JYVÄSKYLÄ STUDIES IN COMPUTING
137

Olga Kuznetsova

Lyapunov Quantities and Limit Cycles in Two-dimensional Dynamical Systems

Analytical Methods, Symbolic Computation and Visualization

JYVÄSKYLÄN YLIOPISTO

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ABSTRACT

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The aim of this work is the development of effective methods of investigation of limit cycles of two-dimensional dynamical systems.

The First Chapter is devoted to the computation of Lyapunov quantities and analysis of "small" limit cycles. Here, for the first time, a complete expression of the 4-th Lyapunov quantity has been obtained. In addition, the symbolic expressions of 5-th, 6-th, and 7-th Lyapunov quantities of Lienard system have been computed. This system describes real electrical and mechanical models and plays an important role in the study of limit cycles of quadratic systems and lower estimates of a possible number of limit cycles in polynomial systems. The development of these new expressions of Lyapunov quantities became possible due to advances in analytical methods of computation, realization of effective algorithms on their basis, and the use of modern software for symbolic computation.

Proceeding from the famous Bautin's method, the technique of computation and analysis of Lyapunov quantities developed in the present work was applied in the Second Chapter to the analysis of the domains of coefficients corresponding to the existence of different configurations of limit cycles of quadratic systems and their visualization. Toward this end, the problem by Kolmogorov concerning the localization and modeling of cycles of quadratic systems is considered in the Second Chapter. Using the method of reduction of quadratic system to Lienard system with discontinuous right-hand side and further development of the asymptotic integration technique made it possible to obtain effective criteria of qualitative behavior of trajectories for sufficiently large initial data. The so developed analytical criteria of global behavior of trajectories, local analysis of Lyapunov quantities, and computational experiments allowed for the investigation of the domains of coefficients, corresponding to the existence of four limit cycles in quadratic system. Furthermore, the methods developed in this work enabled the visualization of three and four limit cycles in quadratic system and computational study of a "cycles dance".

Keywords: two-dimensional dynamical systems, Lyapunov quantities, symbolic computation, limit cycles, computer visualization, hidden oscillations

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LIST OF FIGURES

CONTENTS

LIST OF INCLUDED ARTICLES

PI G.A. Leonov, O.A. Kuznetsova. Evaluation of the first five Lyapunov Exponents for the Lienard system. *Doklady Physics, vol. 54, n. 3, pp. 131–133*, 2009.

PII G.A. Leonov, S.M. Seledzhi, O.A. Kuznetsova, A.A. Fyodorov, E.V. Kudryashova. Periodical oscillations of control systems. Analytical and numerical approach. *Proceedings MATHMOD 09, Vienna, pp. 416–427*, 2009.

PIII G.A. Leonov, O.A. Kuznetsova. Lyapunov Quantities and Limit Cycles of Two-dimensional Dynamical Systems. Analytical Methods and Symbolic Computation. *Regular and Chaotic Dynamics, vol. 15, n. 2–3, pp. 356–379*, 2010.

PIV N.V. Kuznetsov, O.A. Kuznetsova, G.A. Leonov, V.I. Vagaytsev. Hidden attractor in Chua's circuits. *8th International Conference on Informatics in Control, Automation and Robotics (ICINCO), proceedings, vol. 1, pp. 279–283*, 2011.

INTRODUCTION AND THE STRUCTURE OF THE WORK

The present work is devoted to the study of limit cycles of two-dimensional dynamical systems and the computation of symbolic expressions of Lyapunov quantities.

The notion of *limit cycle* appeared first in Poincare's works, where he pointed out the importance of the analysis of limit cycles for the study of nonlinear systems behavior and suggested general methods of their investigation. Further consideration of limit cycles was motivated by as fundamental questions (such as Hilbert 16-th problem on possible number and mutual disposition of limit cycles in two-dimensional polynomial systems) as the problems of investigation of applied systems and the search of effective existence criteria of limit cycles in these systems ([Lyapunov, 1892, Lienard, 1928], *et al.*).

Note that for more than century history (see, references in [Reyn, 1994]) of study of the famous Hilbert 16-th problem it has not been solved even for the simple case of quadratic systems. One of important directions of the solution of this problem is a proof of finiteness of the number of limit cycles. The history of this proof for polynomial systems on plane is connected with the well-known work of Dulac [Dulac, 1923]. However later in the work [Ilyashenko, 1985] the gaps in Dulac's proof were found. Further, these gaps were corrected in the works of Bamon [Bamon, 1985] (for quadratic polynomial systems) and, independently, in the works of Ilyashenko [Ilyashenko, 1991] and Ecalle [Ecalle, 1992] (for general polynomial systems).

The creation of effective methods for the construction of systems with limit cycles was initiated by the works of Bautin [Bautin, 1949, Bautin, 1952], in which for the construction of nested limit cycles there was suggested the effective analytical method based on determining sequential symbolic expressions of *Lyapunov quantities* (being called also *focus values* or *Poincare-Lyapunov constants*). Sequential small perturbations of Lyapunov quantities, used by Bautin, allowed him for the first time to discriminate a class of quadratic systems with three nested limit cycles [Bautin, 1952] (such cycles are obviously called "small" limit cycles). Next, Petrovskii and Landis published the work [Petrovskii & Landis, 1957], in which it was asserted that in quadratic system there can be less than or equal to three limit cycles. However they found then a gap [Petrovskii & Landis, 1959]. Later there were obtained quadratic systems with four limit cycles [Chen & Wang, 1979, Shi, 1980] (three nested "small" limit cycles, obtained by perturbations of weak focus, and one limit cycle, surrounding another focus equilibrium).

Note that the cycles that can be seen by means of computational procedures are called "large" or "normal" limit cycles (this term is due to L.M. Perko [Perko, 1991]). Existence of semistable and nested limit cycles, and trajectory flattening effects [Kudryashova, 2009] make rather difficult computational analysis and visualization of limit cycles and show that a simple search of systems and computational integration are ineffective for the search of limit cycles. For example, academician V.I. Arnold writes [Arnold, 2005]: "To estimate the number of limit cycles of square vector fields on plane, A.N. Kolmogorov had distributed several hundreds of such fields (with randomly chosen coefficients of quadratic expressions) among a few hundreds of students of Mechanics and Mathematics Faculty of Moscow State University as a mathematical practice. Each student had to find the number of limit cycles of his/her field. The result of this experiment was absolutely unexpected: not a single field had a limit cycle!"

This example confirms the necessity of the search of effective analytical methods of investigation limit cycles.

The relevance of investigation of limit cycles of two-dimensional dynamical systems is also affirmed by the fact that with the help of these systems various real models are described in many scientific fields, including mechanics, biology, chemistry, and physics. In these models the important role play limit periodic solutions (called limit cycles). We give here certain well-known examples (see, e.g., [Strogatz, 1994, Jones *et al.*, 2010], *et al.*) of such systems.

Examples of limit cycles in mechanics and electronics.

In studying string oscillations [Rayleigh, 1877] Rayleigh discovered first that in two-dimensional nonlinear dynamical system can arise undamped vibrations without external periodic action (limit cycles). It is considered localization of limit cycle in Rayleigh system

$$\ddot{x} - (a - b\dot{x}^2)\dot{x} + x = 0, \tag{1}$$

for $a = 1, b = 0.1$. In Fig. 1 the limit cycle is localized by two trajectories (each of which trajectories begins in the red, and ends in green), tending to the limit cycle.

The extension of equation (1) is Van der Pol equation [van der Pol, 1926] (describe the nonlinear electrical circuits used in the first radios). It is considered localization of limit cycle in Van der Pol system

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \tag{2}$$

for $\mu = 0.1$ (Fig. 2).

This system, in turn, is a special case of Lienard system [Lienard, 1928], the study of which was motivated by the development of radio and vacuum tube technology. The Lienard system is as follows

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{3}$$

where f, g are smooth differentiable functions. This equation can also be interpreted mechanically (see, e.g., [Perko, 1991]) as the equation of motion for a unit mass subject to a nonlinear damping force $-f(x)\dot{x}$ and a nonlinear restoring force $-g(x)$.

For this class of systems, Lienard has suggested effective criterium for existence of unique limit cycle (see, e.g., [Cesari, 1959]).

Lienard theorem. *System* (3) *has a unique and stable limit cycle surrounding the origin if the following conditions are satisfied:*

FIGURE 1 Localization of limit cycle in Rayleigh system.

1) $g(x) > 0 \quad \forall x > 0$

2) $\lim\limits_{x \to \infty} \int\limits_0^x f(s)ds = \infty$

3) $F(x) = \int\limits_0^x f(s)ds$ *has exactly one positive root at certain value* a, $F(x) < 0$ *for* $0 < x < a$, *and* $F(x) > 0$ *and monotonically increases for* $x > a$.

Examples of limit cycles in biology.

In the year 1925 the famous scholar Alfred Lotka suggested a mathematical model of competition of two species (known as the Lotka-Volterra "predator-prey" model), which is presented in the form of two-dimensional dynamical system [Lotka, 1925]. Lotka showed that all the trajectories of this system are periodic, reflecting what is well-known in nature, the periodicity of variation in the population number [Volterra, 1926]. Note, however, that the periodic trajectories of Lotka-Volterra "predator-prey" model are not limit since this system is conservative.

Further development of mathematical biology led to consideration of different models, in which there were found limit cycles.

For example, in 1980 a two-dimensional system of differential equations

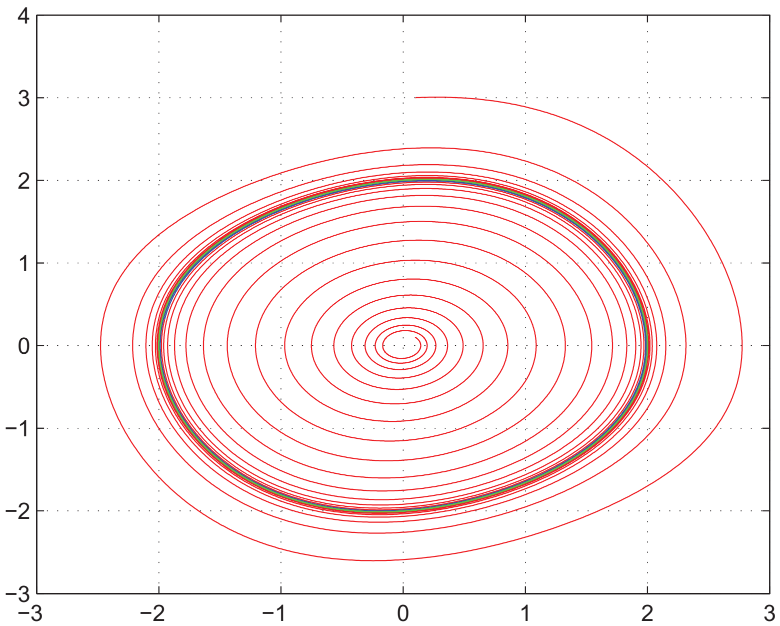

FIGURE 2 Localization of limit cycle in Van der Pol system.

was considered [Odell, 1980], describing another "predator-prey" model

$$\frac{dx}{dt} = x[x(1-x) - y], \quad \frac{dy}{dt} = y(x-a), \tag{4}$$

where $x \geq 0$ is the dimensionless population of the prey, $y \geq 0$ is the dimensionless population of the predator, and $a \geq 0$ is a control parameter. Here for the value of parameter a close to $a_c = \frac{1}{2}$ it appears limit cycle. It is considered localization of limit cycle in system (4) for $a = 0.4$ (Fig. 3).

Besides, in 1979 a two-dimensional differential system was considered, describing a model of respiration of a bacterial culture [Fairen & Velarde, 1979]

$$\frac{dx}{dt} = B - x - \frac{xy}{1+qx^2}, \quad \frac{dy}{dt} = A - \frac{xy}{1+qx^2}, \tag{5}$$

where x, y are the levels of nutrient and oxygen, respectively, and $A, B, q > 0$ are parameters. It turned out that for certain values of parameters A, B, q in the system it can appear a limit cycle. It is considered localization of limit cycle in system (5) for $A = 11, B = 13.4, q = 0.5$ (Fig. 4).

Examples of limit cycles in chemistry.

In 1951 periodic oscillations in chemical reactions were first found experimentally by Boris Belousov. Proceeding with these investigations, Anatol Zhabotin-

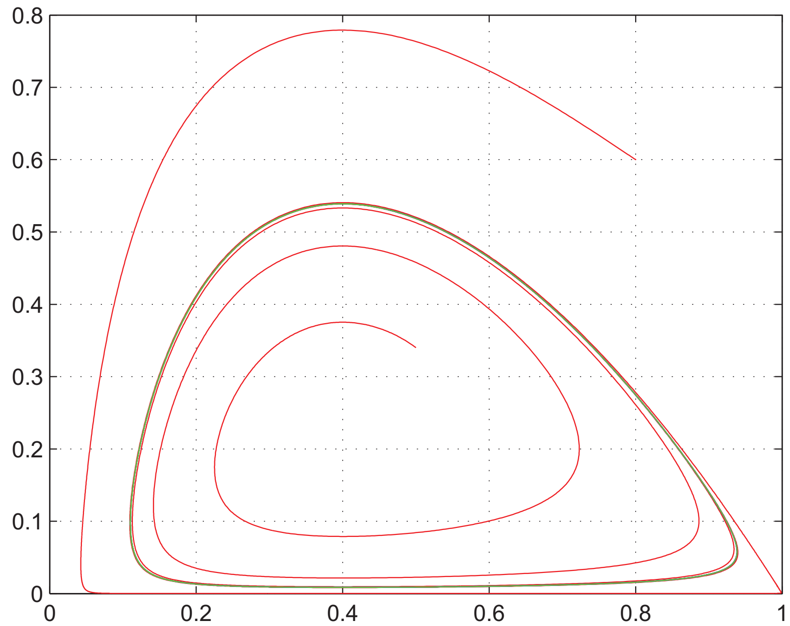

FIGURE 3   Localization of limit cycle in "predator-prey" model.

sky discovered new chemical reactions with autooscillations (called Belousov-Zhabotinsky reactions) and constructed first mathematical models of these reactions.

In 1990 a particularly elegant model of oscillating reaction, the chlorine dioxide-iodine-malonic acid reaction (the counterpart of famous Belousov-Zhabotinsky reaction) was proposed and analyzed [Lengyel *et al.*, 1990]. After suitable nondimensionalization, the model becomes

$$\frac{dx}{dt} = a - x - \frac{4xy}{1+x^2}, \quad \frac{dy}{dt} = bx(1 - \frac{y}{1+x^2}), \tag{6}$$

where x and y are the dimensionless concentrations of I^- and ClO_2^-. The parameters $a, b > 0$ depend on the empirical rate constants and on the concentrations assumed for the slow reactants. It turns out that when $b < b_c = \frac{3a}{5} - \frac{25}{a}$, all trajectories are attracted to a stable limit cycle. It is considered localization of limit cycle in system (6) for $a = 10, b = 2$ (Fig. 5).

In the fundamental biochemical process called glycolysis (see [Chance *et al.*, 1973, Goldbeter, 1980]) living cells obtain energy by breaking down sugar. In intact yest cells glycolysis can proceed in an oscillatory fashion, with the concentrations of various intermediates waxing and waning with a period of several minutes. A simple model of these oscillations can be described by the following

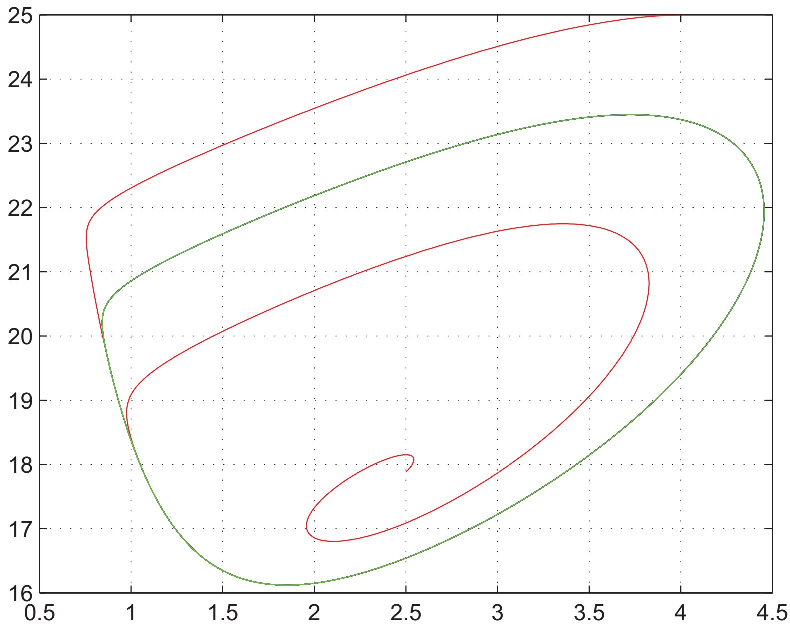

FIGURE 4 Localization of limit cycle in model for respiration of a bacterial culture.

simple dimensionless equations

$$\frac{dx}{dt} = -x + ay + x^2y, \quad \frac{dy}{dt} = b - ay - x^2y, \tag{7}$$

where x and y are the concentrations of adenosine diphosphate and fructose-6-phosphate, and $a, b > 0$ are kinetic parameters. Let $\tau = -\frac{b^4+(2a-1)b^2+(a+a^2)}{a+b^2}$, then for parameters a, b in the region, corresponding to $\tau > 0$, we are guaranteed that the system has a stable limit cycle. It is considered localization of limit cycle in system (7) for $a = \frac{1}{16}, b = \frac{\sqrt{7}}{4}$ (Fig. 6).

