

2.3. Gromov-Hausdorff distance of pointed metric spaces. For defining limits of unbounded metric spaces, we need to consider pointed metric spaces. A *pointed metric space* is a pair (X, x) of a metric space $X = (X, d)$ and a point $x \in X$. Recall that, when we are considering a metric space (X, d) , we tend to refer to it simply as X whenever it is not necessary to specify the distance. Moreover, we denote by d the distance of any metric space we are considering. Likewise, we will denote simply by X a pointed metric space when it is not important to specify the base point. In case we need to denote the base point (and the metric space in discussion is clear), we will use the symbol \star .

For a set A in a metric space X , we denote the δ -neighborhood of A as

$$B(A, \delta) := B_X(A, \delta) := \{x \in X \mid \text{dist}(x, A) < \delta\}.$$

Given two metric spaces $X = (X, d_X)$ and $Y = (Y, d_Y)$, we say that d is *an extension of the distances on $Y \sqcup X$* if d is a semidistance on the set $Y \sqcup X$ (i.e., d might vanish on distinct points), if it coincides with d_X when restricted to X , and if it coincides with d_Y when restricted to Y .

We shall use the Gromov-Hausdorff convergence for pointed metric spaces. We will actually want to have a precise distance giving such a topology. Inspired by the definitions of Gromov and Gabber (see [Gro81, Section 6]), we define the modified Gromov-Hausdorff measurement as follows: for all pointed metric spaces $X = (X, \star_X)$ and $Y = (Y, \star_Y)$, we set

$$\widetilde{\text{Dist}}_{\text{GH}}(X, Y) = \inf \left\{ \varepsilon > 0 \mid \exists \text{ an extension of the distances on } Y \sqcup X : \right. \\ \left. d(\star_X, \star_Y) \leq \varepsilon, B_X\left(\star_X, \frac{1}{\varepsilon}\right) \subseteq B_{Y \sqcup X}(Y, \varepsilon), B_Y\left(\star_Y, \frac{1}{\varepsilon}\right) \subseteq B_{Y \sqcup X}(X, \varepsilon) \right\}.$$

Probably, the function $\widetilde{\text{Dist}}_{\text{GH}}$ is not a distance, since it is not clear whether it satisfies the triangle inequality, as Gromov already pointed out. However, using the following Lemma 2.12, we can easily modify it to be a distance.

Lemma 2.12. *Let X, Y, Z be pointed metric spaces. Then, we have the following:*

- (i) *If both $\widetilde{\text{Dist}}_{\text{GH}}(X, Y), \widetilde{\text{Dist}}_{\text{GH}}(Y, Z) \leq \frac{1}{2}$, then*

$$\widetilde{\text{Dist}}_{\text{GH}}(X, Z) \leq \widetilde{\text{Dist}}_{\text{GH}}(X, Y) + \widetilde{\text{Dist}}_{\text{GH}}(Y, Z).$$

- (ii) *Consequently, the function $\min\{\frac{1}{2}, \widetilde{\text{Dist}}_{\text{GH}}\}$ satisfies the triangle inequality.*

Proof. The claim follows by considering distances of the form

$$d_{X \sqcup Z}(x, z) := \inf_{y \in Y} \{d_{Y \sqcup X}(x, y) + d_{Z \sqcup Y}(y, z)\}$$

defined from extensions $d_{Y \sqcup X}$ and $d_{Z \sqcup Y}$ of the distances on X, Y , and Z . \square

We define the *Gromov-Hausdorff distance* of two pointed metric spaces X, Y as

$$\text{Dist}_{\text{GH}}(X, Y) := \min \left\{ \frac{1}{2}, \widetilde{\text{Dist}}_{\text{GH}}(X, Y) \right\}.$$

Other simple properties that hold for $\widetilde{\text{Dist}}_{\text{GH}}$ and thus for Dist_{GH} are the following.

Lemma 2.13. *Let X be a metric space. Then, we have the following:*

- (i) *For all $\lambda_1 > \lambda_2 > 0$ and all $x \in X$,*

$$\text{Dist}_{\text{GH}}((\lambda_1 X, x), (\lambda_2 X, x)) \leq \sqrt{\frac{\lambda_1}{\lambda_2} - 1}.$$

- (ii) *For all $x, x' \in X$, $\text{Dist}_{\text{GH}}((X, x), (X, x')) \leq d(x, x')$.*
 (iii) *The function $(x, \lambda) \mapsto (\lambda X, x)$ is continuous. In fact, it is Hölder on compact sets of $(0, \infty) \times X$.*

Proof. To easily obtain (i), one can isometrically embed X into the Banach space $L^\infty(X)$. The dilations of the ambient Banach space give a straightforward calculation of the Hausdorff distance of $\lambda_1 X$ and $\lambda_2 X$.

To see (ii), use the original distance as the extension. Claim (iii) follows from (i) and (ii). \square

To define tangent metric spaces, we dilate a metric space and consider the accumulation points of such sequences of dilated spaces. Recall that, whenever the distance and the base point are clear, we simply write X to denote the pointed metric space (X, d, \star) , or we write $d = d_X$ to emphasize that d is the distance on X . Moreover, we denote by λX the pointed metric space obtained by dilating the distance by $\lambda > 0$. Namely, we set $\lambda X := (X, \lambda d_X, \star)$.

Definition 2.14 (Tangent metric spaces). Let (X, d_X) be a metric space, and x a point of it. Then a pointed metric space (Y, d_Y, γ) is said to be a *tangent* of (X, d_X) at x if $(X, \lambda d_X, x)$ accumulate to (Y, d_Y, γ) in the pointed Gromov-Hausdorff topology, as $\lambda \rightarrow \infty$. Namely, there exists a diverging sequence λ_j such that

$$\text{Dist}_{\text{GH}}((X, \lambda_j d_X, x), (Y, d_Y, \gamma)) \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

We denote by $\text{Tan}(X, x)$ the collection of all tangents of (X, d_X) at x .

It is easy to come up with examples of metric spaces with more than one tangent at a given point. Following is an easy criterion (which follows from Lemma 2.13(i)) to conclude that the tangent at a point is unique.

Lemma 2.15. *Let X and Y be two pointed metric spaces. We have the following:*

- (i) *If $nX \rightarrow Y$, as $n \rightarrow \infty$, $n \in \mathbb{N}$, then $\lambda X \rightarrow Y$, as $\lambda \rightarrow \infty$, $\lambda \in \mathbb{R}$.*
- (ii) *More generally, let a_n be a diverging sequence such that $a_{n+1}/a_n \rightarrow 1$, as $n \rightarrow \infty$. If $a_n X \rightarrow Y$, as $n \rightarrow \infty$, then $\lambda X \rightarrow Y$, as $\lambda \rightarrow \infty$, $\lambda \in \mathbb{R}$.*

2.4. Some remarks on the dimensions of limits. In this section, we prove that the dimensions of the tangents bound from below the dimension of a space. Regarding the Assouad dimension, this fact is a quantified version of the well-known result that limits of doubling metric spaces, with uniform constants, are doubling.

Lemma 2.16. *Suppose that a sequence of pointed metric spaces X_j converges to a pointed metric space X_∞ . Let $\beta \geq 0$, $C > 1$, and $\bar{R} > 0$. If, for all j , $\dim_A(X_j, C, \bar{R}) \leq \beta$, then $\dim_A(X_\infty, C, \bar{R}) \leq \beta$.*

Proof. Take $0 < r < R < \bar{R}$ and $x \in X_\infty$. For all $j \in \mathbb{N}$, let d_j be a distance on $X_j \sqcup X_\infty$ extending the distances on X_j and X_∞ such that $d(\star_{X_j}, \star_{X_\infty}) \leq \varepsilon_j$,

$$B_{X_j} \left(\star_{X_j}, \frac{1}{\varepsilon_j} \right) \subseteq B_{X_\infty \sqcup X_j}(X_\infty, \varepsilon_j),$$

and

$$B_{X_\infty} \left(\star_{X_\infty}, \frac{1}{\varepsilon_j} \right) \subseteq B_{X_\infty \sqcup X_j}(X_j, \varepsilon_j)$$

for some sequence ε_j going to 0.

Now, for $j \in \mathbb{N}$ large enough, we have $B_{X_\infty}(x, R) \subseteq B_{X_\infty \sqcup X_j}(X_j, \varepsilon_j)$. Take $x_j \in X_j$ with $d_j(x, x_j) < \varepsilon_j$. For j large enough, we have $R + 2\varepsilon_j < \bar{R}$, and we need at most $C((R + 2\varepsilon_j)/(r - 2\varepsilon_j))^\beta$ points

$$x_{j,i} \in B_{X_j}(x, R + r) \subset B_{X_j}(\star_{X_j}, 1/\varepsilon_j)$$

such that the ball $B_{X_j}(x_j, R + 2\varepsilon_j)$ is covered by the balls $B_{X_j}(x_{j,i}, r - 2\varepsilon_j)$. Select points $\tilde{x}_{j,i} \in X_\infty$ with $d_j(x_{j,i}, \tilde{x}_{j,i}) < \varepsilon_j$. We claim that $\{B_{X_\infty}(\tilde{x}_{j,i}, r)\}_i$ is a cover for the ball $B_{X_\infty}(x, R)$.

