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GENERALIZED DIMENSION DISTORTION UNDER SOBOLEV MAPPINGS

ALEKSANDRA ZAPADINSKAYA

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Jyväskylä, April 2011
Aleksandra Zapadinskaya

LIST OF INCLUDED ARTICLES

This dissertation consists of an introductory part and the following five articles:

- [A] Pekka Koskela, Aleksandra Zapadinskaya, and Thomas Zürcher, *Generalized dimension distortion under planar Sobolev homeomorphisms*, Proc. Amer. Math. Soc. **137** (2009), no. 11, 3815–3821. MR 2529891
- [B] Pekka Koskela, Aleksandra Zapadinskaya, and Thomas Zürcher, *Mappings of finite distortion: generalized Hausdorff dimension distortion*, J. Geom. Anal. **20** (2010), no. 3, 690–704. MR 2610895
- [C] Tapio Rajala, Aleksandra Zapadinskaya, and Thomas Zürcher, *Generalized dimension distortion under mappings of sub-exponentially integrable distortion*, preprint
- [D] Tapio Rajala, Aleksandra Zapadinskaya, and Thomas Zürcher, *Generalized Hausdorff dimension distortion in Euclidean spaces under Sobolev mappings*, preprint
- [E] Aleksandra Zapadinskaya, *Generalized dimension compression under mappings of exponentially integrable distortion*, Cent. Eur. J. Math. **9** (2011), no. 2, 356–363.

The work presented in this dissertation has been carried out at the Department of Mathematics and Statistics of University of Jyväskylä during the period 2006–2010. The author of this dissertation has actively taken part in the research for these articles.

INTRODUCTION

1. Measures and dimensions

This thesis studies estimates on the size of images of certain sets under certain kinds of Sobolev mappings. We will make assumptions on both the size of the preimages and properties of the mappings to obtain these estimates. Both Hausdorff and Minkowski dimensions and Hausdorff measures will be used to characterize the sets.

The generalized Hausdorff measure of a set $A \subset \mathbb{R}^n$ is defined as

$$\mathcal{H}^h(A) = \lim_{\delta \to 0} \mathcal{H}^h_\delta(A),$$

where

$$\mathcal{H}^h_\delta(A) = \inf\Big\{ \sum_{i=1}^{\infty} h(\operatorname{diam} U_i) \colon A \subset \bigcup_{i=1}^{\infty} U_i, \operatorname{diam} U_i \leq \delta \Big\}$$

and $h\colon [0,\infty[\to [0,\infty[$ is a dimension gauge: a non-decreasing function with $\lim_{t\to 0+} h(t) = h(0) = 0$. In the special case where $h(t) = t^\alpha$ for some $\alpha \geq 0$, we have the usual Hausdorff α-dimensional measure, which we simply denote by $\mathcal{H}^\alpha$. The Hausdorff dimension $\dim_{\mathcal{H}} A$ of a set $A \subset \mathbb{R}^n$ is the smallest $\alpha_0 \geq 0$ such that $\mathcal{H}^\alpha(A) = 0$ for any $\alpha > \alpha_0$.

The lower Minkowski dimension $\dim_{\mathcal{M}}(A)$ of a bounded set $A \subset \mathbb{R}^n$ is defined as

$$\dim_{\mathcal{M}}(A) = \inf\{ s\colon \liminf_{\varepsilon\to 0+} N(A,\varepsilon)\varepsilon^s = 0\},$$

where $N(A,\varepsilon)$, $\varepsilon > 0$, denotes the smallest number of balls of radius ε needed to cover A, i.e.

$$N(A,\varepsilon) = \min\Big\{ k\colon A \subset \bigcup_{i=1}^{k} B(x_i,\varepsilon) \text{ for some } \{x_1,\dots,x_k\} \subset \mathbb{R}^n \Big\}.$$

The estimates obtained in this thesis, mentioned in the beginning of the section, will naturally be referred to as “dimension distortion estimates”.

2. Quasiconformal mappings

Let Ω be a domain in $\mathbb{R}^n$. A homeomorphism f in the Sobolev class $W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ is called quasiconformal if there exists a constant $K \in [1,\infty[$ such that the inequality

$$|Df(x)|^n \leq K|J_f(x)|$$

holds almost everywhere in Ω. Here $|Df(x)|$ denotes the operator norm of the formal differential of f at the point $x \in \Omega$, which exists provided all the distributional partial derivatives of f exist, and J_f is the Jacobian of f.

Dimension distortion results for quasiconformal mappings were first obtained by F.W. Gehring and J. Väisälä in [10]. They proved the existence of a constant

$\beta \in [\alpha, 2[$, depending only on $K \in [1, \infty[$ and $\alpha \in [0, 2[$, such that $\dim_{\mathcal{H}} f(E) \leq \beta$ for every planar K-quasiconformal mapping $f\colon \Omega \to \mathbb{R}^2$ and a set $E \subset \Omega$ with $\dim_{\mathcal{H}} E \leq \alpha$. The obtained estimate strongly relies on the higher integrability result of B. Bojarski [5], which shows that for a planar K-quasiconformal mapping f, J_f is locally L^q-integrable for all $q \in [1, p(K)]$, where $p(K) > 1$ is some constant depending only on K.

Later this dimension distortion estimate was generalized to higher dimensions [9]. This time the constant $\beta \in [\alpha, n[$ depends on the dimension n of the underlying space along with K and $\alpha \in [0, n[$. Again, the result is a consequence of the higher integrability of the Jacobian, obtained in the same paper for quasiconformal mappings in $\mathbb{R}^n$ with $n \geq 2$.

