Cosmic Perturbation Theory and Inflation

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Abstract

In this thesis I have reviewed the basic theory of single scalar field cosmological inflation and cosmological perturbation theory. I go through the dynamics of the background Friedmann–Robertson–Walker -spacetime and then study the evolution of perturbations around the background. Cosmological perturbations in general are gauge dependent, so I introduce the well known gauge invariant variables, the Mukhanov-Sasaki variable q and the comoving curvature perturbation $\mathcal{R}$. I calculate the scalar and tensor perturbation power spectra and the spectral parameters finally going through two simple examples, the power law inflation and the Higgs inflation.

Tiivistelmä

Tässä pro gradu -tutkielmassa olen käynyt läpi yhden skalaarikentän synnyttämän kosmisen inflaation teoriaa. Tätä varten olen opiskellut kosmista häiriöteoriaa joka tutkii Friedmann–Robertson–Walker -avaruusajan ympärille kehitettyjen häiriöiden kehitystä inflaation aikana. Kosmiset häiriöt riippuvat mitan valinnasta, joten olen esitellyt hyvin tunnetut mittainvariantit muuttujat, Mukhanovin-Sasakin muuttujan q sekä mukanaliikkuvan kaarevuushäiriön $\mathcal{R}$. Lasken skalaari- ja tensorihäiriöiden tehospektrit sekä relevantit spektriparametrit. Lopuksi käyn läpi kaksi yksinkertaista esimerkkiä, potenssilaki-inflaatio sekä Higgs-inflaatio.

Contents

1 Introduction

The history of the Universe is considered as a thermal history: temperature rises when going backwards towards the birth of the Universe. These following are the several more or less hypothetical epochs in the history of the Universe, from which the four latter are rather well established:

- Planck epoch at temperature corresponding to Planck energy $T \sim 10^{19}\,\mathrm{GeV}$ in the very early Universe at time $t \sim 10^{-43}\,\mathrm{s}$. Quantum gravity is needed to describe conditions at this time.
- Baryogenesis at somewhere between temperatures of $10^{16} < T < 10^{2}\,\mathrm{GeV}$ at $t \sim 10^{-35}\,s$. Asymmetry between matter and antimatter formed.
- Electroweak phase transition when the temperature was of the order of the mass of the weak gauge bosons, $T \sim 10^{2}\,\mathrm{GeV}$. Particles acquired their masses.
- Quark-hadron transition with temperature $T \sim 1\,\mathrm{GeV}$ corresponding to nucleon mass. Protons and neutrons formed. Universe was about 10^{-5} seconds old.
- Nucleosynthesis ~ 3 minutes after Big Bang at nuclear levels. Atomic nuclei and light elements such as deuterium, helium and lithium formed at temperatures of $T \sim 0.1\,\mathrm{MeV}$.
- Recombination at $T \sim 0.1\,\mathrm{eV}$, $t \sim 10^{5}\,\mathrm{y}$ at atomic levels. Photons were able to travel freely when atoms formed from nuclei and electrons. Cosmic microwave background was formed.
- Formation of first stars, galaxies and cosmic large scale structure much after recombination.
- Present day at $T = 2.75\,\mathrm{K} = 10^{-3}\,\mathrm{eV}$. Accelereting expansion of the universe suggesting the beginning of a dark energy dominated era.

The focus of this thesis is in the *inflationary* epoch somewhere at the time between the birth of the Universe and electroweak phase transition. Inflationary epoch was invented to solve some fundamental problems arising from the basic Big Bang -model, but it has proven to have some extremely vital features in addition, such as the capability of explaining the origin of the primordial seeds for the cosmic large scale structure and fluctuations in the cosmic microwave backround.

The inflationary scenario says that during some short epoch in the very early Universe the non-zero vacuum energy density of some unknown field dominated the energy density of all other forms of energy, such as matter or radiation. In the simplest case the inflation is caused by a cosmological constant. A more complete scenario is inflation driven by a slowly rolling scalar field in a potential well. During the inflatory phase the scale factor of the Universe grew exponentially so that initially small patches of space could have been stretched bigger than the current observable Universe.

A mathematical tool called cosmological perturbation theory is essential in order to study the extremely rich phenomea of the inflationary scenario and it's connection to

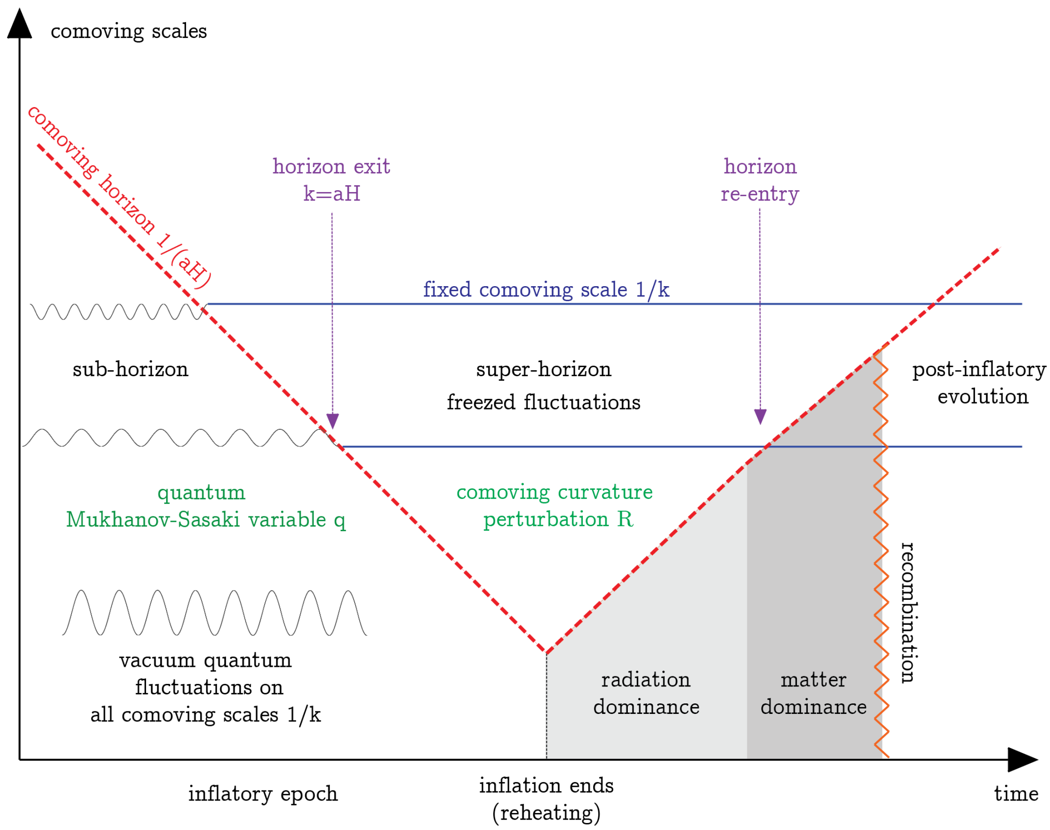

Figure 1: Schematic picture of the evolution of perturbations during inflation. This thesis focuses on the details of this picture in the inflatory epoch: generation of curvature fluctuations from vacuum and the freeze-out of fluctuations outside the horizon.

present day observations. In this perturbative analysis one studies the evolution of small fluctuations around a homogeneous and isotropic background universe. The fluctuations are thought to origin from vacuum quantum fluctuations during inflation and then being stretched to cosmological scales due to exponential expansion. These amplified quantum fluctuations are then thought to transform into classical spacetime/density fluctuations in the early universe.

The detailed mathematics involved in the study of the evolution of the perturbations from the inflationary epoch until today is quite complicated. In particular the above mentioned gauge-dependence, or the dependence of the chosen coordinate system, complicates the things. The outline is to form a gauge invariant perturbation variable as a linear combination of the inflaton fluctuations and metric fluctuations. This is the so called Mukhanov-Sasaki variable q which can be quantized when it's modes are deep inside the inflationary horizon. The variable q is then closely related to the comoving curvature perturbation $\mathcal{R}$ which has a property of staying constant once it's stretched into cosmological scales. As the name implies, the variable $\mathcal{R}$ is again related to the fluctuations of the spatial curvature of the universe and eventually to the density fluctuations. In this thesis I study this process of birth of primordial fluctuations in detail and introduce the observable quantities called power spectrum and the spectral parameters.

All this following trouble is necessary to find an answer to the following question: "How does inflation have anything to do with present day observations". The answer is presented schematically in Figure 1 which shows the evolution of comoving scales as a function of time. The comoving scales themselves stay constant, but the Hubble radius evolves in time. The red dotted line is the comoving Hubble radius (also called the horizon). Solution to the drawbacks of the original Big Bang -model require that the comoving horizon shrinks exponentially fast during an epoch called inflation. All the other fixed scales, such as the typical comoving scale of a galaxy, then exit the shrinking inflationary horizon and re-enter it much after the inflationary epoch when the horizon increases again during radiation- and matter-dominated eras. It is equivalent to say that the *physical* scales are stretched and the *physical* Hubble radius stays constant during inflation. It happens so that all the macroscopic scales stretch well beyond the horizon so that all densities are enormously diluted practically to zero and inside the horizon only the vacuum remains. However the seething vacuum quantum fluctuations are also stretched and they become small classical stochastic density fluctuations on all scales. As the figure suggests, inside the inflationary horizon the vacuum fluctuations are mathematically described by the two-point correlator of the so-called Mukhanov-Sasaki variable q. Outside the horizon a useful quantity is the curvature perturbation $\mathcal{R}$, closely related to q. This thesis focuses on the details of the figure and outline given above.

The first section is a short introduction to basics of cosmology and inflation. The topic of section 3 is cosmological perturbation theory and gauge issues. In section 4 I apply the cosmological perturbation theory to a single scalar field inflation and study the observables obtained that way with two examples. The conventions and some definitions that I've used in this thesis can be found from appendix A. As it happens, this thesis contains no new research. All the theory has been invented slowly from the 70's and the purpose of this work is to get familiar with the issues not currently taught in our University.

2 Background dynamics

I will first briefly introduce the non-perturbed standard cosmological model. More details can be found for example in Weinberg [1] or Mukhanov [2]. The basic formulae obteined here are needed throughout the latter part of the work. Later on when discussing cosmological perturbation theory I refer to this section as *background model*.

The most general spacetime metric obeying the cosmological principle (homogeneity and isotropy) is the Friedmann–Lemaître–Robertson–Walker (FRLW or FRW) metric [3]. In spherical coordinates it can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right), \tag{2.1}$$

where $\mathrm{d}\Omega^2$ is the 2-sphere metric and $k \in \{-1,0,+1\}$ corresponding to open, flat and closed geometries of the spatial hypersurface. Observations obtained from missions such as Planck [4] suggest that the Universe is nearly or exactly flat, so I take $k = 0$ from now on. In cartesian coordinates the FRW-metric is then

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{2.2}$$

For aesthetic reasons it is convenient to define the *conformal flat FRW-metric*

$$\mathrm{d}s^2 = a^2(\tau)[-\mathrm{d}\tau^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2] \quad \text{or} \quad g_{\mu\nu} = a^2\eta_{\mu\nu}, \tag{2.3}$$

where the *conformal time* τ is defined as

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{a(t)} \quad \text{or} \quad \tau = \int_0^t \frac{\mathrm{d}t}{a(t)}. \tag{2.4}$$

The different time derivatives are denoted by

$$' \equiv \frac{\mathrm{d}}{\mathrm{d}\tau} \quad \text{and} \quad \dot{} \equiv \frac{\mathrm{d}}{\mathrm{d}t}. \tag{2.5}$$

The Hubble constant H (or conformal Hubble constant $\mathcal{H}$ respectively) is defined as

$$H \equiv \frac{\dot{a}}{a} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \quad \text{or} \quad \mathcal{H} \equiv \frac{a'}{a} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\tau}. \tag{2.6}$$

The relation between these is $\mathcal{H} = aH$. It is straightforward to show the following handy equalities

$$\frac{a''}{a} = \mathcal{H}^2(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}) \qquad \text{and} \qquad \frac{\mathcal{H}'}{\mathcal{H}^2} = 1 + \frac{\dot{H}}{H^2}. \tag{2.7}$$

The Christoffel symbols are also needed later when I calculate the perturbations of the curvature tensor. The definition is the familiar

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}\left(g_{\alpha\delta,\beta} + g_{\beta\delta,\alpha} - g_{\alpha\beta,\delta}\right) \tag{2.8}$$

which for the metric (2.3) are all zero except

$$\Gamma^{\beta}_{0\alpha} = \mathcal{H}\delta^{\beta}_{\alpha} \quad \text{and} \quad \Gamma^{0}_{\alpha\beta} = \mathcal{H}\delta_{\alpha\beta}. \tag{2.9}$$

The metric determinant is $\sqrt{-g} = a^4(\tau)$. The energy-matter-content of the Universe is described by a perfect fluid which has a stress-energy tensor of the form

$$T^{\mu\nu} = (p+\rho)u^{\mu}u^{\nu} + pg^{\mu\nu}. \tag{2.10}$$

The Einstein field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ then give the *Friedmann equations* (on the left I write the equations in terms of cosmic time and on the right they are in terms of conformal time)

$$H^2 = \frac{8\pi G}{3}\rho \qquad \text{or} \qquad \mathcal{H}^2 = \frac{8\pi G}{3}a^2\rho \tag{2.11}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \qquad \text{or} \qquad \mathcal{H}' = -\frac{4\pi G}{3}a^2(\rho + 3p). \tag{2.12}$$

The *energy-continuity equation* $\nabla_{\mu}T^{0\mu} = 0$ gives

$$\dot{\rho} + 3\frac{\dot{a}}{a}(1+w) = 0. \qquad \text{or} \qquad \rho' + 2\mathcal{H}(1+w)\rho = 0. \tag{2.13}$$

Above I have defined *the equation of state parameter* w and also introduce the *sound speed squared* c^2:

$$w \equiv \frac{p}{\rho}, \qquad c^2 = \frac{\partial p}{\partial \rho}.$$

A number of useful identities can be derived from above equations:

$$\mathcal{H}' = -\frac{1}{2}(1+3w)\mathcal{H}^2 \tag{2.14}$$

$$p' = -3\mathcal{H}(1+w)c^2\rho \tag{2.15}$$

$$\frac{w'}{1+w} = -3\mathcal{H}(c^2 - w). \tag{2.16}$$

A key concept in this thesis is the *horizon*. In the theory of inflation the horizon usually refers to the *comoving Hubble radius* defined by

$$d_H \equiv \frac{1}{aH} = \frac{1}{\mathcal{H}}. \tag{2.17}$$

However, there's another concept of horizon called the *particle horizon* and it's defined to be the distance R_H light could have travelled from the beginning of the Universe until time t. Since light rays follow null paths $\mathrm{d}s^2 = 0$, I get $\mathrm{d}r = \mathrm{d}t/a(t)$ and thus the comoving radius of a particle horizon is

$$R_H = \int_0^t \frac{\mathrm{d}t'}{a(t')} = \int_{\tau_0}^{\tau} \mathrm{d}\tau' = \int_0^a \frac{1}{aH}\,\mathrm{d}\ln a\,. \tag{2.18}$$

The particle horizon is thus same thing as elapsed conformal time, or the logarithmic integral of the Hubble radius. I furthermore introduce the *redshift* z, defined as $1+z = \frac{1}{a}$, which measures the streching of the wavelenght of light due to to expansion of space. The comoving distance between redshifts z_1 and z_2 is

$$d(z_1, z_2) = \int_{z_1}^{z_2} \frac{\mathrm{d}z}{H(z)}. \tag{2.19}$$

These are the basic concepts and definitions needed in the following sections. I'm not going to go any further in presenting the rich phenomena of the unperturbed standard cosmological model, but rather move on towards the motivation and theory of cosmic inflation in next section and slowly towards the cosmic perturbation theory.

2.1 Inflation

The main topic of this section is cosmic inflation: its motivation, embodiment and the extremely useful set of assumptions justifying the approximation scheme called slow roll. I begin by presenting the well known drawbacks of the original Big Bang theory. After that I go through the inflationary scenario as a solution to those problems. I follow discussions from several books such as Mukhanov [2] and Dodelson [5] and one particularly excellent set of lecture notes by Baumann [6].

2.1.1 The Horizon problem

The cosmic microwave background (CMB) has observable temperature inhomogeneties only of the order 10^{-5}. However, the CMB sky consist of several patches that could have not been in causal contact in the standard Big Bang model. The problems is presented schematically in Figure 2. Let's look to this in detail. The particle horizon size at the time of recombination was

$$d_{rec} \equiv d_H(z_{rec}, \infty) = \int_{z_{rec}}^{\infty} \frac{\mathrm{d}z}{H(z)}. \tag{2.20}$$

The distance from us to the recombination surface (lookback horizon) is

$$d_{lookback} \equiv d_H(0, z_{rec}) = \int_{0}^{z_{rec}} \frac{\mathrm{d}z}{H(z)}. \tag{2.21}$$

Using $H(z) = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_\gamma(1+z)^4 + \Omega_\lambda}$ and $z_{rec} \approx 1000$ one can numerically integrate and estimate the number of causally disconnected volumes of space at the time of recombination to be

$$\left[\frac{d_{lookback}}{d_{rec}}\right]^3 \sim 10^5 \gg 1. \tag{2.22}$$

Now a question arises: how can the CMB be so homogenous if the distant parts have never been in causal contact? What has caused the coherent smoothing of the temperature inhomogeneities?

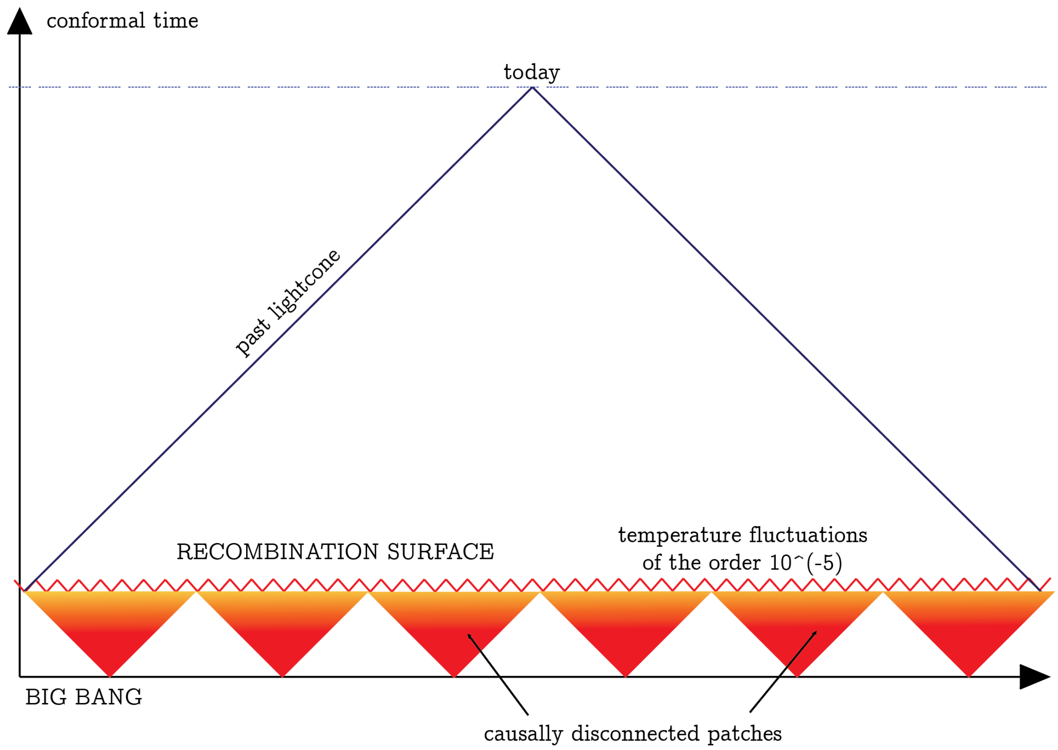

Figure 2: Without inflation the recombination surface consist of $\sim 10^5$ causally disconnected patches which have temperature differences only of order $\delta T/T \sim 10^{-5}$.

There's another way of phrasing the problem. In the Universe there are observed structures of galaxy filaments and walls that are up to 100 Mpc in size. According to measurements the energy density of the Universe today is very close to the critical density $\rho_{\rm crit} \approx 2.78 \cdot 10^{11} h^2 M_\odot/(\text{Mpc})^3$, where $h \approx 0.68$ and $M_\odot$ is the mass of the Sun. The corresponding mass of the observable Universe is then

$$M_{obs} \sim \frac{4\pi\rho_{\rm crit}}{3}(100\,\text{Mpc})^3 \simeq 6 \cdot 10^{17} M_\odot,$$

but the mass of a causal horizon at early times was

$$M_H = \frac{0.11}{\sqrt{g_*}}\left(\frac{\text{MeV}}{T}\right) M_\odot,$$

where g_* is the number of degrees of freedom in the plasma. Sensible values of g_* at the early Universe are $g_* \sim 5 - 100$ and thus $M_H \ll M_{obs}$ when the temperature was large.