Note that in all these models a problem of limit periodical solutions investigation is of great importance since periodical solution means critical state of the system.

In the above examples the localization of limit cycles is performed with the help of standard computational procedure. It consist in that the trajectory, going from small neighborhood of unstable equilibrium, after transient process, is attracted to unique stable limit cycle and "computes" it. However the application of such procedure to visualization of limit cycles in quadratic systems does not allow one to find semistable and nested cycles.

In the present work it is considered the development of effective methods for investigation of limit cycles of two-dimensional dynamical systems and it is

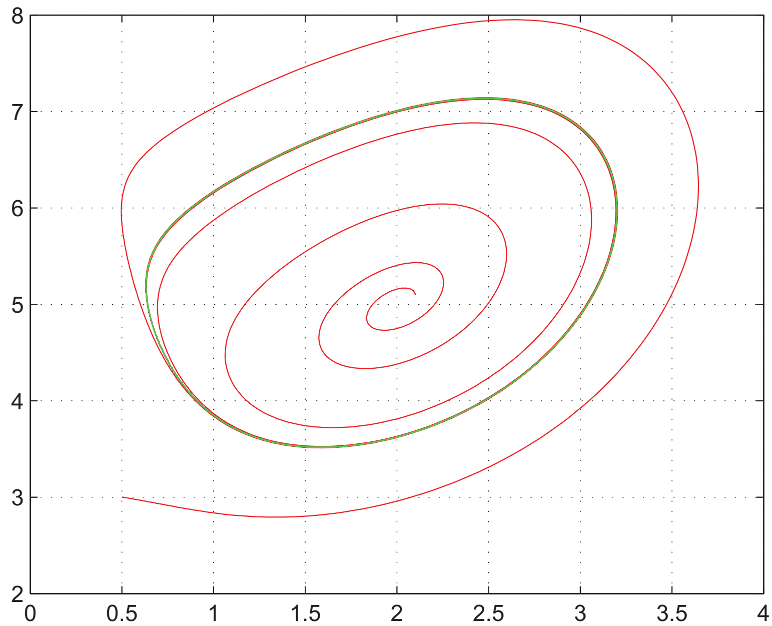

FIGURE 5 Localization of limit cycle in chlorine dioxide-iodine-malonic acid reaction.

extended the theses [Kuznetsov, 2008, Kudryashova, 2009]. The material of dissertation is organized as follows.

The First Chapter is devoted to computation of Lyapunov quantities and the analysis of "small" limit cycles. The method of computation of Lyapunov quantities was proposed first in the works of Poincare [Poincare, 1885] and Lyapunov [Lyapunov, 1892] for the analysis of local behavior of system in the neighborhood of a complicated equilibrium (two purely imaginary eigenvalues of linear part of system). The innovative works of Bautin [Bautin, 1949, Bautin, 1952] are devoted to development of these ideas and the analysis of symbolic expressions of sequential Lyapunov quantities. In these works there was proposed an effective approach to the construction of nested limit cycles and to obtaining the lower estimate of the number of limit cycles in famous Hilbert 16-th problem for two-dimensional polynomial systems. These works of Bautin led to the appearance of a great many of papers and books, devoted to symbolic computation and analysis of Lyapunov quantities for different types of systems ([Serebryakova, 1959, Schuko, 1968, Lloyd, 1988, Lynch, 2005], *et al.*). However substantial progress on the study of general Lyapunov quantities became possible only in the past decade by virtue of modern computer techniques and the software tools of symbolic computation. For example, the symbolic expressions for the first and second Lyapunov quantities were calculated in the 40-50-s of

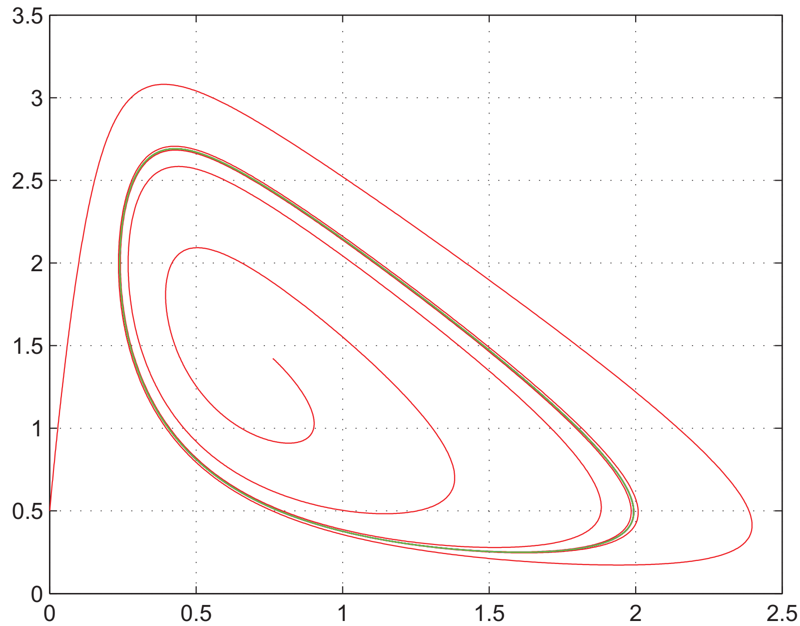

FIGURE 6 Localization of limit cycle in glycolysis.

last century [Bautin, 1949, Serebryakova, 1959], while the expression for the third Lyapunov quantity in general form (in terms of decomposition of right-hand side of system in the original space) was first computed only in 2008 year and occupies a few pages [Kuznetsov & Leonov, 2008[1], Kudryashova, 2009].

In the present work there was obtained first a complete expression for the fourth Lyapunov quantity in general form, which occupies more than 45 pages. In addition it was also computed symbolic expressions of 5-th, 6-th, and 7-th Lyapunov quantities for Lienard system. Note that this equation not only describes real electrical and mechanical models but also plays important role in the study of limit cycles of quadratic systems [Cherkas, 1976, Leonov, 1997] and lower estimate of possible number of limit cycles in polynomial systems [Christopher & Lloyd, 1996, Dumortier & Li, 1996, Han, 1999, Lynch, 2005]. The obtaining of these new expressions of Lyapunov quantities became possible by virtue of the development of analytical methods of computation of Lyapunov quantities, realization of effective algorithms on their basis, and the application of modern methods of symbolic computation [Kuznetsov, 2008, Kudryashova, 2009, Leonov *et al.*, 2011]. The process developed in the work for the computation and analysis of Lyapunov quantities is applied in the Second Chapter of the thesis to analysis of limit cycles of quadratic systems, namely, to analysis of domains of coefficients, corresponding to the existence of different configurations of limit cycles and their visualization.

The Second Chapter is devoted to investigation of "large" limit cycles. The problem of investigation of "large" limit cycles is exceedingly complicated even for the simplest case of quadratic systems. While "small" limit cycles of quadratic systems are considered from the middle of 20-th century [Bautin, 1949, Serebryakova, 1959], the quadratic systems with "small" and "large" limit cycles were found not until much later. For example, in the works [Chen & Wang, 1979, Shi, 1980] it was obtained class of quadratic systems with 3 "small" limit cycles in the neighborhood of one equilibrium and one "large" limit cycle around another equilibrium. In the works [Artes *et al.,* 2006, Leonov, 2009] it is given quadratic systems with two "large" limit cycles. Note that the problem of localization and visualization of "large" cycles remains unsolved, as before, even in the case of quadratic systems. Certain analytical and computational methods of investigation of "large" limit cycles were proposed in the works [Lefschetz, 1957, Blows & Lloyd, 1984[1], Perko, 1990, Rousseau, 1993, Blows & Perko, 1994, Cherkas *et al.,* 2003, Kudryashova *et al.,* 2008, Leonov, 2009].

The reduction used in this work of the quadratic system to the Lienard system with the discontinuous right-hand side and the development of ideas of asymptotic integration made it possible to obtain analytical criteria of qualitative behavior of trajectories for sufficiently large initial data. The obtained analytical criteria of global behavior of trajectories and the local analysis of Lyapunov quantities allowed one to investigate the domains of coefficients, corresponding to the existence of one and two "large" limit cycles in quadratic system. By the obtained analytical results and computational experiments the domains of coefficients, corresponding to the existence of three and four "large" limit cycles, were considered. In addition, visualization was carried out of three and four limit cycles in quadratic system and the computational study of a "cycles dance" (i.e. the varying of limit cycles configuration under varying of system parameters).

Included articles: The present work is based on more than 15 published journal papers and reports at international conferences. In these papers, the statements of problems are due to the supervisors, while the development of analytical methods, computational procedures, algorithms, and computer modeling are due to the author. Results of the first and second chapters were presented in joint reports at international conferences [PII, PIV] and partially published in the articles [PI, PIII], included into Appendix and containing other approaches to computation of Lyapunov quantities and localization of limit cycles.

1 COMPUTATION OF LYAPUNOV QUANTITIES AND "SMALL" LIMIT CYCLES

1.1 Introduction

The first part of work is devoted to symbolic computation of Lyapunov quantities and is based on the surveys [Leonov & Kuznetsova, 2010, Leonov *et al.,* 2011]. A method of computation of Lyapunov quantities (being called also focus values and Poincare-Lyapunov constants) was proposed at the end of the 19th century in the classical works of A. Poincare [Poincare, 1885] and A.M. Lyapunov [Lyapunov, 1892] for analysis of stability of degenerated focus equilibrium. Note that a sign of Lyapunov quantities defines winding/unwinding of solutions of systems in small neighborhoods of equilibrium and stability/instability of equilibrium.

The development of methods for computation and analysis of Lyapunov quantities was encouraged by applied engineering problems (such as the study of oscillations excitation and boundaries of domain of stability) as purely mathematical problems (such as center–focus problem, the cyclicity problem, analysis of dynamical systems stability, and famous Hilbert 16-th problem).

Note that yet in the first half of last century the scholars began to consider the problem of symbolic computation of Lyapunov quantities, i.e. the search of symbolic expressions of Lyapunov quantities in terms of coefficients of the right-hand side of considered dynamical system. However, substantial progress on the study of Lyapunov quantities became possible only in the past decade by virtue of the use of modern software tools of symbolic computation. The symbolic expressions for the first and second Lyapunov quantities were obtained in the 40-50-s of last century by Bautin [Bautin, 1949] and Serebryakova [Serebryakova, 1959] respectively, while the expression for the third Lyapunov quantity in general form (in terms of decomposition of right-hand side of system in the original space) has been computed first only in 2008 year [Kuznetsov & Leonov, 2008[1], Leonov, Kuznetsov & Kudryashova, 2008, Kudryashova, 2009]. The expression for the fourth Lyapunov quantity in general form was obtained by the author in 2009 year and occupies more than 45 pages. The obtaining of expressions for the third and fourth quantities became possible by virtue of development of analytical methods for computation of Lyapunov quantities, realization on their basis of effective algorithms, and applying a modern computational tools.

At present there exist a few methods for computation of Lyapunov quantities and their computer realizations, which permit one to find Lyapunov quantities in the form of symbolic expressions depending on the coefficients of system. These methods are differed by the complexity of realization of algorithms, a space, in which the computation is made, and the compactness of obtained symbolic expressions [Chow & Hale, 1982, Gasull & Prohens, 1997, Li, 2003, Chavarriga & Grau, 2003, Lynch, 2005, Dumortier *et al.,* 2006, Christopher & Li, 2007, Gine, 2007, Yu & Chen, 2008].

The first method for computation of Lyapunov quantities was proposed in the works [Poincare, 1885] and [Lyapunov, 1892]. It consists in sequential construction of Lyapunov function, using the integral of linear part of system. The algorithm of computation of symbolic expressions of Lyapunov quantities, based on this approach, is described below.

Further the different methods of computation of Lyapunov quantities, using the reduction of system to normal forms [Yu, 1998, Li, 2003], were developed. However, in realizing these methods there arises the complexity, connected with the nonuniqueness of process of construction of system normal form. Another approach to the computation of Lyapunov quantities is connected with determining approximations of solutions of system. For example, in the work [Lyapunov, 1892] it is used the passage to the polar coordinates and the procedure of sequential construction of approximations of solution.

In the works [Kuznetsov & Leonov, 2008[1], Kuznetsov & Leonov, 2008[2], Leonov *et al.,* 2008] it was proposed new method for computation of Lyapunov quantities, based on the construction of approximations of solution (in the form of finite sum in terms of powers of initial datum) in the original Euclidean system of coordinates and in the time domain. The advantage of this method is ideological simplicity and visualization power. This approach can also be applied to the solution of problem of determination of isochronous center [Sabatini & Chavarriga, 1999] since it allows one to find the approximation of the return time of trajectory depending upon initial data. The algorithm of computation of symbolic expressions of Lyapunov quantities, based on this method, is described below.

Often for computational speedup and simplification of finite expressions of Lyapunov quantities it is used different modifications of the considered above methods, connected with the transformation of system to complex variables [Schuko, 1968, Gasull *et al.,* 1997, Li, 2003, Yu & Chen, 2008]. For example, based on the method of constructing Lyapunov function in complex domain, in 1968 year it was developed, evidently, the first computer program for computation of Lyapunov quantities [Schuko, 1968].

Note that the computation of symbolic expressions of Lyapunov quantities can also be reduced to application of recurrent formulas [Lynch, 2005] and the use of algebraic methods of construction and investigation of special polynomials (see, for example, [Romanovskii, 1996]).

In the work it is used two methods for computation of Lyapunov quantities: classical method of Poincare-Lyapunov and the method for computation of Lyapunov quantities in Euclidean space and in the time domain. Note that these methods are distinct from each other in principle. The first method is based on the construction of Lyapunov function for a system, while the second one on the search of approximate solution of system. The computation of symbolic expressions of Lyapunov quantities by these two methods permits one to be sure that the obtained results are true.

1.2 Lyapunov quantities

Following the survey [Leonov *et al.*, 2011], introduce Lyapunov quantities. Consider sufficiently smooth two-dimensional system

$$\frac{dx}{dt} = F(x,y), \quad \frac{dy}{dt} = G(x,y), \tag{8}$$

where $F(0,0) = G(0,0) = 0$ (i.e. the point $(0,0)$ is stationary point of system). Now we represent system (8) as

$$\begin{aligned} \frac{dx}{dt} &= f_{10}x + f_{01}y + f(x,y), \\ \frac{dy}{dt} &= g_{10}x + g_{01}y + g(x,y), \end{aligned} \tag{9}$$

where the expansion of functions f and g starts with the terms greater or equal to the second order.

Suppose that in open neighborhood U of radius R_U of the point $(x,y) = (0,0)$, the right-hand side of system has continuous partial derivatives of the n-th order:

$$f(\cdot,\cdot), g(\cdot,\cdot) \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \ \in \mathcal{C}^n(U) \tag{10}$$

and the following representation

$$\begin{aligned} f(x,y) &= \sum_{k+j=2}^{n} f_{kj}x^k y^j + o\big((|x|+|y|)^n\big) = f_n(x,y) + o\big((|x|+|y|)^n\big), \\ g(x,y) &= \sum_{k+j=2}^{n} g_{kj}x^k y^j + o\big((|x|+|y|)^n\big) = g_n(x,y) + o\big((|x|+|y|)^n\big) \end{aligned} \tag{11}$$

is satisfied.