Since the inverse of a K-quasiconformal mapping is also quasiconformal, we obtain a dimension distortion estimate of the form

$$\dim_{\mathcal{H}} E \geq \alpha' \Longrightarrow \dim_{\mathcal{H}} f(E) \geq \beta',$$

where $f\colon \Omega \to \mathbb{R}^n$ is a K-quasiconformal mapping, $E \subset \Omega$ and $\beta' \in]0, n]$ is a constant, depending only on n, K and $\alpha' \in]0, n]$.

The sharp integrability result for the Jacobian of a quasiconformal mapping in the plane was established only in 1994 in [1] together with sharp dimension distortion estimates. It was shown that each K-quasiconformal mapping $f\colon \Omega \to \mathbb{R}^2$, $\Omega \subset \mathbb{R}^2$, is in the class $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ for all $p < \frac{2K}{K-1}$, and hence J_f is locally L^q-integrable for all $q < \frac{K}{K-1}$. These bounds are best possible. Dimension is distorted by a K-quasiconformal mapping f for a set $E \subset \Omega$ in the following way

$$\frac{1}{K}\Big(\frac{1}{\dim_{\mathcal{H}} E} - \frac{1}{2}\Big) \leq \frac{1}{\dim_{\mathcal{H}} f(E)} - \frac{1}{2} \leq K\Big(\frac{1}{\dim_{\mathcal{H}} E} - \frac{1}{2}\Big). \tag{1}$$

The example in [1, Theorem 1.4] provides a quasiconformal mapping, for which equality in (1) is reached.

3. Beltrami equation

The proofs in [1] utilize the Beltrami equation, which has no convenient analogues in higher dimensions. This makes the generalization of the results difficult.

Every K–quasiconformal mapping in the complex plane $f\colon \Omega \to \mathbb{C}$ satisfies the Beltrami equation

$$\overline{\partial} f(z) = \mu(z) \partial f(z), \tag{2}$$

where $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$, $\partial = \frac{1}{2}(\partial_x - i\partial_y)$, and $\mu = \mu_f = \overline{\partial} f / \partial f$ is called the Beltrami coefficient of f. Note that quasiconformality implies $||\mu||_\infty = \frac{K-1}{K+1} < 1$. This equation was studied by Morrey in the late 1930s. In [26] he proved the existence of homeomorphic solutions for a given Beltrami equation (2) with $||\mu||_\infty < 1$. These solutions are K–quasiconformal mappings with $K = \frac{1+||\mu||_\infty}{1-||\mu||_\infty}$. The existence theorem may be stated as follows (see [19, Theorem 11.1.2]).

Theorem 1. *Let μ be a measurable function in a domain $\Omega \subset \mathbb{C}$ and suppose $||\mu||_\infty < 1$. Then there is a quasiconformal mapping $g\colon \Omega \to \mathbb{C}$ whose Beltrami* *coefficient is equal to μ almost everywhere in Ω. Moreover, every $W^{1,2}_{\mathrm{loc}}(\Omega)$ solution to the Beltrami equation (2) is of the form*

$$f(z) = F(g(z)), \tag{3}$$

where $F\colon g(\Omega) \to \mathbb{C}$ is a holomorphic mapping.

The decomposition in (3) is called Stoilow factorization. This tool allows one to remove the injectivity assumption in the planar setting for certain distortion theorems.

The case when $||\mu||_\infty = 1$ was first studied in [25] and further in [7]. In these works, the existence of homeomorphic solutions of (2) was established under certain integrability conditions on $K(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$. We deal with two different cases when $||\mu||_\infty = 1$: mappings of exponentially integrable distortion and mappings of sub-exponentially integrable distortion, defined in Sections 4 and 6, respectively.

A planar mapping $f\colon \Omega \to \mathbb{C}$ of finite distortion with distortion function K (see the next section for the definition) satisfies the Beltrami equation (2) almost everywhere in Ω with

$$\mu(z) = \mu_f(z) = \begin{cases} \overline{\partial} f(z)/\partial f(z), & \text{if } \partial f(z) \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

such that $|\mu(z)| = \frac{K(z)-1}{K(z)+1} < 1$. Conversely, the Beltrami equation (2) with $|\mu(z)| < 1$ for almost every z implies distortion inequality (4) with $K(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$ for almost every z, for its pointwise solutions in $W^{1,1}_{\mathrm{loc}}$.

Mappings of exponentially integrable distortion form a particular class of mappings of finite distortion with $\exp(\lambda K)$ locally integrable for some $\lambda > 0$ (here K is the distortion function of the mapping). The existence of a homeomorphic solution g for the Beltrami equation (2) with $K = \frac{1+|\mu|}{1-|\mu|}$ such that $\exp(\lambda K) \in L^1$ for some $\lambda > 0$ was proved in [7, Théorème 1] and [19, Theorem 11.8.3]. Moreover, every mapping f of λ-exponentially integrable distortion satisfying the same Beltrami equation admits a Stoilow factorization (3) with holomorphic F.

The Beltrami equation (2) with sub-exponentially integrable distortion function $K = \frac{1+|\mu|}{1-|\mu|}$ (see Section 6) also has homeomorphic solutions and every mapping of sub-exponentially integrable distortion admits Stoilow factorization (see [19, Theorem 11.8.2] and [4, Theorems 20.5.1 and 20.5.2]).