2.1.2 The Flatness problem

Let's consider the Friedmann's first equation (2.11) with the curvature term added:

$$H^2 = \frac{1}{3M_P^2}\rho - \frac{k}{a^2}. \tag{2.23}$$

This can be written as

$$\Omega - 1 = \frac{k}{(aH)^2}, \tag{2.24}$$

where $\Omega = \frac{\rho}{\rho_{crit}}$ and $\rho_{crit} \equiv 3M_p^2H^2$ is the critical density. We know that the total fractional energy density of the Universe today is [4]

$$\Omega_0 \simeq 1.02 \pm 0.02,$$

which corresponds to a flat or nearly flat spacetime. But when considering the early era, assuming that the Universe has gone through only matter and radiation dominated epochs, the Friedmann equation reveals that the curvature is a growing function in time:

$$\Omega - 1 = \frac{k}{(aH)^2} \sim \begin{cases} a^2 \sim t, & \text{radiation dom.} \\ a \sim t^{2/3}, & \text{matter dom.} \end{cases}$$

Then at Planck time $t_{PL} \sim 10^{-43}\,$s the quantity was

$$|\Omega - 1|_{t \approx t_{PL}} \simeq \mathcal{O}(10^{-64})|\Omega - 1|_0.$$

Thus at the beginning of the Universe the spatial curvature must've been fine-tuned to a value extremely close to 0 but not exactly 0. From equation (2.24) it is clear that the flatness problem has something to do with the time-evolution of the Hubble radius $(aH)^{-1}$. The flatness problem is often also called the *age problem*: if the initial conditions for a FRW-expansion would have been somewhat 'natural' at Planck time, i.e.

$$\Omega_{PL} \simeq 1 \pm \delta\Omega_{PL},$$

where $\delta\Omega_{PL} \simeq \mathcal{O}(1)$, then in case of positively curved space $k > 0$ the Universe would have recollapsed at time $\sim t_{PL}/\delta\Omega_{PL}$ or in turn cooled down to 3K at same time if $k < 0$ in the negative curvature case.

One way to solve these problems is to assume that the Universe was somewhere in its past dominated by a non-zero vacuum energy. This corresponds to a cosmological constant Λ. When Λ dominates, the scale factor has a de Sitter solution

$$a(t) \sim e^{Ht}.$$

This removes the horizon problem since now every co-moving scale passes the horizon twice: first a given causally connected scale passes the horizon during the de Sitter phase, and afterwards when the de Sitter phase is over the scale returns inside the horizon during the FRLW-phase. The flatness problem is also solved: let us assume that at the onset of inflation

$$\Omega_{inf} - 1 \sim \mathcal{O}(1).$$

Now during the inflation

$$|\Omega - 1| \simeq \frac{1}{(aH)^2} = \frac{1}{(a_{inf}H)^2}e^{-2Ht} \to 0$$

as $t \to \infty$. If the inflation last a time $t = HN$ (N e-foldings), we get

$$|\Omega_{out} - 1| = |\Omega_{inf} - 1|e^{-2N}.$$

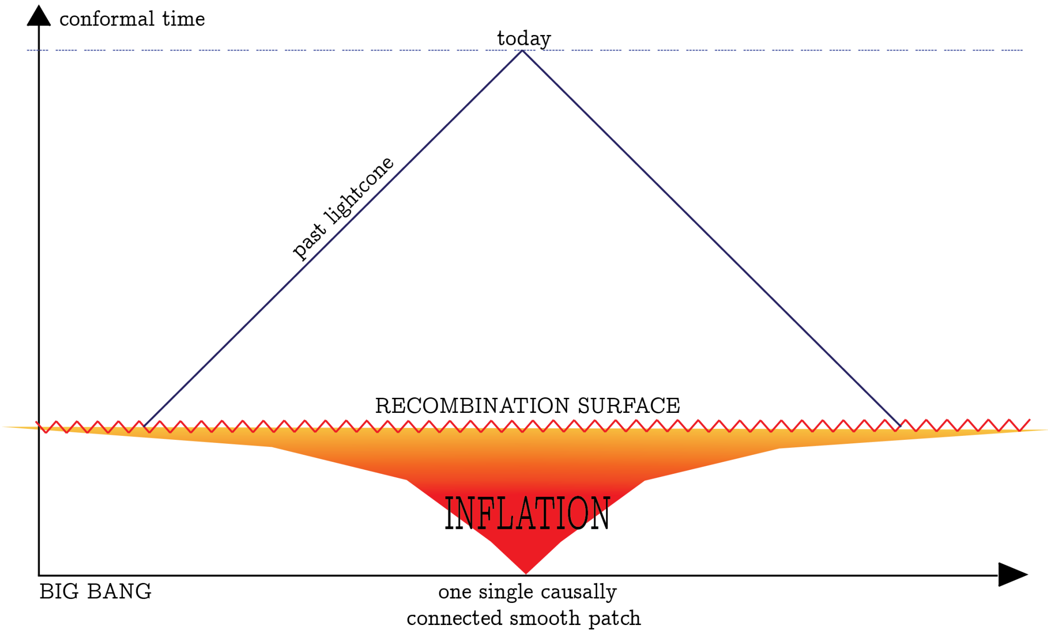

Figure 3: Inflation stretches initially small patches of space exponentially so that the horizon problem is resolved.

So if $t_{out} \simeq t_{PL}$, one needs $e^{-2N} < e^{-60}$, i.e. $N \geq 70$ so that inflation would have enough time to arrange suitable initial conditions for a FRW-expansion. A pure de Sitter phase though is not necessarily required for the inflation to happen. What is needed, is simply an accelerated expansion:

$$\ddot{a} > 0 \quad \Leftrightarrow \quad \rho < -\frac{1}{3}p \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{aH}\right) < 0. \tag{2.25}$$

The last requirement is intuitive from the flatness problem -point of view: for the spatial curvature term to have a non-growing behaviour one needs a shrinking Hubble radius. I define inflation to be equivalent to any of the requirements in equation (2.25). The pioneering authors inventing the theory of inflation were Starobinsky [7], Guth [8] and Linde [9] in the late 70'. Alan Guth proposed that the exponential expansion could be produced by a scalar field trapped in a false vacuum state due to supercooling of the Universe. The false vacuum with high energy density could then act as a cosmological constant. This metastable state could then decay by quantum tunneling which would end the inflation. Guth himself realized that this model had problems with *reheating* of the Universe after inflation. After inflation the Universe is extremely flat, but also extremely empty. One important feature for a theory of inflation is the so-called reheating after the inflation which would produce the needed amount of radiation in the early Universe. In this thesis I'm not going to discuss reheating however. Andrei Linde solved the reheating problem in Guth's model by introducing a field slowly rolling in a potential well so that the potential energy dominates over the kinetic energy of the field. The inflation ends when the field rolls down to the bottom of the potential and starts to oscillate thus transferring it's energy to radiation through decay processes to Standard Model particles. These kinds of models are called "new inflation" opposed to Guths "old inflation". A popular scenario belonging to this category is the "chaotic inflation" occurring near the Planck scale, where inflation actually never ends. It may be manifest in almost every realistic inflationary model proposed nowadays. Amazingly so, the first ever proposed inflationary model by Starobinsky is still after 35 years inside the 1σ-limit of current observations [10]. Starobinsky himself didn't consider the inflationary implications of his theory of quantum corrections to general relativity but realized that a modification of Einstein-Hilbert action to have a Ricci scalar squared term at near quantum gravity scales would lead to a de Sitter -phase of the Universe. This kind of "$R + R^2$" model is very similar to the Higgs inflation model that I'm going to discuss in the last section.

2.2 Inflation from a scalar field

I showed that inflation can be achieved at least with a cosmological constant so that the scale factor gets an exponential solution. A scalar field can quite easily mimic a constant vacuum energy if the potential is sufficiently flat. The requirement $\rho < -\frac{1}{3}p$ for an accelerated expansion can thus be achieved by assuming that the early universe was filled with a scalar field rolling down a potential. Let's examine how this is accomplished in more detail. Take a single scalar field Lagrangian in curved spacetime:

$$\mathcal{L}_\varphi = \frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) \tag{2.26}$$

and the action $S_\varphi = \int \mathcal{L}_\varphi\sqrt{-g}\mathrm{d}x^{\,4}$. The total action is then

$$S = \frac{1}{16\pi G}S_H + S_\varphi = \int \sqrt{-g}\mathrm{d}x^{\,4}\left(\frac{1}{16\pi G}R + \mathcal{L}\varphi\right). \tag{2.27}$$

The Euler-Lagrange equations for the scalar field are

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\right) = 0 \tag{2.28}$$

$$\Rightarrow \quad \frac{\partial V(\varphi)}{\partial\varphi} + \nabla_\mu\nabla^\mu\varphi = 0, \tag{2.29}$$

where $\nabla_\mu\nabla^\mu\varphi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\varphi\right)$. Variation with respect to the metric gives the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{2.30}$$

where

$$T_{\mu\nu} \equiv -2\frac{1}{\sqrt{-g}}\frac{\delta S_\varphi}{\delta g^{\mu\nu}}. \tag{2.31}$$

For a single scalar field we get

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}\mathcal{L}_\varphi. \tag{2.32}$$

When we take the background universe to be the FRW-universe, we have $\sqrt{-g} = a^3(t)$ and the equation of motion is now

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial V(\varphi)}{\partial\varphi} = 0. \tag{2.33}$$

This looks similar to a harmonic oscillator with a friction term proportional to the Hubble constant. As a function of conformal time this reads

$$\varphi'' + 2\mathcal{H}\varphi' + a^2\partial_\varphi V = 0. \tag{2.34}$$

The energy-momentum tensor has components

$$T^0_0 = g^{\alpha 0}T_{0\alpha} = -\frac{1}{2a^2}(\varphi')^2 - V = -\frac{1}{2}\dot{\varphi}^2 - V \equiv -\rho \tag{2.35}$$

$$T^i_0 = 0 \tag{2.36}$$

$$T^i_j = \delta^i_j\left(\frac{1}{2a^2}(\varphi')^2 - V(\varphi)\right) = \delta^i_j\left(\frac{1}{2}\dot{\varphi}^2 - V\right) \equiv \delta^i_j p, \tag{2.37}$$

from which it is easy to show the following useful relations

$$\rho + p = \frac{1}{a^2}(\varphi')^2 = \dot{\varphi}^2 \tag{2.38}$$

$$\rho - p = 2V. \tag{2.39}$$

The equation of state -parameter $w \equiv \frac{p}{\rho}$ is now

$$w = \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)} \quad \text{or} \quad w = \frac{(\varphi')^2 - 2a^2V}{(\varphi')^2 + 2a^2V} \tag{2.40}$$

so that $-1 \leq w \leq 1$. A cosmological constant corresponds to $w = -1$, but as I've said, that is not necessary. With a scalar field the less restrictive requirement $\rho < -\frac{1}{3}p$ can be achieved. For further use introduce the sound speed c^2 which is now, using the equation of motion (2.34),

$$c^2 = \frac{p'}{\rho'} = \frac{2\mathcal{H}\varphi' + 2a^2V'}{3\mathcal{H}\varphi'} = \frac{-1}{3\mathcal{H}}\left(\mathcal{H} + 2\frac{\varphi''}{\varphi'}\right). \tag{2.41}$$

The first Friedmann equation can be written as

$$H^2 = \frac{8\pi G}{3}\rho = \frac{1}{3M_P^2}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right), \tag{2.42}$$

where $M_p^2 = \frac{1}{8\pi G} = 2.436 \cdot 10^{18}\text{GeV}$ is the *reduced Planck mass.* From this Friedmann equation and the equation of motion one can easily derive a useful relation

$$\dot{H} = -4\pi G\dot{\varphi}^2. \tag{2.43}$$

2.3 Slow roll approximation

The condition for inflation is $\rho + 3p = 2\dot{\varphi}^2 - 2V(\varphi) < 0$, from which we get $\dot{\varphi}^2 < V(\varphi)$. On the other hand, the previous condition should be valid sufficiently long time (~ 60 e-foldings) in order to make the universe flat enough. Then it is clear that

- The potential has to be sufficiently slowly changing in the region where the potential dominates ($\ddot{a} > 0$).

- There has to be a minimum of the potential $V(\varphi_{min}) = 0$ where the inflation ends.
- Furthermore, $\dot{\varphi}$ cannot be too large at the beginning.

These conditions can be quantified as *slow-roll conditions*:

$$\dot{\varphi}^2 \ll V \tag{2.44}$$
$$|\ddot{\varphi}| \ll |3H\dot{\varphi}|. \tag{2.45}$$

Using these, one can write the Friedmann equation and the equation of motion as

$$H^2 \approx \frac{1}{3M_P^2}V \qquad \text{and} \qquad 3H\dot{\varphi} \approx -V'. \tag{2.46}$$

These are called the *slow-roll equations* and from now on I use equality signs in the above equations when I have explicitly specified a case where I use the approximation. Taking a time derivative of the above equations one gets

$$\dot{H} = -\frac{(V')^2}{6V} \qquad \text{and} \qquad \ddot{\varphi} = \frac{M_P^2}{3}\left(\frac{V''V'}{V} - \frac{(V')^3}{2V^2}\right). \tag{2.47}$$

It is convenient to define the *slow-roll parameters*

$$\epsilon \equiv \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2 \tag{2.48}$$
$$\eta \equiv M_P^2\frac{V''}{V} \tag{2.49}$$
$$\delta \equiv \eta - \epsilon, \tag{2.50}$$

so that the slow-roll conditions can be written as

$$\epsilon \ll 1, \qquad |\eta| \ll 1. \tag{2.51}$$

The parameter δ proves to be useful later on. As can be seen from the definitions, these parameters describe the slope (ϵ) and the curvature (η) of the potential. Using these parameters the second Friedmann equation (2.12) can be written as

$$\frac{\ddot{a}}{a} = H^2(1-\epsilon). \tag{2.52}$$

From above it is clear that there is inflation as long as $\epsilon < 1$ and a quasi-de Sitter universe when $\epsilon \ll 1$. A pure de Sitter would correspond to $\epsilon = 0$. The parameter η simply tells that when $|\eta| < 1$, the inflation keeps running sufficiently long.

2.3.1 Useful relations for slow roll parameters

From the slow-roll equations and the definition of the slow-roll parameters one can derive many useful identities and results, such as

$$\frac{\dot{H}}{H^2} = -\epsilon \qquad \text{and} \qquad \left(\frac{\dot{\varphi}}{H}\right)^2 = 2M^2\epsilon. \tag{2.53}$$

The derivative of the slow roll ϵ is second order small:

$$\frac{\dot{\epsilon}}{H} = 3\epsilon^2 - 2\epsilon\eta. \tag{2.54}$$

In terms of the conformal time there are identities such as

$$\frac{\mathcal{H}'}{\mathcal{H}^2} = 1 - \epsilon \qquad \text{and} \qquad \frac{a''}{a} = \mathcal{H}^2(2 - \epsilon). \tag{2.55}$$

One could also take (2.53) as the definition of the parameter ϵ and then define the slow-roll parameters as follows. I denote by a subscript H these alternative parameters which are defined only in terms of the Hubble parameter. The definitions are

$$\epsilon_H \equiv \frac{-\dot{H}}{H^2} \qquad \text{and} \qquad \eta_H \equiv \frac{\ddot{H}}{2H\dot{H}},$$

and the equality of the two different parameter sets is true only when they are small. This set of parameters is particularly handy if one wants to calculate the Mukhanov-Sasaki equation in Section 4.2 to second order in slow roll parameters. Weinberg [1] uses these parameters already at first order.

2.3.2 Number of e-foldings

The duration of inflation is usually measured in *e-foldings* defined by

$$\frac{a_{\text{beg}}}{a_{\text{end}}} \equiv e^{-N}. \tag{2.56}$$

For the single scalar field inflation, using slow-roll equations, one finds

$$N = N(\phi_{\text{beg}},\phi_{\text{end}}) = \int_{t_{\text{beg}}}^{t_{\text{end}}} H \mathrm{d}t \; = \frac{1}{M_p^2} \int_{\phi_{\text{beg}}}^{\phi_{\text{end}}} \frac{V}{V'} \mathrm{d}\phi\,. \tag{2.57}$$

The comoving scale corresponding to our current cosmological horizon left the inflatory horizon at $\frac{1}{a_0 H_0} = \frac{1}{(aH)_{\text{H-out}}}$. Using the slow roll equations (2.46) it is straightforward to show that

$$\frac{a_{\text{H-out}}}{a_{\text{end}}} = e^{-N(\phi_{\text{H-out}},\phi_{\text{end}})} = \left(\frac{V_{\text{end}}}{V_{\text{H-out}}}\right)^{\frac{1}{2}} \frac{a_0 H_0}{(aH)_{\text{end}}}, \tag{2.58}$$

and from this it follows that

$$N(\phi_{\text{H-out}}, \phi_{\text{end}}) = \log\left(\frac{a_0 H_o}{(aH)_{\text{H-out}}}\right) + \frac{1}{2}\log\left(\frac{V_{\text{beg}}}{V_{\text{H-out}}}\right). \tag{2.59}$$

Now using a what is called the *instantenous reheating* approximation one can show that

$$\frac{a_0 H_0}{(aH)_{\text{end}}} \simeq 1.7 \cdot 10^{-30} \frac{M_p}{V_{\text{end}}^{1/4}}. \tag{2.60}$$

This assumes that the radiation dominated epoch started instantenously after a_{end}. Plugging this into (2.59) gives

$$N(\phi_{\text{H-out}}, \phi_{\text{end}}) \simeq 68.5 + \frac{1}{4}\log\left(\frac{V_{\text{end}}}{M_p^4}\right) + \frac{1}{2}\log\left(\frac{V_{\text{H-out}}}{V_{\text{end}}}\right). \tag{2.61}$$

This can be easily generalized to get the number of e-foldings from the horizon crossing of an arbitrary scale $k \equiv aH = k(a_0H_0)$ to the end of inflation:

$$N(\phi_{\text{k}}, \phi_{\text{end}}) \simeq 68.5 + \frac{1}{4}\log\left(\frac{V_{\text{end}}}{M_p^4}\right) + \frac{1}{2}\log\left(\frac{V_{\text{k}}}{V_{\text{end}}}\right) - \log k. \tag{2.62}$$

All the information about inflation comes to us from observations of density fluctuations in the scales greater than the galactic scale $k_{galaxy}^{-1} \sim \frac{1}{3}10^{-3}$Mpc [11] and smaller than the scale corresponding to the size of the observable universe today. Fluctuations below that scale are completely washed out by non-linear gravitational effects. The galactic scales then correspond to roughly $\log k_{galaxy} \sim 8$. Then if the inflationary energy scale is say few orders of magnitude lower than the Planck scale $V_{end} \sim V_k \sim 10^{-3}M_p^4$, the number of e-foldings at the horizon-crossing can be roughly approximated as

$$N(\phi_{H-out}) \sim 65, \tag{2.63}$$

and at the galaxy-crossing

$$N(\phi_{galaxy}) \sim 55. \tag{2.64}$$

The relevant interval of e-foldings is then approximately $55 \leq N \leq 65$ before the end of inflation. These numbers are however model dependent. Typically when calculating the specral parameters, discussed in section 4, one uses $50 \leq N \leq 60$.

2.3.3 Example: Polynomial potential

Take the inflaton potential to be

$$V(\phi) = \lambda_p \phi^p, \tag{2.65}$$

where $p \in \mathbb{R}$. For this potential the slow-roll parameters are

$$\epsilon = \frac{M_p^2}{2}\left(\frac{p}{\phi}\right)^2 \quad \text{and} \quad \eta = M_p^2\frac{p(p-1)}{\phi^2}. \tag{2.66}$$

Inflation ends when $\epsilon \simeq 1$, so the field value at the end of inflation is

$$\phi_{\text{end}} = \frac{p}{\sqrt{2}}M_p. \tag{2.67}$$

The number of e-foldings from the k-scale horizon exit to the end of inflation is then

$$N(\phi_k, \phi_{\text{end}}) = \frac{1}{M_p^2}\int_{\phi_{\text{end}}}^{\phi_k}\frac{V}{V'} = \frac{1}{2pM_p^2}\left(\phi_k^2 - \frac{p^2}{2}M_p^2\right). \tag{2.68}$$

Inverting this, one gets

$$\phi_k = \sqrt{2pM_p(N + \frac{p}{4})}, \tag{2.69}$$

so that one can express the slow-roll parameters in terms of e-foldings:

$$\epsilon(N) = \frac{p}{4}\left[N(\phi_k, \phi_{\text{end}}) + \frac{p}{4}\right]^{-1} \tag{2.70}$$

$$\eta(N) = \frac{p-1}{2}\left[N(\phi_k, \phi_{\text{end}}) + \frac{p}{4}\right]^{-1}. \tag{2.71}$$

Taking for example $p = 4$ and $N = 60$, which was the rough estimate for the flatness problem to be solved, the parameters get numerical values of $\epsilon(N = 60) \approx 0.016$ and $\eta(N = 60) \approx 0.025$. Later when discussing the spectral parameters I'm going to re-examine this example.

3 Cosmological perturbation theory

In this section I'll go through the basics of cosmological perturbation theory and the issue of gauge invariance. I'll introduce the conformal Newtonian gauge and derive the first order perturbed Einstein equations in this gauge.

In General relativistic perturbation theory any tensorial quantity is split into a background quantity and perturbations around it. In particular, in cosmology, the background is the *homogenous* and *isotropic* FRW-metric, so that any tensor in the full spacetime (whatever it is) can be written as

$$\mathbf{T}(\tau,\mathbf{x}) \equiv \bar{\mathbf{T}}(\tau) + \delta\mathbf{T}(\tau,\mathbf{x}), \tag{3.1}$$

where

$$\delta\mathbf{T}(\tau,\mathbf{x}) \equiv \sum \frac{\epsilon^n}{n!}\delta_n\mathbf{T}(\tau,\mathbf{x}). \tag{3.2}$$

From here on, I consider only first order perturbations and absorb the ϵ to the definition of the perturbations for convenience. The barred variables refer to background quantities and quantities without bars are all small perturbations. The spacetime is splitted into temporal 1-dimensional *threading* and spatial 3-dimensional hypersurfaces, called *slicings*, of constant conformal time. This is the so-called $(3+1)$-split [13]. The perturbed variables are furthermore decomposed into scalar, vector and tensor parts according to their transformation properties under spatial rotations around the wave-vector in the Fourier space. This is called the SVT-decompostion or Helmholtz decomposition [6]. The scalar, vector and tensor parts are said to have *helicity* of $0, \pm 1, \pm 2$ respectively. I will discuss the helicity of the gravitational waves in Section 4.6 briefly. In what follows I will denote the scalar, vector and tensor parts of different variables with the same letter as the variable itself, only the number of indices change. Without a proof I conclude the following SVT-theorem: the vector perturbations β_i decompose into a scalar and a vector part, namely

$$\beta_i = \beta_i^S + \beta_i^V, \tag{3.3}$$

where

$$\beta_i^S = \nabla_i\beta \quad \text{and} \quad \nabla^i\beta_i^V = 0 \quad \text{with} \quad \beta \in \mathbb{R}. \tag{3.4}$$

A symmetric, traceless 3-tensors γ_{ij} decompose into a scalar, vector and tensor parts:

$$\gamma_{ij} = \gamma_{ij}^S + \gamma_{ij}^V + \gamma_{ij}^T, \tag{3.5}$$

where

$$\gamma_{ij}^S = \left(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\gamma, \quad \gamma \in \mathbb{R} \tag{3.6}$$

$$\gamma_{ij}^V = \frac{1}{2}\left(\nabla_i\gamma_j + \nabla_j\gamma_i\right), \quad \nabla_i\gamma_i = 0, \quad \gamma_i \in \mathbb{R}^2 \tag{3.7}$$

$$\nabla_i\gamma_{ij}^T = 0. \tag{3.8}$$

The scalars α obviously do not decompose into any but scalar part, $\alpha = \alpha^S$. The usefulness of this decomposition follows from the fact that in the first order perturbation theory the scalar, vector and tensor parts evolve independently. That is, the equations of motion following from the Einstein field equations do not mix perturbations of different helicity. In addition, different Fourier modes (different wavenumber k) evolve independently. The first claim follows actually from the rotational invariance of the background, and the latter from it's translational invariance [6]. I do not consider any vector perturbations in this thesis, because they have been shown to have only decaying solutions.