Indeed, take any $y \in B_{X_\infty}(x, R)$ and $y_j \in X_j$ with $d_j(y, y_j) < \varepsilon_j$. Now,

$$d_j(y_j, x_j) < d_j(y_j, y) + d_j(y, x) + d_j(x, x_j) \leq \varepsilon_j + R + \varepsilon_j = R + 2\varepsilon_j,$$

and so $y_j \in B_{X_j}(x_{j,i}, r - 2\varepsilon_j)$ for some i . Because

$$d_j(y, \tilde{x}_{j,i}) < d_j(y, y_j) + d_j(y_j, x_{j,i}) + d_j(x_{j,i}, \tilde{x}_{j,i}) \leq \varepsilon_j + r - 2\varepsilon_j + \varepsilon_j = r,$$

we have $y \in B_{X_\infty}(\tilde{x}_{j,i}, r)$.

Since $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$, with large enough j we obtain a cover of $B_{X_\infty}(x, R)$ with no more than $C(R/r)^\beta$ balls of radius r . \square

The following consequence can be also found in [MT10, Proposition 6.1.5].

Corollary 2.17. *The Assouad dimension of any tangent space of a metric space X does not exceed the Assouad dimension of X .*

Regarding the Nagata dimension, the similar bound is slightly less trivial and is based on Lemma 2.9.

Proposition 2.18. *Suppose that a sequence of pointed metric spaces X_j covers to a pointed metric space X_∞ . Let $0 < s_1 < \infty$ and $0 < c < 1$. Assume that*

$$\dim_{\mathbb{N}}(X_j, c, s) \leq n, \quad \text{for all } 0 < s \leq s_1 \text{ and all } j = 1, 2, \dots .$$

Then, for any $0 < c' < c^2/5$, we have

$$\dim_{\mathbb{N}}(X_\infty, c', s) \leq n, \quad \text{for all } 0 < s < \left(1 + \frac{2}{3}c\right) s_1.$$

Proof. Our aim is to show that for all $0 < c'' < c$, we have

$$(2.2) \quad \dim_{\mathbb{N}}(B_{X_\infty}(\star_{X_\infty}, k), c'', s) \leq n, \quad \text{for all } 0 < s < s_1 \text{ and all } k = 1, 2, \dots .$$

Once we have obtained this, the claim will follow from Lemma 2.9.

Fix an integer $k > 0$ and $0 < c'' < c$. For all $j \in \mathbb{N}$, let d_j be a distance on $X_j \sqcup X_\infty$ extending the distances on X_j and X_∞ such that $d(\star_{X_j}, \star_{X_\infty}) \leq \varepsilon_j$,

$$B_{X_j} \left(\star_{X_j}, \frac{1}{\varepsilon_j} \right) \subseteq B_{X_\infty \sqcup X_j}(X_\infty, \varepsilon_j),$$

and

$$B_{X_\infty} \left(\star_{X_\infty}, \frac{1}{\varepsilon_j} \right) \subseteq B_{X_\infty \sqcup X_j}(X_j, \varepsilon_j)$$

for some sequence ε_j going to 0.

Take $0 < s < s_1$ and $j \in \mathbb{N}$ so that $\varepsilon_j < 1/k$ and

$$(2.3) \quad cs - 2(1 + c)\varepsilon_j \geq c''s.$$

Let $\bigcup_{i=0}^n \mathcal{B}_i$ be an $(s - 2\varepsilon_j)$ -bounded cover of X_j where each \mathcal{B}_i is $c(s - 2\varepsilon_j)$ -separated. Define the new collections of sets \mathcal{B}'_i as

$$\mathcal{B}'_i := \{B_{X_\infty}(\star_{X_\infty}, k) \cap B_{X_\infty \sqcup X_j}(B, \varepsilon_j)\}_{B \in \mathcal{B}_i}.$$

Now, for each $x \in B_{X_\infty}(\star_{X_\infty}, k)$, there exists $x_j \in X$ with $d_j(x, x_j) < \varepsilon_j$. Because $x_j \in B$ for some $i = 0, \dots, n$ and $B \in \mathcal{B}_i$, we have $x \in B'$ for some $i = 0, \dots, n$ and $B' \in \mathcal{B}'_i$. Therefore, $\bigcup_{i=0}^n \mathcal{B}'_i$ is an s -bounded cover of $B_{X_\infty}(\star_{X_\infty}, k)$ where each \mathcal{B}'_i is $c''s$ -separated, by (2.3). \square

Corollary 2.19. *The Nagata dimension of any tangent space of a metric space X does not exceed the Nagata dimension of X .*

As shown by the next example, in general one cannot hope to deduce an upper bound for the dimension of the space by simply looking at the dimensions of the tangents.

Example 2.20. Define a set $X \subset \mathbb{R}$ as

$$X = \{0\} \cup \bigcup_{i=1}^{\infty} \bigcup_{k=1}^i \{2^{-i^2} + k2^{-i^3}\}.$$

Let the distance be induced by the Euclidean distance on \mathbb{R} . Then, X is compact, $\dim_A X = \dim_N X = 1$, and $\text{Tan}(X, x) \subset \{\{0, t\} \subset \mathbb{R} \mid t \in [0, \infty)\}$ for all $x \in X$. Hence, $\dim_A Y = \dim_N Y = 0$, for all $Y \in \text{Tan}(X, x)$. In particular,

$$(2.4) \quad \max\{\dim Y \mid Y \in \text{Tan}(X, x), x \in X\} < \dim X,$$

where \dim is either the Nagata or the Assouad dimension.

Notice that the space in Example 2.20 has multiple tangents at 0, the convergence to the tangents is not uniform, and that in the definition of a tangent space, we keep the base point fixed. Hence, if we want to obtain the Nagata dimension of the space as the maximum of the dimensions of the tangents, we need to either change the notion of tangents (e.g., consider weak tangents) or impose more restrictions on the convergence to the tangents. Regarding this last option, we will study the case of uniformly close tangents, and, in the subsequent Section 4, we will apply our results to sub-Riemannian manifolds.

Observe also that in Example 2.20, if we take a larger class of tangents where we allow the change of the the base point, we obtain equality in (2.4). Simply let $r_i = i2^{-i^3}$ and $x_i = 2^{-i^2}$. This observation is true in compact doubling metric spaces when $\dim_N X = 1$. This is the content of the following proposition, where we consider *weak tangents*, that is, limits of the forms $((1/r_i)X, x_i)$.

Proposition 2.21. *Let X be a doubling metric space with $\dim_{\text{LC}} X \geq 1$. Then, there exists a sequence of points $x_i \in X$ and a sequence $r_i \searrow 0$ such that the sequence $((1/r_i)X, x_i)$ converges to a space with linearly controlled dimension at least 1.*

Proof. For each $x \in X$ and $\delta > 0$, define the iterated δ -neighborhoods of x by setting $N_X^0(x, \delta) := \{x\}$ and

$$N_X^n(x, \delta) := B(N_X^{n-1}(x, \delta), \delta), \quad \text{for } n \geq 1.$$

Define also $N_X^\infty(x, \delta) := \bigcup_{i=1}^\infty N_X^n(x, \delta)$. Take $i \in \mathbb{N}$. Since $\dim_{\text{LC}} X \geq 1$, we claim that there exist a point $x_i \in X$ and a radius $r_i < 1/i$ such that we have $\text{diam}(N_X^\infty(x_i, r_i/i)) > r_i$. Indeed, suppose that this is not true. Then for every $r < 1/i$, the collection $\{N_X^\infty(x, r/i) \mid x \in X\}$ would be an r -bounded $r/(2i)$ -separated cover of X . This would mean that $\dim_{\text{LC}} X = 0$.

By the Gromov Theorem (see [BBI01, Theorem 8.1.10]), because of the doubling assumption, the sequence $(r_i^{-1}X, x_i)$ has a subsequence converging to a pointed metric space (Z, z) . Now, for any $0 < \varepsilon < \frac{1}{2}$, there is $i > 1/\varepsilon$ such that

$$\text{Dist}_{\text{GH}}((r_i^{-1}X, x_i), (Z, z)) < \varepsilon.$$

Consequently, there exists a distance d extending the distances on $r_i^{-1}X$ and Z such that $d(x_i, z) \leq \varepsilon$ and $B_{r_i^{-1}X}(x_i, 1/\varepsilon) \subseteq B_{Z \sqcup r_i^{-1}X}(Z, \varepsilon)$. Now, for each

$$x' \in N_{r_i^{-1}X}^\infty\left(x_i, \frac{1}{i}\right) \cap B_{r_i^{-1}X}(x_i, 2) = N_X^\infty\left(x_i, \frac{r_i}{i}\right) \cap B_X(x_i, 2r_i),$$

there exists $z' \in Z$ with $d(x', z') < \varepsilon$. Thus, $\text{diam}(N_Z^\infty(z, 3\varepsilon)) \geq 1 - 2\varepsilon$. Letting $\varepsilon \searrow 0$ shows that $\dim_{\text{LC}} Z \geq 1$. \square

We do not know if Proposition 2.21 is true with the dimension lower bound replaced by a higher bound. We were unable to find a way to generalize our argument to the higher-dimensional case.