4. Mappings of finite exponentially integrable distortion

Mappings of finite distortion form a natural generalization of the class of quasiconformal mappings, where K may be an unbounded function. More precisely, a continuous mapping $f\colon \Omega \to \mathbb{R}^n$ of the class $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is called a mapping of finite distortion if there exists a measurable function $K\colon \Omega \to [1, \infty[$ such that the inequality

$$|Df(x)|^n \leq K(x) J_f(x) \tag{4}$$

holds for almost every x in Ω. In order to obtain nice properties for the mapping f, the Jacobian J_f is additionally assumed to be locally integrable in Ω. An optimal choice for the function K is

$$K(x) = K_f(x) = \begin{cases} \frac{|Df(x)|^n}{J_f(x)}, & \text{if } J_f(x) \neq 0, \\ 1, & \text{otherwise.} \end{cases}$$

We usually refer to this K, when we speak of the "distortion function" of f.

A special class of mappings of finite distortion is the class of mappings of exponentially integrable distortion, that is, such mappings for which $\exp(\lambda K) \in L^1_{\mathrm{loc}}(\Omega)$ holds for their distortion functions K and some $\lambda > 0$. This assumption allows one to obtain such properties for the mapping as monotonicity, continuity (without an a priori assumption), discreteness, openness and Luzin condition N [18, 23, 22].

It was shown in [15] and [28] that the estimate

$$\dim_{\mathcal{H}} E < n \Longrightarrow \dim_{\mathcal{H}} f(E) < n$$

does not necessarily hold when f is a general mapping of exponentially integrable distortion. It was shown in [15] that there exists a constant $C \geq 1$ depending only on n, such that for any given $\lambda > 0$ and $\varepsilon \in]0, \lambda[$, a mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of $(\lambda/C - \varepsilon)$-exponentially integrable distortion may be constructed so that f sends a set $\mathcal{C}$ of Hausdorff dimension strictly less than n to a set $\mathcal{C}'$ of Hausdorff dimension n. More precisely, the image set $\mathcal{C}'$ has a positive generalized Hausdorff measure $\mathcal{H}^h$ with gauge function $h(t) = t^n \log^{\lambda}(1/t)$.

The paper [15] also gives an estimate for the size of the image of the unit circle $S^1 \subset \mathbb{R}^2$ under a planar mapping of exponentially integrable distortion. The estimate is obtained in terms of a generalized Hausdorff measure: there exists an absolute constant $k > 0$ such that if $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism of λ-exponentially integrable distortion for some $\lambda > 0$ then $\mathcal{H}^h(f(S^1)) < \infty$ for $h(t) = t^2 \log^{kn}(1/t)$.

This thesis deals with estimates of the similar form. In [B], the estimate above was proved for a more general class of pre-images and with a slightly better gauge function for the generalized Hausdorff measure of image sets. The following theorem was established.

Theorem 2. [B, Theorem 1] *Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$, $\Omega \subset \mathbb{R}^2$, be a mapping of locally λ-exponentially integrable distortion, $\lambda > 0$. Set $h_s(t) = t^2 \log^s(1/t)$ for $s \in \mathbb{R}$. If $E \subset \mathbb{R}^2$ satisfies $\dim_{\mathcal{H}}(E) < 2$, then $\mathcal{H}^{h_s}(f(E)) = 0$ for all $s < \lambda$, where $\mathcal{H}^{h_s}$ is the generalized Hausdorff measure associated to h_s.*

The example in [15] mentioned above shows this result is sharp modulo the constant.

The proof is based on the fact that generalized derivatives of mappings of exponentially integrable distortion which are *a priori* assumed to be only locally integrable, actually enjoy higher regularity. Some higher regularity results were proved in [7, 16, 17, 8] and [2] gives the sharp regularity for the planar case. The regularity is stated in terms of Orlicz-Sobolev classes: it was first shown that

$$|Df|^2 \log^{c\lambda - 1}(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega)$$

with a constant c (which may depend on the dimension) for a mapping f of λ-exponentially integrable distortion, and then in [2, Theorem 1.1], it was demonstrated that in the planar case, c may be taken as any number strictly less than 1.

Thus, it suffices to establish a dimension distortion estimate for Orlicz-Sobolev mappings to conclude Theorem 2. Such estimates were obtained in the planar case in [A, Theorem 1.1], [B, Theorem 2] and [27, Theorem 1.1], and the results were generalized to higher dimensions in [D, Theorem 1].

5. Orlicz-Sobolev mappings

An Orlicz function is a continuously increasing function $P\colon [0,\infty) \to [0,\infty)$ such that $P(0) = 0$ and $\lim_{t\to\infty} P(t) = \infty$. Given an Orlicz function P, we denote by $L^P(\Omega)$ the Orlicz class of integrable functions $h : \Omega \to \mathbb{R}$ such that

$$\int_\Omega P(\nu|h|) < \infty$$

for some $\nu = \nu(h) > 0$. An Orlicz-Sobolev class $W^{1,P}(\Omega)$ is the class of mappings $g \in W^{1,1}(\Omega, \mathbb{R}^2)$ which have all first order partial derivatives in the class $L^P(\Omega)$.

Due to the form of the higher regularity condition for the mappings of exponentially integrable distortion, we are particularly interested in dimension distortion estimates for mappings in the Orlicz-Sobolev class $W^{1,P}(\Omega, \mathbb{R}^2)$ with $P(t) = t^2 \log^\lambda(e + t)$ for some $\lambda > 0$. The proof of [B, Theorem 1] relies on the following result.