Consider the matrix of the first approximation of system in zero stationary point:

$$A_{(0,0)} = \begin{pmatrix} f_{10} & f_{01} \\ g_{10} & g_{01} \end{pmatrix}. \tag{12}$$

Introduce the notions

$$\sigma = \mathrm{Tr} A_{(0,0)} = f_{10} + g_{01}, \quad \Delta = \det A_{(0,0)} = f_{10}g_{01} - f_{01}g_{10}$$

and write out eigenvalues of (12)

$$\lambda_{1,2} = -\frac{\sigma}{2} \pm \sqrt{\frac{\sigma^2}{4} - \Delta}.$$

Let the matrix $A_{(0,0)}$ of the first approximation of system has two purely imaginary eigenvalues (i.e. $\sigma = 0$ and $\Delta > 0$). In this case, without loss of generality (i.e. there exists a nonsingular linear change of variable), it can be assumed that

$$f_{10} = 0, \quad f_{01} = -1, \quad g_{10} = 1, \quad g_{01} = 0.$$

Therefore we can consider the following system

$$\begin{aligned} \frac{dx}{dt} &= -y + f(x, y), \\ \frac{dy}{dt} &= x + g(x, y). \end{aligned} \tag{13}$$

The first approximation of system (13) takes the form

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x. \tag{14}$$

The eigenvalues of matrix of the right-hand side of this system are equal to $\pm i$. All the trajectories of system (14) are closed and stationary point $(0, 0)$ is called a center (Fig. 7, a). Let us study the influence of nonlinear terms $f(x, y)$ and $g(x, y)$ on the behavior of trajectories of system (13) in small neighborhood of stationary point. Consider, following the method of Poincare, the intersection of trajectory of system (13) with the straight line $x = 0$.

At time $t = 0$ the trajectory $\big(x(t, h), y(t, h)\big)$ is started from the point $(0, h)$ (h is sufficiently small)

$$\big(x(0, h), y(0, h)\big) = \big(0, h\big). \tag{15}$$

Denote by $T(h)$ return time of trajectory, which is a time between two successive intersections of trajectory with the straight line $x = 0$. Note that for sufficiently small h return time can be found and it is finite since the right-hand side of systems (13) and (14) differ by $o(|x| + |y|)$ in the neighborhood of zero. Then

$$x(T(h), h) = 0 \tag{16}$$

and $y(T(h), h)$ can sequentially be approximated by a series in terms of powers of h:

$$y(T(h), h) = h + \widetilde{L}_2 h^2 + \widetilde{L}_3 h^3 + ... \tag{17}$$

Here the first nonzero coefficient $\widetilde{L}_m$ is called Lyapunov quantity. It determines stability or instability of stationary point and describes a winding/unwinding of trajectory (Fig. 7, b). One can show (see, e.g., [Lyapunov, 1892]) that the first

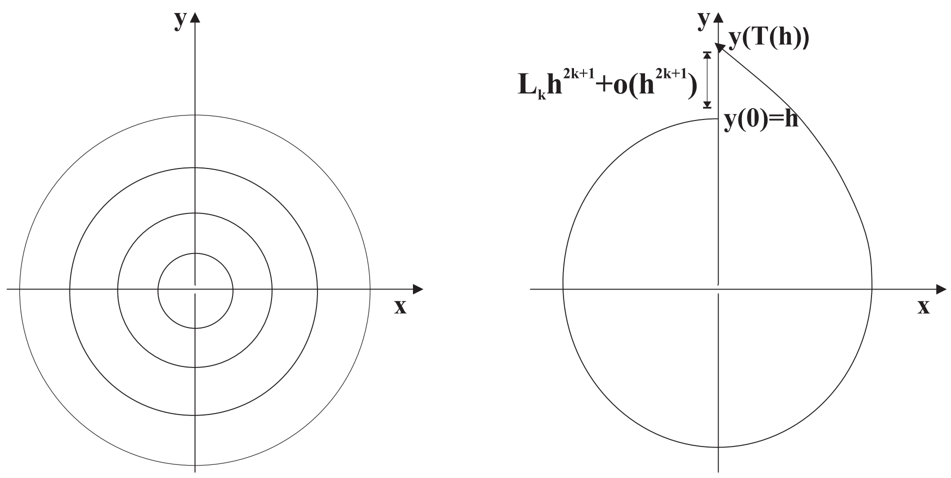

FIGURE 7 a) Center b) Focus: defenition of Lyapunov quantity.

nonzero coefficient have necessarily odd number $m = (2k+1)$. The value $\widetilde{L}_{2k+1}$ is called k-th Lyapunov quantity

$$\mathrm{L}_k = \widetilde{L}_{2k+1}$$

and the equilibrium is a weak focus of k-th order.

In the case of $\sigma \neq 0$ for the complex eigenvalues of the first approximation matrix (12) the notion of Lyapunov quantity is considered similarly. In this case it is introduced the notion of zero Lyapunov quantity $\mathrm{L}_0 = \widetilde{L}_1$ such that

$$y(T(h), h) = (1 + \widetilde{L}_1)h + o(h).$$

Note that $\widetilde{L}_1$ describes exponential increase of solutions of system, being caused by real parts of eigenvalues (similarly to Lyapunov exponents or characteristic exponents, see [Leonov & Kuznetsov, 2007]).

Note, following the work of A.M. Lyapunov [Lyapunov, 1892], a similar procedure for the study of stability can be used for systems of larger dimension (in the case when a linear system has two purely imaginary roots and the rest of roots are negative). Certain results on the computation of Lyapunov quantities for systems of larger dimension can be found, for example, in [Bautin, 1949].

1.3 Classical Poincare-Lyapunov method for Lyapunov quantities computation

Following the classical works [Poincare, 1885, Lyapunov, 1892], we consider the method for Lyapunov quantities computation, based on the construction of Lyapunov function for system (13). Below there are represented two modifications of this method (in the original (Euclidean) space and in the complex space).

1.3.1 Computation of Lyapunov quantities in Euclidean space

In the classical method of Poincare-Lyapunov for the computation of symbolic expressions of Lyapunov quantities, in the neighborhood of zero equilibrium it is necessary to find Lyapunov function $V(x,y)$ for system (13) in the form

$$V(x,y) = V_2(x,y) + V_3(x,y) + ... + V_{n+1}(x,y). \tag{18}$$

Here $V_2(x,y) = \frac{x^2+y^2}{2}$ and $V_k(x,y), \quad k = 3, ..., n+1$, are homogeneous polynomials

$$V_k(x,y) = \sum_{i+j=k} V_{i,j} x^i y^j$$

with unknown coefficients $\{V_{i,j}\}_{i+j=k,\ i,j\geqslant 0}$.

For the derivative of $V(x,y)$ in virtue of system (13) we have

$$\dot{V}(x,y) = \frac{\partial V(x,y)}{\partial x}(-y + \sum_{k+j=2}^{n} f_{kj} x^k y^j) + \frac{\partial V(x,y)}{\partial y}(x + \sum_{k+j=2}^{n} g_{kj} x^k y^j)$$
$$+ o\big((|x| + |y|)^{n+1}\big).$$

Denoting in the above expression the homogeneous terms of order k by $W_k(x,y)$, we obtain

$$\dot{V}(x,y) = W_3(x,y) + ... + W_{n+1}(x,y) + o\big((|x| + |y|)^{n+1}\big).$$

Note that terms of the second degree in $\dot{V}(x,y)$ are canceled out. Then the equations $W_k(x,y) = 0$ for $k = 2p+1$, where $p = 1, ...$, and $W_k(x,y) = w_k(x^2+y^2)^p$ for $k = 2p$, where $p = 2, ...$ (w_k are unknown coefficients) are solved sequentially.

If for certain $k = 2p^* \leq n+1$ the relation $w_k \neq 0$ is satisfied, then the quantity $2\pi w_{2p^*}$ is equal to (p^*-1)-th Lyapunov quantity [Frommer, 1928] of system (13).

Detailed justification of this method can be found, for example, in the works [Lynch, 2005, Leonov & Kuznetsova, 2010].

Below is given a computational algorithm, based on the classical method of Poincare-Lyapunov in Euclidean space, and a program in Matlab, permitting one to realize necessary computation.

Algorithm for computation of Lyapunov quantities

Consider system (13). In place of the functions f and g we will regard their approximate values

$$f_n(x,y) = \sum_{k+j=2}^{n} f_{kj} x^k y^j, \quad g_n(x,y) = \sum_{k+j=2}^{n} g_{kj} x^k y^j, \tag{19}$$

where n is odd ($n = 2m+1$). Then we will search Lyapunov function $V(x,y)$ in the form of polynomial of $(n+1)$-th degree. Impose additional requirements on the coefficients of polynomial:

$$V_{2p,2p+2} + V_{2p+2,2p} = 0, \quad V_{2p,2p} = 0, \quad p = 1,2,... \tag{20}$$

the satisfaction of which provides uniqueness of determining the Lyapunov function $V(x,y)$ (a more detailed consideration of these conditions can be found in the work [Leonov *et al.*, 2011]). Then homogeneous polynomials of formula (18) take the form
$V_2(x,y) = \frac{x^2+y^2}{2}$,
$V_3(x,y) = V_{3,0}x^3 + V_{2,1}x^2y + V_{1,2}xy^2 + V_{0,3}y^3$,
$V_4(x,y) = V_{4,0}x^4 + V_{3,1}x^3y + V_{1,3}xy^3 + V_{0,4}y^4$,
$V_5(x,y) = V_{5,0}x^5 + V_{4,1}x^4y + V_{3,2}x^3y^2 + V_{2,3}x^2y^3 + V_{1,4}xy^4 + V_{0,5}y^5$,
$V_6(x,y) = V_{6,0}x^6 + V_{5,1}x^5y + V_{4,2}x^4y^2 + V_{3,3}x^3y^3 - V_{4,2}x^2y^4 + V_{1,5}xy^5 + V_{0,6}y^6$,
and so on.

For the derivative of $V(x,y)$ in virtue of system (13) we have

$$\dot{V}(x,y) = \frac{\partial V(x,y)}{\partial x}(-y + \sum_{k+j=2}^{n} f_{kj}x^ky^j) + \frac{\partial V(x,y)}{\partial y}(x + \sum_{k+j=2}^{n} g_{kj}x^ky^j)$$
$$+ o\big((|x|+|y|)^{n+1}\big) = \tilde{\dot{V}}(x,y) + o\big((|x|+|y|)^{n+1}\big).$$

In order to find $\tilde{\dot{V}}(x,y)$ we find $\frac{\partial V(x,y)}{\partial x}$ and $\frac{\partial V(x,y)}{\partial y}$:

$$\frac{\partial V(x,y)}{\partial x} = \frac{\partial V_2(x,y)}{\partial x} + \frac{\partial V_3(x,y)}{\partial x} + ... + \frac{\partial V_{n+1}(x,y)}{\partial x} =$$
$$= x + (3V_{3,0}x^2 + 2V_{2,1}xy + V_{1,2}y^2) + (4V_{4,0}x^3 + 3V_{3,1}x^2y + V_{1,3}y^3) + ...$$

$$\frac{\partial V(x,y)}{\partial y} = \frac{\partial V_2(x,y)}{\partial y} + \frac{\partial V_3(x,y)}{\partial y} + ... + \frac{\partial V_{n+1}(x,y)}{\partial y} =$$
$$= y + (V_{2,1}x^2 + 2V_{1,2}xy + 3V_{0,3}y^2) + (V_{3,1}x^3 + 3V_{1,3}xy^2 + 4V_{0,4}y^3) + ...$$

Then

$$\tilde{\dot{V}}(x,y) = (x + 3V_{3,0}x^2 + 2V_{2,1}xy + V_{1,2}y^2 + ...)(-y + f_{20}x^2 + f_{11}xy + f_{02}y^2 + ...)+$$
$$+ (y + V_{2,1}x^2 + 2V_{1,2}xy + 3V_{0,3}y^2 + ...)(x + g_{20}x^2 + g_{11}xy + g_{02}y^2 + ...). \tag{21}$$

Each bracket has degree n. It means that the function obtained is a polynomial of degree $2n$. Eliminating all terms, the summary degree of which exceeds $n+1$, we obtain a polynomial of degree $n+1$, depending only on the coefficients $\{V_{ij}\}_{i+j=2}^{n}$, $\{f_{ij}\}_{i+j=2}^{n}$ and $\{g_{ij}\}_{i+j=2}^{n}$, i.e.

$$\tilde{\dot{V}}(x,y,\ \{V_{ij}\}_{i+j=2}^{n},\ \{f_{ij}\}_{i+j=2}^{n},\ \{g_{ij}\}_{i+j=2}^{n}).$$

Define the function

$$W(x,y) = \sum_{k=2}^{m+1} w_{2k}(x^2+y^2)^k, \tag{22}$$

where w_{2k} are unknown coefficients. The above function is a polynomial of degree $n+1$ in x and y.

Subtracting $W(x,y)$ from $\widetilde{V}(x,y)$ and equating, in the obtained polynomial, to zero all the coefficients of all degrees x and y, we get a system, which is always solvable uniquely [Lynch, 2005]. In this case the first nonzero coefficient of w_{2k} gives the value of $(k-1)$-th Lyapunov quantity up to the constant ($L_{k-1} = 2\pi w_{2k}$).

Equating to zero the coefficients of t-th summary degree x and y of the polynomial $\widetilde{V}(x,y) - W(x,y)$, we get a system, which can be represented in matrix form

$$ML + P = 0. \tag{23}$$

Here M is a $(t+1 \times t+1)$ matrix of real numbers, L and P are the columns of size $t+1$, and L consists of $V_{i,j}$, where $i+j=t$, and w_t (if t is even). Thus, we have a representation

$$L = \begin{pmatrix} V_{t,0} \\ V_{t-1,1} \\ \ldots \\ V_{p+1,p-1} \\ w_t \\ V_{p-1,p+1} \\ \ldots \\ V_{1,t-1} \\ V_{0,t} \end{pmatrix} \text{ in the case } t=2p, \text{ and } L = \begin{pmatrix} V_{t,0} \\ V_{t-1,1} \\ \ldots \\ V_{p+1,p} \\ V_{p,p+1} \\ \ldots \\ V_{1,t-1} \\ V_{0,t} \end{pmatrix} \text{ in the case } t=2p+1.$$

Note that condition (20) imposed on coefficients $V_{i,j}$ makes possible for all t to compile the column L, involving all unknown coefficients $V_{i,j}$, where $i+j=t$. Since at t-th step we assume that $V_{i,j}$, where $i+j<t$, already are known, the column

$$P(\ \{V_{ij}\}_{i+j=2}^{t-1},\ \{f_{ij}\}_{i+j=2}^{t-1},\ \{g_{ij}\}_{i+j=2}^{t-1}) = P(\ \{f_{ij}\}_{i+j=2}^{t-1},\ \{g_{ij}\}_{i+j=2}^{t-1})$$

is also known. Solving linear matrix equation (23) with respect to L, we obtain

$$L = M^{-1}(-P). \tag{24}$$

If t is odd, then we proceed to $(t+1)$-th step, and if even, then we test the obtained value w_t. If $w_t \neq 0$, then it means that Lyapunov quantity is found, if not, then we proceed to $t+1$-th step.