Question 2.22. *Let X be a metric space. Does there exist a sequence of points $x_i \in X$ and a sequence $r_i \searrow 0$ such that the weak tangent of X along the sequences x_i, r_i has exactly the same linearly controlled dimension as X ?*

3. DIMENSION OF UNIFORMLY CLOSE TANGENTS

As we saw in Example 2.20, both the Nagata dimension and the Assouad dimension of a space can in general be strictly larger than the supremum of the dimensions of its tangents. We wonder when such a supremum equals the dimension of the space. A key assumption that we will make, which will still not be enough, is that the convergence to the tangents is uniform, as we now explain. Recall that Dist_{GH} is the distance defined in Section 2.3.

Let X be a metric space. We say that the metric space X has *uniformly close tangents in $K \subset X$* if we have

$$\lim_{\lambda \rightarrow \infty} \text{Dist}_{\text{GH}}((\lambda X, x), \text{Tan}(X, x)) = 0$$

uniformly in $x \in K$. If X has uniformly close tangents in X , we simply say that X has uniformly close tangents. Before investigating what other assumptions we need to make, we list some basic properties of uniformly close tangents.

Lemma 3.1. *Assume that a metric space X has uniformly close tangents. If, for all $x \in X$, there exists only one element $T_x \in \text{Tan}(X, x)$, then the function $x \mapsto T_x$ is continuous.*

We omit the easy proof of the above fact, which is proved exactly as one proves that the uniform limit of continuous maps is continuous, via Lemma 2.13(ii). In general, without the uniqueness assumption on the tangents, one cannot even conclude that the “graph” $\{(x, T) \mid x \in X, T \in \text{Tan}(X, x)\}$ is closed (see, e.g., Example 3.3).

From [LD11], we know that if a complete metric space admits a doubling measure μ , and if the space has unique tangents, then at μ -almost every point the unique tangent is an isometrically homogeneous space admitting dilations. We have the following version of this result.

Proposition 3.2. *Let X be a complete doubling metric space with uniformly close tangents such that, for all $x \in X$, there exists only one element $T_x \in \text{Tan}(X, x)$. Then, for all $x \in X$, the tangent T_x is an isometrically homogeneous space admitting dilations.*

Proof. Since X is a complete doubling metric space, there exists a doubling measure μ on X (see, e.g., [VK84], [LS98], or [KRS12]). By [LD11, Theorem 1.4], we already know that at μ -almost every $x \in X$, the tangent T_x is an isometrically homogeneous space. By Lemma 3.1, we know that all the tangents T_x are Gromov-Hausdorff limits of isometrically homogeneous spaces. Since being isometrically homogeneous is stable under Gromov-Hausdorff convergence, all the tangents are isometrically homogeneous. The fact that the tangents admit dilations follows directly from the fact that $\text{Tan}(X, x)$ are singletons. \square

3.1. Nagata dimension and uniformly close tangents. It turns out that having uniformly close tangents is not enough to tie the Nagata dimension of the space to the Nagata dimension of its tangents. This is shown by the next example.

Example 3.3. Let us define a metric space $X \subset \mathbb{R}^2$. Define for all $n \in \mathbb{N}$ a basic construction piece S_n , in polar coordinates as

$$S_n = \left\{ \left(1, \frac{2\pi k}{n} \right) \mid k = 1, \dots, n \right\}.$$

The set S_n consists of n equally distributed points on the unit circle in \mathbb{R}^2 .

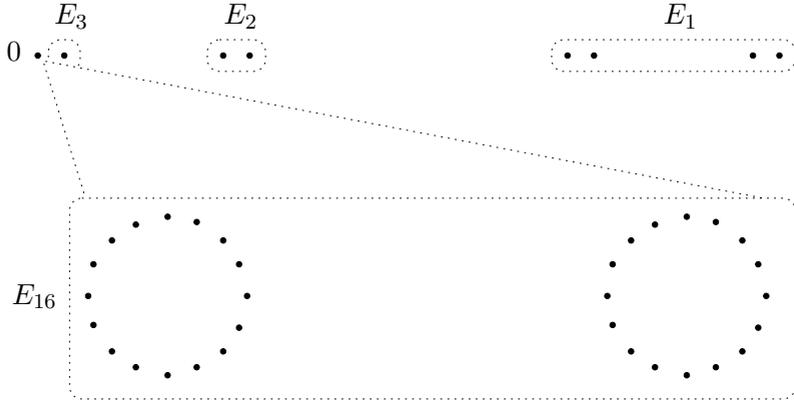


FIGURE 3.1. Illustration of the space X in Example 3.3. It consists of an infinite number of construction pieces E_i , each of which is constructed by selecting i points on a sphere on odd construction steps, and two points on even steps. The magnified illustration of the construction piece E_{16} shows this idea.

Using the sets S_n , define for all $n \in \mathbb{N}$ the set

$$E_n = \left\{ \sum_{i=n}^{\infty} 2^{-i^2} A_i \mid A_i = S_n \text{ if } i \text{ odd, and } A_i = \{(0, 0), (1, 0)\} \text{ if } i \text{ even} \right\},$$

and from these finally the space

$$X = \text{cl} \left(\bigcup_{n=1}^{\infty} E_n + (2^{-n^2}, 0) \right).$$

The construction of X is illustrated in Figure 3.3.

As tangents for every $x \in \text{cl}(E_n + 2^{-n^2})$, we have

$$\text{Tan}(X, x) = \{tS_n \mid t \in (0, \infty)\} \cup \{\{0, t\} \mid t \in [0, \infty)\},$$

and for the origin,

$$\text{Tan}(X, (0, 0)) = \{\{0, t\} \mid t \in [0, \infty)\}.$$

We claim that the space X (with the distance induced by the Euclidean distance on \mathbb{R}^2) has uniformly close tangents. To see this, take $\lambda > 0$ and $x \in X$. Let $i \in \mathbb{N}$ be such that $2^{i^2-i} < \lambda \leq 2^{(i+1)^2-i-1}$. Then, $\text{diam}(\lambda 2^{-j^2} A_j) \leq 2^{-i}$ for all $j \geq i+1$; thus, down to scale 2^{-i} , we can consider the sets $2^{-j^2} A_j$, $j \geq i+1$, to

be just points in the λ -dilated distance. On the other hand, since any two distinct points in A_{i-1} have distance at least $1/(i-1)$ between them, and since we have $\lambda(1/(i-1))2^{-(i-1)^2} \geq (1/(i-1))2^{i-1}$, there is at most one point of $2^{-(i-1)^2}A_{i-1}$ inside a ball of radius $(1/(1-i))2^{i-2}$ in the λ dilated distance. In particular,

$$\text{Dist}_{\text{GH}}((\lambda X, x), \text{Tan}(X, x)) \leq (1-i)2^{2-i}$$

for all $x \in X$. Hence, X has uniformly close tangents.

The Nagata dimension of any of the tangents of X is zero. However, the Nagata dimension and the linearly controlled dimension of the space are one.

In Example 3.3, the tangents of the space have Nagata dimension zero with smaller and smaller constant c as we move the base point towards $(0,0)$. On the other hand, the set of tangents is more than a singleton at every point in the space. In the following Theorem 3.4, we show that if we rule out one of the above mentioned properties—namely, if we require either uniformity of the constant c or the uniqueness of tangents—the linearly controlled dimension of the space is bounded above by the supremum of the dimensions of its tangents.

Theorem 3.4. *Let X be a metric space. Let $K \subset X$ be a subset on which the convergence to tangents is uniform. Assume that $\dim_{\text{LC}} K < \infty$ and that one of the following two situation holds:*

- (i) *Either $\dim_{\text{LC}}(B_Y(\star, 1)) \leq n$, for all $x \in K$ and $Y \in \text{Tan}(X, x)$, with uniform constants c and s ;*
- (ii) *Or, K is compact, and for every $x \in K$, the tangent is unique; i.e., the set $\text{Tan}(X, x)$ is a singleton denoted by $T_x X$, and $\dim_{\text{LC}}(B_{T_x X}(\star, 1)) \leq n$.*

Then, $\dim_{\text{LC}} K \leq n$.

For proving the above theorem, we need the following two lemmas.

Lemma 3.5. *Let (X, x) and (Y, y) be two pointed metric spaces. Suppose that there exist some ε, r, c, s, n such that*

$$\text{Dist}_{\text{GH}}((X, x), (Y, y)) < \varepsilon \quad \text{and} \quad \dim_{\text{N}}(B_Y(y, r), c, s) \leq n.$$

Then, if $r' \leq \min\{1/\varepsilon, r - 2\varepsilon\}$, $c' := (cs - 2\varepsilon)/(s + 2\varepsilon)$, and $s' := s + 2\varepsilon$, we have

$$\dim_{\text{N}}(B_X(x, r'), c', s') \leq n.$$

Proof. Since (X, x) and (Y, y) have distance $< \varepsilon$, we can see them as a subset of a metric space Z such that $d(x, y) \leq \varepsilon$, and $B_X(x, 1/\varepsilon) \subseteq B_Z(Y, \varepsilon)$.