3.1 Gauge transformations

Gauge transformations in the metric perturbation theory are transformations between spesific coordinate systems on the physical perturbed spacetime. On a manifold one could always choose the coordinates in such a way that a coordinate dependent numerical value, e.g. a components of a tensor, gets arbitrary values. The crucial feature of the gauge transformations is that they leave the perturbations *small*, i.e. they do not break the perturbative analysis. In another words, if the perturbations are small in some coordinates, then the gauge transformation is a change from those coordinates into some others, where the perturbations are different but still small. A schematic picture in 2D is shown in Figure 4. All gauges are equally good, so it would be nice to know the relation between

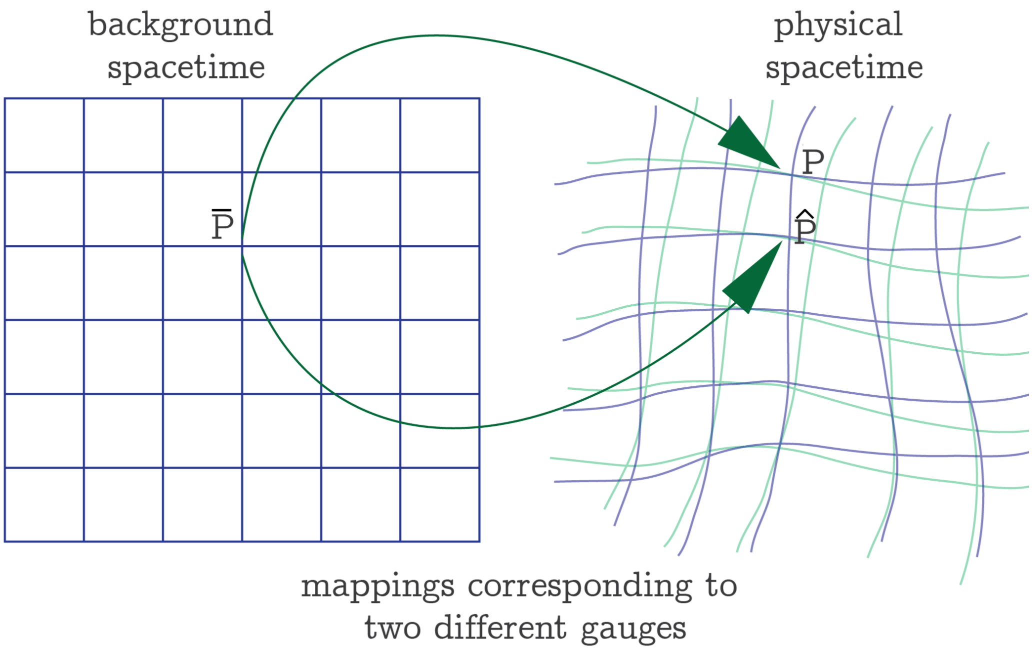

Figure 4: A point on the background manifold does not have unique correspondence to a point on the physical spacetime.

perturbations in different gauges. Start by considering two coordinate systems x^α and $\hat{x}^\alpha$ on a physical manifold $\mathcal{M}$ so that these two coordinate systems correspond to two different gauges. Barred variables always refer to quantities on the background. The coordinates are then by definition related by some gauge transformation vector $\xi = (\xi^0, \xi^i)$, i.e. some four functions ξ^α by

$$x^\alpha \to \tilde{x}^\alpha = x^\alpha + \xi^\alpha, \tag{3.9}$$

where ξ^α is first order small, so that $(\xi^\alpha)^2 \approx 0$. Also the derivatives of ξ^α with respect to both coordinates are assumed to be small. Since ξ^α is taken to be first order small, I can associate with it a fixed value on the background:

$$\xi^\alpha = \xi^\alpha(\bar{x}(\bar{P})). \tag{3.10}$$

A coordinate transformation relates the coordinates on the *same point* on the actual manifold, i.e.

$$\begin{aligned} \hat{x}^\alpha(P) &= x^\alpha(P) + \xi^\alpha \\ \hat{x}^\alpha(\hat{P}) &= x^\alpha(\hat{P}) + \xi^\alpha. \end{aligned} \tag{3.11}$$

Both coordinates are however related to *same point on the background* manifold:

$$x^\alpha(P) = \hat{x}^\alpha(\hat{P}) = \bar{x}^\alpha(\bar{P}). \tag{3.12}$$

Using the last two equations one can relate two distinct points in same coordinates on the physical manifold:

$$\begin{aligned} x^\alpha(\hat{P}) &= x^\alpha(P) - \xi^\alpha \\ \hat{x}^\alpha(\hat{P}) &= \hat{x}^\alpha(P) - \xi^\alpha. \end{aligned} \tag{3.13}$$

Now I define the perturbations in different gauges to be functions on the background manifold in a given background coordinate system $\bar{x}^\alpha$ at a given background point $\bar{P}$:

$$\begin{aligned} \hat{\delta}\mathbf{T} &\equiv \mathbf{T}(\hat{x}^\alpha(\hat{P})) - \bar{\mathbf{T}}(\bar{x}^\alpha(\bar{P})) \\ \delta\mathbf{T} &\equiv \mathbf{T}(x^\alpha(P)) - \bar{\mathbf{T}}(\bar{x}^\alpha(\bar{P})). \end{aligned} \tag{3.14}$$

The gauge choice thus manifests itself as a choice of the coordinates and the point on the physical manifold. The correspondence between perturbations in different gauges is then simply, using (3.14),

$$\hat{\delta}\mathbf{T} = \delta\mathbf{T} + \mathbf{T}(\hat{x}^\alpha(\hat{P})) - \mathbf{T}(x^\alpha(P)). \tag{3.15}$$

Now let's first consider a scalar perturbation $s = \bar{s} + \delta s$. When moving from point P with coordinates x^α to a point $\hat{P}$ with coordinates $\hat{x}^\alpha$, the full scalar s acquires alteration only due to the change in place, not in coordinates. Expanding the new scalar around the old point gives

$$\begin{aligned} s(\hat{x}^\alpha(\hat{P})) &= s(\hat{x}^\alpha(P)) + (\hat{x}^\beta(\hat{P}) - \hat{x}^\beta(P))\frac{\partial}{\partial x^\beta} s(\hat{x}^\alpha(P)) \\ &= s(x^\alpha(P)) - \xi^\beta \frac{\partial}{\partial x^\beta} s(x^\alpha(P)) \\ &= s(x^\alpha(P)) - \xi^0 \bar{s}', \end{aligned} \tag{3.16}$$

where in the last line I used the fact that ξ^α is already first order small, and the background quantity has only τ-dependence due to homogeneity. Using (3.15) then gives

$$\begin{aligned}\hat{\delta} s &= \delta s + s(\hat{x}^\alpha(\hat{P})) - s(x^\alpha(P)) \\ &= \delta s - \xi^0 \bar{s}'.\end{aligned} \tag{3.17}$$

Here it is good to notice that a scalar perturbation is *not* immediately gauge invariant. It is though possible to form linear combinations of different perturbation variables, which are gauge invariant. I will discuss such gauge invariant variables a little later since they prove to be very useful and physically meaningful quantities. Now to get the transformation laws for higher order tensor perturbations I can use the general transformation law of the tensor components. First note that the Jacobian matrix for the infinitesimal transformation (3.9) and its inverse are now

$$\begin{aligned}\frac{\partial x^\alpha}{\partial \hat{x}^\mu} &= \delta^\alpha_\mu - \xi^\alpha_{,\mu} \\ \frac{\partial \hat{x}^\alpha}{\partial x^\mu} &= \delta^\alpha_\mu + \xi^\alpha_{,\mu}.\end{aligned} \tag{3.18}$$

Now take for example the metric tensor. Expanding around the old point gives, similarly as in the case of a scalar,

$$g_{\mu\nu}(x^\delta(\hat{P})) = g_{\mu\nu}(x^\delta(P)) - \xi^\alpha \bar{g}_{\mu\nu,\alpha}(x^\delta(P)). \tag{3.19}$$

On the other hand, making the coordinate transformation (3.9) changes the components of a tensor as

$$\begin{aligned}g_{\mu\nu}(\hat{x}^\delta(\hat{P})) = \frac{\partial x^\alpha}{\partial \hat{x}^\mu}\frac{\partial x^\beta}{\partial \hat{x}^\nu} g_{\alpha\beta}(x^\delta(\hat{P})) &= (\delta^\alpha_\mu - \xi^\alpha_{,\mu})(\delta^\beta_\nu - \xi^\beta_{,\nu})\left[g_{\alpha\beta}(x^\delta) - \xi^\gamma \bar{g}_{\alpha\beta,\gamma}(x^\delta)\right] \\ &\approx g_{\mu\nu}(x^\delta(P)) - \xi^\alpha_{,\mu}\bar{g}_{\alpha\nu} - \xi^\alpha_{,\nu}\bar{g}_{\mu\alpha} - \xi^\gamma \bar{g}_{\mu\nu,\gamma}.\end{aligned} \tag{3.20}$$

Plugging this to the transformation formula (3.15) gives

$$\hat{\delta} g_{\mu\nu} = \delta g_{\mu\nu} - \xi^\alpha_{,\mu}\bar{g}_{\alpha\nu} - \xi^\alpha_{,\nu}\bar{g}_{\mu\alpha} - \xi^\gamma \bar{g}_{\mu\nu,\gamma}. \tag{3.21}$$

From here on, I will not write the different coordinates x^α or $\bar{x^\alpha}$ or points explicitly, but refer to quantities in different gauges with only a tilde above. In the next subsections I collect transformation laws for all different types of perturbations. I make use of the fact that the background spacetime is homogenous and isotropic, so that the background 4-vectors and -tensors are necessarily of the form

$$\bar{w}^\alpha = (\bar{w}^0, \vec{0}), \quad \bar{A}^\mu_\nu = \begin{bmatrix} \bar{A}^0_0 & 0 \\ 0 & \frac{1}{3}\delta^i_j \bar{A}^k_k \end{bmatrix}, \tag{3.22}$$

where all quantities are functions only of the conformal time τ.

Scalars

A generic scalar perturbation δs defined by $s = \bar{s} + \delta s$ changes by

$$\delta s \to \tilde{\delta} s = \delta s - \bar{s}' \xi^0. \tag{3.23}$$

4-vectors

A 4-vector perturbation defined by $w^\mu = \bar{w}^\mu + \delta w^\mu$ changes by

$$\delta w^\mu \to \tilde{\delta} w^\mu = \delta w^\mu + \xi^\mu_{,\nu}\bar{w}^\nu - \bar{w}^\mu_{,\nu}\xi^\nu, \tag{3.24}$$

from which it follows that

$$\begin{cases} \tilde{\delta} w^0 = \delta w^0 + \xi^0_{,0}\bar{w}^0 - \bar{w}^0_{,0}\xi^0 \\ \tilde{\delta} w^i = \delta w^i + \xi^i_{,0}\bar{w}^0 \end{cases}. \tag{3.25}$$

Mixed 4-tensors

A mixed 4-tensor perturbation defined by $A^\mu_\nu = \bar{A}^\mu_\nu + \delta A^\mu_\nu$ changes into

$$\delta A^\mu_\nu \to \tilde{\delta} A^\mu_\nu = \delta A^\mu_\nu + \xi^\mu_{,\rho}\bar{A}^\rho_\nu - \xi^\rho_{,\nu}\bar{A}^\mu_\rho - \bar{A}^\mu_{v,\rho}\xi^\rho, \tag{3.26}$$

from which it follows that

$$\begin{cases} \tilde{\delta} A^0_0 = \delta A^0_0 - \bar{A}^0_{0,0}\xi^0 \\ \tilde{\delta} A^0_i = \delta A^0_i + \frac{1}{3}\xi^0_{,i}\bar{A}^k_k - \xi^0_{,i}\bar{A}^0_0 \\ \tilde{\delta} A^i_0 = \delta A^i_0 + \xi^i_{,0}\bar{A}^k_k - \frac{1}{3}\xi^i_{,0}\bar{A}^0_0 \\ \tilde{\delta} A^i_j = \delta A^i_j - \frac{1}{3}\delta^i_j\bar{A}^k_{k,0}\xi^0 \end{cases}. \tag{3.27}$$

The trace and the traceless parts in particular transform as

$$\begin{cases} \tilde{\delta} A^k_k = \delta A^k_k - \bar{A}^k_{k,0}\xi^0 \\ \tilde{\delta} A^i_j - \frac{1}{3}\tilde{\delta} A^k_k = \delta A^i_j - \frac{1}{3}\delta A^k_k \end{cases}. \tag{3.28}$$

From here one can easily see that the traceless part is gauge invariant.

3.2 Perturbations of the metric tensor

The perturbed metric tensor is defined as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}), \tag{3.29}$$

where $\bar{g}_{\mu\nu} = a^2\eta_{\mu\nu}$ is the flat unperturbed FRW metric and $h_{\mu\nu}$ is a first order small perturbation. The inverse metric is

$$g^{\mu\nu} \equiv \frac{1}{a^2}(\eta^{\mu\nu} - h^{\mu\nu}). \tag{3.30}$$

so that the first order inverse metric perturbation is

$$h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}, \tag{3.31}$$

Performing the SVT-decomposition to the the metric pertubation one gets

$$h_{\mu\nu} = \begin{bmatrix} -2A & -B_i \\ -B_i & -2D\delta_{ij} + 2E_{ij} \end{bmatrix}, \tag{3.32}$$

where $D \equiv -\frac{1}{6}h^i_i$ carries the trace of the spatial perturbation h_{ij}, E_{ij} is a traceless tensor, B_i is called the *shift vector* and A is called the *lapse function*. The inverse is obtained by raising the indicies with $\eta^{\mu\nu}$ and thus

$$h^{\mu\nu} = \begin{bmatrix} -2A & +B_i \\ +B_i & -2D\delta_{ij} + 2E_{ij} \end{bmatrix}. \tag{3.33}$$

Then in terms of the conformal time τ the line-element can be written as

$$\mathrm{d}s^{\,2} = a^2(\tau)\left[(-1-2A)\mathrm{d}\tau^{\,2} - 2B_i\mathrm{d}x^{\,i}\mathrm{d}\tau \,+ [(1-2D)\delta_{ij} + 2E_{ij}]\mathrm{d}x^{\,i}\mathrm{d}x^{\,j}\right]. \tag{3.34}$$

The vector perturbation B_i splits into zero-curl and zero-divergence parts:

$$B_i = -B_{,i} + B_i^V, \tag{3.35}$$

where B is a scalar and $\delta^{ij}B^V_{i,j} = 0$. The tensorial part E_{ij} is divided into scalar, vector and pure tensor parts

$$E_{ij} = E^S_{ij} + E^V_{ij} + E^T_{ij}, \tag{3.36}$$

where

$$E^S_{ij} = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E \tag{3.37}$$

$$E^V_{ij} = -\frac{1}{2}\left(E_{i,j} + E_{j,i}\right) \quad \text{with} \quad \delta^{ij}E_{i,j} = 0, \tag{3.38}$$

i.e. E^S_{ij} is symmetric and traceless, E^V_{ij} is symmetric, traceless and divergenless, and the tensorial part has properties

$$\delta^{ik}E^T_{ij,k} = 0 \quad \delta^{ij}E^T_{ij} = 0, \tag{3.39}$$

i.e. it is tranverse and traceless.

All in all the perturbed metric encompasses 4 scalars (A,B, D, E), 2 vectors (B^V_i, E_i) and one tensor E^T_{ij}. This makes all together 10 degrees of freedom, since scalars each have 1 degree of freedom, vectors have 2 (helicity ±1) and the tensorial part has also 2 degrees of freedom (helicity ±2). The scalar perturbations are the most important and difficult ones. In the following sections we'll see that they couple to the density and pressure perturbations of the stress-energy tensor and they are understood as the prime factor of primordial density perturbations in the early Universe, possibly and most likely explaining the formation of structure and the temperature fluctuations in the cosmic microwave background [1, 2, 5, 11, 12]. As I said earlier, the vector perturbations have been shown to have only decaying solutions so in this thesis I don't discuss them at all. The tensor perturbations on the contrary are interesting because they are believed to be gravity waves and they do not necessarily couple to energy-momentum at all. They evolve independently and could also have also left marks to the CMB.

Gauge transformations of the metric perturbations

When applying the above tensor transformation laws to the metric perturbation one gets

$$\begin{aligned}\hat{\delta} g_{\mu\nu} &= \delta g_{\mu\nu} - \xi^{\rho}{}_{,\mu}\bar{g}_{\rho\nu} - \xi^{\rho}{}_{,\nu}\bar{g}_{\mu\rho} - \xi^0 \bar{g}_{\mu\nu,0} \\ &= \delta g_{\mu\nu} - a^2\left(\xi^{\rho}{}_{,\mu}\eta_{\rho\nu} + \xi^{\rho}{}_{,\nu}\eta_{\mu\rho} + 2\mathcal{H}\eta_{\mu\nu}\xi^0\right), \end{aligned} \tag{3.40}$$

where I used the conformal flat FRW metric $\bar{g}_{\mu\nu} = a^2\eta_{\mu\nu}$ and $\bar{g}_{\mu\nu,0} = 2a'a\eta_{\mu\nu}$. Studying the independent components it is possible to get the transformation laws for the metric perturbations A, B_i, D, E_{ij}. For example

$$\begin{aligned}\hat{\delta} g_{00} \equiv -2a^2\hat{A} &= \delta g_{00} - a^2\left(\xi^{\rho}{}_{,0}\eta_{\rho 0} + \xi^{\rho}{}_{,0}\eta_{0\rho} + 2\mathcal{H}\eta_{00}\xi^0\right) \\ &= -2a^2\left(A - \xi^0{}_{,0} - \mathcal{H}\xi^0\right), \end{aligned} \tag{3.41}$$

from which it's possible to identify a transformation law

$$\hat{A} = A - \xi^0{}_{,0} - \mathcal{H}\xi^0. \tag{3.42}$$

Similar analysis for the other perturbations gives

$$\hat{B}_i = B_i + \xi^i{}_{,0} - \xi^0{}_{,i} \tag{3.43}$$

$$\hat{D} = D - \frac{1}{3}\xi^k{}_{,k} + \mathcal{H}\xi^0 \tag{3.44}$$

$$\hat{E}_{ij} = E_{ij} - \frac{1}{2}(\xi^i{}_{,j} + \xi^j{}_{,i}) + \frac{1}{3}\delta_{ij}\xi^k{}_{,k} \tag{3.45}$$

3.3 Perturbations of the energy tensor

The background energy tensor is necessarily of the form

$$\bar{T}^{\mu\nu} = (\bar{\rho} + \bar{p})\bar{u}^{\mu}\bar{u}^{\nu} + \bar{p}\bar{g}^{\mu\nu}, \tag{3.46}$$

and due to homogenuity $\bar{\rho} = \bar{\rho}(\tau)$ and $\bar{p} = \bar{p}(\tau)$. Due to isotropy, the fluid is at rest in the background: $\bar{u}^{\mu} = (\bar{u}^0,0,0,0)$. We know also that

$$\bar{u}^{\mu}\bar{u}_{\mu} = a^2\eta_{\mu\nu}\bar{u}^{\mu}\bar{u}^{\nu} = -a^2(\bar{u}^0)^2 = -1, \tag{3.47}$$

so that

$$\bar{u}^{\mu} = \frac{1}{a}(1,\vec{0}), \quad \bar{u}_{\mu} = a(-1,\vec{0}). \tag{3.48}$$

The total energy tensor is then divided into background and perturbation:

$$T^{\mu\nu} = \bar{T}^{\mu\nu} + \delta T^{\mu\nu}. \tag{3.49}$$

We define the density, pressure and velocity perturbations:

$$\rho = \bar{\rho} + \delta\rho \tag{3.50}$$

$$p = \bar{p} + \delta p \tag{3.51}$$

$$u^i = \bar{u}_i + \delta u^i = \delta u^i \equiv \frac{1}{a}v^i, \tag{3.52}$$

where we used $\bar{u}_i = 0$. Next we write the velocities in terms of v^i:

$$u^\mu = \bar{u}^\mu + \delta u^\mu \equiv \frac{1}{a}(1 + a\delta u^0, v^1, v^2, v^3) \tag{3.53}$$
$$u_\mu = \bar{u}_\mu + \delta u_\mu \equiv (-a + \delta u_0, \delta u_1, \delta u_2, \delta u_3), \tag{3.54}$$

which are related by $u^\mu = g^{\mu\nu} u_\nu$ and $u^\mu u_\nu = -1$. Using the general perturbed metric one finds (to first order)

$$u_0 = g_{0\mu} u^\mu = -a - a^2 \delta u^0 - 2aA, \tag{3.55}$$

from which it follows that

$$\delta u_0 = -(a^2 \delta u^0 + 2aA). \tag{3.56}$$

Similarly

$$u_i = \delta u_i = g_{i\mu} u^\mu = -aB_i + av_i. \tag{3.57}$$

Furthermore, from $u^\mu u_\mu = -1$ I get

$$\delta u^0 = -\frac{A}{a}. \tag{3.58}$$

Thus the total 4-velocities are

$$u^\mu = \frac{1}{a}(1 - A, v_i) \tag{3.59}$$
$$u_\mu = a(-1 - A, v_i - B_i). \tag{3.60}$$

Then the energy tensor is

$$T^\mu_\nu = \bar{T}^\mu_\nu + \delta T^\mu_\nu \tag{3.61}$$
$$= \begin{bmatrix} -\bar{\rho} & 0 \\ 0 & \bar{p}\delta^i_j \end{bmatrix} + \begin{bmatrix} -\delta\rho & (\bar{\rho} + \bar{p})(v_i - B_i) \\ -(\bar{\rho} + \bar{p})v_i & \delta p \delta^i_j + \Sigma_{ij} \end{bmatrix}, \tag{3.62}$$

where I have defined the spatial perturbation as a sum of a perfect and non-perfect fluid components

$$\delta T^i_j \equiv \delta p \delta^i_j + \Sigma_{ij} \equiv \bar{p}\left(\frac{\delta p}{\bar{p}} + \Pi_{ij}\right). \tag{3.63}$$

Here both Σ_{ij}, Π_{ij} are symmetric and traceless so that I can write the pressure perturbation as a trace

$$\delta p \equiv \frac{1}{3}\delta T^k_k \tag{3.64}$$

and define the *anisotropic stress* as the traceless part

$$\Sigma_{ij} \equiv \delta T^i_j - \frac{1}{3}\delta^i_j \delta T^k_k. \tag{3.65}$$

The velocity perturbation v_i and the 3-tensor Π_{ij} can also be decomposed by the SVT-decomposition.