Note that terms of the second degree in $\widetilde{V}(x,y)$ are canceled out (see formula (21)). Writing the expression for $\widetilde{V}(x,y)$ of 3-th degree, we obtain

$$x^3(f_{20} + V_{2,1}) + x^2y(-3V_{3,0} + f_{11} + g_{20} + 2V_{1,2}) +$$

$$+xy^2(f_{02} - 2V_{2,1} + g_{11} + 3V_{0,3}) + y^3(-V_{1,2} + g_{02}).$$

Since the polynomial $W(x,y)$ contains only terms of even degrees, then we set equal to zero the coefficients of the third degrees of polynomial, i.e. we get

$$\begin{aligned} f_{20} + V_{2,1} &= 0, \\ -3V_{3,0} + f_{11} + g_{20} + 2V_{1,2} &= 0, \\ f_{02} - 2V_{2,1} + g_{11} + 3V_{0,3} &= 0, \\ -V_{1,2} + g_{02} &= 0. \end{aligned} \tag{25}$$

In this case the matrix M and the columns L and P take the form

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} V_{3,0} \\ V_{2,1} \\ V_{1,2} \\ V_{0,3} \end{pmatrix},$$

$$P = \begin{pmatrix} f_{20} \\ g_{20} + f_{11} \\ f_{02} + g_{11} \\ g_{02} \end{pmatrix}.$$

Then we have

$$L = \begin{pmatrix} V_{3,0} \\ V_{2,1} \\ V_{1,2} \\ V_{0,3} \end{pmatrix} = M^{-1}(-P) = \frac{1}{3}\begin{pmatrix} 0 & -1 & 0 & -2 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 2 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -f_{20} \\ -g_{20} - f_{11} \\ -f_{02} - g_{11} \\ -g_{02} \end{pmatrix}.$$

The solution of the equation is as follows

$$V_{3,0} = \frac{1}{3}(f_{11} + 2g_{02} + g_{20}),\ V_{2,1} = -f_{20},\ V_{1,2} = g_{02},\ V_{0,3} = -\frac{1}{3}(f_{02} + 2f_{20} + g_{11}).$$

Passing to the next step, we set equal to zero the coefficients of the polynomial $\widetilde{V}(x,y) - W(x,y)$ of the fourth degrees. Then we obtain the following system:

$$\begin{aligned} f_{30} - w_4 + V_{3,1} + V_{2,1}g_{20} + 3V_{3,0}f_{20} &= 0, \\ 3V_{3,0}f_{11} + 2V_{1,2}g_{20} + 2V_{2,1}f_{20} + g_{30} - 4V_{4,0} + f_{21} + V_{2,1}g_{11} &= 0, \\ g_{21} + 3V_{0,3}g_{20} + V_{2,1}g_{02} + 2V_{2,1}f_{11} + 3V_{3,0}f_{02} - 2w_4 + & \\ +3V_{1,3} - 3V_{3,1} + f_{12} + V_{1,2}f_{20} + 2V_{1,2}g_{11} &= 0, \\ 4V_{0,4} + g_{12} + f_{03} + 2V_{1,2}g_{02} + 3V_{0,3}g_{11} + V_{1,2}f_{11} + 2V_{2,1}f_{02} &= 0, \\ 3V_{0,3}g_{02} - w_4 + V_{1,2}f_{02} + g_{03} - V_{1,3} &= 0. \end{aligned} \tag{26}$$

Here the matrix M and the columns L and P take the following form

$$M = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & -1 & -1 & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} V_{4,0} \\ V_{3,1} \\ w_4 \\ V_{1,3} \\ V_{0,4} \end{pmatrix},$$

$$P = \begin{pmatrix} f_{30} + V_{2,1}g_{20} + 3V_{3,0}f_{20} \\ V_{2,1}g_{11} + 3V_{3,0}f_{11} + 2V_{1,2}g_{20} + 2V_{2,1}f_{20} + f_{21} + g_{30} \\ V_{2,1}g_{02} + V_{1,2}f_{20} + 3V_{3,0}f_{02} + 2V_{1,2}g_{11} + 2V_{2,1}f_{11} + 3V_{0,3}g_{20} + g_{21} + f_{12} \\ 2V_{2,1}f_{02} + 3V_{0,3}g_{11} + V_{1,2}f_{11} + g_{12} + f_{03} + 2V_{1,2}g_{02} \\ g_{03} + 3V_{0,3}g_{02} + V_{1,2}f_{02} \end{pmatrix}.$$

Having performed the substitution of the obtained expressions for coefficients $V_{3,0}$, $V_{2,1}$, $V_{1,2}$, and $V_{0,3}$ in P and having solved the system (23), we obtain the coefficients $V_{4,0}$, $V_{3,1}$, w_4, $V_{1,3}$, $V_{0,4}$. Here the coefficient w_4 is equal to the expression

$$\frac{1}{8}(3f_{30} - 2f_{20}g_{20} + f_{20}f_{11} + g_{21} - g_{20}g_{11} + f_{02}f_{11} + 2g_{02}f_{02} + f_{12} - g_{02}g_{11} + 3g_{03}),$$

and if w_4 is not equal to zero, then $2\pi w_4$ is a first Lyapunov quantity of original system. If $w_4 = 0$, then one have to consider coefficients of greater degrees. The algorithm is ended when the first nonzero w_{2k} is found, either in putting all w_{2k} from formula (22) to zero. Note that the putting of all w_{2k} from (22) to zero can mean that it is necessary to consider more precise approximation of the functions $f(x,y)$ and $g(x,y)$ than $f_n(x,y)$ and $g_n(x,y)$, described by formula (19).

Matlab Programming code for computation of Lyapunov quantities

```
%Example of LQ computation for general Lienard system.
clear all; syms x y g1_0
N=7; %number of Lyapunov quantities
NS = 2*N+1;
gxy=0;
for n=2:NS %form function g
  gxy=gxy+sym(['g',int2str(n),'_',int2str(0)],'real')*x^n+ ->
      sym(['g',int2str(n-1),'_',int2str(1)],'real')*x^(n-1)*y;
end;
    %form system of equations
dx = -y; dy = g1_0*x+gxy;
subs(dy,x,x/sqrt(g1_0));
```

```
dy=dy/sqrt(g1_0);

KVxy_n(1:NS+2,1:NS)=0*x;
KVxy_n(1,1) = 'W2'; KVxy_n(2,1) = 1/2; KVxy_n(3,1) = 1/2;
Vxy = (1/2)*(x^2+y^2);
Wxy=KVxy_n(1,1)*(x^2+y^2);

for n=3:NS+1 %form V(x,y) and W(x,y)
  for iy=0:n
    ix = n - iy;
    if not ((iy == ix)&&(mod(ix,2) == 0))&&not((iy-ix==2) ->
        && (mod(iy,2)==0))
      KVxy_n(iy+1,n-1) = sym(['V',int2str(ix),'_',->
          int2str(iy)],'real');
      Vxy = Vxy + KVxy_n(iy+1,n-1)*x^ix*y^iy;
    else
      KVxy_n(iy+1,n-1) = sym(['W',int2str(n)],'real');
      Wxy = Wxy+ KVxy_n(iy+1,n-1)*(x^2+y^2)^(n/2);
      if (iy-ix==2)
        Vxy = Vxy -KVxy_n(n-1,iy+1) *x^ix*y^iy;
      end;
    end;
  end;
end;

dVxy = simplify(diff(Vxy, x)*dx+diff(Vxy, y)*dy);%form dV(x,y)
dVxy_Wxy_n(1:NS,1:NS+2) = 0*x*y; dVxy_Wxy = dVxy-Wxy;
for n=2:NS+1 %form dV(x,y)-W(x,y)
  for iy=0:n
   ix = n - iy;
    dVxy_Wxy_n(n-1,iy+1)=simplify(subs((diff(diff((dVxy_Wxy),->
      x,ix)/factorial(ix),y,iy)/factorial(iy)),[x y],[0 0]));
  end;
end;

New_KVxy_n(1:NS+2,1:NS) = 0*x;
for n=3:NS+1 %define new coefficients of V(x,y)
  clear M; M(1:n+1,1:n+1) = 0*x;
  for ix=1: n+1 %form matrix M
    M(1:n+1,ix) = diff(dVxy_Wxy_n(n-1, 1:n+1),KVxy_n(ix,n-1));
  end;

  clear P; P(1:n+1,1) =0*x;
  for ix=1: n+1 %form column P
    P(ix,1)=-dVxy_Wxy_n(n-1,ix)+M(ix,1:n+1)*KVxy_n(1:n+1,n-1);
  end;     %form column of new coefficients of V(x,y)
  New_KVxy_n(1:n+1,n-1)= simplify(M^(-1)*P);
end;

L(1:N)=0*x; G(1:N)=0*x; %define L_i and g_2i,1
for i=1:N
  L(i) = New_KVxy_n(i+2+mod(i+1,2),2*i+1);
  for n= NS+1: -1:4
    for ix= 0: n-1 %put new coefficients of V(x,y) in L_i
```

```
    L(i)=subs(L(i),KVxy_n(ix+1,n-2),New_KVxy_n(ix+1,n-2),0);
    end;
  end;

  for n=1:i-1 %put previous g_2n,1 in L_i
    L(i) = subs(L(i),['g',num2str(2*n),'_1'],G(n),0);
  end;
  L(i) = simplify(L(i));
  G(i) =solve(L(i),['g',num2str(2*i),'_1']); %define g_2i,1
end;
```

1.3.2 Computation of Lyapunov quantities in complex space

The well-known modification of the Poincare-Lyapunov method is the passage to the complex variables (see, for example, [Schuko, 1968, Gasull *et al.*, 1997]). In this case at first the real variables (x, y) are changed to the complex variable (u, v). Further the method practically repeats the Poincare-Lyapunov method in Euclidean space. Note that in passing to the complex variables certain symbolic expressions take more simple forms than in the case of original variables.

Consider system (13). Let us make the change of variables:

$$u = \frac{x + iy}{2}, \quad v = \frac{x - iy}{2}. \tag{27}$$

Then the system takes the form

$$\begin{aligned} \frac{du}{dt} &= iu + \hat{f}(u, v), \\ \frac{dv}{dt} &= -iv + \hat{g}(u, v). \end{aligned} \tag{28}$$

Here $\hat{f}$ and $\hat{g}$ are smooth complex-valued functions of the form

$$\begin{aligned} \hat{f}(u, v) &= \sum_{k+j=2}^{n} \hat{f}_{kj} u^k v^j + o\big((|u| + |v|)^n\big), \\ \hat{g}(u, v) &= \sum_{k+j=2}^{n} \hat{g}_{kj} u^k v^j + o\big((|u| + |v|)^n\big), \quad \hat{f}_{kj}, \hat{g}_{kj} \in \mathbb{C}. \end{aligned} \tag{29}$$

In the neighborhood of zero state we will seek Lyapunov function $V(u, v)$ in the form

$$V(u, v) = uv + V_3(u, v) + \ldots + V_{n+1}(u, v), \tag{30}$$

where $V_q(u, v)$ are homogeneous polynomials

$$V_q(u, v) = \sum_{k+j=q} V_{k,j} u^k v^j, \quad q = 3, \ldots, n+1$$

with unknown coefficients $\{V_{k,j}\}_{k+j=q,\, k,j \geqslant 0} \in \mathbb{C}$.

Note that uv is an integral of the first approximation of the system (28)

$$\frac{du}{dt} = iu, \quad \frac{dv}{dt} = -iv$$

since

$$(uv)' = \frac{\partial(uv)}{\partial u}(iu) + \frac{\partial(uv)}{\partial v}(-iv) = v(iu) + u(-iv) = 0.$$

For the derivative of $V(u,v)$ in virtue of system (28) we have

$$\dot{V}(u,v) = \frac{\partial V(u,v)}{\partial u}(iu + \sum_{k+j=2}^{n} \hat{f}_{kj}u^k v^j) + \frac{\partial V(u,v)}{\partial v}(-iv + \sum_{k+j=2}^{n} \hat{g}_{kj}u^k v^j)$$
$$+ o\big((|u| + |v|)^{n+1}\big).$$

Denoting in this expression the homogeneous terms of order k by $W_k(u,v)$ and taking into account that $(uv)' = v\hat{f}(u,v) + u\hat{g}(u,v) = o\big((|u| + |v|)^2\big)$, we obtain

$$\dot{V}(u,v) = W_3(u,v) + ... + W_{n+1}(u,v) + o\big((|u| + |v|)^{n+1}\big) =$$
$$= \widetilde{V}(u,v) + o\big((|u| + |v|)^{n+1}\big).$$

Here the expression $W_3(u,v)$ depends only on unknown coefficients of the form V_3

$$W_3(u,v) = \frac{\partial V_3(u,v)}{\partial u}(iu) + \frac{\partial V_3(u,v)}{\partial v}(-iv).$$

From the relation

$$W_3(u,v) = 0$$

it is always possible to express the coefficients of the form V_3 by the expansion coefficients of the functions $\hat{f}$ and $\hat{g}$ (see, for example, the work [Schuko, 1968, Gasull *et al.*, 1997]. Introduce the relation

$$W_4(u,v) = w_4(uv)^2.$$

Here w_4 is unknown coefficient. From the above relation, coefficient w_4 and the coefficients of the form V_4 can always be expressed in terms of coefficients of the functions $\hat{f}$ and $\hat{g}$. In this case if $w_4 = 0$, we continue the above-mentioned procedure for $k = 5, ...,$ solving sequentially equations $W_k(u,v) = 0$ for $k = 2p+1$ and $W_k(u,v) = w_k(uv)^p$ for $k = 2p$.

If for certain $k = 2p^* \le n+1$ the relation $w_k \ne 0$ is satisfied, the quantity $2\pi w_{2p^*}$ is equal to $(p^* - 1)$-th Lyapunov quantity [Frommer, 1928] of system (28) and, by that, of system (13) obtained from system (28) by the nonsingular change of variable.

The form $W_4(u,v)$ (and also $W_6, W_8, ...$) is similar to the form $W_4(x,y)$ (and also $W_6, W_8, ...$) considered previously in describing Poincare-Lyapunov method in the original coordinates since in virtue of (27) we have

$$x^2 + y^2 = uv.$$

Below is given the computational algorithm, based on the described above method for computation of Lyapunov quantities, and program in Matlab, permitting one to realize necessary computation.

The algorithm for computation of Lyapunov quantities

The algorithm for computation of Lyapunov quantities by the Poincare-Lyapunov method in complex space repeats practically the algorithm described above. Below is pointed out only certain distinctions of this modification.

In place of the functions $\hat{f}$ and $\hat{g}$ from system (28) it is considered their approximate values

$$\hat{f}_n(u,v) = \sum_{k+j=2}^{n} f_{kj} u^k v^j, \quad \hat{g}_n(u,v) = \sum_{k+j=2}^{n} g_{kj} u^k v^j, \tag{31}$$

where n is odd ($n = 2m+1$).

On the coefficients of polynomial $V(u,v)$ also additional requirements (20) are imposed.

The function $W(u,v)$ is as follows

$$W(u,v) = \sum_{k=2}^{m+1} w_{2k} (uv)^k, \tag{32}$$

where w_{2k} are unknown coefficients. In $W(u,v)$ there are half as many addends than in $W(x,y)$.

The matrix M, defined from equation (23), has diagonal form, i.e. more simple form in comparison to the matrix M from the algorithm, described above.

The diagonal elements of this matrix are as follows $M(s,s) = (\tilde{n} + 1 - 2s)i$ (where i is $\sqrt{-1}$) for all $s \leq \tilde{n}$ if the size of matrix $\tilde{n}$ is even. If $\tilde{n}$ is odd, then the element, placed at the intersection of diagonals, is equal to -1 and the rest of elements have the same form.

For example, for $\tilde{n} = 5$ and $\tilde{n} = 6$ the matrix M has the form

$$M_5 = \begin{pmatrix} 4i & 0 & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2i & 0 \\ 0 & 0 & 0 & 0 & -4i \end{pmatrix},$$

$$M_6 = \begin{pmatrix} 5i & 0 & 0 & 0 & 0 & 0 \\ 0 & 3i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -3i & 0 \\ 0 & 0 & 0 & 0 & 0 & -5i \end{pmatrix}.$$

Further, in the same way as in the algorithm described above, linear matrix equation (23) with respect to L will be obtained.

Then at t-th step of algorithm if t is odd, then one passes to $(t+1)$-th step, if t is even, then one tests the value w_t. In this case if $w_t \neq 0$, then it means that Lyapunov quantity is obtained and the algorithm is ended, if $w_t = 0$, then one passes to $(t+1)$-th step.