We claim that

$$(3.1) \quad B_X(x, r') \subseteq B_Z(B_Y(y, r), \varepsilon).$$

Indeed, $B_X(x, r') \subseteq B_X(x, 1/\varepsilon) \subseteq B_Z(Y, \varepsilon)$, and hence, for all $x' \in B_X(x, r')$, there is $y' \in X$ such that $d(x', y') \leq \varepsilon$. Such a y' is such that $d(y', y) \leq r' + 2\varepsilon$. Hence, $B_X(x, r') \subseteq B_Z(B_Y(y, r' + 2\varepsilon), \varepsilon) \subseteq B_Z(B_Y(y, r), \varepsilon)$.

Let \mathcal{U} be an s -bounded cover of $B_Y(\gamma, r)$ such that $\mathcal{U} = \mathcal{U}_0 \sqcup \cdots \sqcup \mathcal{U}_n$ with each \mathcal{U}_j cs -separated. Set $\mathcal{V}_j := \{B_Z(U, \varepsilon) \cap X \mid U \in \mathcal{U}_j\}$. Clearly, \mathcal{V}_j are $(s + 2\varepsilon)$ -bounded and $(cs - 2\varepsilon)$ -separated. Since (3.1), the family $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_n$ gives a cover of $B_X(x, r')$. \square

Lemma 3.6. *For every choice of constants $\bar{n}, n \in \mathbb{N}, \bar{s}, s_0 \in (0, \infty], a, b > 0$, and $c, \bar{c} \in (0, 1)$ there exists $\varepsilon \in (0, \frac{1}{8})$ such that the following holds.*

Let $K \subset X$. Suppose that

$$(3.2) \quad \dim_{\mathbb{N}}(K, \bar{c}, s) \leq \bar{n}, \quad \text{for all } 0 < s \leq \bar{s},$$

and that there exists $r_\varepsilon > 0$ such that, for all $x \in K$, for all $r \in (0, r_\varepsilon)$, and for all $s \in (0, s_0)$,

$$(3.3) \quad \dim_{\mathbb{N}} \left(B \left(x, \frac{r}{2} \right), \frac{cas - b\varepsilon}{as + b\varepsilon}, r(as + b\varepsilon) \right) \leq n.$$

Then,

$$\dim_{\text{LC}} K \leq n.$$

Proof. Take $0 < r < \min\{r_\varepsilon, \bar{s}\}$. From (3.2), we have an $r/2$ -bounded cover $\mathcal{U} = \mathcal{U}_0 \sqcup \cdots \sqcup \mathcal{U}_{\bar{n}}$ of K where each \mathcal{U}_j is $\bar{c}r/2$ -separated. By (3.3), we have, for all $U \in \mathcal{U}$,

$$\dim_{\mathbb{N}} \left(U, \frac{cas - b\varepsilon}{as + b\varepsilon}, r(as + b\varepsilon) \right) \leq n, \quad \forall 0 < s < s_0.$$

For all $j = 0, \dots, \bar{n}$, setting $U^j := \bigcup_{U \in \mathcal{U}_j} U$, from Lemma 2.6 we get

$$\dim_{\mathbb{N}} \left(U^j, \min \left\{ \frac{cas - b\varepsilon}{as + b\varepsilon}, \frac{\bar{c}}{2(as + b\varepsilon)} \right\}, r(as + b\varepsilon) \right) \leq n, \quad \forall 0 < s < s_0.$$

We may assume that $2(as + b\varepsilon) < 1$ by decreasing s and ε to be small enough. Set $s_\varepsilon := ((b(2 + c))/(ac))\varepsilon$. Then, one can easily check that, if $s > s_\varepsilon$,

$$\frac{acs - b\varepsilon}{as + b\varepsilon} \geq \frac{c}{2}.$$

Indeed, one just needs to verify that, by substituting the value of ε in terms of s_ε , the above inequality is equivalent to $s > s_\varepsilon$. Therefore, setting $c_1 = \min\{c/2, \bar{c}\}$, we have that, for all s between s_ε and s_0 and all $j = 0, \dots, \bar{n}$, we have

$$\dim_{\mathbb{N}}(U^j, c_1, r(as + b\varepsilon)) \leq n.$$

From Corollary 2.8, it then follows that

$$\dim_{\mathbb{N}} \left(K, \frac{c_1^{\bar{n}}}{5^{\bar{n}-1}}, r(as + b\varepsilon) \right) \leq n,$$

for all

$$(3.4) \quad \left(2 + \frac{3}{c_1}\right)^{\bar{n}-1} r(as_\varepsilon + b\varepsilon) \leq r(as + b\varepsilon) \leq r(as_0 + b\varepsilon).$$

In order to have $\dim_{\text{LC}} X \leq n$, it now suffices to select the constant ε so that there exists some s satisfying (3.4). This is the case if we take

$$\varepsilon \leq as_0 \left(\left(2 + \frac{3}{c_1}\right)^{\bar{n}-1} \left(\frac{b(2+c)}{c} + b \right) - b \right)^{-1}. \quad \square$$

Proof of Theorem 3.4. We shall prove the claims by checking that the assumptions of Lemma 3.6 hold. First of all, the assumption $\dim_{\text{LC}} X < \infty$ implies that there exist \bar{n} , \bar{s} , \bar{c} such that (3.2) is satisfied in both cases of the theorem.

Since by assumption the tangents are uniformly close in K , for any $\varepsilon > 0$ there exists λ_ε such that, for all $\lambda \geq \lambda_\varepsilon$ and all $x \in K$,

$$(3.5) \quad \text{Dist}_{\text{GH}}((\lambda X, x), \text{Tan}(X, x)) < \varepsilon.$$

We now consider separately the two situations assumed in the hypothesis of the theorem. In the first case, we will show that (3.3) holds. Thus, let $\varepsilon \in (0, \frac{1}{8})$ be fixed. In this case, we are assuming that we already have c and s_0 such that, for all $x \in K$ and for all $Y \in \text{Tan}(X, x)$,

$$(3.6) \quad \dim_{\text{N}}(B_Y(\star, 1), c, s) \leq n, \quad \text{for all } 0 < s < s_0.$$

By Lemma 3.5, from (3.5) and (3.6), since $\frac{1}{2} < \min\{1/\varepsilon, 1 - 2\varepsilon\}$, we have

$$\dim_{\text{N}} \left(B_{\lambda X} \left(x, \frac{1}{2} \right), \frac{cs - 2\varepsilon}{s + 2\varepsilon}, s + 2\varepsilon \right) \leq n, \quad \forall \lambda \geq \lambda_\varepsilon \text{ and } s \in (0, s_0).$$

With respect to the distance of X , the equation reads as

$$\dim_{\text{N}} \left(B_X \left(x, \frac{1}{2\lambda} \right), \frac{cs - 2\varepsilon}{s + 2\varepsilon}, \frac{1}{\lambda}(s + 2\varepsilon) \right) \leq n, \quad \forall \lambda \geq \lambda_\varepsilon \text{ and } s \in (0, s_0),$$

which is (3.3) with $r_\varepsilon := 1/\lambda_\varepsilon$ (set $r = 1/\lambda$). By Lemma 3.6, the first case then follows.

Regarding the second case, we have that

$$(3.7) \quad \dim_{\text{N}}(B_{T_x X}(\star, 1), c_x, s) \leq n$$

for some c_x depending on $x \in K$ and for all $s > 0$, since unique tangents admit dilations. Take now $x \in K$, and let $\varepsilon_x \in (0, \frac{1}{8})$ be the ε in Lemma 3.6 given by

the constants $s_0 = \infty$, $a = 4$, $b = 12$, and $c = c_X$. We abbreviate $\lambda_x := \lambda_{\varepsilon_x}$. From (3.5) and (3.7), Lemma 3.5 implies that, since $\frac{1}{2} < \min\{1/\varepsilon_x, 1 - 2\varepsilon_x\}$,

$$\dim_{\mathbb{N}} \left(B_{\lambda X} \left(x, \frac{1}{2} \right), \frac{c_X s - 2\varepsilon_x}{s + 2\varepsilon_x}, s + 2\varepsilon_x \right) \leq n \quad \forall \lambda \geq \lambda_x \text{ and all } s > 0.$$

Thus,

$$\dim_{\mathbb{N}} \left(B_X \left(x, \frac{1}{2\lambda} \right), \frac{c_X s - 2\varepsilon_x}{s + 2\varepsilon_x}, \frac{1}{\lambda}(s + 2\varepsilon_x) \right) \leq n \quad \forall \lambda \geq \lambda_x \text{ and all } s > 0.$$