Theorem 3. [B, Theorem 2] *Let Ω be an open set in $\mathbb{R}^2$ and $f\colon \Omega \to f(\Omega) \subset \mathbb{R}^2$ be a homeomorphism in $W^{1,2}(\Omega; \mathbb{R}^2)$ with $|Df|^2 \log^\lambda(e + |Df|) \in L^1_{\text{loc}}(\Omega)$ for some $\lambda > 0$. Then, for $E \subset \mathbb{R}^2$, we have*

$$\dim_{\mathcal{H}}(E) < 2 \Longrightarrow \mathcal{H}^{h_\lambda}(f(E)) = 0$$

for $h_\lambda(t) = t^2 \log^\lambda\left(\frac{1}{t}\right)$.

This theorem, stated for homeomorphisms, actually only uses the monotonicity of f. Recall that a continuous real-valued function $u\colon \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$) is said to be monotone if for every ball $B \subset \Omega$ we have

$$\sup_{\partial B} f = \sup_B f \qquad \text{and} \qquad \inf_{\partial B} f = \inf_B f.$$

A mapping $f\colon \Omega \to \mathbb{R}^n$ is called monotone if its component functions are monotone. However, the assumption of being a homeomorphism is not too restrictive in the case of planar mappings of finite distortion because of Stoilow factorization.

As one can see, the direct combination of Theorem 3 with the higher regularity result of Theorem 1.1 in [2] would give us Theorem 2 with $s < \lambda - 1$ instead of $s < \lambda$. The gap is bridged with the help of the so-called "minimal decomposition" of the Beltrami coefficient, proved in [7, Proposition 3] for the quasiconformal case and generalized to the case of mappings of exponentially integrable distortion by [2, Corollary 4.4]. We also use Stoilow factorization to obtain homeomorphic mappings. Eventually, the initial mapping f is decomposed into a holomorphic mapping, a quasiconformal mapping and a homeomorphism with finite distortion, having better integrability properties than the distortion of the initial mapping. We employ higher regularity from [1] to obtain a generalized Hausdorff measure distortion estimate for the quasiconformal mapping in question.

In the proof of Theorem 3, we estimate the generalized Hausdorff measure of the image using the $5r$–covering theorem. The covering sets are the images of the balls which cover the initial set E. First, we bound the diameters of the images of these balls from above by the integrals of the differential of our mapping f over larger annuli. This is done simply using properties of monotone Sobolev mappings. Passing from a larger annulus to a smaller ball is possible thanks to the following concept of a maximal operator. Assume that $\Omega \subset \mathbb{R}^2$ is a square and $h\colon \Omega \to \mathbb{R}$ is nonnegative and integrable. The maximal operator $\mathcal{M}_\Omega$ is defined by

$$\mathcal{M}_\Omega h(x) = \sup\Big\{ \fint_Q h\, dx\colon\ x \in Q \subset \Omega \Big\},$$

where the supremum is taken over all subsquares of Ω containing the given point $x \in \Omega$. Direct estimations bound the diameters of the images of the initial balls by the integrals of $\mathcal{M}_\Omega |Df|$ over the balls which are five times smaller and pairwise disjoint by the $5r$–covering theorem. Having obtained this estimate, we are able to establish an arbitrarily small bound for the sum

$$\sum \operatorname{diam}^2 f(B) \log^\lambda \left(\frac{1}{\operatorname{diam} f(B)} \right)$$

over the initial covering balls. The estimations take advantage of the size of the initial set E and the same Orlicz-type integrability for $\mathcal{M}_\Omega |Df|$ as for $|Df|$. The latter is by [12, Lemma 5.1].

A better estimate than the one in Theorem 3 was established in [A] for the Minkowski dimension. The following was shown.

Theorem 4. [A, Theorem 1.1] *Let Ω and Ω' be open sets in $\mathbb{R}^2$ and $f\colon \Omega \to \Omega'$ a homeomorphism of class $W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ with*

$$|Df|^2 \log^{\lambda-1}(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega)$$

for some $\lambda > 0$. Then

$$\mathcal{H}^{h_\lambda}(f(E)) = 0$$

for $h_\lambda(t) = t^2 \log^\lambda \left(\frac{1}{t}\right)$ and every set $E \subset \Omega$ of lower Minkowski dimension $\dim_{\mathcal{M}}(E)$ strictly less than two.

The sharpness of this theorem is demonstrated by the previously mentioned example in [15, Proposition 5.1]. Indeed, it may be shown that the constructed homeomorphism f, sending a set of Hausdorff (and Minkowski) dimension strictly less than n to a set of positive generalized Hausdorff measure $\mathcal{H}^h$ with gauge function $h(t) = t^n \log^\lambda(1/t)$, belongs to the class $W^{1,P}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ with $P(t) = t^n \log^s(e+t)$ for all $s < \lambda - 1$ (see [A, Section 2]).

The proof of the theorem relies on the Hölder continuity estimates for the inverse of a Sobolev mapping. In order to establish such estimates for f, we use the fact that f^{-1} belongs to the class $BV_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ under our assumptions [14, Theorem 1.2]. We apply the obtained Hölder continuity while estimating the Lebesgue measure of the image with the help of the inequality $|f(A)| \leq \int_A J_f$ from [22, Lemma 3.2]. We further use the higher integrability of the Jacobian of an Orlicz-Sobolev mapping from [20, Corollary 9.1] to continue with the estimations. Finally, the Besicovitch covering theorem allows us to make conclusions on the generalized Hausdorff measure of the image based on the obtained estimate for its Lebesgue measure.