Matlab Programming code for computation of Lyapunov quantities

```
%Example of LQ computation for general Lienard system
clear all; syms z w x y g1_0 'real'
N=7; %number of Lyapunov quantities
NL = 2*N+2;

gxy=0;
for n=2:NL-1 %form function g
  gxy=gxy+sym(['g',int2str(n),'_',int2str(0)],'real')*x^n->
  +sym(['g',int2str(n-1),'_',int2str(1)],'real')*x^(n-1)*y;
end;
    %form system of equations
dx = -y; dy = g1_0*x+gxy;
subs(dy,x,x/sqrt(g1_0));
dy=dy/sqrt(g1_0);
    %pass from (x,y) to the complex variables (z,w)
dz=1i*z; dw=-1i*w;
dz = (dx-1i*dy)/2; dz = subs(dz,x,(z+w),0); %form dz and dw
dz = simplify(subs(dz,y,(z-w)*1i,0)); dz=collect(dz,[w,z]);
dw=dw+(dz-1i*z)';

Vzw = z*w; KVzw_n(1:NL+1,1:NL-1)=0*z;
KVzw_n(2,1)=1; Wzw=z*w; %form V(z,w) and W(z,w)
for n=3:NL
  for iy=0:n
    ix = n- iy;
    if not((iy==ix)&&(mod(ix,2)==0))&&not((iy-ix==2)->
        &&(mod(iy,2)==0))
      KVzw_n(iy+1,n-1)=sym(['V',int2str(ix),'_',int2str(iy)]);
      Vzw = Vzw + KVzw_n(iy+1,n-1)*z^ix*w^iy;
    else
      KVzw_n(iy+1,n-1)= sym(['W',int2str(n)]);
      Wzw = Wzw+KVzw_n(iy+1,n-1)*(z*w)^(n/2);
      if (iy-ix==2)
        Vzw = Vzw -KVzw_n(n+1-iy,n-1)*z^ix*w^iy;
      end;
    end;
  end;
end;

dVzw=simplify(diff(Vzw,z)*dz + diff(Vzw,w)*dw); %form dV(z,w)
dVzw_Wzw_n(1:NL-1,1:NL+1)=0*z; dVzw_Wzw = simplify(dVzw-Wzw);
for n=2:NL %form dV(z,w)-W(z,w)
  for iy=0:n
    ix = n - iy;
    dVzw_Wzw_n(n-1,iy+1) = subs((diff(diff(dVzw_Wzw, z, ix)->
        /factorial(ix),w,iy)/factorial(iy)),[z w], [0 0],0);
  end;
end;

New_KVzw_n(1:NL+1,1:NL-1) = 0*z; %matrix of new coefficients
New_KVzw_n(1,1) = 0; New_KVzw_n(2,1) = 1; New_KVzw_n(3,1) = 0;
```

```
for n=3:NL %define new coefficients of V(z,w)
  clear M; M(1:n+1,1:n+1) = 0*z;
  for ix=1: n+1 %form matrix M
    if (2*ix-2-n==0) M(ix,ix) = -1;
    else M(ix,ix) =(-2*ix+2+n)*1i;
    end
  end;

  clear P; P(1:n+1,1) =0*z;
  for ix=1: n+1 %form column P
    P(ix,1)=-dVzw_Wzw_n(n-1,ix)+M(ix,1:n+1)*KVzw_n(1:n+1,n-1);
  end;   %form column of coefficients of V(z,w)
  New_KVzw_n(1:n+1,n-1)= simplify(M^(-1)*P);
end;

L(1:N)=0*x; G(1:N)=0*x; %define Lyap. quant. and coeff. g_2j,1
for j=1:N
  L(j) = simplify(New_KVzw_n(j+2+mod(j+1,2),2*j+1));
  for n= NL: -1:4
    for ix= 0: n-1 %put new coefficients of V(z,w) in L_j
      L(j)=subs(L(j),KVzw_n(ix+1,n-2),New_KVzw_n(ix+1,n-2),0);
    end;
    L(j) = simplify(L(j));
  end;

  for n = 1:j-1 %put expressions of the previous g_2n,1 in L_j
    L(j) = subs(L(j),['g',num2str(2*n),'_1'],G(n),0);
  end;
  L(j) = factor(expand((L(j))));
  G(j)=solve(L(j),['g',num2str(2*j),'_1']);%determine g_2j,1
end;
```

1.4 Computation of Lyapunov quantities in Euclidean space and in the time domain

The method of computation of Lyapunov quantities in Euclidean space and in the time domain is based on the search of approximate solution of system (13) with initial data (15).

Following the work [Leonov, Kuznetsov & Kudryashova, 2008], in place of condition (10) one requires the strengthened condition

$$f(\cdot,\cdot), g(\cdot,\cdot) \in \mathcal{C}^{n+1}(U), \tag{33}$$

where U is open neighborhood of radius R_U of point $(x,y)=(0,0)$.

Then for sufficiently small h a solution of system can be represented as

$$\begin{aligned} x(t,h) &= x_{h^n}(t,h) + o(h^n) = \sum_{k=1}^{n} \widetilde{x}_{h^k}(t)h^k + o(h^n), \\ y(t,h) &= y_{h^n}(t,h) + o(h^n) = \sum_{k=1}^{n} \widetilde{y}_{h^k}(t)h^k + o(h^n), \end{aligned} \tag{34}$$

where $o(h^n)$ is a uniformly bounded with respect to t function for sufficiently small h. Here for determining the coefficients $\widetilde{x}_{h^k}(t), \widetilde{y}_{h^k}(t)$ it is necessary to substitute the representation of solution (34) in system (13) and to integrate sequentially the obtained differential equations of the corresponding degrees h.

Similarly, the return time $T(h)$ of the first intersection of the solution

$$\big(x(t,h), y(t,h)\big)$$

with the half-line $\{x = 0,\ y > 0\}$ can be represented as

$$T(h) = 2\pi + \Delta T(h),$$

where $\Delta T(h) = \sum\limits_{k=1}^{n} \widetilde{T}_k h^k + o(h^n)$ and $\widetilde{T}_k = \frac{1}{k!}\frac{d^k T(h)}{dh^k}$.

Putting in (34) $t = T(h)$, one can sequentially obtain the coefficients $\widetilde{T}_k$ from relation (16).

Having defined the crossing time $T(h)$ and using the representation

$$y(T(h),h) = \sum_{k=1}^{n} \widetilde{y}_k h^k + o(h^n),$$

one can sequentially obtain the coefficients $\widetilde{y}_k$.

If $\widetilde{y}_k = 0$ for $k = 2,..,2m$ and $\widetilde{y}_{2m+1} \neq 0$, then $\widetilde{y}_{2m+1}$ is m-th Lyapunov quantity L_m.

This method and the computational algorithm, based on this method, are described in detail in the works [Kuznetsov, 2008, Leonov & Kuznetsova, 2010, Leonov *et al.*, 2011].

Below is given a program in Matlab, permitting one to realize the computation of Lyapunov quantities in Euclidean space and in time domain. Remark that this program represents the improvement of the code given in the work [Kuznetsov, 2008] since it realizes the computation of Lyapunov quantities for more general case.

Matlab Programming code for the computation of Lyapunov quantities

```
%Example of LQ computation for general Lienard system.
clear all; syms x y h t g1_0 'real'
m = 7; %the number of computed Lyapunov quantities
NL =2*m+1;%degree of function g
gxy=0; %form function g
for n=2:NL
gxy=gxy+sym(['g',int2str(n),'_',int2str(0)],'real')*x^n+ ->
sym(['g',int2str(n-1),'_',int2str(1)],'real')*x^(n-1)*y;
end;
%set equal to 1 the coefficient of x in the second equation
gxy=subs(gxy,x,x/sqrt(g1_0));
gxy=gxy/sqrt(g1_0);

```

```
xt_s(1:NL-1)=0*h; yt_s(1:NL-1)=0*h; xth_s =0*t; yth_s =0*t;
for n=1:NL %form x(t) and y(t)
  xt_s(n) = sym(['xt_',int2str(n)],'real');
  yt_s(n) = sym(['yt_',int2str(n)],'real');
  xth_s = xth_s + xt_s(n)*h^n; yth_s = yth_s+ yt_s(n)*h^n;
end

sT_h_cur = 0;
for i = 1:NL-1 %form T(h)
  sT_h(i,1) = sym(['T',int2str(i)],'real');
  sT_h_cur = sT_h_cur + sT_h(i,1)*h^i;
end;
ugt(1:NL) = 0*t; xt(1:NL)=0*t; yt(1:NL)=0*t;
xt(1)=-sin(t); yt(1)=cos(t); xt_cur=xt(1)*h; yt_cur=yt(1)*h;
for i=2:NL  %find the expansion of g(t)
  ugt_s = subs(diff(subs(gxy, [x y], [xth_s ->
      yth_s]),h,i)/factorial(i),h,0);
  ugt(i) = subs(ugt_s, [xt_s yt_s], [xt yt]);
  uIt = diff(ugt(i),t);
  Iucos=int(cos(t)*uIt,t); Iucos_t0=Iucos - subs(Iucos,t,0);
  Iusin=int(sin(t)*uIt,t); Iusin_t0=Iusin - subs(Iusin,t,0);
  ug0 = subs(ugt(i),t,0); %find expansion of x(t) and y(t)
xt(i)=simplify(cos(t)*(ug0+Iucos_t0)+Iusin_t0*sin(t)-ugt(i));
yt(i)=simplify(sin(t)*(ug0+Iucos_t0)-Iusin_t0*cos(t));
  xt_cur = xt_cur + xt(i)*h^i; yt_cur = yt_cur + yt(i)*h^i;
end;

xh_cur = subs(xt_cur,t,2*pi); %find x(2pi)
for k = 1:NL
  xh_cur = xh_cur +subs(diff(xt_cur,k,t),t,2*pi)* ->
      *sT_h_cur^k/factorial(k);
end;

for k = 1:NL %find coefficients of x(2pi)
  xh(k,1) = subs(diff(xh_cur,k,h)/factorial(k),h,0);
end;

xh_temp = xh; T_cur = 0;
for k = 2:NL %find coeff. of T(h) and put them in x(t)
  T(k-1,1) = solve(xh_temp(k,1),sT_h(k-1,1));
  T_cur = T_cur + T(k-1,1)*h^(k-1);
  xh_temp = subs(xh_temp,sT_h(k-1,1),T(k-1,1));
end;

yh_cur = subs(yt_cur,t,2*pi); %find y(2pi)
for k = 1:NL
  yh_cur = yh_cur + subs(diff(yt_cur,k,t),t,2*pi)* ->
      *T_cur^k/factorial(k);
end;

for k = 1:NL %find coefficients of y(2pi)
  yh(k,1) = subs(diff(yh_cur,k,h)/factorial(k),h,0);
end;
yh = factor(yh);
```

```
for i=1:m % find L_i and g_2i,1
  L(i) = factor(yh(1+2*i,1));
  for n=1:i-1 %put previous g_2n,1 in L_i
    L(i) = subs(L(i),['g',num2str(2*n),'_1'],G(n),0);
  end;
  L(i) = simplify(L(i));
  G(i) = solve(L(i),['g',num2str(2*i),'_1']);
end;
```

1.5 Symbolic expressions of Lyapunov quantities

In the previous sections the methods for computation of symbolic expressions of Lyapunov quantities are described. Note that in despite of the existence of effective algorithms for computation of these expressions, the problem of determining Lyapunov quantities is still far from being resolved in the general case. The reason is in the computational complexity of determining Lyapunov quantities for systems of high degrees, as the problem of independently putting n first Lyapunov quantities to zero (see, for example, [Romanovskii, 1993]) that is necessary for determining of $(n+1)$-th quantity.

Below are given certain expressions of Lyapunov quantities (for general polynomial systems, Lienard systems, and quadratic systems), obtained by the author. The computation of these quantities was made by two different methods: classical Poincare-Lyapunov method and the method of computation of Lyapunov quantities in Euclidean space and in time domain, and it was used modern programming tools for symbolic computation. The formulas below obtained by two independent methods coincide. It permits one to be sure that they are true.

1.5.1 Lyapunov quantities of polynomial systems of general form

For the testing of computational technique, used in the work, by this technique it was obtained the known results for the general symbolic expressions L_1 and L_2, obtained in 40-50s of 20th century by N.N. Bautin [Bautin, 1949] and N.N. Serebryakova [Serebryakova, 1959], respectively, and also the symbolic expression L_3, obtained in the year 2008 in the works [Kuznetsov & Leonov, 2008[1]].

Next, within the framework of the present work there were obtained new results. By the algorithms, based on Poincare-Lyapunov method, the method for computation of Lyapunov quantities in Euclidean space and in the time domain, and computational software tool Matlab, for the system of the form (13) it was computed the fourth Lyapunov quantity of general form:

$$L_4 = \frac{\pi}{259200}(106920 g_{30} g_{02} f_{11} g_{20} g_{13} + \dots - 42525 f_{30} f_{70} + 78975 f_{30} g_{16}).$$

The expression for the fourth Lyapunov quantity occupies over 45 pages and it

can be found in internet
(http://sites.google.com/site/okuznetsovamath/home/4lyapqu.pdf).

1.5.2 Lyapunov quantities of Lienard system

Wide class of polynomial systems (involving quadratic systems) can be reduced to the form of Lienard systems, for which the study of Lyapunov quantities is a more simple problem [Blows & Lloyd, 1984[2], Christopher & Lloyd, 1996, Dumortier & Li, 1996, Christopher & Lynch, 1999, Han, 1999, Lynch, 2005].

In the present work, for the testing, it was repeated the known results for symbolic expressions of the first four Lyapunov quantities [Leonov, Kuznetsov & Kudryashova, 2008, Leonov & Kuznetsova, 2009] and then were obtained fifth, sixth, and seventh Lyapunov quantities of Lienard system [Kuznetsova, 2010[1]], the symbolic expressions of which are given below.

Consider the right-hand side of system (9) of special form

$$f_{10} = 0,\ f_{01} = -1,\ f(x,y) \equiv 0,$$
$$g_{01} = 0,\ g(x,y) = g_{x1}(x)y + g_{x0}(x),$$

where

$$g_{x1}(x) = g_{11}x + g_{21}x^2 + \dots, \quad g_{x0}(x) = g_{20}x^2 + g_{30}x^3 + \dots .$$

Then we obtain a general Lienard system

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = g_{10}x + g_{x1}(x)y + g_{x0}(x). \tag{35}$$

Here in order to the matrix of the first approximation in zero have two purely imaginary eigenvalues it is necessary that the following condition $g_{10} > 0$ is satisfied.

Since the above methods of the computation of Lyapunov quantities are used only for systems (35), in which $g_{10} = 1$, in system (35) it is necessary to make the following change of variables (it is realized in all three used in above programming codes):

$$t \to \sqrt{g_{10}}t, \quad x \to \sqrt{g_{10}}x.$$

Then the coefficient of x in the second equation of the system becomes equal to 1. This change of variables does not change y since

$$y = -\frac{dx}{dt} \to -\frac{\sqrt{\frac{1}{g_{10}}}dx}{\sqrt{\frac{1}{g_{10}}}dt} = y.$$

In this case expansion (17) is not changed, which means that the expressions for Lyapunov quantities in the original and new variable coincide.