Now, take $y \in B_X(x, 1/(4\lambda_x))$ so that $B_X(y, 1/(4\lambda_x)) \subseteq B_X(x, 1/(2\lambda_x))$, and hence

$$\dim_{\mathbb{N}} \left(B_X \left(y, \frac{1}{4\lambda_x} \right), \frac{c_X s - 2\varepsilon_x}{s + 2\varepsilon_x}, \frac{1}{\lambda_x}(s + 2\varepsilon_x) \right) \leq n \quad \text{for all } s > 0.$$

Multiplying the distance by $4\lambda_x$, we get

$$(3.8) \quad \dim_{\mathbb{N}} \left(B_{4\lambda_x X} (y, 1), \frac{c_X s - 2\varepsilon_x}{s + 2\varepsilon_x}, 4(s + 2\varepsilon_x) \right) \leq n \quad \text{for all } s > 0.$$

Take $r \in (0, (4\lambda_x)^{-1})$. We need to compare $B_{4\lambda_x X}(y, 1)$ and $B_{(1/r)X}(y, 1)$. Now we are going to use uniqueness of the tangents. Indeed, by the triangle inequality with $T_y X$, from (3.5) we have

$$(3.9) \quad \text{Dist}_{\text{GH}} \left((4\lambda_x X, y), \left(\frac{1}{r} X, y \right) \right) < 2\varepsilon_x,$$

since $1/r \geq 4\lambda_x \geq \lambda_x$. Again from Lemma 3.5, by (3.9) and (3.8), since we have $\frac{1}{2} < \min\{1/(2\varepsilon_x), 1 - 4\varepsilon_x\}$,

$$\dim_{\mathbb{N}} \left(B_{(1/r)X} \left(y, \frac{1}{2} \right), \frac{\frac{c_X s - 2\varepsilon_x}{s + 2\varepsilon_x} 4(s + 2\varepsilon_x) - 4\varepsilon_x}{4(s + 2\varepsilon_x) + 4\varepsilon_x}, 4(s + 2\varepsilon_x) + 4\varepsilon_x \right) \leq n;$$

that is,

$$\dim_{\mathbb{N}} \left(B_{(1/r)X} \left(y, \frac{1}{2} \right), \frac{4c_X s - 12\varepsilon_x}{4s + 12\varepsilon_x}, 4s + 12\varepsilon_x \right) \leq n.$$

Finally, multiplying the distance by r , we then get (3.3) for x replaced by any $y \in B_X(x, 1/(4\lambda_x))$, and with $r_\varepsilon := 1/(4\lambda_x)$, $s_0 = \infty$, $a = 4$, $b = 12$, and $c = c_X$. By Lemma 3.6, we then have

$$\dim_{\text{LC}} \left(B_X \left(x, \frac{1}{4\lambda_x} \right) \right) \leq n.$$

By compactness of K , there exists a finite collection of balls $\{B_X(x, 1/(4\lambda_x))\}_x$ covering K . Using Corollary 2.8, we then conclude that $\dim_{\text{LC}} K \leq n$, which finishes the proof of the second case. \square

Proof of Theorem 1.2. Since Y is relatively compact, the set $K := \text{cl } Y$ is compact. Note that $\dim_{\text{N}} K = \dim_{\text{N}} Y$. By Theorem 3.4 with assumption (ii),

$$\dim_{\text{N}} Y \leq \sup\{\dim_{\text{N}} T_x X \mid x \in K\}.$$

Take now a point x in the interior of Y . Then, by Corollary 2.19, we have $\dim_{\text{N}} T_x Y \leq \dim_{\text{N}} Y$. \square

3.2. Assouad dimension and uniformly close tangents. Let us now state the analog of the first part of Theorem 3.4 for Assouad dimension.

Theorem 3.7. *Let X be a metric space with uniformly close tangents. Let $C > 1$ and $\alpha \geq 0$. Assume that, for all $x \in X$ and for all $(Y, \gamma) \in \text{Tan}(X, x)$, the Assouad dimension of Y is less than or equal to α with constant C . Then, there exists $R > 0$ such that*

$$\dim_{\text{A}}(B(x, R)) \leq \alpha, \quad \text{for all } x \in X.$$

In fact, Theorem 3.7 is an immediate consequence of the following proposition where we only need to have a cover of the balls in the tangents centered at the base point.

Proposition 3.8. *Let X be a metric space with uniformly close tangents. Assume that there are constants $C > 1$ and $\alpha \geq 0$ such that, for all $x \in X$, for all $(Y, \gamma) \in \text{Tan}(X, x)$, and for all $\delta \in (0, 1)$, we need at most $C(2/\delta)^\alpha$ balls of radius $\delta/2$ to cover the ball $B(\gamma, 2)$. Then, there exists a constant $R' > 0$ such that, for all $\beta > \alpha$, there exists $C' > 1$ for which*

$$\dim_{\text{A}}(B(x, R'), C', R') \leq \beta, \quad \text{for all } x \in X.$$

The analog of the second part of Theorem 3.4 is the following.

Theorem 3.9. *Let X be a metric space with uniformly close unique tangents in a compact subset $K \subset X$. Then,*

$$\dim_{\text{A}} K \leq \sup\{\dim_{\text{A}} Y \mid x \in K, Y \in \text{Tan}(X, x)\}.$$

Again, as for the Nagata dimension, we have to make one of the two assumptions on the tangents, given in the two previous theorems, in order to be able to get a bound on the Assouad dimension of the space from the Assouad dimensions of its tangents. The following example, similar to Example 3.3, shows that the assumptions are necessary.

Example 3.10. In Example 3.3, we defined the space X as a subset of the Euclidean plane. The construction had three stages: we first defined S_n , then, by using it, E_n , and finally X . Here, we replace S_n by a set $\{0, 1\}^n$. We want the

points in this set to be equidistant, and so we consider $\{0, 1\}^n \subset \mathbb{R}^n$ with the maximum norm.

We define $E_n \subset \mathbb{R}^n$ as

$$E_n = \left\{ \sum_{i=n}^{\infty} a_i 2^{-i^2} \mid a_i \in \{0, 1\}^n \text{ if } i \text{ odd, and } a_i \in \{0, 1\} \text{ if } i \text{ even} \right\},$$

and using it, we define $X \subset \mathbb{R}^{\mathbb{N}}$ as

$$X = \text{cl} \left(\bigcup_{n=1}^{\infty} (E_n + 2^{-n^2}) \right),$$

where the embedding of different dimensional \mathbb{R}^n to $\mathbb{R}^{\mathbb{N}}$ are understood by identifying $x \in \mathbb{R}^n$ with $(x, 0, 0, \dots) \in \mathbb{R}^{\mathbb{N}}$. We equip X with the supremum distance of $\mathbb{R}^{\mathbb{N}}$.

We have

$$\text{Tan}(E_n, x) = \{\{0, t\}^n \mid t \in [0, \infty)\} \cup \{\{0, t\} \mid t \in [0, \infty)\}$$

for all $n \in \mathbb{N}$ and $x \in E_n$, and

$$\text{Tan}(X, 0) = \{\{0, t\} \mid t \in [0, \infty)\}.$$

As in Example 3.3, we have uniformly close tangents essentially because

$$\{\{0, t\} \mid t \in [0, \infty)\} \subset \text{Tan}(E_n, x)$$

for all $x \in X$. This set of tangents takes care of the scales where other sets of the form $\{0, 1\}^n$ cannot yet be seen.

Take any $\alpha > 0$. For any $x \in X$ and any $Y \in \text{Tan}(X, x)$, the tangent Y has only finite number of points, and hence $\dim_A Y \leq \alpha$. However, for any $n \in \mathbb{N}$, the set $\{0, 1\}^n$ needs 2^n balls of radius $r < 1$ to cover it. Therefore, for any $C > 0$ there are arbitrarily small balls $B(x, r)$ in X that cannot be covered by less than $C2^\alpha$ balls of radius $r/2$. Thus, $\dim_A X \geq \alpha$. This was true for any $\alpha > 0$, and so,

$$\dim_A X = \infty \quad \text{and} \quad \sup\{\dim_A Y \mid x \in X, Y \in \text{Tan}(X, x)\} = 0.$$

Hence, the two assumptions in Theorem 3.7 and in Theorem 3.9 are necessary.

Before proving Proposition 3.8 we provide a lemma.

Lemma 3.11. *Assume that we have $\text{Dist}_{\text{GH}}((X, x), (Y, y)) < \delta/4$ for some $\delta \in (0, 1)$, and that, for some $L \in \mathbb{N}$, we have that the ball $B_Y(y, 2)$ can be covered with L balls of radius $\delta/2$. Then, the ball $B_X(x, 1)$ can be covered with L balls of radius δ .*

Proof. Set $\varepsilon = \delta/4$. Since (X, \mathcal{x}) and (Y, \mathcal{y}) have distance $< \varepsilon$, we can see them as a subset of a metric space Z such that $d(\mathcal{x}, \mathcal{y}) \leq \varepsilon$, $B_X(\mathcal{x}, 1/\varepsilon) \subseteq B_Z(Y, \varepsilon)$, and $B_Y(\mathcal{y}, 1/\varepsilon) \subseteq B_Z(X, \varepsilon)$.