A sharp dimension distortion estimate in the plane for Hausdorff dimensions was proved in [27]:

Theorem 5. [27, Theorem 1.1] *Let Ω be an open set in $\mathbb{R}^2$ and $f\colon \Omega \to f(\Omega)$ a homeomorphism of the class $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$ with*

$$|Df|^2 \log^{\lambda-1}(e+|Df|) \in L^1_{\mathrm{loc}}(\Omega)$$

for some $\lambda > 0$. Then, with $h_\lambda(t) = t^2 \log^\lambda \frac{1}{t}$,

$$\mathcal{H}^{h_\lambda}(f(E)) = 0$$

for every set $E \subset \Omega$ such that $\dim_{\mathcal{H}} h(E) < 2$.

The proof follows the same strategy as in Theorem 4, but the coverings used in the proof are dealt with in a finer way.

Planar results mentioned in this section were generalized to higher dimensions in [D]:

Theorem 6. [D, Theorem 1] *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ a continuous map in $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with $|Df|^n \log^\lambda(e+|Df|) \in L^1_{\mathrm{loc}}(\Omega)$ for some $\lambda \in \mathbb{R}$. Then, with $h_\gamma(t) = t^n \log^\gamma \left(\frac{1}{t}\right)$,*

$$\dim_{\mathcal{H}}(E) < n \Longrightarrow \mathcal{H}^{h_\gamma}(f(E)) = 0,$$

if one of the following cases occurs:

(i) $\lambda > n-1$ and $\gamma < \lambda - n + 1$.
(ii) f is monotone, $\lambda > 0$, and $\gamma \leq \lambda$.
(iii) f is a homeomorphism, $f^{-1} \in W^{1,p}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ for some $p > n-1$, $\lambda > -1$, and $\gamma \leq \lambda + 1$.

The items (*ii*) and (*iii*) are generalizations of Theorems 3 and 5, respectively. The proofs repeat the same main steps. The assumption $f^{-1} \in W^{1,p}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ in (*iii*) is dictated by the techniques of the proof since we wish to obtain a Hölder continuity estimate for f^{-1} analogous to the one we used in the proofs of Theorems 4 and 5. The proof of (*i*) relies on the estimate from [21, Theorem 3.2] on the oscillation of a Sobolev function on balls. Further estimations on the images of balls use techniques similar to the ones in the proof of (*ii*). This time, the diameters of the images of the balls are bounded by [21, Theorem 3.2] by integrals over the same balls, so we do not have to use maximal operators to pass to smaller balls. We use the Besicovitch covering theorem instead of the $5r$–covering theorem to obtain our balls, which do not overlap too many times.

We do not know if the estimates in (*i*) and (*ii*) are sharp. However, the same example as above shows sharpness of the estimate in (*iii*).

As in the planar case, once we have higher regularity for mappings of finite distortion, we may combine it with the obtained dimension distortion estimates for Orlicz-Sobolev mappings. A mapping of exponentially integrable distortion is monotone, its inverse was proved to have nice regularity properties [13], and the higher regularity is demonstrated in [8, Theorem 1.1]. All this gives us the following corollary:

Corollary 1. [D, Corollary 2] *Let $\Omega \subset \mathbb{R}^n$ be a domain and $\lambda > 0$. There exist positive constants c_1 and c_2 depending only on n such that if $f : \Omega \to \mathbb{R}^n$ is of λ-exponentially integrable distortion and satisfies*

(i) $\lambda > 1/c_1$ and $\gamma \le c_1\lambda - 1$ or

(ii) f is homeomorphic and $\gamma \le c_2\lambda$,

then the following implication is true:

$$\dim_{\mathcal{H}}(E) < n \Longrightarrow \mathcal{H}^{h_\gamma}(f(E)) = 0,$$

where $h_\gamma(t) = t^n \log^\gamma\left(\frac{1}{t}\right)$.

The example above demonstrates sharpness of (ii) modulo the constant.

6. Mappings of sub-exponentially integrable distortion

The assumption of exponential integrability for the distortion function may be relaxed by replacing it with a more general Orlicz condition. That is, one may assume $e^{\mathcal{A}(K)} \in L^1_{\mathrm{loc}}$ with some smooth increasing function $\mathcal{A}\colon [1,\infty[\to [0,\infty[$ such that

$$\int_1^\infty \frac{\mathcal{A}(t)}{t^2} dt = \infty, \tag{5}$$

where K is a distortion function of a mapping of finite distortion $f\colon \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ (see [4, Section 20.5]). In particular, when $\mathcal{A}(t) = p\frac{t}{1+\log t} - p$, for some $p > 0$, such a mapping f is called a mapping of sub-exponentially integrable distortion. Mappings of sub-exponentially integrable distortion are continuous, open, discrete and satisfy Luzin condition N [24].

The higher regularity for these mappings in general dimensions was studied in [6]. An Orlicz-Sobolev condition established there has the following form in the case of our particular $\mathcal{A}$: $|Df| \in L^{P_\beta}_{\mathrm{loc}}(\Omega)$ for a mapping $f\colon \Omega \to \mathbb{R}^n$ of sub-exponentially integrable distortion and all $\beta < cp$, where

$$P_\beta(t) = \frac{t^n}{\log(e+t)\log^{1-\beta}(\log(e^e+t))} \tag{6}$$

and $c > 0$ is a constant depending only on n. This result was sharpened for the planar case in [11]. The constant c is removed and the higher regularity condition becomes plainly

$$\frac{|Df|^2}{\log(e+|Df|)\log^{1-\beta}(\log(e^e+|Df|))} \in L^1_{\mathrm{loc}}(\Omega)$$

for a planar mapping of sub-exponentially integrable distortion $f\colon \Omega \to \mathbb{R}^2$ and all $\beta < p$.