The symbolic expressions L_1, L_2, L_3, L_4, L_5 can be found in the work [Leonov & Kuznetsova, 2009]. For example, for the first Lyapunov quantity we have

$$L_1 = \frac{\pi}{4(g_{10})^{5/2}} \left(g_{21} g_{10} - g_{11} g_{20} \right).$$

If $g_{21} = \frac{g_{11} g_{20}}{g_{10}}$, then $L_1 = 0$ and L_2 can be obtained. From the condition $L_2 = 0$ the coefficient g_{41} can be determined. Repeating this procedure for L_3, L_4, L_5 and the coefficients g_{61}, g_{81}, g_{101}, for L_6 we have

$$\begin{aligned}
L_6 = {} & \frac{\pi}{130636800 (g_{10})^{\frac{33}{2}}} \times \\
& (18243225\, g_{20} g_{30} g_{91} + 6081075\, g_{50} g_{11} g_{80} + 32837805\, g_{30}^2 g_{40} g_{11} g_{50} \\
& - 36486450\, g_{20} g_{30} g_{11} g_{90} - 4209975\, g_{11} g_{120} - 987566580\, g_{20}^4 g_{30}^2 g_{11} g_{40} \\
& + 4975425\, g_{30} g_{11} g_{100} + 313913600\, g_{20}^9 g_{11} g_{30} - 52123500\, g_{20}^2 g_{50} g_{11} g_{60} \\
& + 18243225\, g_{20} g_{11} g_{110} - 4378374\, g_{30} g_{40}^3 g_{11} - 31274100\, g_{20} g_{30} g_{40} g_{11} g_{60} \\
& - 340540200\, g_{20}^2 g_{30}^3 g_{11} g_{40} + 1230028800\, g_{20}^7 g_{30}^2 g_{11} + 364864500\, g_{20}^3 g_{30}^4 g_{11} \\
& - 556395840\, g_{20}^6 g_{30} g_{11} g_{40} + 18243225\, g_{20} g_{30}^5 g_{11} + 1245944700\, g_{20}^5 g_{30}^3 g_{11} \\
& - 10945935\, g_{40} g_{50}^2 g_{11} + 151351200\, g_{20}^3 g_{50}^2 g_{11} - 43783740\, g_{20} g_{40}^2 g_{11} g_{50} \\
& - 36486450\, g_{20} g_{30} g_{50} g_{51} + 418377960\, g_{20}^2 g_{30} g_{40} g_{11} g_{50} \\
& - 20270250\, g_{20}^2 g_{30} g_{11} g_{80} - 15637050\, g_{30} g_{50} g_{11} g_{60} - 264864600\, g_{20}^5 g_{71} \\
& + 54729675\, g_{20} g_{30} g_{50}^2 g_{11} - 245675430\, g_{20}^2 g_{30} g_{40} g_{51} - 194594400\, g_{20}^3 g_{30} g_{71} \\
& + 10945935\, g_{30} g_{40} g_{71} + 109459350\, g_{20}^2 g_{40} g_{71} + 18243225\, g_{20} g_{50} g_{71} \\
& + 100772100\, g_{20}^2 g_{30}^2 g_{11} g_{60} + 178378200\, g_{20}^3 g_{30} g_{40}^2 g_{11} + 4378374\, g_{40}^3 g_{31} \\
& - 21891870\, g_{30} g_{40} g_{11} g_{70} - 743242500\, g_{20}^3 g_{30}^2 g_{11} g_{50} + 4209975\, g_{121} \\
& + 84648564\, g_{20} g_{30}^2 g_{40}^2 g_{11} + 81081000\, g_{20}^3 g_{91} - 72972900\, g_{20} g_{30}^3 g_{11} g_{50} \\
& - 100772100\, g_{20}^2 g_{30} g_{60} g_{31} - 18243225\, g_{20} g_{30}^2 g_{71} - 10945935\, g_{40} g_{91} \\
& - 1239638400\, g_{20}^5 g_{30} g_{11} g_{50} + 7818525\, g_{30}^3 g_{11} g_{60} + 556395840\, g_{20}^6 g_{40} g_{31} \\
& - 1230028800\, g_{20}^7 g_{30} g_{31} + 18243225\, g_{20} g_{90} g_{31} + 10945935\, g_{30}^3 g_{40} g_{31} \\
& - 178378200\, g_{20}^3 g_{40}^2 g_{31} - 364864500\, g_{20}^3 g_{30}^3 g_{31} - 18243225\, g_{20} g_{30}^4 g_{31} \\
& - 1245944700\, g_{20}^5 g_{30}^2 g_{31} + 454053600\, g_{20}^5 g_{50} g_{31} - 470870400\, g_{20}^7 g_{11} g_{50} \\
& - 10945935\, g_{30}^4 g_{11} g_{40} - 313913600\, g_{20}^9 g_{31} + 7818525\, g_{60} g_{11} g_{70} \\
& + 264864600\, g_{20}^5 g_{11} g_{70} + 43783740\, g_{20} g_{40}^2 g_{51} + 52123500\, g_{20}^2 g_{60} g_{51} \\
& + 7818525\, g_{30} g_{60} g_{51} + 299999700\, g_{20}^3 g_{30}^2 g_{51} + 18243225\, g_{20} g_{70} g_{51} \\
& + 18243225\, g_{20} g_{30}^3 g_{51} + 785584800\, g_{20}^5 g_{30} g_{51} - 4975425\, g_{100} g_{31} \\
& - 18243225\, g_{20} g_{111} - 111969000\, g_{20}^4 g_{60} g_{31} - 113513400\, g_{20}^3 g_{70} g_{31} \\
& - 7818525\, g_{30}^2 g_{60} g_{31} + 7818525\, g_{50} g_{60} g_{31} + 20270250\, g_{20}^2 g_{80} g_{31} \\
& + 6081075\, g_{30} g_{80} g_{31} - 18243225\, g_{20} g_{50}^2 g_{31} + 111969000\, g_{20}^4 g_{30} g_{11} g_{60} \\
& - 6081075\, g_{80} g_{51} + 470870400\, g_{20}^7 g_{51} + 54729675\, g_{20} g_{30}^2 g_{11} g_{70} \\
& - 84648564\, g_{20} g_{30} g_{40}^2 g_{31} - 7818525\, g_{60} g_{71} + 308107800\, g_{20}^3 g_{30} g_{11} g_{70} \\
& - 109459350\, g_{20}^2 g_{40} g_{11} g_{70} - 36486450\, g_{20} g_{50} g_{11} g_{70} \\
& + 410810400\, g_{20}^4 g_{40} g_{11} g_{50} - 172702530\, g_{20}^2 g_{40} g_{50} g_{31} \\
& + 987566580\, g_{20}^4 g_{30} g_{40} g_{31} + 340540200\, g_{20}^2 g_{30}^2 g_{40} g_{31} \\
& - 21891870\, g_{30} g_{40} g_{50} g_{31} + 54729675\, g_{20} g_{30}^2 g_{50} g_{31} \\
& + 443242800\, g_{20}^3 g_{30} g_{50} g_{31} - 410810400\, g_{20}^4 g_{40} g_{51} - 151351200\, g_{20}^3 g_{50} g_{51} \\
& - 10945935\, g_{30}^2 g_{40} g_{51} + 10945935\, g_{40} g_{50} g_{51} + 31274100\, g_{20} g_{40} g_{60} g_{31} \\
& + 10945935\, g_{40} g_{70} g_{31} - 36486450\, g_{20} g_{30} g_{70} g_{31} + 10945935\, g_{40} g_{11} g_{90} \\
& - 81081000\, g_{20}^3 g_{11} g_{90} - 6081075\, g_{30}^2 g_{11} g_{80}).
\end{aligned}$$

If g_{121} is defined from the equation $L_6 = 0$, then L_7 takes the form

$$
\begin{aligned}
L_7 = {} & \frac{1}{263363788800(g_{10})^{\frac{39}{2}}}(-70338578310 g_{20} g_{40}^2 g_{11} g_{70} \\
& + 1820376558000 g_{20}^7 g_{71} + 10343908575 g_{30} g_{40} g_{91} + 84283699500 g_{20}^2 g_{30}^2 g_{11} g_{80} \\
& - 793945152000 g_{20}^6 g_{30} g_{11} g_{60} + 17239847625 g_{20} g_{30} g_{111} \\
& + 1800776577600 g_{20}^4 g_{40} g_{50} g_{31} - 132993110250 g_{20}^2 g_{50} g_{60} g_{31} \\
& + 1024685461800 g_{20}^4 g_{40} g_{11} g_{70} - 4326563241000 g_{20}^5 g_{30} g_{50} g_{31} \\
& + 17239847625 g_{20} g_{50} g_{91} - 34479695250 g_{20} g_{30} g_{90} g_{31} + 31031725725 g_{30}^2 g_{40} g_{11} g_{70} \\
& + 10913935032000 g_{20}^7 g_{30}^2 g_{31} - 980390911500 g_{20}^4 g_{30}^2 g_{11} g_{60} \\
& + 143803315656 g_{20}^2 g_{30} g_{40}^3 g_{11} - 103439085750 g_{20} g_{30}^2 g_{50}^2 g_{11} \\
& + 125432307000 g_{20}^4 g_{30} g_{11} g_{80} - 4701776625 g_{100} g_{51} + 710026317000 g_{20}^3 g_{30} g_{50} g_{51} \\
& + 10343908575 g_{30}^5 g_{11} g_{40} - 2157483328000 g_{20}^{11} g_{11} g_{30} + 3447969525 g_{141} \\
& - 555761606400 g_{20}^5 g_{91} - 294226732800 g_{20}^2 g_{40} g_{50} g_{51} + 1585759495320 g_{20}^3 g_{30} g_{40}^2 g_{31} \\
& - 7388506125 g_{30}^2 g_{60} g_{51} - 376978001400 g_{20}^2 g_{30} g_{40} g_{71} - 396972576000 g_{20}^4 g_{60} g_{51} \\
& + 3097553659200 g_{20}^4 g_{30} g_{40} g_{51} - 9070477416000 g_{20}^9 g_{30}^2 g_{11} \\
& - 43418875500 g_{20}^2 g_{50} g_{11} g_{80} - 272006484750 g_{20}^3 g_{30} g_{91} + 51719542875 g_{20} g_{30}^2 g_{50} g_{51} \\
& + 156307951800 g_{20}^2 g_{40} g_{91} + 7833275950500 g_{20}^5 g_{30}^2 g_{11} g_{50} - 17239847625 g_{20} g_{30}^2 g_{91} \\
& - 7388506125 g_{60} g_{91} + 294226732800 g_{20}^2 g_{40} g_{50}^2 g_{11} - 10048368330 g_{30} g_{40}^2 g_{11} g_{60} \\
& - 17239847625 g_{20} g_{131} + 5746615875 g_{30}^3 g_{11} g_{80} + 108547188750 g_{20}^3 g_{111} \\
& + 31031725725 g_{30}^2 g_{40} g_{50} g_{31} + 270320810760 g_{20} g_{30} g_{40}^2 g_{11} g_{50} \\
& - 130037707800 g_{20} g_{30} g_{40} g_{60} g_{31} + 17239847625 g_{20} g_{110} g_{31} \\
& + 321810489000 g_{20}^2 g_{30} g_{50} g_{11} g_{60} - 83736402750 g_{20}^2 g_{60} g_{11} g_{70} \\
& - 223479506250 g_{20}^3 g_{50} g_{71} + 10343908575 g_{40} g_{50} g_{71} - 10343908575 g_{30}^2 g_{40} g_{71} \\
& - 3506712709500 g_{20}^5 g_{30}^2 g_{51} + 220670049600 g_{20} g_{30}^2 g_{40}^2 g_{31} \\
& - 357275318400 g_{20}^3 g_{40} g_{60} g_{31} + 31031725725 g_{30} g_{40} g_{50}^2 g_{11} \\
& - 34479695250 g_{20} g_{30} g_{50} g_{71} - 4997257164900 g_{20}^4 g_{30}^2 g_{40} g_{31} \\
& - 7388506125 g_{50}^2 g_{11} g_{60} + 413756343000 g_{20}^3 g_{30} g_{11} g_{90} \\
& - 41375634300 g_{30}^3 g_{40} g_{11} g_{50} + 646072867440 g_{20}^3 g_{40}^2 g_{11} g_{50} \\
& + 4144197657600 g_{20}^8 g_{30} g_{11} g_{40} + 5746615875 g_{30} g_{80} g_{51} - 180060630750 g_{20}^3 g_{70} g_{51} \\
& + 10343908575 g_{40} g_{70} g_{51} + 597648051000 g_{20}^2 g_{30} g_{40} g_{11} g_{70} - 10343908575 g_{40} g_{111} \\
& + 9001158566400 g_{20}^6 g_{30}^2 g_{11} g_{40} - 26051325300 g_{20} g_{30} g_{40} g_{11} g_{80} \\
& + 4701776625 g_{30} g_{100} g_{31} + 17239847625 g_{20} g_{50}^3 g_{11} - 4723292574000 g_{20}^5 g_{30}^4 g_{11} \\
& - 17239847625 g_{20} g_{30}^6 g_{11} - 670438518750 g_{20}^3 g_{30}^5 g_{11} - 1820376558000 g_{20}^7 g_{11} g_{70} \\
& + 130037707800 g_{20} g_{30}^2 g_{40} g_{11} g_{60} + 1941079140000 g_{20}^3 g_{30}^3 g_{11} g_{50} \\
& + 86199238125 g_{20} g_{30}^4 g_{11} g_{50} + 43418875500 g_{20}^2 g_{80} g_{51} + 7388506125 g_{50} g_{60} g_{51} \\
& + 17239847625 g_{20} g_{11} g_{130} + 555761606400 g_{20}^5 g_{11} g_{90} + 3978426375 g_{30} g_{11} g_{120} \\
& - 108547188750 g_{20}^3 g_{11} g_{110} + 10343908575 g_{40} g_{11} g_{110} - 141749858250 g_{20}^3 g_{90} g_{31} \\
& + 17762267250 g_{20}^2 g_{100} g_{31} - 3447969525 g_{11} g_{140} + 66989122200 g_{20} g_{40} g_{60} g_{51} \\
& + 5746615875 g_{50} g_{80} g_{31} + 10343908575 g_{40} g_{90} g_{31} - 17762267250 g_{20}^2 g_{30} g_{11} g_{100} \\
& - 34479695250 g_{20} g_{30} g_{11} g_{110} + 11962343250 g_{20} g_{60}^2 g_{31} - 18205279092 g_{30} g_{40}^3 g_{31} \\
& - 1746017713440 g_{20}^5 g_{30} g_{40}^2 g_{11} + 4701776625 g_{50} g_{11} g_{100} \\
& - 1585759495320 g_{20}^3 g_{30}^2 g_{40}^2 g_{11} + 10343908575 g_{30}^3 g_{40} g_{51} - 5746615875 g_{30}^2 g_{80} g_{31} \\
& + 814742428500 g_{20}^5 g_{70} g_{31} - 996080085000 g_{20}^3 g_{30}^2 g_{11} g_{70} \\
& - 68959390500 g_{20} g_{30}^3 g_{11} g_{70} - 66989122200 g_{20} g_{40} g_{50} g_{11} g_{60} \\
& - 11493231750 g_{30} g_{50} g_{11} g_{80} - 125432307000 g_{20}^4 g_{80} g_{31} + 7388506125 g_{60} g_{70} g_{31} \\
& + 7388506125 g_{30}^3 g_{60} g_{31} - 220670049600 g_{20} g_{30}^3 g_{40}^2 g_{11} - 1227079854000 g_{20}^5 g_{50}^2 g_{11}
\end{aligned}
$$

$$
\begin{aligned}
&+ 10048368330g_{40}^2g_{60}g_{31} - 143803315656g_{20}^2g_{40}^3g_{31} + 9070477416000g_{20}^9g_{30}g_{31} \\
&+ 793945152000g_{20}^6g_{60}g_{31} + 403540137000g_{20}^3g_{50}g_{11}g_{70} \\
&- 712580368500g_{20}^2g_{30}^3g_{40}g_{31} - 3236224992000g_{20}^9g_{51} \\
&- 111714212610g_{20}g_{40}^2g_{50}g_{31} - 156307951800g_{20}^2g_{40}g_{11}g_{90} \\
&- 34479695250g_{20}g_{50}g_{11}g_{90} + 51719542875g_{20}g_{30}^2g_{11}g_{90} \\
&- 11962343250g_{20}g_{30}g_{60}^2g_{11} - 646072867440g_{20}^3g_{40}^2g_{51} - 17239847625g_{20}g_{50}^2g_{51} \\
&- 983309827500g_{20}^3g_{30}g_{50}^2g_{11} - 68959390500g_{20}g_{30}^3g_{50}g_{31} \\
&- 1353647295000g_{20}^3g_{30}^2g_{50}g_{31} - 262702440000g_{20}^2g_{30}^3g_{11}g_{60} \\
&- 14777012250g_{30}g_{50}g_{60}g_{31} + 396972576000g_{20}^4g_{50}g_{11}g_{60} \\
&+ 22165518375g_{30}^2g_{50}g_{11}g_{60} + 4723292574000g_{20}^5g_{30}^3g_{31} + 17239847625g_{20}g_{30}^5g_{31} \\
&+ 670438518750g_{20}^3g_{30}^4g_{31} + 551675124000g_{20}^3g_{30}g_{70}g_{31} \\
&+ 980390911500g_{20}^4g_{30}g_{60}g_{31} - 5746615875g_{80}g_{71} - 4144197657600g_{20}^8g_{40}g_{31} \\
&- 10343908575g_{30}^4g_{40}g_{31} + 26051325300g_{20}g_{40}g_{80}g_{31} \\
&- 1024685461800g_{20}^4g_{40}g_{71} + 444404961000g_{20}^3g_{30}^2g_{71} + 17239847625g_{20}g_{30}^3g_{71} \\
&+ 103439085750g_{20}g_{30}g_{50}g_{11}g_{70} + 83736402750g_{20}^2g_{60}g_{71} \\
&+ 4997257164900g_{20}^4g_{30}^3g_{11}g_{40} + 712580368500g_{20}^2g_{30}^4g_{11}g_{40} \\
&- 6324209892000g_{20}^7g_{30}g_{51} + 3303693993600g_{20}^6g_{40}g_{51} + 9378477108g_{40}^3g_{51} \\
&+ 357275318400g_{20}^3g_{30}g_{40}g_{11}g_{60} - 20687817150g_{30}g_{40}g_{50}g_{51} \\
&+ 273283510500g_{20}^3g_{50}^2g_{31} - 10343908575g_{40}g_{50}^2g_{31} - 587431845000g_{20}^3g_{30}^3g_{51} \\
&- 17239847625g_{20}g_{30}^4g_{51} - 84283699500g_{20}^2g_{30}g_{80}g_{31} \\
&+ 1746017713440g_{20}^5g_{40}^2g_{31} - 3978426375g_{120}g_{31} + 2157483328000g_{20}^{11}g_{31} \\
&- 3245221980000g_{20}^7g_{50}g_{31} - 14777012250g_{30}g_{60}g_{11}g_{70} \\
&- 2671537869000g_{20}^5g_{30}g_{11}g_{70} + 17239847625g_{20}g_{70}g_{71} \\
&+ 1856795440500g_{20}^5g_{30}g_{71} + 7388506125g_{30}g_{60}g_{71} + 70338578310g_{20}g_{40}^2g_{71} \\
&+ 262702440000g_{20}^2g_{30}^2g_{60}g_{31} - 9001158566400g_{20}^6g_{30}g_{40}g_{31} \\
&+ 7388506125g_{60}g_{11}g_{90} - 10913935032000g_{20}^7g_{30}^3g_{11} + 5746615875g_{70}g_{11}g_{80} \\
&- 4701776625g_{30}^2g_{11}g_{100} + 864291027600g_{20}^2g_{30}g_{40}g_{50}g_{31} \\
&+ 17239847625g_{20}g_{90}g_{51} + 1227079854000g_{20}^5g_{50}g_{51} \\
&+ 9569431872000g_{20}^7g_{30}g_{11}g_{50} - 3303693993600g_{20}^6g_{40}g_{11}g_{50} \\
&- 7388506125g_{30}^4g_{11}g_{60} - 34479695250g_{20}g_{30}g_{70}g_{51} - 9378477108g_{40}^3g_{11}g_{50} \\
&+ 3236224992000g_{20}^9g_{11}g_{50} - 20687817150g_{40}g_{50}g_{11}g_{70} \\
&- 20687817150g_{30}g_{40}g_{11}g_{90} - 20687817150g_{30}g_{40}g_{70}g_{31} \\
&+ 18205279092g_{30}^2g_{40}^3g_{11} - 220670049600g_{20}^2g_{40}g_{70}g_{31} \\
&- 34479695250g_{20}g_{50}g_{70}g_{31} - 1448147200500g_{20}^2g_{30}^2g_{40}g_{11}g_{50} \\
&- 4898330236800g_{20}^4g_{30}g_{40}g_{11}g_{50} - 188817378750g_{20}^2g_{30}g_{60}g_{51} \\
&- 158606598150g_{20}g_{30}g_{40}^2g_{51} + 583856172900g_{20}^2g_{30}^2g_{40}g_{51} \\
&- 17239847625g_{20}g_{70}^2g_{11} + 51719542875g_{20}g_{30}g_{50}^2g_{31} \\
&+ 51719542875g_{20}g_{30}^2g_{70}g_{31}).
\end{aligned}
$$