Fluctuations in the energy density are said to be *adiabatic* if locally the total density $\rho_i = \bar{\rho}_i + \delta\rho_i$ of a given fluid component is the same as in the background but at a slightly different time:

$$\delta\rho_i(\tau, \mathbf{x}) \equiv \rho_i(\tau + \delta\tau(\mathbf{x})) - \bar{\rho}_i(\tau) = \bar{\rho}_i'\delta\tau(\mathbf{x}). \tag{3.66}$$

This implies

$$\delta\tau = \frac{\delta\rho_i}{\bar{\rho}_i'} = \frac{\delta\rho_j}{\bar{\rho}_j'} \quad \text{for all } i \text{ and } j. \tag{3.67}$$

Fluctuations that are not purely adiabatic are said to possess *isocurvature modes*, and these kind of modes can emerge easily from multi-field inflation. Recent measurements are consistent with no deviations from adiabatic initial conditions [10].

Gauge transformations of the energy tensor

The above energy perturbations obey the gauge transformation rules presented in Section 3.1:

$$\begin{cases} \tilde{\delta T}^0_0 = -\tilde{\delta}\rho = -\delta\rho + \bar{\rho}'\xi^0 \\ \tilde{\delta T}^i_0 = -(\bar{\rho} + \bar{p})\tilde{v}_i = -(\bar{\rho} + \bar{p})v_i - \xi^i_{,0}(\bar{\rho} + \bar{p}) \\ \dfrac{1}{3}\tilde{\delta T}^k_k = \tilde{\delta}p = \dfrac{1}{3}(\delta T^k_k - \bar{T}^k_{k,0}\xi^0) = \delta p - \bar{p}'\xi^0 \\ \tilde{\Sigma}_{ij} = \Sigma_{ij}. \end{cases} \tag{3.68}$$

Or equivalently:

$$\begin{cases} \tilde{\delta}\rho = \delta\rho - \bar{\rho}'\xi^0 \\ \tilde{\delta}p = \delta p - \bar{p}'\xi^0 \\ \bar{v}_i = v_i + \xi^i_{,0} \\ \tilde{\delta} \equiv \dfrac{\tilde{\delta}\rho}{\bar{\rho}} = \delta - \dfrac{\bar{\rho}'}{\bar{\rho}}\xi^0 = \delta + 3\mathcal{H}(1 + w)\xi^0, \end{cases} \tag{3.69}$$

where δ is the *relative energy density perturbation.* The last equality follows from (2.13).

3.4 Scalar perturbations

Consider now purely scalar perturbations. The shift vector is now $B_i = B_{,i}$, the velocity perturbation is $v_i = v_{,i}$ and the tensor perturbation is $E_{ij} = E_{,ij}$, where B,v and E are scalars. The most general metric for the perturbed universe involving only scalar perturbations is then

$$\mathrm{d}s^2 = a^2(\tau)\left\{(-1 - 2A)\mathrm{d}\tau^2 + 2B_{,i}\mathrm{d}x^i\mathrm{d}\tau + \left[(1 - 2D)\delta_{ij} + 2(\partial_i\partial_j + \frac{1}{3}\nabla^2)E\right]\mathrm{d}x^i\mathrm{d}x^j\right\}. \tag{3.70}$$

I now define the *curvature perturbation* as

$$\psi \equiv D + \frac{1}{3}\nabla^2 E, \tag{3.71}$$

so that the metric takes the form

$$\mathrm{d}s^{\,2} = a^2(\tau)\left[(-1-2A)\mathrm{d}\tau^{\,2} + 2B_{,i}\mathrm{d}x^{\,i}\mathrm{d}\tau\, + ((1-2\psi)\delta_{ij} + 2E_{,ij})\,\mathrm{d}x^{\,i}\mathrm{d}x^{\,j}\right]. \tag{3.72}$$

The name of the curvature perturbation comes from the fact [14, 15] that it allows to write the Ricci tensor of the 3-dimensional spatial hypersurfaces for the metric (3.70) in the form

$${}^{(3)}R = \frac{4}{a^2}\nabla^2\psi. \tag{3.73}$$

Even if one started from pure scalar perturbations, the gauge transformation generated by the the vector field $\xi^\alpha = \xi^\alpha(\tau, x^i) = (\xi^0, \xi^i)$ could induce some vector perturbations to the metric. But since in first order theory the scalar and vector parts do not couple, the induced vectors are considered as pure gauge, i.e. not physical. Thus to avoid pure gauge degrees of freedom, I divide the spatial part of ξ^i as usual to zero-divergence and zero-curl parts

$$\xi^i = \xi^i_\perp - \xi^{,i} \tag{3.74}$$

with $\xi^i_{\perp,i} = 0$ (divergenless) and ξ is a scalar. The pure vector part $\xi^i_\perp$ is the one responsible of the undesired pure vector gauge degrees of freedom which would nonetheless decay away, so I drop the term $\xi^i_\perp$ and focus only on the scalar parts ξ^0 and ξ. One can then write

$$\begin{cases} \tau = \tau + \xi^0 \\ x^i = x^i - \delta^{ij}\xi_{,i} \end{cases} \quad \text{the scalar gauge transformations.} \tag{3.75}$$

Applying the above transformations to the scalar perturbations one obtains in this special case:

$$\hat{A} = A - {\xi^0}_{,0} - \mathcal{H}\xi^0 \tag{3.76}$$
$$\hat{B} = B + \xi^0 + \xi_{,0} \tag{3.77}$$
$$\hat{D} = D - \frac{1}{3}\nabla^2\xi + \mathcal{H}\xi^0 \tag{3.78}$$
$$\hat{E} = E + \xi \tag{3.79}$$
$$\hat{\psi} = \psi + \mathcal{H}\xi^0. \tag{3.80}$$

Now I choose a specific gauge, the *conformal Newtonian gauge*. I perform all the upcoming calculations using it. We get to the conformal Newtonian gauge by setting $B = E = 0$. This can be effected by the gauge choice:

$$\xi^0 = -B + E' \quad \text{and} \quad \xi = -E. \tag{3.81}$$

This choice of gauge gives (denote by superscript 'N' the conformal Newtonian gauge)

$$A^N = A + (B - E')' + \mathcal{H}(B - E') \tag{3.82}$$

$$D^N = \psi^N = D + \frac{1}{3}\nabla^2 E + \mathcal{H}(-B + E') = \psi - \mathcal{H}(B - E') \tag{3.83}$$

The remaining metric perturbations are thus A^N and ψ^N. The conformal Newtonian gauge provides an elegant way for obtaining gauge-invariant field equations [16]. First one has to note that by making linear combinations of the metric perturbations one can construct gauge invariant variables. Two simplest ones are the *Bardeen potentials* Ψ and Φ defined by

$$\Psi \equiv A + \frac{1}{a}\left[a(B - E')\right]' \tag{3.84}$$

$$\Phi \equiv \psi - \mathcal{H}(B - E'). \tag{3.85}$$

In the conformal Newtonian gauge the remaining metric variables thus coincide with the gauge-invariant metric perturbations: $A^N = \Psi$ and $\psi^N = \Phi$. The scalar perturbed metric in terms of the Bardeen potentials is then simply, replacing A^N and ψ^N with Ψ and Φ:

$$g_{\mu\nu} = a^2 \begin{bmatrix} -(1+2\Psi) & 0 \\ 0 & (1-2\Phi)\delta_{ij} \end{bmatrix} \quad \text{and} \quad g^{\mu\nu} = a^{-2} \begin{bmatrix} -1+2\Psi & 0 \\ 0 & (1+2\Phi)\delta_{ij} \end{bmatrix}. \tag{3.86}$$

Or writing only the perturbations, the following section in mind,

$$h_{\mu\nu} = \begin{bmatrix} -2\Psi & 0 \\ 0 & -2\Phi\delta_{ij} \end{bmatrix} \quad \text{and} \quad h^{\mu\nu} = \begin{bmatrix} -2\Psi & 0 \\ 0 & -2\Phi\delta_{ij} \end{bmatrix}. \tag{3.87}$$

3.5 Scalar perturbations on the curvature tensors in the conformal Newtonian gauge

Perturbations on the connection $\Gamma^\mu_{\nu\lambda} = \bar{\Gamma}^\mu_{\nu\lambda} + \delta\Gamma^\mu_{\nu\lambda}$ are easy to figure out by expanding the the total Christoffel symbol $\Gamma^\mu_{\nu\lambda}$ to first order in the metric perturbation. The background symbols and the perturbations are then easy to separate resulting

$$\bar{\Gamma}^\mu_{\nu\lambda} = \frac{1}{2}\bar{g}^{\mu\rho}\left(\partial_\lambda \bar{g}_{\nu\rho} + \partial_\nu \bar{g}_{\lambda\rho} - \partial_\rho \bar{g}_{\nu\lambda}\right) \tag{3.88}$$

$$\delta\Gamma^\mu_{\nu\lambda} = \frac{1}{2}\bar{g}^{\mu\rho}\left(\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\lambda} - \partial_\rho h_{\lambda\nu} - 2h_{\rho\sigma}\bar{\Gamma}^\sigma_{\nu\lambda}\right). \tag{3.89}$$

The only non-vanishing components of the unperturbed connection in the FLRW-universe are

$$\bar{\Gamma}^\mu_{0\nu} = \mathcal{H}\delta^\mu_\nu \quad \text{and} \quad \bar{\Gamma}^0_{\mu\nu} = \mathcal{H}\delta_{\mu\nu}, \tag{3.90}$$

and thus the perturbed connection coefficients are

$$\begin{aligned} \delta\Gamma^0_{00} &= \Psi' \\ \delta\Gamma^0_{0k} &= \Psi_{,k} \\ \delta\Gamma^0_{ij} &= -\left[\Phi' + 2\mathcal{H}(\Phi + \Psi)\right]\delta_{ij} \\ \delta\Gamma^i_{00} &= \Psi_{,i} \\ \delta\Gamma^i_{0j} &= -\Phi'\delta_{ij} \\ \delta\Gamma^i_{jk} &= \Phi_{,i}\delta_{jk} - \Phi_{,j}\delta_{ik} - \Phi_{,k}\delta_{ij}. \end{aligned} \tag{3.91}$$

The Ricci tensor $R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu}$ is

$$\begin{aligned} R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = \bar{R}_{\mu\nu} &+ \delta\Gamma^{\alpha}_{\nu\mu,\alpha} - \delta\Gamma^{\alpha}_{\alpha\mu,\nu} + \bar{\Gamma}^{\alpha}_{\alpha\beta}\delta\Gamma^{\beta}_{\nu\mu} \\ &+ \bar{\Gamma}^{\beta}_{\nu\mu}\delta\Gamma^{\alpha}_{\alpha\beta} - \bar{\Gamma}^{\alpha}_{\nu\beta}\delta\Gamma^{\beta}_{\alpha\mu} - \bar{\Gamma}^{\beta}_{\alpha\mu}\delta\Gamma^{\alpha}_{\nu\beta}. \end{aligned} \tag{3.92}$$

A straightforward calculation gives

$$\begin{aligned} R_{00} &= -3\mathcal{H}' + 3\Phi'' + \nabla^2\Psi + 3\mathcal{H}(\Psi' + \Phi') \\ R_{0i} &= 2(\Phi' + \mathcal{H}\Psi)_{,i} \\ R_{ij} &= (\mathcal{H}' + 2\mathcal{H}^2)\delta_{ij} \\ &\quad + \left[-\Phi'' + \nabla^2\Phi - \mathcal{H}(\Psi' + 5\Phi') - (2\mathcal{H}' + 4\mathcal{H}^2)(\Phi + \Psi)\right]\delta_{ij} \\ &\quad + (\Phi - \Psi)_{,ij}. \end{aligned} \tag{3.93}$$

Now we need to raise the second index of our Ricci tensor (since our stress-energy tensor is in the mixed form). This gives

$$R^{\mu}_{\nu} = g^{\mu\alpha}R_{\alpha\nu} = (\bar{g}^{\mu\alpha} + \delta g^{\mu\alpha})(\bar{R}_{\alpha\nu} + \delta R_{\alpha\nu}) = \bar{R}^{\mu}_{\nu} + \delta g^{\mu\alpha}\bar{R}_{\alpha\nu} + \bar{g}^{\mu\alpha}\delta R_{\alpha\nu}. \tag{3.94}$$

One gets

$$\begin{aligned} R^0_0 &= \frac{1}{a^2}3\mathcal{H}' + \frac{1}{a^2}\left[-6\mathcal{H}'\Psi - 3\Phi'' - \nabla^2\Psi - 3\mathcal{H}(\Psi' + \Phi')\right] \\ R^0_i &= -\frac{2}{a^2}\left(\Phi' + \mathcal{H}\Psi\right)_{,i} \\ R^i_0 &= \frac{2}{a^2}\left(\Phi' + \mathcal{H}\Psi\right)_{,i} \\ R^i_j &= \frac{1}{a^2}(\mathcal{H}' + 2\mathcal{H}^2)\delta^i_j \\ &\quad + \frac{1}{a^2}\left[-\Phi'' + \nabla^2\Phi - \mathcal{H}(\Psi' + 5\Phi') - (2\mathcal{H}' + 4\mathcal{H}^2)\Psi\right]\delta^i_j \\ &\quad + \frac{1}{a^2}(\Phi - \Psi)_{,ij}. \end{aligned} \tag{3.95}$$

The Ricci scalar is then

$$\begin{aligned} R = R^0_0 + R^i_i &= \frac{6}{a^2}(\mathcal{H}' + \mathcal{H}^2) \\ &+ \frac{1}{a^2}\left[-12(\mathcal{H}' + \mathcal{H}^2)\Psi - 6\Phi'' + 2\nabla^2(2\Phi - \Psi) - 6\mathcal{H}(\Psi' + 3\Phi')\right]. \end{aligned} \tag{3.96}$$

And finally, the Einstein tensor $G^{\mu}_{\nu} = R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R$ is

$$\begin{aligned} G^0_0 &= -\frac{3}{a^2}\mathcal{H}^2 + \frac{2}{a^2}\left[3\mathcal{H}\Phi' + 3\mathcal{H}^2\Psi - \nabla^2\Phi\right] \\ G^0_i &= R^0_i \\ G^i_0 &= R^i_0 \\ G^i_j &= \frac{1}{a^2}(-2\mathcal{H}' - \mathcal{H}^2)\delta^i_j \\ &\quad + \frac{1}{a^2}\left[2\Phi'' + \nabla^2(\Psi - \Phi) + 2\mathcal{H}(\Psi' + 2\Phi') + 2(2\mathcal{H}' + \mathcal{H}^2)\Psi\right]\delta^i_j \\ &\quad - \frac{1}{a^2}(\Psi - \Phi)_{,ij}. \end{aligned} \tag{3.97}$$

These are thus the components of the Einstein tensor in the conformal Newtonian gauge. In other gauges they look different.

3.6 Einstein equations in the conformal Newtonian gauge

The linearized Einstein equations are

$$\bar{G}_{\mu\nu} + \delta G_{\mu\nu} = 8\pi G[\bar{T}_{\mu\nu} + \delta T_{\mu\nu}], \tag{3.98}$$

from which it is possible to obtain the perturbation equations by assuming the background equation $\bar{G}_{\mu\nu} = 8\pi G\bar{T}_{\mu\nu}$ to hold. The perturbation equation is then just

$$\delta G_{\mu\nu} = 8\pi G\delta T_{\mu\nu}. \tag{3.99}$$

Before I write the perturbed Einstein equations I need the stress-energy tensor in the conformal Newtonian gauge with pure scalar perturbations. In this gauge $v_i^N = -v_{,i}^N$ and $B_i^N = B_{,i}^N = 0$. I also need to compose the traceless tensor part Π_{ij} in a usual way to extract the scalar part. I get

$$\Pi_{ij} = \Pi_{ij}^S + \Pi_{ij}^V + \Pi_{ij}^T, \tag{3.100}$$

where

$$\Pi_{ij}^S = (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\Pi, \tag{3.101}$$

and Π is a scalar. Dropping the vectorial and tensorial parts Π_{ij}^V and Π_{ij}^T allows one to write

$$\delta T^{\mu}_{\nu} = \begin{bmatrix} -\delta\rho^N & -(\bar{\rho}+\bar{p})v_{,i}^N \\ (\bar{\rho}+\bar{p})v_{,i}^N & \delta p^N\delta^i_j + \bar{p}(\Pi_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\Pi) \end{bmatrix}, \tag{3.102}$$

where the the superscript N indicates that we are in a conformal Newtonian gauge. Now the perturbed Einstein equations $\delta G^{\mu}_{\nu} = 8\pi G\delta T^{\mu}_{\mu}$ become

$$3\mathcal{H}(\Phi' + \mathcal{H}\Psi) - \nabla^2\Phi = -4\pi Ga^2\delta\rho^N \tag{3.103}$$

$$(\Phi' + \mathcal{H}\Psi)_{,i} = 4\pi Ga^2(\bar{\rho}+\bar{p})v_{,i}^N \tag{3.104}$$

$$\begin{aligned}&[\Phi'' + \frac{1}{2}\nabla^2(\Psi-\Phi) + \mathcal{H}(\Psi+2\Phi)' + \Phi(2\mathcal{H}' + \mathcal{H}^2)]\delta^i_j - \frac{1}{2}(\Psi-\Phi)_{,ij} \\ &= 4\pi Ga^2[\delta p^N\delta^i_j + \bar{p}(\Pi_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\Pi)].\end{aligned} \tag{3.105}$$

The first equation refers to the (0,0)-component, second one to the (0,i)-component and the third one to the (i,j)-component. The spatial derivatives on the both sides of the second equation can be dropped since otherwise the equality would only up to a function that is linear in coordinates. The coordinate dependent function could then get arbitrary large values and break the perturbative analysis. One can then conclude that

$$\Phi' + \mathcal{H}\Psi = 4\pi Ga^2(\bar{\rho}+\bar{p})v^N. \tag{3.106}$$

This can be subtracted from the first equation resulting a generalization of the usual Poisson equation

$$\nabla^2\Psi = 4\pi G[a^2\delta\rho^N + 3\mathcal{H}a^2(\bar{\rho}+\bar{p})v^N]. \tag{3.107}$$

The spatial Einstein equation (4.19) can be split into trace and traceless part: trace of δ^i_j is 3 and the trace of $(\Psi - \Phi)_{,ij}$ is $\nabla^2(\Psi - \Phi)$. Then the trace of (i,j)-component is

$$\Phi'' + \mathcal{H}(\Psi + 2\Phi)' + \Psi(2\mathcal{H}' + \mathcal{H}^2) + \frac{1}{3}\nabla^2(\Psi - \Phi) = 4\pi G a^2 \delta p^N \tag{3.108}$$

and the part without trace is

$$(\partial_i\partial_j - \frac{1}{3}\delta^i_j\nabla^2)(\Psi - \Phi) = 8\pi G a^2 \bar{p}(\partial_i\partial_j - \frac{1}{3}\delta^i_j\nabla^2)\Pi. \tag{3.109}$$

3.7 Summary

In this section I introduced cosmological perturbation theory. I presented the real physical spacetime as a non-homogenous and non-isotropic manifold such that it deviates only little from the FRW-spacetime. I then splitted the tensorial quantities of the physical spacetime into a background part plus perturbation part. Considering the perturbations to be first order small, I concluded that the scalar, vector and tensor perturbations evolve independently and can be Fourier transformed. An issue of gauge dependence rises from the fact that perturbations around a given background are not uniquely defined. I performed the SVT-decomposition to the perturbations of the metric tensor and the stress-energy tensor. By choosing the so called conformal Newtonian gauge I reduced the number of independent scalar metric perturbations from 4 to 2 and noticed that the remaining quantities coincide with gauge invariant Bardeen potentials. After some algebra I obtained ordinary differential equations for the first order perturbations from Einstein field equations in the conformal Newtonian gauge. In next section I'm going to apply the tools and equations of cosmic perturbation theory to a single scalar field inflation.