Paper [C] takes advantage of these higher regularity results. We apply techniques that are similar to the ones used in the proofs of Theorems 4, 5 and 6 (iii). In the case of the higher regularity of the form (6), these techniques give us a dimension distortion estimate of the following form, where $h_{n,\beta}(t) = t^n(\log\log(1/t))^\beta$ and K_f is the distortion function of f:

Theorem 7. [C, Theorem 1] *There exists a constant $c > 0$, which depends only on the dimension n of the underlying space, such that for every homeomorphism of finite distortion $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$, with*

$$e^{\frac{K_f}{1+\log K_f}} \in L^p_{\mathrm{loc}}(\Omega)$$

for some $p > 0$, we have $\mathcal{H}^{h_{n,\beta}}(f(E)) = 0$ for all $\beta < cp$, whenever $E \subset \Omega$ is such that $\dim_{\mathcal{H}} E < n$.

Due to the general form of all auxiliary results we use in the proof of the theorem above, one could extend this theorem to a case of a more general function $\mathcal{A}$ in the Orlicz integrability condition for the distortion function, in particular, when $\mathcal{A}$ is given by

$$\mathcal{A}_{p,k}(t) = \frac{pt}{1 + \log(t)\log(\log(e-1+t))\cdots\log(\ldots(\log(e^{e^{\cdot^{\cdot^{\cdot^{e}}}}} - 1 + t))\ldots)} - p$$

with some $p > 0$ and $k \in \mathbb{N}$, where k means that the last logarithmic expression is a k-th iterated logarithm.

When $n = 2$, the assumption that f is a homeomorphism is not necessary due to Stoilow factorization:

Theorem 8. [C, Theorem 2] *Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$, $\Omega \subset \mathbb{R}^2$, be a mapping of finite distortion with*

$$e^{\frac{K_f}{1+\log K_f}} \in L^p_{\mathrm{loc}}(\Omega).$$

Then $\mathcal{H}^{h_{2,\beta}}(f(E)) = 0$ for all $\beta < p$, whenever $E \subset \Omega$ is such that $\dim_{\mathcal{H}} E < 2$.

Paper [C] provides an example to show that Theorems 7 and 8 are sharp modulo the constants:

Example 1. [C, Example 1] There exists a constant $C \geq 1$ depending only on n, such that for any $\beta > 0$ and $\varepsilon \in]0, \beta[$, we may construct sets $\mathcal{C}$, $\mathcal{C}' \subset [0,1]^n$, satisfying $\dim_{\mathcal{H}} \mathcal{C} < n$ and $\mathcal{H}^{h_{n,\beta}}(\mathcal{C}') > 0$, and a mapping of finite distortion $f \in W^{1,1}([0,1]^n; \mathbb{R}^n)$, such that

$$e^{\frac{K_f}{1+\log K_f}} \in L^{\frac{1}{C}\beta - \varepsilon}([0,1]^n)$$

and $f(\mathcal{C}) = \mathcal{C}'$.

The construction is analogous to the one in [15, Proposition 5.1]. The image and the pre-image sets $\mathcal{C}$, $\mathcal{C}' \subset [0,1]^n$ are Cantor sets which are obtained as Cartesian products of Cantor sets on the real line with themselves. The Cantor set on the real line used for construction of the pre-image set $\mathcal{C}$ is a “standard” self-similar Cantor set, obtained as an infinite intersection of families of equal length intervals, such that the length of each interval of the next generation differs from the length of the previous generation intervals by a fixed factor. The construction in the case of the image set $\mathcal{C}'$ is very similar, but the above mentioned factor is not fixed. This allows one to obtain a set of positive generalized Hausdorff measure with the required gauge.

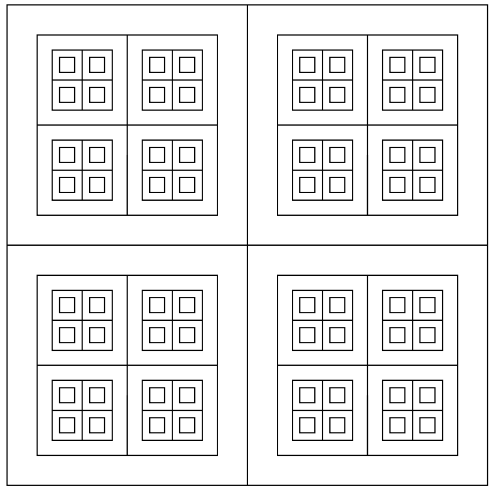

FIGURE 1. "Cubical frames" used in the construction.

Eventually, the cube $[0,1]^n$ on both the pre-image and the image side becomes split into so-called "cubical frames" (see Figure 1), that is, sets of the form

$$\{x \in \mathbb{R}^n \colon r < |x|_\infty < R\},$$

where $|\cdot|_\infty$ denotes the maximal norm defined as $|x|_\infty = \max\{|x_i|\}_{i=1}^n$ for $x \in \mathbb{R}^n$. The frames are pairwise disjoint and cover the cube $[0,1]^n$ up to a set of Lebesgue measure zero. The behaviour of our mapping f is essentially defined by its form on those frames, where it looks like

$$(a|x|_\infty + b)\frac{x}{|x|_\infty}$$

up to a translation (here, the numbers a and b are adjusted so that the pre-image frame is mapped exactly onto the image frame, inner boundary to inner boundary).