1.5.3 Lyapunov quantities of quadratic system

Let us write the expressions for the first three Lyapunov quantities of the following system

$$\begin{aligned} \frac{dx}{dt} &= x^2 + xy + y, \\ \frac{dy}{dt} &= a_2x^2 + b_2xy + c_2y^2 + \alpha_2 x + \beta_2 y. \end{aligned} \tag{36}$$

These expressions are used for the construction of 1, 2 and 3 "small" limit cycles around zero equilibrium.

Note that system (36) can be is obtained from the quadratic system of general form

$$\begin{aligned} \frac{dx}{dt} &= a_1x^2 + b_1xy + c_1y^2 + \alpha_1 x + \beta_1 y, \\ \frac{dy}{dt} &= a_2x^2 + b_2xy + c_2y^2 + \alpha_2 x + \beta_2 y, \end{aligned} \tag{37}$$

where $a_j, b_j, c_j, \alpha_j, \beta_j$ are real numbers (the reduction to this special form is described in more detail in [Leonov & Kuznetsova, 2010]).

The first Lyapunov quantity then has the form
$L_1(0) = \frac{-\pi}{4(-\alpha_2)^{\frac{5}{2}}}(\alpha_2(b_2c_2 - 1) - a_2(b_2 + 2))$.

In the case when the conditions

$$\alpha_2 = \frac{a_2(2 + b_2)}{b_2c_2 - 1}, \quad b_2c_2 - 1 \neq 0$$

are satisfied we obtain $L_1(0) = 0$ and the second Lyapunov quantity takes the form

$$\begin{aligned} L_2(0) = & \frac{\pi(b_2 - 3)(b_2\,c_2 - 1)^{\frac{5}{2}}}{24(-a_2)^{\frac{7}{2}}(2 + b_2)^{\frac{7}{2}}} \times \\ & \times ((c_2b_2 + b_2 - 2c_2)(c_2b_2 - 1) - a_2(c_2 - 1)(1 + 2c_2)^2). \end{aligned}$$

If condition $b_2 = 3$ is satisfied, we obtain $L_2(0) = 0$ and the third Lyapunov quantity is as follows

$$L_3(0) = \frac{\pi\sqrt{5}(3c_2 - 1)^{\frac{9}{2}}}{500000(-a_2)^{\frac{9}{2}}}(c_2 - 2)(4c_2^3a_2 - 3c_2^2 - 3a_2c_2 - 8c_2 - a_2 + 3).$$

It should be noted that the expressions L_1 and L_2 can be put to zero otherwise. However, in these cases one will get $L_3 = 0$.

In putting L_3 to zero, we obtain $L_4 = L_5 = ... = 0$.

The expressions obtained are used for the study of "small" limit cycles of quadratic systems in the next chapter.

Note that for the perturbations of "small" cycles around a nonzero point it is necessary to compute in this point the expressions of Lyapunov quantities. For this we will consider the reduction of system (36) to Lienard system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -f(x)y - g(x). \tag{38}$$

The reduction to Lienard system is described in more detail in [Leonov & Kuznetsova, 2010].

A Lienard system (38) (and, therefore, quadratic system (36)) has equilibria in all points, for which the condition $g(x) = 0$ is satisfied. It is known (see, for example, [Ye, 1986]) that in quadratic system the limit cycles surround only the equilibria of focus type. In quadratic system it can be less than or equal to two such equilibria, so the equation

$$0 = g(x) = ((-a_2 + b_2 - c_2)x^4 - (2a_2 - b_2 + \alpha_2 - \beta_2)x^3 - \\ - (a_2 + 2\alpha_2 - \beta_2)x^2 - \alpha_2 x)\frac{|x+1|^{2q}}{(x+1)^3}$$

has only 2 roots ($x = 0$ and x_0 to the left of the line of discontinuity $x = -1$). In this case it can be considered limit cycles in the points $x = 0$ and x_0.

The first Lyapunov quantity L_1 in the point x_0 has the form

$L_1(x_0) = -\pi(x_0+1)^{2c_2+4}[4a_2 + 2\alpha_2 - 4\beta_2 + 2a_2b_2 - 2a_2\beta_2 - 2b_2\beta_2 + 20a_2x_0 +$
$4\alpha_2x_0 - 12b_2x_0 - 8\beta_2x_0 + 2{\beta_2}^2c_2 + 34a_2{x_0}^2 + 24a_2{x_0}^3 + 6a_2{x_0}^4 + 2\alpha_2{x_0}^2 - 30b_2{x_0}^2 -$
$6{b_2}^2x_0 - 24b_2{x_0}^3 - 6b_2{x_0}^4 - 4\beta_2{x_0}^2 + 4{\beta_2}^2x_0 + 24c_2{x_0}^2 + 24c_2{x_0}^3 + 6c_2{x_0}^4 + 2{\beta_2}^2 -$
$18{b_2}^2{x_0}^2 - 20{b_2}^2{x_0}^3 - 10{b_2}^2{x_0}^4 - 2{b_2}^2{x_0}^5 + 2{\beta_2}^2{x_0}^2 - 72{c_2}^2{x_0}^3 - 60{c_2}^2{x_0}^4 - 14{c_2}^2{x_0}^5 +$
$72{c_2}^3{x_0}^4 - {c_2}^2{x_0}^6 + 42{c_2}^3{x_0}^5 + 5{c_2}^3{x_0}^6 - 28{c_2}^4{x_0}^5 - 8{c_2}^4{x_0}^6 + 4{c_2}^5{x_0}^6 + 24a_2{c_2}^2{x_0}^2 +$
$80a_2{c_2}^2{x_0}^3 + 93a_2{c_2}^2{x_0}^4 - 16a_2{c_2}^3{x_0}^3 + 42a_2{c_2}^2{x_0}^5 - 40a_2{c_2}^3{x_0}^4 + 5a_2{c_2}^2{x_0}^6 - 32a_2{c_2}^3{x_0}^5 +$
$4a_2{c_2}^4{x_0}^4 - 8a_2{c_2}^3{x_0}^6 + 8a_2{c_2}^4{x_0}^5 + 4a_2{c_2}^4{x_0}^6 + 21\alpha_2{c_2}^2{x_0}^2 + 24\alpha_2{c_2}^2{x_0}^3 - 10\alpha_2{c_2}^3{x_0}^2 +$
$9\alpha_2{c_2}^2{x_0}^4 - 22\alpha_2{c_2}^3{x_0}^3 - 14\alpha_2{c_2}^3{x_0}^4 + 4\alpha_2{c_2}^4{x_0}^3 - 2\alpha_2{c_2}^3{x_0}^5 + 8\alpha_2{c_2}^4{x_0}^4 + 4\alpha_2{c_2}^4{x_0}^5 +$
$12{b_2}^2c_2{x_0}^2 - 66b_2{c_2}^2{x_0}^3 + 31{b_2}^2c_2{x_0}^3 - 111b_2{c_2}^2{x_0}^4 + 27{b_2}^2c_2{x_0}^4 - 51b_2{c_2}^2{x_0}^5 +$
$34b_2{c_2}^3{x_0}^4 + 9{b_2}^2c_2{x_0}^5 - 6b_2{c_2}^2{x_0}^6 + 45b_2{c_2}^3{x_0}^5 + {b_2}^2c_2{x_0}^6 + 11b_2{c_2}^3{x_0}^6 - 6b_2{c_2}^4{x_0}^5 -$
$6b_2{c_2}^4{x_0}^6 - 36\beta_2{c_2}^2{x_0}^2 - 12{\beta_2}^2c_2{x_0}^2 - 6{\beta_2}^2{c_2}^2x_0 - 42\beta_2{c_2}^2{x_0}^3 - 10{\beta_2}^2c_2{x_0}^3 - 3\beta_2{c_2}^2{x_0}^4 +$
$28\beta_2{c_2}^3{x_0}^3 - 2{\beta_2}^2c_2{x_0}^4 + 3\beta_2{c_2}^2{x_0}^5 + 33\beta_2{c_2}^3{x_0}^4 + 5\beta_2{c_2}^3{x_0}^5 - 6\beta_2{c_2}^4{x_0}^4 - 6\beta_2{c_2}^4{x_0}^5 -$
$2\alpha_2b_2c_2 - 2a_2\beta_2c_2 + 3\alpha_2\beta_2c_2 + 10a_2b_2x_0 - 10a_2\beta_2x_0 - 12a_2c_2x_0 - 6\alpha_2c_2x_0 + 2b_2\beta_2x_0 +$
$12\beta_2c_2x_0 - 9{b_2}^2{c_2}^2{x_0}^3 - 21{b_2}^2{c_2}^2{x_0}^4 - 15{b_2}^2{c_2}^2{x_0}^5 + 2{b_2}^2{c_2}^3{x_0}^4 - 3{b_2}^2{c_2}^2{x_0}^6 + 4{b_2}^2{c_2}^3{x_0}^5 +$
$2{b_2}^2{c_2}^3{x_0}^6 - 12{\beta_2}^2{c_2}^2{x_0}^2 - 6{\beta_2}^2{c_2}^2{x_0}^3 + 2{\beta_2}^2{c_2}^3{x_0}^2 + 4{\beta_2}^2{c_2}^3{x_0}^3 + 2{\beta_2}^2{c_2}^3{x_0}^4 +$
$3\alpha_2\beta_2{c_2}^2 + 20a_2b_2{x_0}^2 + 20a_2b_2{x_0}^3 + 10a_2b_2{x_0}^4 + 2a_2b_2{x_0}^5 - 20a_2\beta_2{x_0}^2 - 20a_2\beta_2{x_0}^3 -$
$10a_2\beta_2{x_0}^4 - 2a_2\beta_2{x_0}^5 - 54a_2c_2{x_0}^2 - 84a_2c_2{x_0}^3 - 55a_2c_2{x_0}^4 - 14a_2c_2{x_0}^5 - a_2c_2{x_0}^6 -$
$11\alpha_2c_2{x_0}^2 + 6\alpha_2{c_2}^2x_0 - 4\alpha_2c_2{x_0}^3 + \alpha_2c_2{x_0}^4 + 16b_2\beta_2{x_0}^2 + 20b_2\beta_2{x_0}^3 + 10b_2\beta_2{x_0}^4 +$
$2b_2\beta_2{x_0}^5 + 48b_2c_2{x_0}^2 + 100b_2c_2{x_0}^3 + 67b_2c_2{x_0}^4 + 16b_2c_2{x_0}^5 + b_2c_2{x_0}^6 + 6\beta_2c_2{x_0}^2 -$
$2{\beta_2}^2c_2x_0 - 16\beta_2c_2{x_0}^3 - 12\beta_2c_2{x_0}^4 - 2\beta_2c_2{x_0}^5 - 24a_2b_2c_2{x_0}^2 - 37a_2b_2c_2{x_0}^3 - 27a_2b_2c_2{x_0}^4 -$
$9a_2b_2c_2{x_0}^5 - a_2b_2c_2{x_0}^6 - 9\alpha_2b_2c_2{x_0}^2 + 3\alpha_2b_2{c_2}^2x_0 - 5\alpha_2b_2c_2{x_0}^3 - \alpha_2b_2c_2{x_0}^4 + 3a_2\beta_2c_2{x_0}^2 +$
$6a_2\beta_2{c_2}^2x_0 + 13a_2\beta_2c_2{x_0}^3 + 11a_2\beta_2c_2{x_0}^4 + 3a_2\beta_2c_2{x_0}^5 + 15\alpha_2\beta_2c_2{x_0}^2 + 9\alpha_2\beta_2{c_2}^2x_0 +$
$9\alpha_2\beta_2c_2{x_0}^3 - 2\alpha_2\beta_2{c_2}^3x_0 + 2\alpha_2\beta_2c_2{x_0}^4 + 21b_2\beta_2c_2{x_0}^2 + 3b_2\beta_2c_2{x_0}^3 - 9b_2\beta_2c_2{x_0}^4 -$
$3b_2\beta_2c_2{x_0}^5 + 6a_2b_2{c_2}^2{x_0}^2 + 21a_2b_2{c_2}^2{x_0}^3 + 27a_2b_2{c_2}^2{x_0}^4 - 2a_2b_2{c_2}^3{x_0}^3 + 15a_2b_2{c_2}^2{x_0}^5 -$
$6a_2b_2{c_2}^3{x_0}^4 + 3a_2b_2{c_2}^2{x_0}^6 - 6a_2b_2{c_2}^3{x_0}^5 - 2a_2b_2{c_2}^3{x_0}^6 + 9\alpha_2b_2{c_2}^2{x_0}^2 + 9\alpha_2b_2{c_2}^2{x_0}^3 -$
$2\alpha_2b_2{c_2}^3{x_0}^2 + 3\alpha_2b_2{c_2}^2{x_0}^4 - 6\alpha_2b_2{c_2}^3{x_0}^3 - 6\alpha_2b_2{c_2}^3{x_0}^4 - 2\alpha_2b_2{c_2}^3{x_0}^5 + 21a_2\beta_2{c_2}^2{x_0}^2 +$
$27a_2\beta_2{c_2}^2{x_0}^3 - 2a_2\beta_2{c_2}^3{x_0}^2 + 15a_2\beta_2{c_2}^2{x_0}^4 - 6a_2\beta_2{c_2}^3{x_0}^3 + 3a_2\beta_2{c_2}^2{x_0}^5 - 6a_2\beta_2{c_2}^3{x_0}^4 -$
$2a_2\beta_2{c_2}^3{x_0}^5 + 9\alpha_2\beta_2{c_2}^2{x_0}^2 + 3\alpha_2\beta_2{c_2}^2{x_0}^3 - 6\alpha_2\beta_2{c_2}^3{x_0}^2 - 6\alpha_2\beta_2{c_2}^3{x_0}^3 - 2\alpha_2\beta_2{c_2}^3{x_0}^4 -$
$15b_2\beta_2{c_2}^2{x_0}^2 - 33b_2\beta_2{c_2}^2{x_0}^3 - 21b_2\beta_2{c_2}^2{x_0}^4 + 4b_2\beta_2{c_2}^3{x_0}^3 - 3b_2\beta_2{c_2}^2{x_0}^5 + 8b_2\beta_2{c_2}^3{x_0}^4 +$
$4b_2\beta_2{c_2}^3{x_0}^5 - 6a_2b_2c_2x_0 - 7\alpha_2b_2c_2x_0 - 4a_2\beta_2c_2x_0 + 11\alpha_2\beta_2c_2x_0 + 12b_2\beta_2c_2x_0]/[8(\alpha_2 +$
$2a_2x_0 + 2\alpha_2x_0 - 2\beta_2x_0 + 5a_2{x_0}^2 + 4a_2{x_0}^3 + a_2{x_0}^4 + \alpha_2{x_0}^2 - 3b_2{x_0}^2 - 4b_2{x_0}^3 - b_2{x_0}^4 -$