4 Perturbations from inflation

The strategy now is to take a single scalar field, introduce perturbations to it and then determine the perturbed stress-energy tensor in terms of scalar field perturbation. That done, one can plug everything to the perturbed Einstein equations and solve the evolution of a quantum scalar field during and after inflation. Take now a scalar field $\varphi(\tau, \mathbf{x}) = \bar{\varphi}(\tau) + \delta\varphi(\tau, \mathbf{x})$ with action

$$\int \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right)\sqrt{-g}\mathrm{d}x^{\,4} \tag{4.1}$$

and the scalar perturbed FRW-metric in the conformal Newtonian gauge:

$$\mathrm{d}s^{\,2} = a^2(\tau)\left[(-1-2\Psi)\mathrm{d}\tau^{\,2} + (1-2\Phi)\delta_{ij}\mathrm{d}x^{\,i}\mathrm{d}x^{\,j}\right]. \tag{4.2}$$

The Klein-Gordon equation for the scalar field is

$$\partial_\mu\partial^\mu\varphi - V_\varphi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\varphi) - V_\varphi = 0, \tag{4.3}$$

where $V_\varphi \equiv \frac{\partial V}{\partial\varphi}$ and where the metric determinant is

$$\sqrt{-g} = a^4(1+2\Psi-6\Phi)^{\frac{1}{2}} \approx a^4(1+\Psi-3\Phi). \tag{4.4}$$

Expansion of the potential to first order gives

$$V(\varphi) = V(\bar{\varphi}+\delta\varphi) = V(\bar{\varphi}) + \frac{\partial V}{\partial\varphi}\delta\varphi. \tag{4.5}$$

The derivative becomes

$$V_\varphi = \frac{\partial V}{\partial\varphi}(\bar{\varphi}+\delta\varphi) = \frac{\partial V}{\partial\varphi}(\bar{\varphi}) + \frac{\partial^2 V}{\partial\varphi^2}\delta\varphi \equiv \bar{V}_\varphi + \bar{V}_{\varphi\varphi}\delta\varphi. \tag{4.6}$$

Plugging everything in to the Klein-Gordon, one can extract the zeroth order part (the background equation), which is

$$\bar{\varphi}'' + 2\mathcal{H}\bar{\varphi}' + a^2\bar{V}_\varphi = 0. \tag{4.7}$$

The linear order part is a bit more tedious and one has to use equation (4.6) and the background equation (4.7) to get

$$\delta\varphi'' + 2\mathcal{H}\delta\varphi' - \nabla^2\delta\varphi - (\Psi' + 3\Phi')\bar{\varphi}' + a^2\bar{V}_{\varphi\varphi}\delta\varphi + 2a^2\Psi\bar{V}_\varphi = 0. \tag{4.8}$$

I call this *the field perturbation equation* and use it later to reduce the number of scalar degrees of freedom in the Einstein equations. The stress-energy tensor to first order is

$$\begin{aligned} T^\mu_\nu = {} & g^{\alpha\mu}\partial_\alpha\bar{\varphi}\partial_\nu\bar{\varphi} + g^{\alpha\mu}\partial_\alpha\bar{\varphi}\partial_\nu(\delta\varphi) + g^{\alpha\mu}\partial_\alpha(\delta\varphi)\partial_\nu\bar{\varphi} - \frac{1}{2}\delta^\mu_\nu\partial_\rho\bar{\varphi}g^{\rho\alpha}\partial_\alpha\bar{\varphi} \\ & - \delta^\mu_\nu\partial_\rho\bar{\varphi}g^{\rho\alpha}\partial_\alpha(\delta\varphi) - \delta^\mu_\nu V(\bar{\varphi}) - \delta^\mu_\nu V_\varphi\delta\varphi, \end{aligned} \tag{4.9}$$

from which the background energy tensor is just the familiar

$$\bar{T}^{\mu}_{\nu} = \partial^{\mu}\bar{\varphi}\partial_{\nu}\bar{\varphi} - \frac{1}{2}\delta^{\mu}_{\nu}\partial_{\rho}\bar{\varphi}\partial^{\rho}\bar{\varphi} - \delta^{\mu}_{\nu}V(\bar{\varphi}) \tag{4.10}$$

and the first order perturbations are

$$\delta T^0_0 = \frac{1}{a^2}\left[\Psi(\bar{\varphi}')^2 - \bar{\varphi}'\delta\varphi' - a^2 V_{\varphi}\delta\varphi\right] \tag{4.11}$$

$$\delta T^0_i = -\frac{1}{a^2}\bar{\varphi}'\delta\varphi_{,i} \tag{4.12}$$

$$\delta T^i_0 = \frac{1}{a^2}\bar{\varphi}'\delta\varphi_{,i} \tag{4.13}$$

$$\delta T^i_j = -\frac{1}{a^2}\delta^i_j\left[\Psi(\bar{\varphi}')^2 - \bar{\varphi}'\delta\varphi' + a^2 V_{\varphi}\delta\varphi\right]. \tag{4.14}$$

The gauge of the scalar field perturbations is still to be fixed. As the metric is in conformal Newtonian gauge, I choose the same gauge for the field. The field $\delta\varphi$ transforms as a scalar and the conformal Newtonian gauge is defined by $\xi^0 = -B + E'$, so I get

$$\delta\varphi \to \delta\varphi_N = \delta\varphi + \bar{\varphi}'(B - E').$$

So again, in the conformal Newtonian gauge the scalar field perturbation coincides with the gauge-invariant field perturbation. To avoid a huge mess of sub- and superscripts in which we are spiraling towards with our equations, I still continue to denote the field in conformal Newtonian gauge just by $\delta\varphi$, instead of $\delta\varphi^{(\mathrm{gi})}$ or $\delta\varphi_N$.

Before moving onwards, it is good to notice one property of the stress-energy tensor for a scalar field: it's spatial part is diagonal, meaning that it has no anisotropic stress, i.e. the the traceless part of the energy tensor is

$$\Pi_{ij} = (\partial_i\partial_j - \frac{1}{3}\Delta_{ij}\nabla^2)\Pi = 0. \tag{4.15}$$

Then from the traceless spatial component of the Einstein equations I get that

$$(\partial_i\partial_j - \frac{1}{3}\delta^i_j\nabla^2)(\Psi - \Phi) = 0. \tag{4.16}$$

This implies that $\Psi - \Phi$ is either null or a quadratic function of the spatial coordinates. The latter is impossible since the perturbations could become arbitrarily large for large coordinate values. I conclude that they have to be equal, i.e. $\Psi = \Phi$, and I can write the Einstein equations only in terms of Ψ and the field $\delta\varphi$. The resulting equations are (componentwise in the following order: (0,0), (0,i) and (i,j))

$$3\mathcal{H}\Psi' + (3\mathcal{H}^2 - \nabla^2)\Psi = 4\pi G\left[\Psi(\bar{\varphi}')^2 - \bar{\varphi}'\delta\varphi' - a^2 V_{\varphi}\delta\varphi\right] \tag{4.17}$$

$$(\Psi' + \mathcal{H}\Psi)_{,i} = 4\pi G\bar{\varphi}'\delta\varphi_{,i} \tag{4.18}$$

$$\Psi'' + 3\mathcal{H}\Psi' + (2\mathcal{H}' + \mathcal{H}^2)\Psi = -4\pi G\left[\Psi(\bar{\varphi}')^2 - \bar{\varphi}'\delta\varphi' + a^2 V_{\varphi}\delta\varphi\right] \tag{4.19}$$

These are accompanied with the field perturbation equation which now reads

$$\delta\varphi'' + 2\mathcal{H}\delta\varphi' - \nabla^2\delta\varphi - 4\Psi'\bar{\varphi}' + a^2\bar{V}_{\varphi\varphi}\delta\varphi + 2a^2\Psi\bar{V}_{\varphi} = 0. \tag{4.20}$$

These are the most general equations governing the evolution of scalar perturbations of the metric coupled to a scalar field driving the inflation. Combining the equations above suitably one can extract different second order differential equations to analyze the behaviour of the perturbations at different stages of inflation. Next I'll derive an equation for the gravitational potential Ψ.

Evolution equation for the gravitational potential

Next I proceed towards a single second order differential equation for the potential Ψ only. The equation proves to be useful when I substantiate the non-evolution of the curvature perturbation on superhorizon regime. It also finally provides a way to determine a second order differential equation for the Mukhanov-Sasaki variable. I'm coming back to these issues later. Here I mainly follow Mukhanov, Feldman and Brandenberger [16]. This derivation requires the background field equation (4.7) and the identity

$$\mathcal{H}^2 - \mathcal{H}' = 4\pi G(\bar{\varphi}')^2, \tag{4.21}$$

which is a combination of the Friedmann equations in the background FRW-universe with a homogenous and isotropic scalar field. One can subtract the first Einstein equation (4.17) from the third one (4.19) and use the second equation (4.18) to express $\delta\varphi$ in terms of Ψ. After that one can use (4.7) and (4.21) and finally obtain a second order differential equation for Ψ only

$$\Psi'' - \nabla^2\Psi + 2(\mathcal{H} - \frac{\bar{\varphi}''}{\bar{\varphi}'})\Psi' + 2(\mathcal{H}' - \mathcal{H}\frac{\bar{\varphi}''}{\bar{\varphi}'})\Psi = 0. \tag{4.22}$$

This can be also written in the form

$$\Psi'' - \nabla^2\Psi + 2\left(\frac{a}{\bar{\varphi}'}\right)'\left(\frac{a}{\bar{\varphi}'}\right)^{-1}\Psi' + 2\bar{\varphi}'\left(\frac{\mathcal{H}}{\bar{\varphi}'}\right)'\Psi = 0. \tag{4.23}$$

Using the equation of state parameter and a relation

$$c^2 - w = \frac{-2}{3\mathcal{H}^2}\left(\mathcal{H}' - \mathcal{H}\frac{\bar{\varphi}''}{\bar{\varphi}'}\right) \tag{4.24}$$

the above equation can be written in a yet another form

$$\Psi'' - \nabla^2\Psi + 3\mathcal{H}(1 + c^2)\Psi' - 2\mathcal{H}(c^2 - w)\Psi = 0. \tag{4.25}$$

When I furthermore define a quantity $u = \frac{a}{\bar{\varphi}'}\Psi$, I can write the above equation simply

$$u'' - \nabla^2 u - \frac{\theta''}{\theta}u = 0, \quad \text{where} \quad \theta \equiv \frac{\mathcal{H}}{a\bar{\varphi}'}. \tag{4.26}$$

Finally, in Fourier space this reads

$$u''_{\mathbf{k}} + k^2 u_{\mathbf{k}} - \frac{\theta''}{\theta}u_{\mathbf{k}} = 0. \tag{4.27}$$

This is a quite generic form of an equation of motion when studying single field inflation. Though it is not yet quite useful when studying perturbations inside the horizon, but rather useful for post-inflatory evolution for example when studying the structure formation. However, one can use this to show the non-evolution of the comoving curvature perturbation $\mathcal{R}$ and this is the topic of the next section.

4.1 Comoving curvature perturbation

When discussing perturbations at superhorizon scales, it is extremely useful to deal with a new variable called the *comoving curvature perturbation* $\mathcal{R}$. As the name implies, the variable $\mathcal{R}$ is defined to be the spatial curvature perturbation defined in (3.71) in the comoving gauge. The comoving gauge is defined to be the gauge in which the velocity perturbation and the shift vector both vanish, i.e. $v = B_i = 0$. This can be effected by choosing

$$\xi = v - B. \tag{4.28}$$

The curvature perturbation then is

$$\mathcal{R} \equiv \psi^C = \psi + \mathcal{H}(v - B). \tag{4.29}$$

As an exercise, I'm next going to show how the comoving gauge is defined when dealing with a scalar field, working on a little bit backwards manner to find out what is ξ^0 in (4.29) in terms of the scalar field perturbation. For a scalar field, the velocity perturbation can be defined as

$$v_i = -\frac{\delta T^0_i}{\bar{\rho} + \bar{p}}, \tag{4.30}$$

so the vanishing of v can be accomplished by setting $\delta T^0_i \equiv 0$ with a gauge transformation. The transformation law for δT^0_i reveals that this is possible when we choose

$$\begin{aligned} \xi^0_{,i} &= \frac{-\delta T^0_i}{\bar{\rho} + \bar{p}} = \frac{\bar{\varphi}' \delta\varphi_{,i}}{(\bar{\varphi}')^2} = \frac{\delta\varphi_{,i}}{\bar{\varphi}'} \\ \Leftrightarrow \quad \xi^0 &= \frac{\delta\varphi}{\bar{\varphi}'}. \end{aligned} \tag{4.31}$$

This again implies that in the comoving gauge the scalar field perturbation is turned off, $\delta\varphi^C = 0$. Thus the comoving curvature perturbation in a Universe filled by a scalar field is defined by

$$\mathcal{R} = \psi + \mathcal{H}\frac{\delta\varphi}{\bar{\varphi}'}. \tag{4.32}$$

One can also choose a gauge called *spatially flat gauge* (denoted by subscript Q) in which $\psi_Q \equiv 0$ and hence:

$$\mathcal{R} = \mathcal{H}\frac{\delta\varphi_Q}{\bar{\varphi}'}. \tag{4.33}$$

Playing with different gauges can thus be a source of maximal confusion, but it can be turned into a huge benefit as well. However, the gauge invariant quantites are the only ones with a physical significance. Now let's study what the Einstein equations reveal of the curvature perturbation $\mathcal{R}$. Consider now a general perfect fluid perturbation. In the conformal Newtonian gauge we had $B^N = 0$ and v^N can be solved to be, using the Einstein equation (3.107) with $\Psi = \Phi$,

$$v^N = \frac{\mathcal{H}\Psi + \Psi'}{4\pi G a^2 \bar{\rho}(1 + w)}. \tag{4.34}$$

Then in this gauge it is possible to relate $\mathcal{R}$ and the Bardeen potential Ψ. Going to conformal Newtonian gauge gives

$$\mathcal{R} = \psi^N + \mathcal{H}(v^N - B^N) = \Psi + \mathcal{H}v^N. \tag{4.35}$$

Using (4.34) and (2.11) I get

$$\mathcal{R} = \Psi + \frac{2}{3}\frac{\Psi' + \mathcal{H}\Psi}{\mathcal{H}(1+w)}. \tag{4.36}$$

Now taking a time derivative of this gives

$$\begin{aligned}\frac{3}{2}(1+w)\mathcal{H}\mathcal{R}' &= \Psi'' + \frac{3}{2}(1+w)\Psi' + \mathcal{H}'\Psi + \mathcal{H}\Psi' - (\Psi' + \mathcal{H}\Psi)\left[\frac{\mathcal{H}'}{\mathcal{H}} + \frac{w'}{1+w}\right] \\ &= \Psi'' + \left[\frac{3}{2}(1+w)\mathcal{H} + \mathcal{H} - \frac{\mathcal{H}'}{\mathcal{H}} - \frac{w'}{1+w}\right]\Psi' - \frac{\mathcal{H}w'}{1+w}\Psi \\ &= \Psi'' + 3\mathcal{H}(1+c^2)\Psi' - 3\mathcal{H}(c^2 - w)\Psi \\ &= \nabla^2\Psi, \end{aligned} \tag{4.37}$$

where I used (2.14) and (4.22). In Fourier space this is

$$\frac{3}{2}(1+w)\mathcal{R}'_{\mathbf{k}} = -\frac{k^2}{\mathcal{H}}\Psi_{\mathbf{k}}. \tag{4.38}$$

From above equation it's clear that on superhorizon scales, where $k \ll \mathcal{H}$, the right hand side becomes neglible and thus $\mathcal{R}$ stays constant. The constancy of $\mathcal{R}$ on superhorizon scales is a crucial fact in cosmological perturbation theory because that enables us to connect perturbations generated during inflation to perturbations after the inflation even if we don't know anything about the physical mechanisms inside the horizon during the end of inflation and reheating. $\mathcal{R}_{\mathbf{k}}$ stays constant no matter what and enters the horizon again at much later times. There are also various other proofs for the constancy of $\mathcal{R}$, for example the rigorous and long proof of Weinberg [1], which actually states a bit more: whatever the constituents of the universe, there is always two independet adiabatic physical scalar solutions to the Einstein equations in the Newtonian gauge for which $\mathcal{R}_{\mathbf{k}}$ stays constant far outside the horizon, and there is a tensor mode for which the amplitude stays constant outside the horizon.

In next Section I'm going to derive an equation of motion for a canonical variable q, which is related to the comoving curvature perturbation by a multiplication with a factor of $z = \frac{a\bar{\varphi}'}{\mathcal{H}}$, i.e.

$$q = z\mathcal{R} = a\left[\delta\varphi + \frac{\mathcal{H}}{\bar{\varphi}'}\psi\right]. \tag{4.39}$$

4.2 Mukhanov-Sasaki equation

The perturbations of the early Universe are believed to originate from quantum fluctuations in the extremely hot plasma at the beginning of inflation. During inflation the comoving Hubble radius shrinks exponentially and thus the sub-Hubble scales $k \gg aH$

eventually exit the horizon and become super-Hubble $k \ll aH$. The reason we can make any predictions of inflation, is the fact that some perturbations were subject to certain conservation laws during inflation and up to relatively recent times [1].

The perturbation equations are second order differential equations and need initial conditions, and the intuitive idea is that they are set by the quantum nature of the perturbations at the beginning of inflation. I claim that really it is the variable $q = a\left[\delta\varphi + \frac{\mathcal{H}}{\bar{\varphi}'}\psi\right]$ that can be quantized by well known means. To get a grasp of the linearized dynamics I expand the action of a single scalar field to second order in perturbations in an *unperturbed* FRW-background. This way we I get an evolution equation similar to the full perturbed Klein-Gordon equation (4.20). A difference arises since I'm dropping the metric perturbations. Dropping the metric pertubations is fine if one wants to compute the spectral parameters to first order in slow roll, since the zeroth order power spectrum gives the first order spectral index. In fact, in spatially flat gauge the metric perturbations are suppressed by a slow-roll parameter ϵ relative to the inflaton perturbations. Thus, to leading order in slow-roll approximation the following derivation is valid.

So what I do next, is that I derive an equation of motion for the variable q to zeroth order in slow roll by neglecting the gravitational perturbations completely. This is the way how it's usually done in introductory texts. After that I introduce some methods how one could solve the full equation of motion which is sometimes called *the Mukhanov-Sasaki equation.* I adapt one of these methods and show how the *full* Mukhanov-Sasaki equation can be derived from the gauge invariant perturbed Einstein field equations in a fairly straightforward and easy manner. Done that, I solve the equation to first order in slow roll and show how the variable q is quantized in an expanding background spacetime. From quantization I get explicit initial conditions for the equation of motion and thus fix the amplitude of the primordial perturbation q.

Second order action neglecting gravity

I start by expanding in terms of a variable $q = q(\tau, \mathbf{x})$ defined by $\varphi(\tau, \mathbf{x}) = \bar{\varphi}(\tau) + \delta\varphi = \bar{\varphi}(\tau) + q/a$. As I mentioned, at this point I ignore the metric perturbations. The action becomes

$$\begin{aligned} S_\varphi &= \int \mathrm{d}x^{\,4} a^4 \left[\frac{1}{2}\frac{1}{a^2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V\right] \\ &= \int \mathrm{d}x^{\,4} \Big[\frac{1}{2}a^2\left(\bar{\varphi}' + (q/a)'\right)^2 - \frac{1}{2}a^2\left(\bar{\varphi}_{,i} + (q/a)_{,i}\right)^2 \\ &\quad - a^4\left(V(\bar{\varphi}) + V_\varphi(q/a) + \frac{1}{2}V_{\varphi\varphi}(q/a)^2 + \dots\right)\Big], \end{aligned}$$

where $\mathrm{d}x^{\,4} \equiv \mathrm{d}x^{\,3}\mathrm{d}\tau$. The part of the action that is proportional to first powers of q is

$$S_1 = \int \mathrm{d}x^{\,4}\left[aq'\bar{\varphi}' - qa'\bar{\varphi}' - a^3V_\varphi q\right].$$

Integrating the first term by parts and dropping the boundary term gives

$$S_1 = -\int \mathrm{d}x^{\,4}\left[\bar{\varphi}'' + 2\mathcal{H}\bar{\varphi}' + a^2V_\varphi\right]aq = 0$$

by the background Klein-Gordon equation (4.7). The quadratic action is

$$S_2 = \frac{1}{2}\int \mathrm{d}x^{\,4}\left[(q')^2 - 2\mathcal{H}q'q - (\nabla q)^2 + (\mathcal{H}^2 - a^2V_{\varphi\varphi})q\right].$$

Using $q'q = \frac{1}{2}(q^2)'$ and integrating $\mathcal{H}(q^2)'$ by parts and again dropping the boundary term gives

$$\begin{aligned} S_2 &= \frac{1}{2}\int \mathrm{d}x^{\,4}\left[(q')^2 - (\nabla q)^2 + (\mathcal{H}' - \mathcal{H}^2 - a^2V_{\varphi\varphi})q\right] \\ &= \frac{1}{2}\int \mathrm{d}x^{\,4}\left[(q')^2 - (\nabla q)^2 + (\frac{a''}{a} - a^2V_{\varphi\varphi})q\right]. \end{aligned}$$

Varying this action with respect to q and going into Fourier space gives an equation of motion

$$q''_{\mathbf{k}} + k^2 q_{\mathbf{k}} - \left(\frac{a''}{a} - a^2V_{\varphi\varphi}\right)q_{\mathbf{k}} = 0, \tag{4.40}$$

where $q = a\delta\varphi$ is called the *Mukhanov-Sasaki variable* in spatially flat gauge where $\psi = 0$. Dropping the potential term and looking at the modes deep inside the horizon, where $k \gg \frac{a''}{a}$ this reduces to the familiar Minkowski space Klein-Gordon equation

$$q''_{\mathbf{k}} + k^2 q_{\mathbf{k}} = 0. \tag{4.41}$$

The Full Mukhanov-Sasaki equation

As I told in the introduction of this section, the equation what I just obtained is not the most general one. There are various ways of deriving the full equation, each one being algebraically quite involved. Here's a list of few methods:

(1) This is called the gauge invariant approach, in which one simply combines the equations (4.17) - (4.20) and the background equations suitably in order to get an equation for q, similarly as we did for the gravitational potential Ψ. The problem is that the equations are not so simple and there's quite many of them, so without a guideline it can be difficult.