The integrability assumption in (5) is essential if one wishes to obtain dimension distortion estimates for mappings of finite distortion. Indeed, Section 5 of [22] provides a construction of a homeomorphism f of finite distortion K with $e^{\mathcal{A}(K)} \in L^1_{\text{loc}}$ for some function $\mathcal{A} \colon [1,\infty[\to [0,\infty[$ such that

$$\int_1^\infty \frac{\mathcal{A}(t)}{t^2} dt < \infty,$$

and f maps a set of Hausdorff dimension strictly less than the dimension n of the underlying space to a set of positive Lebesgue measure. More precisely, $\mathcal{A}$ is taken as $\mathcal{A}(t) = p\frac{t}{\log^2(e+t)} - p$ for some particular $p > 0$. See [24] for refined constructions.

7. INVERSE ESTIMATE

By now, we have discussed generalizations for the estimate

$$\dim_{\mathcal{H}} E \leq \alpha \Longrightarrow \dim_{\mathcal{H}} f(E) \leq \beta,$$

obtained for the case of quasiconformal mappings in [10, 9, 1]. A counterpart for the inverse estimate

$$\dim_{\mathcal{H}} E \geq \alpha' \Longrightarrow \dim_{\mathcal{H}} f(E) \geq \beta'$$

is given in [E] for the mappings of exponentially integrable distortion. It is again formulated in terms of generalized Hausdorff measures:

Theorem 9. [E, Theorem 1] *There exists a constant $c_n > 0$, depending only on n, such that for any homeomorphic mapping $f\colon \Omega \to \mathbb{R}^n$ of λ-exponentially integrable distortion, we have $\mathcal{H}^{h_{n,\lambda,s,c_n}}(f(E)) > 0$ for each set $E \subset \mathbb{R}^n$ such that $\mathcal{H}^s(E) > 0$, where $h_{n,\lambda,s,c_n}(t) = \exp(-c_n \lambda^{\frac{1}{n}} s \log^{\frac{n-1}{n}} \frac{1}{t})$.*

This estimate follows from the modulus of continuity of the inverse of a mapping of exponentially integrable distortion, obtained in [15]. The estimate is asymptotically sharp modulo the constant c_n as the dimension s tends to zero. This is demonstrated by the following example.

Example 2. [E, Example 1] For any $s \in]0, n[$ and $\lambda > 0$ there exists a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ of μ-exponentialy integrable distortion for all $\mu \in]0, \lambda[$, mapping a set $\mathcal{C}$ of positive s–dimensional Hausdorff measure to a set $\mathcal{C}'$ such that $\mathcal{H}^h(\mathcal{C}') = 0$ for all $h(t) = \exp(-C' \log^{(n-1)/n}(1/t))$ with

$$C' > C(\lambda, s) = \begin{cases} \left(\frac{n}{n-1}\right)^{\frac{n-1}{n}} \frac{\lambda^{\frac{1}{n}} s}{(n-s)^{\frac{1}{n}} (1-s)^{\frac{n-1}{n}}}, & 0 < s < 1 \\ \left(\frac{n}{n-1}\right)^{\frac{n-1}{n}} \frac{\lambda^{\frac{1}{n}} s \log^{\frac{n-1}{n}} m(s)}{(n-s)^{\frac{1}{n}} \log 2}, & 1 \leq s < n, \end{cases} \tag{7}$$

where $m(s) = (\lceil 2^{\frac{1}{n-s}} n^{\frac{s}{2(n-s)}} \rceil)^n$ (here, $\lceil a \rceil$ for a number $a \in \mathbb{R}$, denotes the smallest integer greater or equal to a).

Note that the dependence $C(\lambda, s)$ behaves asymptotically like $\frac{n^{(n-2)/n}}{(n-1)^{(n-1)/n}} \lambda^{1/n} s$ when s is small.

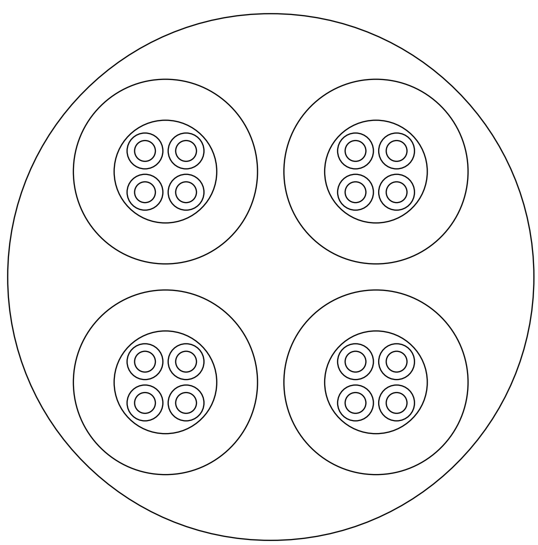

FIGURE 2. Cantor set in Example 2

The construction uses the same idea as Example 1, but this time, the Cantor sets are constructed with the help of balls rather than cubes, similarly to the way it is done in [3, Theorem 7.2]. The limit sets are obtained as infinite intersections of families of balls of equal size (see Figure 2). The fact that now our mapping f "lives" on "spherical annuli" not "cubical frames" allows us to take it as a standard radial stretching on those annuli, that is, a mapping of the form

$$\left|\frac{x}{r}\right|^{\alpha-1}(x-x_0)+x_0',$$

where the numbers r and α depend on the radii of annuli, and x_0 and x_0' are the centres of the pre-image and the image annuli, respectively.

Finally, [E] gives the following corollary to Theorem 9.