By assumption, there are points $\mathcal{y}_1, \dots, \mathcal{y}_L \in Y$ such that

$$B_Y(\mathcal{y}, 2) \subseteq \bigcup_{j=1}^L B_Y\left(\mathcal{y}_j, \frac{\delta}{2}\right).$$

We may assume that $d(\mathcal{y}, \mathcal{y}_j) \leq 2 + \delta/2$. Since $2 + \delta/2 < 1/\varepsilon$, for all j there is $x_j \in X$ with $d_Z(x_j, \mathcal{y}_j) < \varepsilon$. We claim that

$$B_X(\mathcal{x}, 1) \subseteq \bigcup_{j=1}^L B_X(x_j, \delta).$$

Indeed, pick $x' \in B_X(\mathcal{x}, 1)$. Then, there is $\mathcal{y}' \in B_Y(\mathcal{y}, 1 + 2\varepsilon)$ such that $d_Z(x', \mathcal{y}') < \varepsilon$. Since $1 + 2\varepsilon < 2$, there exists j such that $\mathcal{y}' \in B_Y(\mathcal{y}_j, \delta/2)$. Therefore, $d(x', x_j) \leq d(x', \mathcal{y}') + d(\mathcal{y}', \mathcal{y}_j) + d(\mathcal{y}_j, x_j) < \varepsilon + \delta/2 + \varepsilon = \delta$. \square

Proof of Proposition 3.8. Fix $\beta > \alpha$. Let $\delta \in (0, 1)$ be such that we have $(1/\delta)^\beta = C(4/\delta)^\alpha =: L$. By assumption, there exists $\lambda_0 > 1$ such that, for all $\lambda > \lambda_0$ and for all $\mathcal{x} \in X$, there exists $(Y, \mathcal{y}) \in \text{Tan}(X, \mathcal{x})$ such that

$$\text{Dist}_{\text{GH}}((\lambda X, \mathcal{x}), (Y, \mathcal{y})) < \frac{\delta}{4}.$$

Let R and r such that $0 < r < R < 1/\lambda_0$. Fix N such that

$$\delta^N \leq \frac{r}{R} \leq \delta^{N-1}.$$

We will apply Lemma 3.11 N times. First, since $1/R > \lambda_0$, by Lemma 3.11, we have that, in the contracted metric space $(1/R)X$, we need at most $L := C(4/\delta)^\alpha$ balls of radius δ to cover $B_{(1/R)X}(\mathcal{x}, 1)$. Such balls are of the form

$$B_{(1/R)X}(x', \delta) = B_{(1/(\delta R))X}(x', 1).$$

Each of these balls needs at most L balls of radius δ to cover it in the space $(1/(\delta R))X$. Iterating, we need at most L^N balls of radius δ , with respect to the distance of $(1/(\delta^{N-1}R))X$, to cover

$$B_{(1/R)X}(\mathcal{x}, 1) = B_X(\mathcal{x}, R).$$

Such sets are balls of radius $\delta^N R$ for X . In other words, we covered $B_X(\mathcal{x}, R)$ with balls of radius r , since $r > \delta^N R$. The number of these balls is bounded by

$$L^N = \left(\frac{1}{\delta}\right)^{\beta N} = \left(\frac{1}{\delta}\right)^\beta \left(\frac{1}{\delta^{N-1}}\right)^\beta \leq \frac{1}{\delta^\beta} \left(\frac{R}{r}\right)^\beta.$$

Applying Lemma 2.5 with $C_\beta := 1/\delta^\beta$, we conclude. \square

Proof of Theorem 3.9. Denote by $(T_x X, \star)$ the unique element in $\text{Tan}(X, x)$, for $x \in K$. Take

$$\beta > \alpha > \sup\{\dim_A T_x X \mid x \in K\}.$$

Fix some $x \in K$, and let $1 < C_x < \infty$ be such that, for any $0 < r < R < \infty$ and $y \in T_x X$, we need at most $C_x(R/r)^\alpha$ balls of radius r to cover the ball $B_{T_x X}(y, R)$. Let $\delta_x \in (0, 1)$ be such that $(1/(16\delta_x))^\beta = C_x(4/\delta_x)^\alpha =: L$. By assumption, there exists $\lambda_x > 1$ such that, for all $\lambda > \lambda_x$ and for all $y \in K$, we have

$$\text{Dist}_{\text{GH}}((\lambda X, y), (T_y X, \star)) < \frac{\delta_x}{4}.$$

We aim to show that, for any $z \in B(x, 1/(4\lambda_x)) \cap K$ and $0 < r < 1/(4\lambda_x)$, we need at most L balls of radius $8\delta_x r$ to cover the ball $B(z, r)$. Fix such z and r . By the definition of C_x , the ball $B_{T_x X}(\star, 2)$ needs at most L balls of radius $\delta_x/2$ to cover it. Since $2\lambda_x > \lambda_x$, we have

$$\text{Dist}_{\text{GH}}((2\lambda_x X, x), (T_x X, \star)) < \frac{\delta_x}{4},$$

and so by Lemma 3.11, we need at most L balls of radius $\delta_x/(2\lambda_x)$ to cover the ball $B(x, 1/(2\lambda_x))$. Since $B(z, 1/(4\lambda_x)) \subset B(x, 1/(2\lambda_x))$, the same collection of balls covers $B(z, 1/(4\lambda_x))$.

Using the fact that the tangent of X at z is unique, and the triangle inequality, we get

$$\begin{aligned} & \text{Dist}_{\text{GH}}((r^{-1}X, z), (8\lambda_x X, z)) \\ & \leq \text{Dist}_{\text{GH}}((r^{-1}X, z), (T_z X, \star)) + \text{Dist}_{\text{GH}}((T_z X, \star), (8\lambda_x X, z)) < \frac{\delta_x}{2}. \end{aligned}$$

Since the ball $B_{8\lambda_x X}(z, 2) = B_X(z, 1/(4\lambda_x))$ needed at most L balls of the type $B_{8\lambda_x X}(x', 4\delta_x) = B_X(x', \delta_x/(2\lambda_x))$ to cover it, by Lemma 3.11 we need no more than L balls of radius $8\delta_x r$ to cover the ball $B(z, r)$.

We now continue as in the proof of Proposition 3.8. Let R and r such that $0 < r < R < 1/(4\lambda_x)$. Fix N such that

$$(16\delta_x)^N \leq \frac{r}{R} \leq (16\delta_x)^{N-1}.$$

First, since $R < 1/(4\lambda_x)$, by the above considerations we need at most L balls of radius $8\delta_x R$ to cover the set $B(x, R) \cap K$, and hence at most L balls of radius $16\delta_x R$ centered at $B(x, R) \cap K$. Each of these balls, since they are all centered at $B(x, R) \cap K$, needs at most L balls of radius $(16\delta_x)^2 R$ centered at $B(x, R) \cap K$ to

cover it. Continuing inductively N times, we obtain a cover of $B(x, R) \cap K$ with at most L^N balls of radius $(16\delta_x)^N R$. Again, we estimate the number

$$L^N = (16\delta_x)^{-\beta N} = (16\delta_x)^{-\beta} ((16\delta_x)^{1-N})^\beta \leq (16\delta_x)^{-\beta} \left(\frac{R}{r}\right)^\beta.$$

Thus, $\dim_A(B(x, 1/(4\lambda_x)) \cap K) \leq \beta$. By compactness of K , we have $\dim_A K \leq \beta$. Letting α and β tend to $\sup\{\dim_A T_x X \mid x \in K, \text{Tan}(X, x) = \{T_x X\}\}$ then finishes the proof. \square

Proof of Theorem 1.4. Since Y is relatively compact, $K := \text{cl } Y$ is compact. Note that $\dim_A K = \dim_A Y$. By Theorem 3.9,

$$\dim_A Y \leq \sup\{\dim_A T_x X \mid x \in K\}.$$

Take now a point x in the interior of Y . Then, by Corollary 2.17, we have $\dim_A T_x Y \leq \dim_A Y$. \square

4. NAGATA DIMENSION OF CARNOT GROUPS AND EQUIREGULAR SUB-RIEMANNIAN MANIFOLDS

Let M be a smooth manifold. Let Δ be a smooth subbundle of the tangent bundle of M . Denote by $\Gamma(\Delta) \subset \text{Vec}(M)$ the $C^\infty(M)$ -module of the smooth sections of Δ . One says that Δ satisfies the *bracket-generating condition* if

$$\bigcup_{j \in \mathbb{N}} \Delta_q^j = T_q M, \quad \forall q \in M,$$

where

$$(4.1) \quad \Delta_q^j := \text{span}\{[X_1, [X_2, \dots, [X_{j-1}, X_j]]](q) \mid X_i \in \Gamma(\Delta)\} \subseteq T_q M,$$

for all $q \in M, j \in \mathbb{N}$.