(2) This is very similar to the previous one, and it is called the gauge dependent approach. By expressing the gauge invariant quantities Ψ and $\delta\varphi$ in some spesific gauge and again combining the equations suitably one could look for an equation for q also expressed in that same gauge. The problem here is that the gauge choice can be made either wisely or badly so that the equations appear either simple or horrible. For further details see [11] and [17].

(3) Third way is to start from the full action

$$S = \int\left[\frac{1}{16\pi G}R + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right]\sqrt{-g}\,\mathrm{d}x^{\,4}$$

and expand the Ricci tensor to second order in perturbations. This calculation is quite tedious and involves lots of integrations by parts and lenghty expressions, but is essentially the same as our calculation above. Casting the second order action in a form where only q appears, one obtains an equation of motion by varying with respect to q.

(4) Fourth possibility is to again derive a quadratic action, but instead of straightforward manipulations of the perturbations, one could make use of the covariant ADM-formalism. In this method one foliates the spacetime into three dimensional spatial hypersurfaces of constant t with metric γ_{ij}, so that the metric separates into spatial and time components in a following way:

$$\mathrm{d}s^2 = -N^2(t,\mathbf{x})\mathrm{d}t^2 + \gamma_{ij}\left[(\mathrm{d}x^i + N^i(t,\mathbf{x})\mathrm{d}t)(\mathrm{d}x^j + N^j(t,\mathbf{x})\mathrm{d}t)\right].$$

The functions N and N^i are called the lapse function and shift vector respectively. These functions are in fact just Lagrange multipliers and obey some constraint equations which can be solved order by order in perturbation theory. Solving the functions to first order in perturbations and substituting to the full action one obtains a quadratic action for q (or actually for the curvature pertubation $\mathcal{R}$). Details can be found e.g. from [6] or from section 10.1 in [16] .

I'm going to use the gauge-invariant method (1), following [12]. First write equation (4.37) into form (remember that $q = z\mathcal{R}$)

$$\frac{\bar{\varphi}'}{2M_p^2}\frac{z}{a}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{q}{z}\right) = \nabla^2\Psi, \tag{4.42}$$

where I defined $z \equiv \frac{a\bar{\varphi}'}{\mathcal{H}}$ and used

$$1 + w = \frac{2(\bar{\varphi}')^2}{(\bar{\varphi}')^2 + 2a^2V} = \frac{(\bar{\varphi}')^2}{2M_p^2\mathcal{H}^2}. \tag{4.43}$$

Next take the second Einstein equation (4.18) and write it in a form

$$\frac{\mathcal{H}}{a}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{a^2}{\mathcal{H}}\Psi\right) = \frac{a\bar{\varphi}'}{2M_p^2}\left[\delta\varphi + \frac{\bar{\varphi}'}{\mathcal{H}}\Psi\right] = \frac{\bar{\varphi}'}{2M_p^2}q. \tag{4.44}$$

Operating with ∇^2 on both sides of (4.44) yields

$$\frac{\mathcal{H}}{a}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{a^2}{\mathcal{H}}\nabla^2\Psi\right) = \frac{\bar{\varphi}'}{2M_p^2}\nabla^2 q, \tag{4.45}$$

and using (4.42) gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[z^2\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{q}{z}\right)\right] = z\nabla^2 q \qquad \text{i.e} \qquad q'' - \nabla^2 q - \frac{z''}{z}q = 0. \tag{4.46}$$

This is the full Mukhanov-Sasaki equation with no slow roll approximations made and it is an exact equation for the gauge invariant perturbation variable. The methods (3) and (4) described earlier both yield the same action

$$S_2 = \int \mathrm{d}\tau\,\mathrm{d}x\frac{1}{2}\left[(q')^2 - (q_{,i})^2 + \frac{z''}{z}q^2\right], \tag{4.47}$$

and variation of this with respect to q leads to the same equation of motion that I just got. In Fourier space the e.o.m is

$$q''_{\mathbf{k}} + \left(k^2 - \frac{z''}{z}\right) q_{\mathbf{k}} = 0, \tag{4.48}$$

which is again analoguous to an harmonic oscillator with time-dependent mass term

$$m^2_{\text{eff}}(\tau) = -\frac{z''}{z} = -\frac{\mathcal{H}}{a\bar{\varphi}'}\frac{d^2}{d\tau^2}\left[\frac{a\bar{\varphi}'}{\mathcal{H}}\right]. \tag{4.49}$$

Using the relation $q = z\mathcal{R}$ it is now easy to derive an e.o.m for $\mathcal{R}$ only:

$$\mathcal{R}''_{\mathbf{k}} + 2\frac{z'}{z}\mathcal{R}'_{\mathbf{k}} + k^2\mathcal{R}_{\mathbf{k}} = 0. \tag{4.50}$$

4.3 Mukhanov-Sasaki equation to 1^{st} order in slow-roll parameters.

I shall first compute the effective mass $\frac{z''}{z}$ to order in slow-roll parameters. This can be done either brute force, by calculating blindly the second derivative of z, or smartly as I shall do next. First compute

$$\begin{aligned}
\frac{z'}{\mathcal{H}z} &= \frac{1}{\mathcal{H}z}\left[\frac{a'\bar{\varphi}'}{\mathcal{H}} + \frac{a\bar{\varphi}''}{\mathcal{H}} - \frac{a\bar{\varphi}'\mathcal{H}'}{\mathcal{H}^2}\right] \\
&= 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} + \frac{\bar{\varphi}''}{\mathcal{H}\bar{\varphi}'} \\
&= \epsilon + \frac{\bar{\varphi}''}{\mathcal{H}\bar{\varphi}'},
\end{aligned} \tag{4.51}$$

where I used the relation (2.55). Next I identify the third slow-roll parameter

$$\delta \equiv \eta - \epsilon = 1 - \frac{\bar{\varphi}''}{\mathcal{H}\bar{\varphi}'} = 1 + \epsilon - \frac{z'}{\mathcal{H}z}. \tag{4.52}$$

The derivative of a slow-roll parameter is second order small, so using the previous equation on can write

$$\delta' = \epsilon' - \frac{z''}{\mathcal{H}z} + \frac{z'\mathcal{H}'}{\mathcal{H}^2 z} + \frac{(z')}{\mathcal{H}z^2} = \mathcal{O}(\epsilon^2, \eta^2) \approx 0. \tag{4.53}$$

And since also $\epsilon' = \mathcal{O}(\epsilon^2, \eta^2)$, I get

$$\frac{z''}{\mathcal{H}z} = \frac{z'\mathcal{H}'}{\mathcal{H}^2 z} + \frac{(z')}{\mathcal{H}z^2} + \mathcal{O}(\epsilon^2, \eta^2). \tag{4.54}$$

Expanding to first order in slow-roll parameters one gets

$$\begin{aligned}
\frac{z''}{z} &= \mathcal{H}^2\left[(1-\epsilon)(1+\epsilon+\delta) + (1+\epsilon-\delta)^2\right] \\
&= \mathcal{H}^2(2 + 2\epsilon - 3\delta) \\
&= \mathcal{H}^2(2 + 5\epsilon - 3\eta).
\end{aligned} \tag{4.55}$$

Writing the relation (2.55) in a following way

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\mathcal{H}}\right) = \frac{-\mathcal{H}'}{\mathcal{H}^2} = \epsilon - 1 \tag{4.56}$$

one can integrate and solve

$$\mathcal{H}^2 = \frac{1}{\tau^2(\epsilon-1)^2} \approx \frac{1}{\tau^2}(1+2\epsilon). \tag{4.57}$$

Plugging in all that we now know, the Mukhanov-Sasaki equation reduces to

$$q''_{\mathbf{k}} + (k^2 - \frac{2+9\epsilon-3\eta}{\tau^2})q_{\mathbf{k}} = 0. \tag{4.58}$$

This can be written in a rather not-so-illuminating, but simple way as

$$q''_{\mathbf{k}} + (k^2 - \frac{\nu^2-\frac{1}{4}}{\tau^2})q_{\mathbf{k}} = 0, \tag{4.59}$$

where to first order

$$\nu = \frac{3}{2} + 3\epsilon - \eta. \tag{4.60}$$

Next one can define a new function y such that $q \equiv y\sqrt{-\tau}$ so that the Mukhanov-Sasaki equation can finally be written as

$$\tau^2\frac{d^2}{d\tau^2}y(\tau) + \tau\frac{\mathrm{d}}{\mathrm{d}\tau}y(\tau) + \left(k^2\tau^2 - \nu^2\right)y(\tau) = 0, \tag{4.61}$$

which is nothing but the usual Bessel equation. The solutions for this equation with the correct are Hankel functions of first- and second kind:

$$H_\nu^{(1)}(k\tau) \equiv J_\nu(k\tau) + iN_\nu(k\tau) \quad \text{and} \quad H_\nu^{(2)}(k\tau) \equiv J_\nu(k\tau) - iN_\nu(k\tau)$$

for $\tau > 0$ and

$$H_\nu^{(1)}(-k\tau) \equiv J_\nu(-k\tau) + iN_\nu(-k\tau) \quad \text{and} \quad H_\nu^{(2)}(-k\tau) \equiv J_\nu(-k\tau) - iN_\nu(-k\tau)$$

for $\tau < 0$. Here J_ν and N_ν are Bessel and Neumann functions respectively. The solution for q is then

$$q_{\mathbf{k}}(\tau) = \sqrt{-\tau}\left[C_1 H_\nu^{(1)}(-k\tau) + C_2 H_\nu^{(2)}(-k\tau)\right]. \tag{4.62}$$

The conformal time runs from $-\infty$ to 0, so early times correspond to $-k\tau \to \infty$ and late times to $-k\tau \to 0$. For real arguments $-k\tau \in \mathbb{R}$ we have $H_\nu^{(1)}(-k\tau) = H_\nu^{(2)}(-k\tau)^*$. The asymptotic behaviour of the Hankel functions is (see e.g. reference [17])

$$\lim_{-k\tau\to\infty} H_\nu^{(1)}(-k\tau) = \sqrt{\frac{2}{\pi k(-\tau)}}e^{i\left[-k\tau-(\nu+\frac{1}{2})\frac{\pi}{2}\right]} \tag{4.63}$$

$$\lim_{-k\tau\to 0} H_\nu^{(1)}(-k\tau) = -i\frac{(\nu-1)!}{\pi}\left(\frac{2}{-k\tau}\right)^\nu = \sqrt{\frac{2}{\pi}}e^{-i\frac{\pi}{2}}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}(-k\tau)^{-\nu}. \tag{4.64}$$

The solution for q at early times then becomes

$$\lim_{-k\tau\to\infty} q_{\mathbf{k}}(\tau) = \sqrt{\frac{2}{\pi k}}\left(C_1 e^{-ik\tau} + C_2 e^{ik\tau}\right), \tag{4.65}$$

where I dropped the constant phase factors $e^{\pm i(\nu+\frac{1}{2})\frac{\pi}{2}}$ since I'll be eventually calculating two-point correlations and so the phase factors are irrevelant. The coefficients C_1, C_2 will be determined by the initial condition set by the quantum nature of the modes that were deep inside the horizon at the beginning of inflation. At late times the solution for q is

$$\lim_{-k\tau\to 0} q_{\mathbf{k}}(\tau) = \sqrt{\frac{2}{\pi}}\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} 2^{\nu-\frac{3}{2}} k^{-\nu}(-\tau)^{-\nu+\frac{1}{2}}\left(C_1 e^{-i\frac{\pi}{2}} + C_2 e^{i\frac{\pi}{2}}\right), \tag{4.66}$$

from which it's clear that the time-depence of the perturbations at late times really is different than in the early times.

4.4 Quantization of the perturbations

Now that we have an action for a gauge-invariant perturbation q, we can start thinking of quantizing the field. From the action we can find out the canonical conjugate momentum π to be

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial q'} = q'.$$

Next thing is to promote the canonical variables to operators $\hat{\pi}$ and $\hat{q}$ and impose equal time commutation relations

$$[\hat{q}(\tau,\mathbf{x}), \hat{q}(\tau,\mathbf{x}')] = [\hat{\pi}(\tau,\mathbf{x}), \hat{\pi}(\tau,\mathbf{x}')] = 0, \qquad [\hat{q}(\tau,\mathbf{x}), \hat{\pi}(\tau,\mathbf{x}')] = i\delta(\mathbf{x}-\mathbf{x}'),$$

so that in Fourier space we also have

$$[\hat{q}_{\mathbf{k}}(\tau), \hat{\pi}_{\mathbf{k}'}(\tau)] = i\delta(\mathbf{k}-\mathbf{k}') \tag{4.67}$$

Then expand the operators $\hat{q}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$ in terms of the creation and annihilation operators

$$\hat{q}_{\mathbf{k}} = q_{\mathbf{k}}(\tau)\hat{a}_{\mathbf{k}} + q^*_{\mathbf{k}}(\tau)\hat{a}^\dagger_{\mathbf{k}},$$

where the function $q_{\mathbf{k}}(\tau)$ and its complex conjugate are two properly normalized linearly independent solutions to the classical Mukhanov-Sasaki equation. The field operator in position space can be thus written as

$$\hat{q}(\tau,\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{\sqrt{2\pi}}[q_{\mathbf{k}}(\tau)\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + q^*_{\mathbf{k}}(\tau)\hat{a}^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}]. \tag{4.68}$$

The operators $\hat{a}, \hat{a}^\dagger$ then obey the commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta(\mathbf{k}-\mathbf{k}') \qquad \text{and} \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0. \tag{4.69}$$

Bunch-Davies vacuum

The time-depence of the frequency $\omega_{\mathbf{k}}(\tau)$ of the perturbations in our dynamical space-time tells us that the vacuum is not unique, since the minimum-energy-state depends on time. However, at the beginning of inflation (large negative value of τ) all relevant modes had wavelenghts well inside the horizon

$$\frac{k}{aH} \sim |k\tau| \gg 1, \quad \text{i.e.} \quad k \gg |\frac{1}{\tau}|.$$

For the effective mass this means

$$m_{\text{eff}}^2(\tau) = k^2 - \frac{z''}{z} \approx k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2} \to k^2, \tag{4.70}$$

so deep inside the horizon the modes have time independent frequencies and they behave as in Minkowski space. The Mukhanov-Sasaki equation then reduces to the Klein-Gordon equation

$$q''_{\mathbf{k}} + k^2 q_{\mathbf{k}} = 0. \tag{4.71}$$

This has the usual positive frequency plane wave solution $q_{\mathbf{k}}(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}$, where the normalization is chosen such that the canonical commutation relation (4.67) holds. This nice feature provides us an initial condition for the quantum fluctuations in a form of a unique physical vacuum, the *Bunch-Davies vacuum*:

$$\lim_{-\tau \to \infty} q_k = \frac{1}{\sqrt{2k}} e^{-ik\tau}. \tag{4.72}$$

When comparing the above result to the solution (4.65) obtained earlier, one can identify

$$C_1 = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad C_2 = 0, \tag{4.73}$$

so the first order solution to the mode functions applying during slow roll is

$$q_{\mathbf{k}}(\tau) = \sqrt{\frac{\pi}{4}} \sqrt{-\tau} H_\nu^{(1)}(-k\tau). \tag{4.74}$$

The expectation value $\langle 0|\hat{q}_{\mathbf{k}}|0\rangle$ vanishes, but the zero-point fluctuations of the vacuum state can be computed as

$$\begin{aligned}
\langle 0|\hat{q}_{\mathbf{k}}\hat{q}_{\mathbf{k}'}|0\rangle &= \langle 0|(q_{\mathbf{k}}\hat{a}_{\mathbf{k}} + q^*_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}})(q_{\mathbf{k}'}\hat{a}_{\mathbf{k}'} + q^*_{\mathbf{k}'}\hat{a}^\dagger_{\mathbf{k}'})|0\rangle \\
&= q_{\mathbf{k}} q^*_{\mathbf{k}'} \langle 0|\hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}'}|0\rangle \\
&= q_{\mathbf{k}} q^*_{\mathbf{k}'} \langle 0|[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}]|0\rangle \\
&= |q_{\mathbf{k}}|^2 \delta(\mathbf{k} - \mathbf{k}'),
\end{aligned} \tag{4.75}$$

where $|q_{\mathbf{k}}|^2 = |q_{\mathbf{k}}(\tau)|^2$.

4.5 Primordial power spectrum

The *power spectrum* is defined to be the dimensionless Fourier transformation of the real space two-point correlation function. The explicit definition for scalar field power spectrum is

$$\langle 0|\hat{q}(\tau,\mathbf{x})\hat{q}(\tau,\mathbf{y})|0\rangle \equiv \int \frac{\mathrm{d}^3\mathbf{k}}{\sqrt{2\pi}} \left[\frac{2\pi^2}{k^3}\mathcal{P}_q(\tau,\mathbf{k})\right] e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}, \tag{4.76}$$

so in another words

$$\mathcal{P}_q(\tau,\mathbf{k}) = |q_\mathbf{k}(\tau)|^2 \frac{k^3}{2\pi^2}. \tag{4.77}$$

The power spectrum of the fluctuations is thus related to the quantum mechanical expectation value of the field. However, it is commonly assumed that the quantum fluctuations become classical in some way as they are stretched beyond the horizon, so that the the expectation value of the quantum field is identified with a variance of a classical stochastic variable.

The power spectrum contains the information of two-point correlations on a given scale. A pure Gaussian distribution has no scale-dependence and thus all the information of the distribution is contained in the power spectrum. It is a useful quantity to consider since the curvature perturbations can be related to density and temperature fluctuations in the post-inflatory Universe which lead to the temperature fluctuations in the cosmic microwave background that can be measured today. Thus the primordial power spectrum that I'm about to calculate can be related to late-time observables.

The power spectrum of q at late times is, using (4.66),

$$\lim_{-\tau\to 0} \mathcal{P}_q(\tau,\mathbf{k}) = \frac{2^{2\nu-3}}{(2\pi)^2}\left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right)^2 k^{-2\nu+3}(-\tau)^{1-2\nu}, \tag{4.78}$$

so the spectrum has an explicit time-depence:

$$\mathcal{P}_q(\tau,\mathbf{k}) \propto (-\tau)^{1-2\nu} = (-\tau)^{-2-6\epsilon+2\eta}. \tag{4.79}$$

Time dependent spectrum at superhorizon scales is of no use, since the physics beyond the horizon during the end of inflation is unknown. However, it can be shown that the comoving curvature perturbation $\mathcal{R}$ introduced in Section 4.1 stays nearly constant on superhorizon scales. To make contact of the quantum fluctuations of q at early times to the fluctuations of $\mathcal{R}$ when they re-enter the horizon at late times, we have to calculate the power spectrum of $\mathcal{R}$ at the moment of horizon exit, after which the fluctuations freeze out beyond the horizon. After re-entering the horizon at much later times after the inflation, the evolution of the spectrum can be again calculated classically, using the perturbed Einstein equations with different kinds of fluids present at the post-inflationary Universe. The spectrum of the curvature fluctuations is

$$\mathcal{P}_\mathcal{R}(\tau,\mathbf{k}) = |\mathcal{R}_\mathbf{k}|^2\frac{k^3}{2\pi^2} = |\frac{1}{z}q_\mathbf{k}|^2\frac{k^3}{2\pi^2} = \frac{1}{z^2}\mathcal{P}_q(\tau,\mathbf{k}). \tag{4.80}$$

At late times ($k \ll \mathcal{H}$) this becomes

$$\lim_{-\tau \to 0} \mathcal{P}_{\mathcal{R}}(\tau, k) = \frac{2^{2\nu-3}}{(2\pi)^2} \left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} \right)^2 k^{-2\nu+3} (-\tau)^{1-2\nu} \frac{1}{z^2}. \tag{4.81}$$

Next I will show that both $\mathcal{R}$ and $\mathcal{P}_{\mathcal{R}}$ are constants in the superhorizon scales to first order in slow-roll. From (4.52) and (4.57) one gets that

$$\frac{\mathrm{d}z / \mathrm{d}\tau}{z} = \frac{-1}{(1-\epsilon)\tau}(1 + 2\epsilon - \eta) \quad \Rightarrow \quad z \propto (-\tau)^{\frac{1}{2}-\nu}. \tag{4.82}$$

Then at late times $q \propto (-\tau)^{\frac{1}{2}-\nu}$, and thus to first order in slow-roll

$$\mathcal{R} = \frac{1}{z} q \propto (-\tau)^0 \propto \text{constant}. \tag{4.83}$$

This is exactly the same result that we obtained earlier using the Einstein equations in Section 4.1. By the same argument

$$\lim_{-\tau \to 0} \mathcal{P}_{\mathcal{R}}(\tau, k) \propto \text{constant}. \tag{4.84}$$

The spectrum has a scale dependence

$$\mathcal{P}_{\mathcal{R}}(\tau, k) \propto k^{-2\nu+3} \propto k^{2\eta - 6\epsilon} \equiv k^{n_s - 1}, \tag{4.85}$$

where the parameter $n_s = 1 + 2\eta - 6\epsilon$ is called the *spectral index*. To 0^{th} order in slow roll the spectrum is scale independent. It is usually said that the spectrum is nearly *scale invariant*, or that it is nearly a *Harrison-Zeldovich* spectrum [1]. I choose to evaluate the spectrum at the time of horizon exit for each scale separately, allthough the spectrum is not constant immediately after that but only at late times. At horizon exit the comoving wavenumber equals the horizon size, $k = \mathcal{H}$. I denote the time of horizon exit with t_*. In slow-roll expansion we had the equation (4.57), so inversion of that gives

$$-\tau = \frac{1}{(1-\epsilon)\mathcal{H}}. \tag{4.86}$$

Plugging these in to the expression of $\mathcal{P}_{\mathcal{R}}$ one gets the spectrum of the comoving curvature perturbation at horizon exit, which is also called the *primordial power spectrum*:

$$\mathcal{P}_{\mathcal{R}}(k = \mathcal{H}) = \frac{2^{2\nu-3}}{(2\pi)^2} \left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} \right)^2 \frac{\mathcal{H}^4}{(a\bar{\varphi}')^2} (1-\epsilon)^{2\nu-1} \Big|_{t=t_*}. \tag{4.87}$$

In terms of the ordinary Hubble constant instead of the conformal one, this is

$$\mathcal{P}_{\mathcal{R}}(k = aH) = \frac{2^{2\nu-3}}{(2\pi)^2} \left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} \right)^2 \frac{H^4}{\dot{\bar{\varphi}}^2} (1-\epsilon)^{2\nu-1} \Big|_{t=t_*}. \tag{4.88}$$

The primordial spectrum has no time dependence anymore, since it is evaluated at the time of horizon crossing. However, it still has the same scale dependence as the original spectrum since the Hubble rate varies slightly in scale in quasi-de Sitter scenario. The *scalar amplitude* is defined to be $A_s \equiv \sqrt{\mathcal{P}_{\mathcal{R}}} \sim |\mathcal{R}_{\mathbf{k}}|^2$ at horizon crossing. To zeroth order in slow-roll $\nu = \frac{3}{2}$ and so A_s can be approximated as

$$A_s \approx \frac{H^2}{2\pi\dot{\bar{\varphi}}} = \frac{3H^3}{2\pi V'} = \frac{1}{2\pi\sqrt{3}M_P^3}\frac{V^{3/2}}{V'}, \tag{4.89}$$

where I used the slow roll equations (2.46). Measurement of the scalar amplitude then possibly provide a way to fix parameters of the potential V for a given field theoretical model. We'll see an example of this at the end of this section when considering the specific Higgs inflation model.