Corollary 2. *Let $f\colon \Omega\to\mathbb{R}^n$ be a homeomorphic mapping of exponentially integrable distortion. Then we have $\mathcal{H}^h(f(E))>0$ for all $E\subset\Omega$ such that $\dim_{\mathcal{H}} E>0$, where $h(t)=\exp((\log^{-1}\log\frac{1}{t})\log^{\frac{n-1}{n}}\frac{1}{t})$.*

This is a simple modification to Theorem 9, which sacrifices accuracy in order to get rid of the uncertain constant c_n. The expression $\log^{-1}\log\frac{1}{t}$ here may be replaced with any function ψ such that $\psi(t)\to 0$ and $\psi(t)\log^{\frac{n-1}{n}}\frac{1}{t}\to\infty$ as $t\to 0$.

8. Further investigation

Another question which may be studied is: How big should the pre-image set be if we wish that its image under a homeomorphism of finite distortion have Hausdorff dimension n? We are able to answer this question for mappings of exponentially integrable distortion in the plane using already familiar techniques. We will see that the fact that a pre-image is of dimension two is not enough to guarantee dimension two on the image side. However, there is some finer scale, indicating that certain sets are sufficiently large.

In examining this problem, we make use of the area distortion result for the inverse of a planar mapping of exponentially integrable distortion proved in [7, Théorème 1]. More precisely, for $E\subset\mathbb{R}^2$ we use the estimate

$$|f(E)|\geq C\exp\Big(-\frac{A}{\lambda}\log^2\Big(2+\frac{1}{|E|}\Big)\Big),$$

where f is a planar homeomorphism of λ–exponentially integrable distortion, C is a constant depending on the data, and A is an absolute constant. This estimate implies higher integrability of the Jacobian of f^{-1}. In particular, $J_{f^{-1}}\in L^P_{\mathrm{loc}}$ with $P(t)=t\exp(\beta\log^{1/2}(e+t))$, for all $\beta<(\lambda/A)^{1/2}$. Such higher regularity in combination with techniques of [27] now applied to the inverse, gives a dimension distortion estimate

$$\mathcal{H}^h(E)>0\Longrightarrow \dim_{\mathcal{H}} f(E)\geq s$$

for any $h(t)=t^2\exp(C_2\log^{1/2}(1/t))$ with $C_2<\big(2\frac{\lambda}{A}(2-s)\big)^{1/2}$.

For fixed $C_2>0$ and $s\in]0,2[$, a construction of the type of [15, Proposition 5.1] with $\mathcal{C}\subset[0,1]^2$ on the pre-image side and $\mathcal{C}'\subset[0,1]^2$ on the image side, such that

$\mathcal{H}^h(\mathcal{C}) > 0$ for $h(t) = t^2 \exp(C_2 \log^{1/2}(1/t)$ and $\dim_{\mathcal{H}}(\mathcal{C}') = s$, gives a mapping with λ–exponentially integrable distortion for all $\lambda > 0$ such that

$$C_2 > \Big(\frac{\lambda}{B}\big(2^{\frac{2}{s}-1} - 1\big)\Big)^{1/2},$$

where B is an absolute constant. We see that this example demonstrates asymptotical sharpness modulo a constant of the estimate above as s approaches 2.

As in [E], we may give a statement for the boundary case, making the estimate rougher. That is, for any planar mapping with exponentially integrable distortion f, we have $\dim_{\mathcal{H}} f(E) = 2$, if $E \subset \mathbb{R}^2$ is such that $\mathcal{H}^h(E) > 0$ with $h(t) = t^2 \exp(\psi(t) \log^{1/2}(1/t))$ for some function ψ such that $\psi(t) \to 0$ and $\psi(t) \log^{\frac{n-1}{n}} \frac{1}{t} \to \infty$ as $t \to 0$.

We are also able to give an answer to the opposite question: How small should the pre-image set be if we wish that its image under a mapping of finite distortion have Hausdorff dimension zero? This time, we involve higher integrability from [2, Theorem 1.1] and a dimension distortion estimate of the form

$$\mathcal{H}^h(E) = 0 \Longrightarrow \mathcal{H}^\beta(f(E)) = 0, \text{ where } h(t) = |\log t|^{-(1+\alpha)\beta(n+\beta)/n^2},$$

proved in [3, Theorem 5.1] for a compact $E \subset \Omega$ and a monotone mapping $f\colon \Omega \to \mathbb{R}^n$ in the Orlicz-Sobolev class $W^{1,P}(\Omega, \mathbb{R}^n)$ with $P(t) = t^n \log^\alpha(e+t)$, $\alpha \geq 0$. As in the proof of Theorem 2, we apply the decomposition from [2, Corollary 4.4] to obtain a better estimate. All this gives the following result: if $f\colon \Omega \to \mathbb{R}^2$, $\Omega \subset \mathbb{R}^2$, is a mapping of λ–exponentially integrable distortion for some $\lambda > 0$, and $E \subset \Omega$ is such that $\mathcal{H}^h(E) = 0$ with $h(t) = |\log t|^{-\frac{\lambda s}{2-s}+\varepsilon}$ for some $s \in]0,2[$ and $\varepsilon \in]0, \frac{\lambda s}{2-s}[$ then $\dim_{\mathcal{H}} f(E) < s$. Obviously, in order to get dimension zero on the image side, it is enough to assume $\mathcal{H}^h(E) = 0$ for the pre-image set with any function h tending to zero as t approaches zero more slowly than $\log^{-\gamma}(1/t)$ for each $\gamma > 0$, such as $h(t) = |\log|\log t||^{-1}$, for instance. This time constructions of the types described above do not give possible sharpness or even asymptotical sharpness of the obtained estimate.

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