A sub-Riemannian manifold is a triple (M, Δ, \mathbf{g}) , where M is a connected smooth manifold, Δ is a bracket-generating subbundle of the tangent bundle of M , and \mathbf{g} is a Riemannian metric tensor restricted to Δ .

A sub-Riemannian manifold has a natural structure of metric space, where the distance is the so-called *Carnot-Carathéodory distance*

$$d_{\text{cc}}(p, q) = \inf \left\{ \int_0^T \sqrt{\mathbf{g}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt \mid \gamma : [0, T] \rightarrow M \text{ is a Lipschitz curve, } \right. \\ \left. \gamma(0) = p, \gamma(T) = q, \dot{\gamma}(t) \in \Delta_{\gamma(t)} \text{ almost everywhere in } [0, T] \right\}.$$

As a consequence of Chow-Rashevsky Theorem, such a distance is always finite and induces on M the original topology.

The sub-Riemannian manifold is called *equiregular* if the dimensions of the spaces Δ_q^i , $i \in \mathbb{N}$, as defined in (4.1), do not depend on the point q .

Very particular sub-Riemannian manifolds are the Carnot groups. Let us briefly recall that a *Carnot group* is a stratified Lie group endowed with a left-invariant Carnot-Carathéodory metric, where the subbundle is the first stratum of the Lie algebra. See [LD13] for an introduction to Carnot groups from a point of view of metric geometry.

Theorem 4.1 (Mitchell-Bellaïche [Bel96]). *Let M be an equiregular and sub-Riemannian manifold equipped with its Carnot-Carathéodory distance d_{cc} . Then, at every point, the tangent space is a Carnot group of same topological dimension. Moreover, the convergence to the tangents is uniform on compact sets.*

We will be able to deduce the local Nagata dimension of an equiregular sub-Riemannian manifold, since we know the Nagata dimension of Carnot groups, by a result originally proved by Urs Lang and the first-named author. Many thanks go to Lang for giving the permission to include here a short proof.

Theorem 4.2 (Lang and Le Donne). *The Nagata dimension of a Carnot group equals its topological dimension.*

Proof. Let G be a Carnot group of topological dimension n . Let d_{cc} be the Carnot-Carathéodory distance, and let d_R be a left-invariant Riemannian distance. It follows from Lemma 2.10 (see Example 2.11) that (G, d_R) has linearly controlled dimension n , with, say, parameters c, \bar{s} . Since d_{cc} and d_R give the same topology on G , we can fix a scale $s_0 \in (0, \bar{s}]$ such that $B^{d_R}(e, cs_0) \subseteq B^{d_{cc}}(e, 1)$, where e is the identity element of G ; furthermore, there exists $\rho > 0$ such that $B^{d_{cc}}(e, \rho) \subseteq B^{d_R}(e, s_0)$. Since both d_{cc} and d_R are left-invariant, this means that $d_R(p, q) \leq cs_0$ implies $d_{cc}(p, q) \leq 1$, and $d_{cc}(p, q) \leq \rho$ implies $d_R(p, q) \leq s_0$. Now, let \mathcal{B} be a cs_0 -bounded cover (with s_0 -multiplicity at most $n+1$) of (G, d_R) . With respect to d_{cc} , \mathcal{B} is 1-bounded and has ρ -multiplicity at most $n+1$. For every $\lambda > 0$, there exists a dilation of (G, d_{cc}) by the factor λ , that is, a bijection $\delta_\lambda: G \rightarrow G$ such that $d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p, q)$ for all $p, q \in G$. If we apply δ_λ to the members of \mathcal{B} , we obtain a λ -bounded cover of (G, d_{cc}) with $\lambda\rho$ -multiplicity at most $n+1$. Thus, (G, d_{cc}) has Nagata dimension at most n , with constant $1/\rho$. On the other hand, from [LS05, Theorem 2.2], we have $\dim_N(G, d_{cc}) \geq n$. \square

Proof of Corollary 1.3. Carnot-Carathéodory distances are locally doubling (a fact that is proved in [NSW85]; however, notice that Theorem 1.4 gives an alternative proof). From [LS05] (or from Theorem 1.1), we deduce that the Nagata dimension of a compact set of a sub-Riemannian manifold is finite. Hence, from Theorem 1.2, together with Theorem 4.1 and Theorem 4.2, we then have Corollary 1.3. \square

Remark 4.3. In the case that the sub-Riemannian manifold is not equiregular, one can still use the fact that the set of regular points for the subbundle is an

open dense subset of the manifold. Here, a point p is called *regular* if it has a neighborhood on which the subbundle is equiregular. Hence, we have that there exists a neighborhood of p whose Nagata dimension equals the topological dimension of the manifold.

Remark 4.4. Probably the most famous Carnot-Carathéodory space that is not an equiregular sub-Riemannian manifold is the Grushin plane. In this example, the dimension of Δ_q varies with q . Also, in this case the Nagata dimension is equal to the topological one. Indeed, recently Meyerson gave simple examples of quasi-symmetric maps between the Grushin plane and the Euclidean plane. In the usual coordinates, an example of such maps is $F(x, y) = (x|x|, y)$ (see [Mey11]). On the other hand, Lang and Schlichenmaier in [LS05] proved that the Nagata dimension is a quasi-symmetric invariant. Therefore, the Nagata dimension of the Grushin plane is two, like the topological dimension.

On compact sets, a Carnot-Carathéodory distance d_{cc} and a Riemannian distance d_R on G satisfy the inequalities

$$(4.2) \quad k(d_{cc})^m \leq d_R \leq d_{cc},$$

for some $m \in \mathbb{N}$ and $k \in (0, 1)$ (see [NSW85]). Because snowflaking a distance preserves the Nagata dimension, it is natural to wonder if partial snowflaking as in (4.2) preserves it as well, giving an alternative method of proof for Theorem 4.2 and Corollary 1.3. We give an example showing that this is not the case.

Example 4.5. We introduce a topological space X with two proper distances d_1 and d_2 with the property that $d_1^2 \leq d_2 \leq d_1$, but $\dim_N(X, d_1) \neq \dim_N(X, d_2)$.

The space X is of the form $\{p_{n,j} \mid n \in \mathbb{N}, j = 0, \dots, n\}$ endowed with the discrete topology. The distances are

$$d_1(p_{n,j}, p_{n',j'}) = \begin{cases} \left| \frac{1}{2^n} - \frac{1}{2^{n'}} \right|, & \text{if } n \neq n', \\ \frac{1}{2^n}, & \text{if } n = n', j \neq j' \\ 0, & \text{if } n = n', j = j' \end{cases}$$

and

$$d_2(p_{n,j}, p_{n',j'}) = \begin{cases} \left| \frac{1}{2^n} - \frac{1}{2^{n'}} \right|, & \text{if } n \neq n', \\ \frac{|j - j'|}{n2^n}, & \text{if } n = n'. \end{cases}$$

One can easily check that these are distance functions, which satisfy $d_1^2 \leq d_2 \leq d_1$, and that $\dim_N(X, d_1) = 0$ but $\dim_N(X, d_2) \neq 0$. The only calculation that is not completely straightforward is that $\dim_N(X, d_2) \neq 0$. For doing this, one assumes that the dimension is 0 with respect to some constant c . Then, we fix $n \in \mathbb{N}$, and let $s = 1/(n2^n)$. Take a cs -bounded cover with s -multiplicity at most 1. This last

property implies that $p_{n,0}$ and $p_{n,1}$, which have distance s , need to be contained in the same element U of the cover. Likewise, $p_{n,j} \in U$, for all $j = 0, \dots, n$. Thus, $1/2^n \leq \text{diam } U \leq cs = c1/(n2^n)$, and so $c > n$ for any $n \in \mathbb{N}$, a contradiction.

5. ASSOUD DIMENSION BOUNDS NAGATA DIMENSION

Denoting by $\dim X$ the topological dimension of a metric space X and by $\dim_{\text{H}} X$ its Hausdorff dimension, recall that one has the chain of inequalities

$$\dim X \leq \dim_{\text{H}} X \leq \dim_{\text{A}} X.$$

From the bound proven in this section, we will conclude that we also have the inequalities

$$\dim X \leq \dim_{\text{N}} X \leq \dim_{\text{A}} X,$$

where the first inequality is obtained in [LS05]. In [LS05], Lang and Schlichenmaier also proved the inequality $\dim_{\text{N}} X \leq 3^{\dim_{\text{A}} X} - 1$. We optimally improve it. We should also point out that, in general, there is no relation between \dim_{N} and \dim_{H} . As examples, on the one hand we have $\dim_{\text{N}} \mathbb{Q} = 1 > 0 = \dim_{\text{H}} \mathbb{Q}$, and on the other hand $\dim_{\text{N}}(\mathbb{R}, |\cdot|^{1/2}) = 1 < 2 = \dim_{\text{H}}(\mathbb{R}, |\cdot|^{1/2})$. After the proof, in Theorem 5.1 we show how the argument can be modified to give a bounded-scale version of the same result.