4.6 Gravitational waves from inflation

If we consider a general perturbed metric with only tensor perturbations present, the line element takes the form

$$\mathrm{d}s^2 = a^2(\tau)\left[-\mathrm{d}\tau^2 + (\delta_{ij} + 2E_{ij})\,\mathrm{d}x^i\mathrm{d}x^j\right],$$

where I have now dropped the scalar and vector degrees of freedom in the decomposition (3.36) of the tensor perturbation E_{ij} such that the remaining function is a transverse and traceless tensor:

$$\delta^{ik}E_{ij,k} = 0 \quad \delta^{ij}E_{ij} = 0. \tag{4.90}$$

Keeping the above properties in mind one can calculate the Christoffel symbols (3.88) for tensor perturbations. Careful calculation gives the non-zero components

$$\begin{aligned}
\delta\Gamma^0_{ij} &= 2\mathcal{H}E_{ij} + E'_{ij} \\
\delta\Gamma^i_{0j} &= E'_{ij} \\
\delta\Gamma^i_{jk} &= \partial_k E_{ij} + \partial_j E_{ik} - \partial_i E_{jk}.,
\end{aligned} \tag{4.91}$$

Furthermore, the non-zero components of the Ricci tensor (3.92) turn out to be

$$\delta R_{ij} = 2\left(\mathcal{H}' + 2\mathcal{H}^2\right)E_{ij} + 2\mathcal{H}E'_{ij} + E''_{ij} - \nabla^2 E_{ij}. \tag{4.92}$$

The mixed Ricci tensor (3.94) is then

$$\delta R^i_j = \frac{1}{a^2}\left(E''_{ij} + 2\mathcal{H}E'_{ij} - \nabla^2 E_{ij}\right). \tag{4.93}$$

It is easy to assure that the Ricci scalar does not acquire any tensorial part from the perturbations. The only non-trivial component of the Einstein equations is then

$$E''_{ij} + 2\mathcal{H}E'_{ij} - \nabla^2 E_{ij} = 8\pi G a^2 \Sigma_{ij}, \tag{4.94}$$

where the right-hand-side is the transverse and traceless part of the energy momentum tensor, the anisotropic stress (3.65). In Section 4 I showed from the perturbed energy-momentum tensor (4.9) that a single scalar field has no anisotropic stress, i.e. $\Sigma_{ij} = 0$.

The resulting equation is an evolution equation for gravitational waves generated during inflation:

$$E''_{ij} + 2\mathcal{H}E'_{ij} - \nabla^2 E_{ij} = 0. \tag{4.95}$$

The Fourier transformation of the tensor perturbations is defined as

$$E_{ij}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}k^{\,3}}{\sqrt{2\pi}} E_{ij}(\tau, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{4.96}$$

Due to the symmetry and the tracelessness and transversality condition the Fourier components satisfy the conditions

$$E_{ij}(\tau, \mathbf{k}) = E_{ji}(\tau, \mathbf{k}), \qquad E_{ii}(\tau, \mathbf{k}) = 0, \qquad k^i E_{ij}(\tau, \mathbf{k}) = 0. \tag{4.97}$$

The spatial tensor E_{ij} has 9 components, but the above constraints reduce the number of independent components to 2 (three constraints come from the symmetricity, one from the trace constraint and three more from the requirement of the tensor to be transverse to the wave vector). By rotating the spatial coordinates I can always choose the wave-vector k to point in the positive z-direction, so that

$$E_{11} = -E_{22}, \qquad E_{12} = E_{21} \quad \text{and} \quad E_{i3} = E_{3i} = 0. \tag{4.98}$$

Naming the components

$$h_1 \equiv E_{11}(\tau, \mathbf{k}) \quad \text{and} \quad h_2 \equiv E_{12}(\tau, \mathbf{k}) \tag{4.99}$$

The gravitational waves can be written as

$$E_{ij}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}k^{\,3}}{\sqrt{2\pi}} E_{ij}(\tau, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} = \int \frac{\mathrm{d}k^{\,3}}{\sqrt{2\pi}} \sum_{\gamma=1,2} h_\gamma e^\gamma_{ij} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{4.100}$$

where the cartesian *polarization tensors* are

$$\mathbf{e}^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}^2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.101}$$

A field which transforms under a rotation $R(\theta)$ by

$$\chi \xrightarrow{R(\theta)} e^{ih\theta}\chi$$

is said to have helicity h [18]. Now, for gravitational waves one can change into circular polarization basis defined by left- and right-handed polarizations

$$\mathbf{e}^{+,\times} \equiv \mathbf{e}^1 \pm i\mathbf{e}^2 \quad \text{where} \quad \begin{cases} + \text{ for } +\text{-polarization} \\ - \text{ for } \times\text{-polarization} \end{cases}$$

and in this basis it is easy to show that under a rotation $R(\theta)$ around the z-axis the polarization tensors transform as $\mathbf{e}' = R^T(\theta)\mathbf{e}R(\theta)$, i.e.

$$e^{L,R}_{ij} \xrightarrow{R(\theta)} R^a_i R^b_j e^{L,R}_{ab} = e^{\pm 2i\theta} e^{L,R}_{ij}.$$

Thus gravitational waves with circular polarization have helicity ± 2. The polarization tensors are then orthonormal in a following sense:

$$\sum_{i,j,\gamma} e_{ij}^{\gamma} e_{ij}^{*\gamma} = 2. \tag{4.102}$$

Plugging in the components (1,1) and (1,2) of (4.100) into the Einstein equation (4.95), one obtains equations for the functions h_+ and $h_\times$ in Fourier space:

$$h_\gamma'' + 2\mathcal{H}h_\gamma' + k^2 h_\gamma = 0, \qquad \gamma = +, \times. \tag{4.103}$$

I drop the subscripts + and − from now on, but keep in mind that there's an equation for both polarizations. Defining a new function $v = ah$, I get (4.103) into form

$$v'' + (k^2 - \frac{a''}{a})v = 0. \tag{4.104}$$

This looks very similar to what I've already seen before when studying scalar perturbations, although the Mukhanov-Sasaki equation for scalar perturbations was much harder to derive and solve. It can be shown that an equation of motion like (4.104) rises from an action of the form

$$\begin{aligned} S &= \frac{1}{2}M_P^2 \int \mathrm{d}\tau\, \mathrm{d}\mathbf{x}^3 a^2 \left[\frac{1}{2}(E_i'^j)^2 - \frac{1}{2}(\partial_k E_j^i)^2\right] \\ &= \frac{1}{2}M_P^2 \int \mathrm{d}\tau\, \mathrm{d}\mathbf{x}^3 \frac{1}{2} \sum_{\lambda=+,\times} \left[(v_\lambda')^2 + \mathcal{H}^2 v_\lambda^2 - 2\mathcal{H}v_\lambda' v_\lambda - (\nabla v_\lambda)^2\right] \\ &= \int \mathrm{d}\tau\, \mathrm{d}\mathbf{x}^3 \frac{1}{2} \sum_{\lambda=+,\times} \left[(v_\lambda')^2 - (\nabla v_\lambda)^2 + \frac{a''}{a} v_\lambda^2\right]. \end{aligned} \tag{4.105}$$

The first line comes from expanding the Einstein-Hilbert action to second order in E_{ij} [16]. In getting the last line I used the identity $v_\lambda' v_\lambda = \frac{1}{2}(v_\lambda^2)'$ and then integrated by parts to get $2\mathcal{H}v_\lambda' v_\lambda \to -\mathcal{H}' v_\lambda^2$. Then I used that $\frac{a''}{a} = \mathcal{H}^2 + \mathcal{H}'$. I also included the Planck mass into the definition of v coming from the Einstein-Hilbert action. The variable

$$v \equiv \frac{aM_P}{\sqrt{2}} h \tag{4.106}$$

can be thus quantized similarly as a scalar field. The arbitrary looking $\sqrt{2}$ at the denominator is due to the polarization tensor normalization (4.102). If I would have chosen the normalization to be $\sum_{i,j,\lambda} e_{ij}^\lambda e_{ij}^\lambda = 1$ I would have gotten a factor of 2 in the denominator of (4.106).

Power spectrum for gravitational waves to 1^{st} order in slow roll

The damping term in the equation of motion (4.104) to first order in slow roll is, using (2.55) and (4.57)

$$\frac{a''}{a} = \mathcal{H}^2(2 - \epsilon) \approx \frac{1}{\tau^2}(1 + 2\epsilon)(2 - \epsilon) \approx \frac{1}{\tau^2}(2 + 3\epsilon). \tag{4.107}$$

The e.o.m then again reduces to Bessel equation

$$v + (k^2 - \frac{\alpha^2 - \frac{1}{4}}{\tau^2})v = 0, \tag{4.108}$$

where

$$\alpha = \sqrt{\frac{9}{4} + 3\epsilon} \approx \frac{3}{2} + \epsilon. \tag{4.109}$$

Solutions to this are again the Hankel functions. Quantization of v provides again an initial condition and it proceeds exactly as before when considering the scalar perturbations, so the solution is

$$v_{\mathbf{k}}(\tau) = \sqrt{\frac{-\pi\tau}{4}} H_\alpha^{(1)}(-k\tau). \tag{4.110}$$

The power spectrum for a single canonically normalized polarization is then

$$\mathcal{P}_v(\tau, \mathbf{k}) = |v_{\mathbf{k}}(\tau)|^2 \frac{k^3}{2\pi^2}, \tag{4.111}$$

so it is just the same as for the Mukhanov-Sasaki variable. At horizon crossing when $k = \mathcal{H}$, quoting (4.87) without the $1/z^2$ factor, it is

$$\Delta_v = \frac{2^{2\alpha-3}}{(2\pi)^2} \left(\frac{\Gamma(\alpha)}{\Gamma(\frac{3}{2})}\right)^2 \mathcal{H}^{4(1-\alpha)}(1-\epsilon)^{(1-2\alpha)}\Big|_{t=t_*}. \tag{4.112}$$

The *tensor mode power spectrum* is defined as

$$\begin{aligned} \mathcal{P}_T(\tau, \mathbf{k}) &\equiv \frac{k^3}{2\pi^2}|E_{ij}|^2 = \frac{k^3}{2\pi^2} E_{ij}E_{ij}^* = \frac{k^3}{2\pi^2}\left(2|h_+|^2 + 2|h_\times|^2\right) \\ &= \frac{4k^3}{2\pi^2}|h|^2 = \frac{8k^3}{2\pi^2 a^2 M_P^2}|v_{\mathbf{k}}(\tau)|^2 = \frac{8}{a^2 M_P^2}\mathcal{P}_v(\tau, \mathbf{k}). \end{aligned} \tag{4.113}$$

This spectrum has a scale dependence

$$\mathcal{P}_T \propto k^{-2\alpha+3} = k^{-2\epsilon} \equiv k^{n_t}, \tag{4.114}$$

where $n_t = -2\epsilon$ is the tensor spectral index. The *primordial tensor power spectrum* is then the spectrum evaluated at the time of horizon crossing ($k = \mathcal{H}$):

$$\Delta_T = \left(\frac{2^\alpha}{2\pi M_P}\right)^2 \left(\frac{\Gamma(\alpha)}{\Gamma(\frac{3}{2})}\right)^2 \frac{\mathcal{H}^{4(1-\alpha)}}{a^2}(1-\epsilon)^{(1-2\alpha)}\Big|_{t=t_*}. \tag{4.115}$$

4.7 Spectral parameters

What I calculated in the previous sections were the power spectra for scalar and tensor perturbations. The most important parameters of these spectra are the spectral indeces n_s and n_t, which are defined to be

$$n_s - 1 \equiv \frac{\mathrm{d}\ln\mathcal{P}_\mathcal{R}}{\mathrm{d}\ln k} \quad \text{and} \quad n_t \equiv \frac{\mathrm{d}\ln\mathcal{P}_T}{\mathrm{d}\ln k}. \tag{4.116}$$

The spectral index is said to be *scale free* if it does not depend on k, i.e. $n(k) = \text{constant}$. In that case the power spectrum is a power law $\mathcal{P} \propto k^n$. That is exactly what I got to first order in slow-roll parameters and the results were

$$n_s = 1 + 2\eta - 6\epsilon \tag{4.117}$$

$$n_t = -2\epsilon. \tag{4.118}$$

Note that the tensor spectral index was defined without the -1 that was in the definition of n_s. This is just a common convention. The spectral indeces are close to values $n_s \approx 1$ and $n_t \approx 0$ in the slow-roll scenario corresponding to nearly scale-invariant spectra. The conclusion then is that a single scalar field inflation generally produces a nearly scale-invariant spectrum. Another important parameter is the *tensor-to-scalar ratio* r which is defined as

$$r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\mathcal{R}}, \tag{4.119}$$

and with my results this becomes, to first order in slow-roll,

$$r = 8\left(\frac{\varphi'}{\mathcal{H}M_P}\right)^2 \left(\frac{\Gamma(\alpha)}{\Gamma(\nu)}\right)^2 4^{\alpha-\nu} k^{2(\nu-\alpha)}(-\tau)^{2(\nu-\alpha)} \approx 8\left(\frac{\varphi'}{\mathcal{H}M_P}\right)^2 = 16\epsilon, \tag{4.120}$$

where $\nu = \alpha = \frac{3}{2}$ in the exponents and in the gamma functions, and finally I used the identity (2.53). The tensor spectral tilt and the parameter r are both proportional to the slow roll parameter ϵ, so they are not independent of each other. This provides what is called *the consistency relation*:

$$r = -8n_t. \tag{4.121}$$

Violation of this relation could imply that there was more than one field driving the inflation or a single field with non-canonical kinetic term [10].

The most recent measured constraints for the scalar spectral parameters evaluated at a pivot scale of $0.05\,\mathrm{Mpc}^{-1}$ with 68% confidence level are from Planck mission from 2015 [10]:

$$n_s = 0.9677 \pm 0.0060 \tag{4.122}$$

$$\frac{\mathrm{d}n_s}{\mathrm{d}\ln k} = -0.0033 \pm 0.0074 \tag{4.123}$$

and for the tensor-to-scalar ratio with 95% confidence level

$$r_{0.002} < 0.11. \tag{4.124}$$

The subscript 0.002 indicates that r is evaluated at a pivot scale $0.002\,\mathrm{Mpc}^{-1}$. The running of the spectral index $\frac{\mathrm{d}n}{\mathrm{d}\ln k}$ (and the running of the running and so on) is defined by parametrizing the spectrum by

$$\mathcal{P}_\mathcal{R}(k) = \frac{k^3}{2\pi^2}|\mathcal{R}_\mathbf{k}|^2 \equiv A_S \left(\frac{k}{k_*}\right)^{n_s - 1 + \frac{1}{2}\frac{\mathrm{d}n_s}{\mathrm{d}\ln k}\ln(k/k_*) + \frac{1}{6}\frac{d^2 n_s}{\mathrm{d}\ln k^2}(\ln(k/k_*))^2 + \ldots} \tag{4.125}$$

$$\mathcal{P}_T(k) = \frac{k^3}{2\pi^2}|E^{ij}_\mathbf{k}|^2 \equiv A_T \left(\frac{k}{k_*}\right)^{n_t + \frac{1}{2}\frac{\mathrm{d}n_t}{\mathrm{d}\ln k}\ln(k/k_*) + \ldots}. \tag{4.126}$$

The amplitude A_s of the scalar power spectrum is measured to be

$$A_s = (2.198 \pm 0.104) \cdot 10^{-9}. \tag{4.127}$$

4.8 Examples

Now that I got the general prediction for the spectral parameters for a single field inflation, I can compare the theory to observations. I introduce two models: the polynomial potential and the Higgs inflation. The first one is straightforward to analyze and some of the work was done already in Section 2.3.3. The Higgs inflation requires a bit more algebra, but the same procedure applies nevertheless.

4.8.1 Polynomial potential

In section 2.3.3 I got exact results for the slow-roll parameters for an inflaton potential $V(\phi) = \lambda_p \phi^p$. The results were

$$\epsilon = \frac{p}{4\,(N + p/4)} \quad \text{and} \quad \eta = \frac{p-1}{2\,(N + p/4)}. \tag{4.128}$$

The spectral index and the tensor-to-scalar ratio are then

$$n_s = 1 + 2\eta - 6\epsilon = 1 - \frac{\frac{1}{2}p + 1}{N + p/4} \tag{4.129}$$

$$r = 16\epsilon = \frac{4p}{N + p/4}. \tag{4.130}$$

Plot of the predictions for values $N = 50 - 60$ and potentials with $p = 1,2,\frac{2}{3},3,\frac{4}{3},4$ is seen in the Figure 6. For illustrative purposes I have used a Matlab code to produce the numerically solved plots shown in Figure 5.

4.8.2 Higgs inflation

Since the Standard model Higgs is the only scalar particle know today, it would be nice if it could serve as an inflaton. Indeed this can be accomplished with a non-minimal coupling of the Higgs field to gravity. Such a theory was proposed by Bezrukov and Shaposhnikov in 2008 [19] and has been widely studied since. The theory is minimal in a sense that it doesn't introduce any new fields, only one extra coupling with interaction strenght ξ. The total action is

$$S_J = \int \mathrm{d}x^4 \sqrt{-g} \left[\mathcal{L}_{SM} - \frac{1}{2} M^2 R - \xi H^\dagger H R \right], \tag{4.131}$$

where the Higgs doublet in the unitary gauge is

$$H = \begin{pmatrix} h/\sqrt{2} \\ 0 \end{pmatrix} \tag{4.132}$$

and the Planck mass is $M_P^2 = M^2 + \xi v^2$ with $v \approx 246$ GeV being the Higgs field vacuum expectation value. Neglecting all other interaction but the Higgs field and gravity for now, the relevant action becomes

$$S_J = \int \mathrm{d}x^4 \sqrt{-g} \left[\frac{M^2 + \xi h^2}{2} R - \frac{1}{2} g^{\mu\nu} (\partial_\mu h)(\partial_\nu h) + \frac{\lambda}{4} (h^2 - v^2)^2 \right]. \tag{4.133}$$

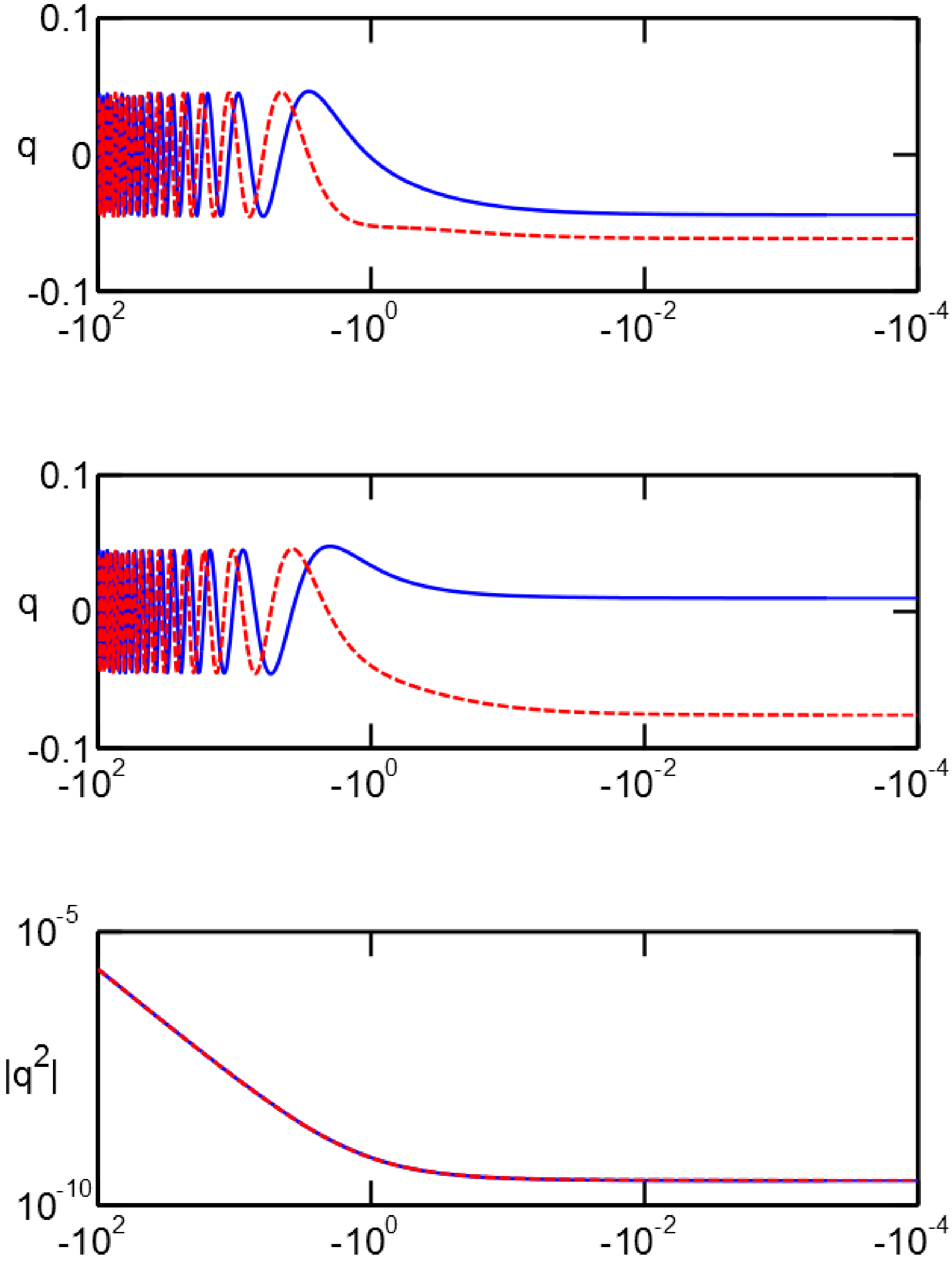

Figure 5: Evolution of the quantum field q as a function of conformal time for a potential $V = \frac{1}{2}m^2\bar{\varphi}^2$. The first plot shows the real part of the field, second one the imaginary part and last one the amplitude squared. The blue line shows the numerical solution of the full Mukhanov-Sasaki equation, the red dotted line shows the evolution in slow roll approximation. Inflation ends at $\tau = -1$.

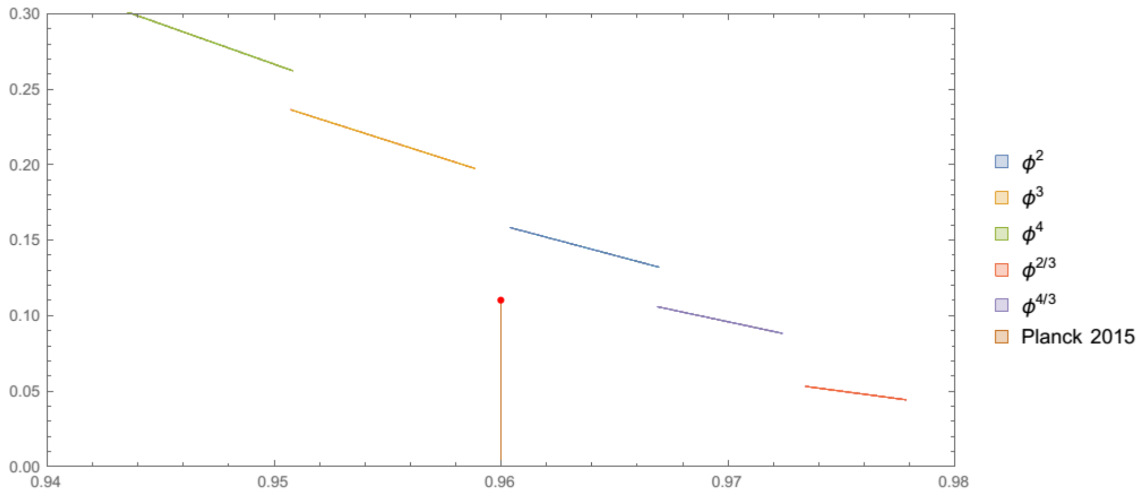

Figure 6: (n_s,r)-plane for polynomial potential. The Planck data is shown as values $n_s = 0.96, r < 0.11$.

In order to find out the inflatory predictions of this model one has to make a conformal transformation to the metric. Detailed study of the conformal transformation and the transformation properties of different entities can be found for example from [20] and [21]. The J in the subscript of the action refers to the so-called *Jordan frame*, or the *physical frame*. After making a conformal transformation

$$g_{\mu\nu}(x) \to \hat{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \tag{4.134}$$

the action is said to be in the *Einstein frame* where the transformation can be chosen so that it depends on the coordinates through fields. By approriate choice of Ω the gravity decouples from the other fields. The effect of the conformal transformation to the metric with upper indices, metric determinant and the Ricci scalar is

$$g^{\mu\nu} \to \hat{g}^{\mu\nu} = \frac{1}{\Omega^2} g^{\mu\nu} \tag{4.135}$$

$$\sqrt{-g} \to \sqrt{-\hat{g}} = \Omega^4 \sqrt{g} \tag{4.136}$$

$$R \to \hat{R} = \frac{1}{\Omega^2}\left[R - 6g^{\alpha\beta}\nabla_\alpha\nabla_\beta \log\Omega - 6g^{\alpha\beta}(\nabla_\alpha \log\Omega)(\nabla_\beta \log\Omega)\right] \tag{4.137}$$

First write the action as

$$S_J = \int \mathrm{d}x^{\,4}\sqrt{-g}\left[\frac{M_P^2}{2}\Omega^2 R - \frac{1}{2}g^{\mu\nu}(\partial_\mu h)(\partial_\nu h) + \frac{\lambda}{4}(h^2 - v^2)^2\right], \tag{4.138}$$

where

$$\Omega^2 = \frac{M^2 + \xi h^2}{M_P^2} = 1 + \frac{\xi(h^2 - v^2)}{M_P^2}. \tag{4.139}$$

Then make the conformal transformation to the Einstein frame with the above definition of Ω and get

$$S_E = \int \mathrm{d}x^{\,4}\sqrt{-\hat{g}}\left[\frac{M_P^2}{2}\hat{R} - \frac{1}{2}\Omega^{-2}\hat{g}^{\mu\nu}(\partial_\mu h)(\partial_\nu h) + \frac{\lambda}{4}\Omega^{-4}(h^2 - v^2)^2\right.$$
$$\left. + 3M_P^2\hat{g}^{\alpha\beta}\left[\nabla_\alpha\nabla_\beta \log\Omega + (\partial_\alpha \log\Omega)(\partial_\beta \log\Omega)\right]\right]. \tag{4.140}$$

In the second line the first term

$$\hat{g}^{\alpha\beta}\nabla_\alpha\nabla_\beta \log\Omega = \nabla_\alpha\partial^\alpha \log\Omega \tag{4.141}$$

is a surface term and can be dropped. The second term with derivatives of $\Omega = \Omega(h)$ becomes

$$3M_P^2\hat{g}^{\alpha\beta}\left(\partial_\alpha \log\Omega\right)\left(\partial_\beta \log\Omega\right) = \frac{3\xi^2h^2}{M_P^2\Omega^4}\hat{g}^{\alpha\beta}(\partial_\alpha h)(\partial_\beta h), \tag{4.142}$$

so that the action can be written as

$$S_E = \int \mathrm{d}x^4\sqrt{-\hat{g}}\left[\frac{M_P^2}{2}\hat{R} - \frac{1}{2}\frac{M_P^2\Omega^2 + 6\xi^2h^2}{M_P^2\Omega^4}\hat{g}^{\mu\nu}(\partial_\mu h)(\partial_\nu h) + \frac{\lambda}{4}\Omega^{-4}(h^2-v^2)^2\right]. \tag{4.143}$$

The Ricci scalar is now completely uncoupled from the Higgs field, but the Higgs kinetic term has acquired a non-canonical form. To get rid of such a nuisance, one can define a new field $\chi = \chi(h)$ so that

$$(\partial_\mu h)(\partial_\nu h) = \left(\frac{\partial\chi}{\partial h}\right)^{-2}(\partial_\mu\chi)(\partial_\nu\chi), \tag{4.144}$$

where $\chi(h)$ is defined as

$$\frac{\partial\chi}{\partial h} = \sqrt{\frac{M_P^2\Omega^2 + 6\xi^2h^2}{M_P^2\Omega^4}}. \tag{4.145}$$

With this field redefinition one gets

$$S_E = S_E \;= \int \mathrm{d}x^4\sqrt{-\hat{g}}\left[\frac{M_P^2}{2}\hat{R} - \frac{1}{2}\hat{g}^{\mu\nu}(\partial_\mu\chi)(\partial_\nu\chi) + \frac{1}{\Omega^4(\chi)}\frac{\lambda}{4}(h^2(\chi)-v^2)^2\right]. \tag{4.146}$$

Next task is to calculate the potential for the new field in order to find out the slow-roll parameters, and that task requires solving the differential equation (4.145). The potential is usually solved in two approximation regimes:

(i) Small field approximation: $h \ll \frac{M_p}{\sqrt{\xi}}$. In this limit $\Omega^2(\chi) \approx 1$ and thus (4.145) can be easily integrated to give $\chi \approx h$. The potential then reduces to the familiar Standard Model potential

$$V(\chi) \approx \frac{\lambda}{4}(\chi^2 - v^2)^2. \tag{4.147}$$

(ii) Large field approximation: $h \gg \frac{M_p}{\sqrt{\xi}} \gg v$. In this limit $\Omega^2 \approx \frac{\xi h^2}{M_P^2}$ so that

$$\frac{\partial\chi}{\partial h} \approx \frac{\sqrt{6}M_P}{h} \quad \text{and} \quad \chi \approx \sqrt{6}M_P\ln(Ch) \tag{4.148}$$

and furthermore

$$h \approx C \exp\left(\frac{\chi}{\sqrt{6}M_P}\right). \tag{4.149}$$

I choose $C = M_P/\sqrt{\xi}$ so that the large field approximation is satisfied when $\chi \gg \sqrt{6}M_p$. The potential then gets an asymptotically flat form

$$V(\chi) \approx \frac{\lambda M_P^4}{4\xi^2}(1 - e^{-2\chi/\sqrt{6}M_P})^2. \tag{4.150}$$

The calculation of the derivatives $\frac{\partial V}{\partial \chi}$ and $\frac{\partial^2 V}{\partial \chi^2}$ is now a straightforward task, so that the slow roll parameters in the large field approximation are

$$\epsilon(\chi) \approx \frac{4}{3}e^{-4\chi/\sqrt{6}M_P} = \frac{4M_P^4}{3\xi^2 h^4(\chi)} \tag{4.151}$$

$$\eta(\chi) \approx -\frac{4}{3}e^{-2\chi/\sqrt{6}M_P} = -\frac{4M_P^2}{3\xi h^2(\chi)} = \sqrt{\frac{4\epsilon}{3}}. \tag{4.152}$$

Next thing is to calculate the number of e-foldings. For that one needs the value of the field at the end of inflation when $\epsilon \equiv 1$. One gets

$$h_{end} = \left(\frac{4}{3}\right)^{1/4} \frac{M_P}{\sqrt{\xi}} \approx 1.07\frac{M_P}{\sqrt{\xi}}. \tag{4.153}$$

Number of e-foldings is

$$N = \frac{1}{M_P^2}\int_{h_{end}}^{h} \frac{V}{\partial V/\partial \chi}\,\mathrm{d}\chi \approx \frac{3}{4M_P^4}e^{2\chi(h)/\sqrt{6}M_p}\Bigg|_{h_{end}}^{h} = \frac{3\xi}{4M_P^2}\left(h^2 - h_{end}^2\right), \tag{4.154}$$

or inversily

$$h^2 = \frac{4M_P^2}{3\xi}(N + \sqrt{\frac{3}{4}}). \tag{4.155}$$

Plugging this to the expression of the slow roll parameters one can write the spectral index and the tensor to scalar ratio as a function of e-foldings

$$n_s = 1 + 2\eta - 6\epsilon = 1 - \frac{2(N + \sqrt{3/4}) + 18/4}{(N + \sqrt{3/4})^2} \tag{4.156}$$

$$r = 16\epsilon = \frac{12}{(N + \sqrt{3/4})^2}. \tag{4.157}$$

For the value $N = 60$ I get

$$n_s = 0.97 \quad \text{and} \quad r = 0.003, \tag{4.158}$$

which are in good agreement with the Planck results. The measured value for the scalar spectrum amplitude introduced in (4.89) fixes the value of ξ. For the potential under consider one gets

$$A_s \approx \frac{\sqrt{2}(1.07)^2}{16\pi}\frac{\sqrt{\lambda}}{\xi}. \tag{4.159}$$

The Higgs mass constraint $m_h \approx \sqrt{2\lambda}v = 125\,\mathrm{GeV}$ gives $\lambda = 0.13$, so that $\xi \sim 5 \cdot 10^6$.

5 Summary

In first section I went through the basic theory of cosmology and the drawbacks of basic Big Bang theory. As a solution to those discomforts, a scenario called cosmic inflation has been slowly invented by pioneers such as Guth, Linde and Starobinsky. Following their footsteps I went through the basic idea of modern inflationary scenario and the slow roll approximation. After that I dwelled into the subject of cosmological perturbations in section 2. I introduced perturbations around a Friedmann–Robertson–Walker -spacetime and discussed the issue of gauge transformations. I found out the form of the metric and stress-energy tensor to first order in perturbation theory. After that I introduced the conformal Newtonian gauge and using that computed the curvature tensors to first order. What I finally got was perturbed Einstein equations for perfect fluid perturbations and the two remaining metric perturbations, the gauge invariant Bardeen potentials Φ and Ψ.

In section 3 I considered single scalar field perturbations in the perturbed FRW-background. I found out the perfect fluid perturbations in terms of the scalar field fluctuation $\delta\varphi$ which proved to coincide with the gauge invariant field fluctuation in conformal Newtonian gauge. I noticed that in the single scalar field case the traceless part of Einstein equations reduce the number of metric fluctuations to one, i.e. i got that $\Psi = \Phi$. Hence the remaining Einstein equations were coupled differential equations for gauge invariant variables Ψ and $\delta\varphi$ only.

I derived a single second order differential equation for the metric perturbation alone and used that to show that a convenient variable in superhorizon scales is the comoving curvature perturbation $\mathcal{R}$ which stays constant on those scales. I showed that deep inside the horizon then a practical variable is the Mukhanov-Sasaki variable $q = z\mathcal{R}$. Both variables are linear combinations of the field and the metric perturbation, and by an appropriate choice of gauge one can make either $\delta\varphi$ or ψ vanish, but not both at the same time. I sticked on the gauge invariant form and derived an exact evolution equation for q. I solved the equation, which proved to be nothing but the Bessel equation, to first order in slow roll and got an initial condition from the fact that q can be quantized when the modes are deep inside the horizon. Quantization in an expanding background space was possible since there's a unique choice of vacuum, the Bunch-Davies vacuum, as a Minkowskian limit when the modes had wavelenghts much less than the Hubble radius.

I introduced the scalar power spectrum as a dimensionless Fourier transformation of the two-point correlator for scalar field fluctuations. As the wavelenghts of the modes stretch beyond the horizon scale I assumed that the quantum fluctuations become stochastic variations of a classical field. At horizon exit I changed my variable to be the curvature perturbation. Constancy of the curvature perturbation on superhorizon scales provides a way to connect inflationary fluctuations to fluctuations much later after the end of inflation. I obtained the spectral parameters called spectral index and the tensor-to-scalar ratio for a generic single scalar field inflation. The spectral parameters turned out to be proportional to the slow-roll parameters, and concluded that observations provide a way to constrain the form of the inflatory potential. I didn't discuss the post-inflatory evolution of the fluctuations, but that is not necessary when focusing only to field theoretical model building.

Finally I went through two basic examples of specific models, the power law potential and the Higgs inflation. Power law inflation assumes that the inflaton potential is of the power law -form, not taking a stand to the essence of the inflaton field. Higgs inflation assumed only a non-minimal coupling of the Higgs field to gravity, so that the only amendment to the Standard Model of particle physics is one extra parameter, the interaction strenght between Higgs and the Ricci scalar. Then a conformal transformation to the metric and re-definition of the Higgs field revealed the inflatory features at large field values. Power law potential did not produce results close to the best-fit values measured by Planck six year measurement. Higgs inflation however provides good results.

What I did not discuss was the known and interesting problem of Higgs inflation, called the unitarity violation: the inflating energy regime of Higgs inflation is beyond the cut-off scale of the effective field theoretical prescription. That particular nuisance has caused a lot of headache in the cosmologist community lately. That is one topic I'm going to orientate myself in the future research. Also the dynamics of other fields during inflation and the connection between inflaton and dark matter is something that has only recently been started to study in detail. There are also fundamental questions considering the basic idea of inflation such as:

- What field should play the role of inflaton?
- What set the form of the effective inflationary potential?
- Why was the field high on the potential at the beginning of inflation?

A major effort for future observational cosmological experiments is to measure the effects of primordial tensor modes, possibly encoded in the polarization of the CMB. Measurement of the tensor spectral index would help to rule out majority of the inflationary models with the help of the consistency relation. Several collaborations are rushing to get a first glimpse of the B-mode polarization caused by inflation generated gravitational waves. In few years we'll hopefully get results from that field too.

In doing this thesis I have slowly learned the subject and can now with a confidence say that I see at least a glimpse of the big picture. Cosmological perturbation theory and inflation have been fascinating topics to learn and will hopefully provide many sensations of success and wonder in my future research.

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Appendices

A Conventions and definitions

The spacetime metric has a signature $(- + ++)$. When used as a tensor index Greek letters $\alpha, \beta, \gamma, \ldots$ take values from 0,1,2,3 and usually refer to four-vectors, as contrary to Latin letters i,j,k,$\ldots$ which are usually used as spatial tensor indeces and take values 1,2,3. The following convention to Fourier transformation and its inverse are used:

$$\begin{aligned} f(t, \mathbf{x}) &= \int \frac{\mathrm{d}\mathbf{x}^3}{\sqrt{2\pi}} f_{\mathbf{k}}(t) e^{+i\mathbf{k}\cdot\mathbf{x}} \\ f_{\mathbf{k}}(t) &= \int \frac{\mathrm{d}\mathbf{k}^3}{\sqrt{2\pi}} f(t, \mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}. \end{aligned}$$

Since every Fourier transformation in this thesis is applied to a differential equation, the only visible effect of such a transformation is that the Laplace operator transforms as

$$\nabla^2 \to -k^2$$

and quantities in Fourier space get a subscript $\mathbf{k}$ indicating their dependence of the wave-vector. The physical wavelenght is defined as

$$\lambda(t) = \frac{2\pi a}{k},$$

where k is the comoving wavenumber. I define perturbations to have wavelenghts inside the Hubble horizon when $\lambda \ll \frac{1}{H}$ and thus $k \gg aH$. Perturbations satisfying this condition are also called *sub-horizon*. Respectively, perturbations with $\lambda \gg \frac{1}{H}$ and thus $k \ll aH$ are called *superhorizon*. I use the reduced Planck mass $M_P^2 = 1/8\pi G$.

References

[1] S. Weinberg, *Cosmology*. Oxford University Press, 2008.

[2] V. Mukhanov, *Physical Foundations of Cosmology*. Cambridge University Press, 2005.

[3] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*. Benjamin Cummings, 2003.

[4] **Planck** Collaboration, R. Adam *et. al.*, "Planck 2015 results. I. Overview of products and scientific results," `1502.01582`.

[5] S. Dodelson, *Modern Cosmology*. Academic Press. Academic Press, 2003.

[6] D. Baumann, "The physics of inflation." A course for graduate students in particle physics and cosmology in 2011, University of Cambridge, Department of Applied Mathematics & Theoretical Physics, 2011.

[7] A. A. Starobinsky, "A New Type of Isotropic Cosmological Models Without Singularity," *Phys.Lett.* **B91** (1980) 99–102.

[8] A. H. Guth, "The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems," *Phys.Rev.* **D23** (1981) 347–356.

[9] A. D. Linde, "A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems," *Phys.Lett.* **B108** (1982) 389–393.

[10] **Planck** Collaboration, P. Ade *et. al.*, "Planck 2015 results. XX. Constraints on inflation," `1502.02114`.

[11] D. Lyth and A. Liddle, *The Primordial Density Perturbation: Cosmology, Inflation and the Origin of Structure*. Cambridge University Press, 2009.

[12] D. Gorbunov and V. Rubakov, *Introduction to the Theory of the Early Universe: Cosmological Perturbations and Inflationary Theory*. World Scientific, 2011.

[13] R. L. Arnowitt, S. Deser, and C. W. Misner, "The Dynamics of general relativity," *Gen.Rel.Grav.* **40** (2008) 1997–2027, `gr-qc/0405109`.

[14] D. Baumann, "Cosmology." Lecture notes on Cosmology.

[15] K. A. Malik and D. Wands, "Cosmological perturbations," *Physics Reports* **475** (2009), no. 1–4 1 – 51.

[16] V. F. Mukhanov, H. A. Feldman, and R. Brandenberger, "Theory of cosmological perturbations," *Physics Reports* **215** (1992) 203–333.

[17] H. Kurki-Suonio, "Cosmological perturbation theory, part 2." Lecture notes for a course of cosmological perturbation theory given at the University of Helsinki 2010, 2010.

[18] S. Weinberg, *Gravitation and Cosmology.* John Wiley & Sons, Inc., 1972.

[19] F. Bezrukov and M. Shaposhnikov, "The standard model higgs boson as the inflaton," *Physics Letters B* **659** (2008), no. 3 703 – 706.

[20] R. Wald, *General Relativity.* Physics, astrophysics. University of Chicago Press, 1984.

[21] V. Faraoni, E. Gunzig, and P. Nardone, "Conformal transformations in classical gravitational theories and in cosmology," *Fund.Cosmic Phys.* **20** (1999) 121, `gr-qc/9811047`.