Proof of Theorem 1.1. Let X be a metric space. We prove $\dim_{\text{N}} X \leq \dim_{\text{A}} X$. Without loss of generality, we may assume $\dim_{\text{A}} X < \infty$. Take α so that we have $\dim_{\text{A}} X < \alpha < \lfloor \dim_{\text{A}} X \rfloor + 1$. From the definition of the Assouad dimension, we know there is some constant $C > 0$ such that, for any $x \in X$ and $0 < r < R < \infty$,

$$(5.1) \quad \text{We need at most } C \left(\frac{R}{r}\right)^{\alpha} \text{ balls} \\ \text{of radius } r \text{ to cover the set } B(x, R) \cap X.$$

We shall fix $r \ll R$ to be determined later in terms of C and α only. The idea of the proof is the following. First, decompose $X =: X_0$ as

$$X_0 = X_1 \sqcup Y_1^1 \sqcup Y_2^1 \sqcup \dots$$

with the properties that $\text{diam}(Y_n^1) < 2R$, $\text{dist}(Y_n^1, Y_m^1) > r$, for all $n \neq m$, and that all balls $B(x, R) \cap X_1$ (with x in X_1) need at most $C_1(R/r)^{\alpha-1}$ balls of radius r to cover them. Here, C_1 is a constant depending only on α and C . Hence, roughly speaking, the subset X_1 has codimension one on scales R and r . Then, we will iterate the decomposition $X_{k-1} = X_k \sqcup Y_1^k \sqcup Y_2^k \sqcup \dots$ with similar properties for Y_n^k . As soon as $k > \alpha$ and both R and r are properly chosen, the subset X_k is empty. The collection $\{Y_n^k \mid k = 1, \dots, \lfloor \alpha \rfloor + 1, n = 1, \dots, N_k\}$ gives a cover showing that the Nagata dimension is less than α .

Take $R > 0$, and let $\{x_i\}_{i=1}^{N_1} \subset X$ be a maximal $R/4$ -separated net of points. (It might be that $N_1 = \infty$. However, recall that metric spaces with finite Assouad dimension are separable, and hence separated nets are countable.)

We notice that, in particular, the balls $B(x_i, R/2)$ cover the set X , and we take $0 < r < R/4$. For each $x_n \in \{x_i\}$, there exists by (5.1) a collection $\mathcal{B} = \{B(y_i, r)\}$ of at most $C(R/r)^\alpha$ balls covering the larger ball $B(x_n, R)$. Let $k \in \mathbb{N}$ be such that $kr < R/2 \leq (k+1)r$. Consider the annular regions

$$A_{n,i} := B(x_n, R - ir) \setminus B(x_n, R - (i+1)r).$$

The collection $\{A_{n,i}\}_{i=0}^{k-1}$ is disjointed, and

$$\bigcup_{i=0}^{k-1} A_{n,i} = B(x_n, R) \setminus B(x_n, R - kr) \subset B(x_n, R) \setminus B\left(x_n, \frac{R}{2}\right).$$

Observe that any ball $B \in \mathcal{B}$ intersects at most three of the annular regions $A_{n,i}$.

Therefore, there exists some $0 \leq k_n \leq k-1$ such that the annulus A_{n,k_n} meets at most $(3/k)C(R/r)^\alpha$ balls of the collection \mathcal{B} . Notice that we can bound

$$\frac{3}{k}C\left(\frac{R}{r}\right)^\alpha \leq \frac{3r}{R/2-r}C\left(\frac{R}{r}\right)^\alpha \leq 12C\left(\frac{R}{r}\right)^{\alpha-1}.$$

Then, in particular, we have the following:

$$(5.2) \quad \text{For all } n, \text{ the number of balls needed} \\ \text{to cover } A_{n,k_n} \text{ is at most } 12C\left(\frac{R}{r}\right)^{\alpha-1}.$$

Set $X_1 := \bigcup_n A_{n,k_n}$. Now, take a point $x \in X_1$. We claim that

$$(5.3) \quad \text{The ball } B(x, R) \text{ intersects at most } C16^\alpha \text{ annular regions } A_{i,k_i}.$$

Indeed, the number of annular regions that $B(x, R)$ meets is less than the number of balls in \mathcal{B} that it meets. Hence, we need to estimate the number of elements of the net $\{x_i\}_i$ in $B(x, 2R)$. By (5.1), we need at most $C(2R/(R/8))^\alpha$ balls of radius $R/8$ to cover the ball $B(x, R)$. Since the elements of the net are $R/4$ separated, any two of them are in different such balls of radius $R/8$. Hence, the number of x_i inside $B(x, 2R)$ is at most

$$C\left(\frac{2R}{R/8}\right)^\alpha = C16^\alpha.$$

Therefore, by combining (5.2) and (5.3), we see we need at most $C_1(R/r)^{\alpha-1}$ balls of radius r to cover $X_1 \cap B(x, R)$, where $C_1 := C^216^\alpha12$.

Define, for all n ,

$$Y_n^1 := B(x_n, R - (k_n + 1)r) \setminus \left(X_1 \cup \bigcup_{i=1}^{n-1} B(x_i, R - k_i r) \right).$$

What we have now obtained is a decomposition of X into disjointed collection $\{Y_n^1\}_{n=1}^{N_1} \cup \{X_1\}$ with Y_n^1 having diameter less than $2R$ and the property that $\text{dist}(Y_i^1, Y_j^1) \geq r$ for any $i \neq j$.

Repeating the above argument, we can decompose X_1 into disjointed collection $\{Y_n^2\}_{n=1}^{N_2} \cup \{X_2\}$, with Y_n^2 again having diameter less than $2R$, with the property that $\text{dist}(Y_i^2, Y_j^2) \geq r$ for any $i \neq j$, and with X_2 such that, for any point $x \in X_2$, we need at most $C_2(R/r)^{\alpha-2}$ balls of radius r to cover the set $X_2 \cap B(x, R)$, where C_2 depends only on C and α .

We continue this for $m = \lfloor \alpha \rfloor + 1$ steps so that we have a decomposition of X into disjointed collection

$$\bigcup_{k=1}^m \{Y_n^k\}_{n=1}^{N_k} \cup \{X_m\},$$

with all Y_n^k having diameter less than $2R$ and $\text{dist}(Y_i^k, Y_j^k) \geq r$ for any $i \neq j$. But now, at the last iteration (step m), when we have arrived to (5.2) we notice that there exists an annular region that intersects at most

$$12C_{m-1} \left(\frac{R}{r} \right)^{\alpha-m}$$

balls of the cover. Provided that we have chosen the ratio r/R to be small enough from the beginning, this means that the annular region is empty, since $\alpha - m < 0$. The conclusion is that $X_m = \emptyset$.

Hence, we have

$$X = \bigcup_{k=1}^m \bigcup_{n=1}^{N_k} Y_n^k.$$

Now, given any $x \in X$, for any $k = 1, \dots, m$, the ball $B(x, r/2)$ intersects at most one set from each collection $\{Y_n^k\}_{n=1}^{N_k}$. Thus, $\dim_N X \leq \lfloor \alpha \rfloor \leq \dim_A X$. \square

The previous proof generalizes to the case of local dimensions of metric spaces that admit well-ordered separated nets. For example, this is the case in separable metric spaces. Nevertheless, if we assume the Axiom of Choice, any net can be well ordered.

Theorem 5.1. *Let X be a metric space. Let $\alpha \geq 0$, $C > 0$, and $R > 0$. Assume that, for all $x \in X$, we have $\dim_A(B(x, R), C, R) \leq \alpha$. Then, $\dim_{LC} X \leq \alpha$.*

Proof. The strategy of the proof is very similar to the one of Theorem 1.1. Hence, we only explain the differences. Instead of having a countable maximal $R/4$ -separated net of points $\{x_i\}_i$, we now have just a maximal $R/4$ -separated net $\{x_j\}_{j \in J}$, where J is a general set of indices.

By the Well-Ordering Theorem, the set J admits a total order such that every nonempty subset of J has a least element for this ordering. (This result is also known as Zermelo's theorem. It follows easily from Zorn's lemma, and it is actually equivalent to it.) For all $j \in J$, we select an annulus A_{j,k_j} as in the proof of Theorem 1.1. Then, we set $X := \bigcup_{j \in J} A_{j,k_j}$, and

$$Y_j^1 := B(x_j, R - (k_j + 1)r) \setminus \left(X_1 \cup \bigcup_{i < j} B(x_i, R - k_i r) \right).$$

The only nontrivial property to check is that $\{Y_j^1\}_{j \in J}$ together with X_1 is a cover. Pick $x \in X$. Set

$$E_x := \{j \in J \mid x \in B(x_j, R - k_j r)\}.$$

The set E_x is nonempty, since $R - k_j r > R/2$. By the well-ordering, we have a least element $j_x \in E_x$. Thus, x is in $B(x_{j_x}, R - k_{j_x} r)$ but not in any of the $B(x_i, R - k_i r)$ for $i < j_x$. If x is not in X_1 (and hence, not in $A_{j_x, k_{j_x}}$), we have that x is in $Y_{j_x}^1$.

The rest of the proof is exactly the same. □

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