$2\beta_2 {x_0}^2 + 4c_2 {x_0}^3 + c_2 {x_0}^4 - 2{c_2}^2 {x_0}^4 - 2\alpha_2 c_2 x_0 - 2a_2 c_2 {x_0}^2 - 4a_2 c_2 {x_0}^3 - 2a_2 c_2 {x_0}^4 - 4\alpha_2 c_2 {x_0}^2 - 2\alpha_2 c_2 {x_0}^3 + 2b_2 c_2 {x_0}^3 + 2b_2 c_2 {x_0}^4 + 2\beta_2 c_2 {x_0}^2 + 2\beta_2 c_2 {x_0}^3)^{\frac{1}{2}}(\alpha_2 + 2a_2 x_0 + 2\alpha_2 x_0 - 2\beta_2 x_0 + 5a_2 {x_0}^2 + 4a_2 {x_0}^3 + a_2 {x_0}^4 + \alpha_2 {x_0}^2 - 3b_2 {x_0}^2 - 4b_2 {x_0}^3 - b_2 {x_0}^4 - 2\beta_2 {x_0}^2 + 4c_2 {x_0}^3 + c_2 {x_0}^4 - 2{c_2}^2 {x_0}^4 - 2\alpha_2 c_2 x_0 - 2a_2 c_2 {x_0}^2 - 4a_2 c_2 {x_0}^3 - 2a_2 c_2 {x_0}^4 - 4\alpha_2 c_2 {x_0}^2 - 2\alpha_2 c_2 {x_0}^3 + 2b_2 c_2 {x_0}^3 + 2b_2 c_2 {x_0}^4 + 2\beta_2 c_2 {x_0}^2 + 2\beta_2 c_2 {x_0}^3)^2].$

Here β_2 is obtained from the condition $L_0 = 0$ and has the form

$$\beta_2 = -\frac{2\,x_0 + b_2\,x_0 + b_2\,{x_0}^2 - 2\,c_2\,{x_0}^2 + {x_0}^2}{x_0 + 1}.$$

The value x_0 is a root of the function $g(x)$ (a solution of 3-d degree equation):

$$x_0 = (((\tfrac{\alpha_2}{2(a_2-b_2+c_2)} + \tfrac{(2a_2+\alpha_2-b_2-\beta_2)^3}{27(a_2-b_2+c_2)^3} - \tfrac{(a_2+2\alpha_2-\beta_2)(2a_2+\alpha_2-b_2-\beta_2)}{6(a_2-b_2+c_2)^2})^2 -$$

$$-(\tfrac{(2a_2+\alpha_2-b_2-\beta_2)^2}{9(a_2-b_2+c_2)^2} - \tfrac{a_2+2\alpha_2-\beta_2}{3(a_2-b_2+c_2)})^3)^{\frac{1}{2}} - \tfrac{(2a_2+\alpha_2-b_2-\beta_2)^3}{27(a_2-b_2+c_2)^3} - \tfrac{\alpha_2}{2(a_2-b_2+c_2)} +$$

$$+\tfrac{(a_2+2\alpha_2-\beta_2)(2a_2+\alpha_2-b_2-\beta_2)}{6(a_2-b_2+c_2)^2})^{\frac{1}{3}} - \tfrac{2a_2+\alpha_2-b_2-\beta_2}{3(a_2-b_2+c_2)} + (\tfrac{(2a_2+\alpha_2-b_2-\beta_2)^2}{9(a_2-b_2+c_2)^2} - \tfrac{a_2+2\alpha_2-\beta_2}{3(a_2-b_2+c_2)})/$$

$$/((((\tfrac{\alpha_2}{2(a_2-b_2+c_2)} + \tfrac{(2a_2+\alpha_2-b_2-\beta_2)^3}{27(a_2-b_2+c_2)^3} - \tfrac{(a_2+2\alpha_2-\beta_2)(2a_2+\alpha_2-b_2-\beta_2)}{6(a_2-b_2+c_2)^2})^2 -$$

$$-(\tfrac{(2a_2+\alpha_2-b_2-\beta_2)^2}{9(a_2-b_2+c_2)^2} - \tfrac{a_2+2\alpha_2-\beta_2}{3(a_2-b_2+c_2)})^3)^{\frac{1}{3}} - \tfrac{(2a_2+\alpha_2-b_2-\beta_2)^3}{27(a_2-b_2+c_2)^3} - \tfrac{\alpha_2}{2(a_2-b_2+c_2)} +$$

$$+\tfrac{(a_2+2\alpha_2-\beta_2)(2a_2+\alpha_2-b_2-\beta_2)}{6(a_2-b_2+c_2)^2})^{\frac{1}{3}}).$$

1.6 Lyapunov quantities and "small" limit cycles

If $L_{1,\dots,n-1} = 0$ and $L_n \neq 0$, then by use of the well-known Bautin's technique (see, for example, [Lynch, 2005]) one can construct in the general case n "small" limit cycles with the help of small perturbations of coefficients of system [Bautin, 1949, Bautin, 1952]. By formula (17) the ordinate of solution of considered system can be presented:

$$y(T(h), h) = h + L_1 h^3 + L_2 h^5 + \dots \tag{39}$$

Suppose that $L_1 = 0$ and the first nonzero Lyapunov quantity $L_2 > 0$. Following technique of N.N. Bautin , with the help of small disturbance of coefficients of considered system one can try for disturbed system the inequalities

$$\widetilde{L}_1 < 0, \quad \widetilde{L}_2 > 0, \qquad |\widetilde{L}_1| << |\widetilde{L}_2|$$

to be satisfied. Then for sufficiently small initial data $h = {r_0}^I$ the trajectories of the disturbed system are winded around a stationary point, while for certain initial data $h = {r_0}^{II}$ (${r_0}^{II} >> {r_0}^I$) the trajectories of the system are unwinded. Thus, for these perturbations one can obtain a "small" unstable limit cycle round zero equilibria.

Similarly, disturbing a few first Lyapunov quantities, in virtue of the smallness of perturbations and continuous dependence of solutions on parameter in the perturbed system one can obtain a few "small" limit cycles.

For example, for quadratic system this technique permits one to construct 3 "small" limit cycles if the coefficients of system are fitted in such a way that $L_{1,2} = 0$ and $L_3 \neq 0$. Recall that if the coefficients are fitted in such a way that $L_{1,2,3} = 0$, then $L_{4,5,\ldots} = 0$.

2 INVESTIGATION OF "LARGE" LIMIT CYCLES IN QUADRATIC SYSTEM

2.1 Introduction

The appearance of modern computers permits one to use numerical modeling of complicated nonlinear dynamical systems and to obtain new information on the structure of their trajectories. However the possibilities of a "simple" approach, based on the construction of trajectories by numerical integration of the considered differential equations, turned out to be highly limited [Arnold, 2005, PIV]. For the analytical investigation of bifurcations of limit cycles there exist different methods such as the investigation of Poincare mapping, the investigation of Poincare-Mel'nikov and Abel integrals, and the averaging method. However the "small" parameters, used for numerical construction of limit cycles on the basis of these methods, often makes the task of numerical analysis of limit cycles rather difficult, especially in the case of nested cycles. In the present work the method of disturbance of Lyapunov quantities together with the method of asymptotical integration [Leonov, 2009] allow one to obtain the conditions of existence of four limit cycles in quadratic systems: three large limit cycles in the case of a weak focus of first order, two large limit cycles in the case of a weak focus of second order, one large limit cycle in the case of a weak focus of third order. The existence conditions, obtained here, have very simple form and generalize widely known theorem of Shi [Shi, 1980]. The development of the method of asymptotical integration permits one to find an example of a quadratic system, for which four large limit cycles can be computed.

2.2 One and two "large" limit cycles in quadratic system

Consider the reduced form of quadratic system (36). For investigation of large limit cycles, following [Leonov & Kuznetsova, 2010], one can make use of the reduction of quadratic system (36) to Lienard system of special form with discontinuous right-hand side (38), which permits one to apply the method of asymptotic integration of trajectories. This method is described in more detail in the work [Leonov & Kuznetsova, 2010].

Further we give the main results on the existence of large limit cycles, obtained in the present work.

2.2.1 Criteria of existence of one and two large limit cycles

Theorem 1 [Leonov & Kuznetsova, 2010] *Suppose that the following conditions on the coefficients of system (36) are satisfied:*

$$4a_2(c_2 - 1) > (b_2 - 1)^2, \tag{40}$$

$$b_2 c_2 > 1, \tag{41}$$

$$b_2 \in (1, 3), \tag{42}$$

$$c_2 \in (1/3, 1), \tag{43}$$

and it is used small disturbance of parameters

$$\begin{aligned} &\beta_2 \in (0, \varepsilon) \\ &\alpha_2 \in \left(\frac{a_2(2+b_2)}{b_2c_2 - 1}, \frac{a_2(2+b_2)}{b_2c_2 - 1} + \delta \right), \end{aligned} \tag{44}$$

where $0 < \varepsilon \ll \delta \ll 1$*, then in system (36) there exist 4 limit cycles (2 "small" and 2 "large").*

Note that the domain of parameters Ω, defined by these conditions, has infinite Lebesgue measure. However this domain is small with respect to the parameters β_2 and α_2.

In the limit case when $b_2 = 3$, around zero the third "small" cycle is in place of a "large" cycle. Then, following the line of reasoning in the work [Leonov & Kuznetsova, 2010], we obtain that system (36) has 4 limit cycles (3 "small" at zero point and 1 "large" at the point $x_1 < -1$) if conditions (40), (43), (44) are satisfied and

$$b_2 \in (3 - \mu, 3). \tag{45}$$

Here $1 \gg \mu \gg \delta \gg \varepsilon \geq 0$.

The domain of undisturbed parameters under the above conditions is two-dimensional. This domain has the form

$$\{c_2 \in (1/3, 1), \quad a_2(c_2 - 1) > 1\}. \tag{46}$$

In Fig. 8 are shown the domain, described by the well-known Shi theorem [Shi, 1980] (hatched), and the domain (46)(grey). It corresponds to the existence of one "large" and three "small" limit cycles.

Thus, the domain obtained covers the Shi domain entirely and coincides with the domain obtained analytically for the same case in the work [Artes & Llibre, 1997]. This confirms the efficiency of the method of asymptotic integration.

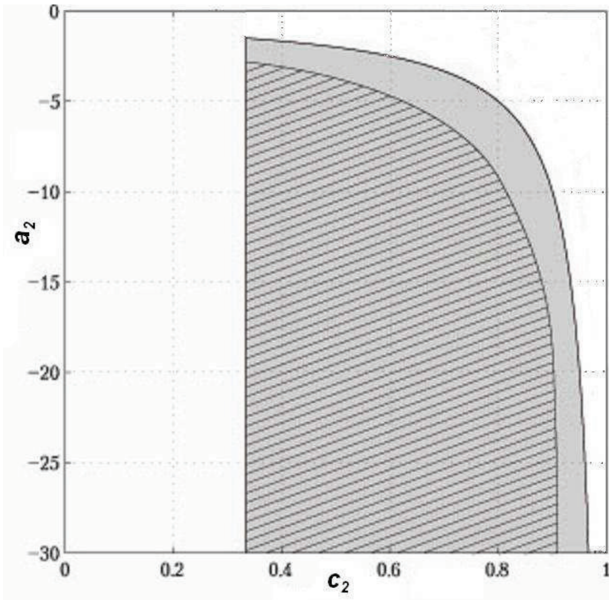

FIGURE 8 Domain of parameters corresponding to the existence of one "large" limit cycle.

Below are given the illustrations of one "large" limit cycle at the point $x_1 < -1$ (by green color). In Fig. 9, 10, 11 are shown "large" cycles for quadratic systems with parameters

$$\alpha_2 = \frac{a_2(2+b_2)}{b_2c_2-1}, \quad \beta_2 = 0, \quad b_2 = 3$$

and

$$c_2 = 0.5, \quad a_2 = \frac{(b_2-1)^2}{4(c_2-1)} - 1;$$

$$c_2 = 0.7, \quad a_2 = \frac{(b_2-1)^2}{4(c_2-1)} - 10;$$

$$c_2 = 0.9, \quad a_2 = \frac{(b_2-1)^2}{4(c_2-1)} - 30,$$

respectively.

In Fig. 12, 13, 14, 15 are shown 2 "large" cycles in quadratic systems, corresponding to the following coefficients

$$\alpha_2 = \frac{a_2(2+b_2)}{b_2c_2-1}, \quad \beta_2 = 0, \quad a_2 = \frac{(b_2-1)^2}{4(c_2-1)} - 10, \quad c_2 = \frac{1}{b_2} + 0.1$$

and

$$b_2 = 2; 1.7; 1.5; 1.3,$$

respectively.

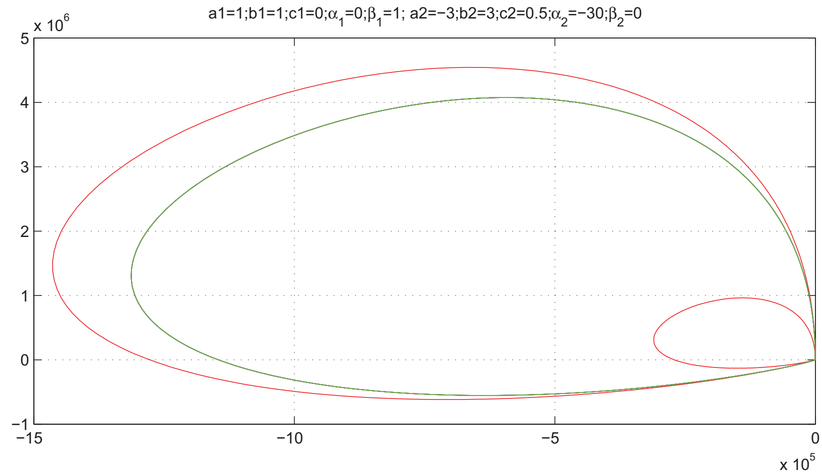

FIGURE 9 Visualization of one large limit cycle. $c_2 = 0.5$.

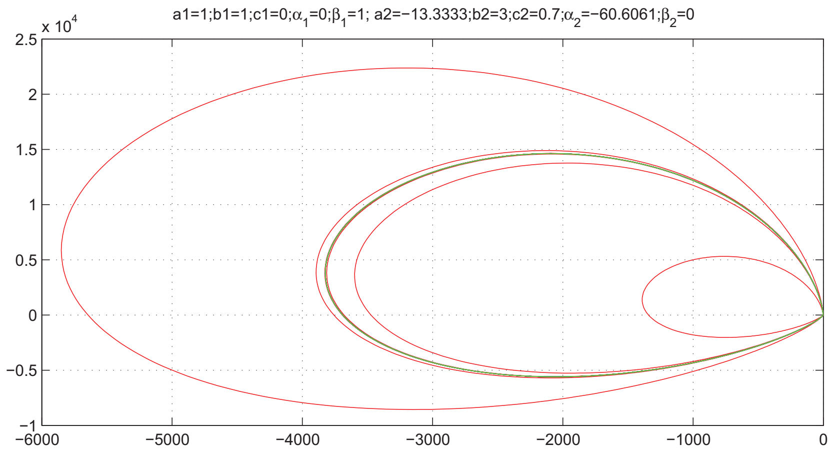

FIGURE 10 Visualization of one large limit cycle. $c_2 = 0.7$.

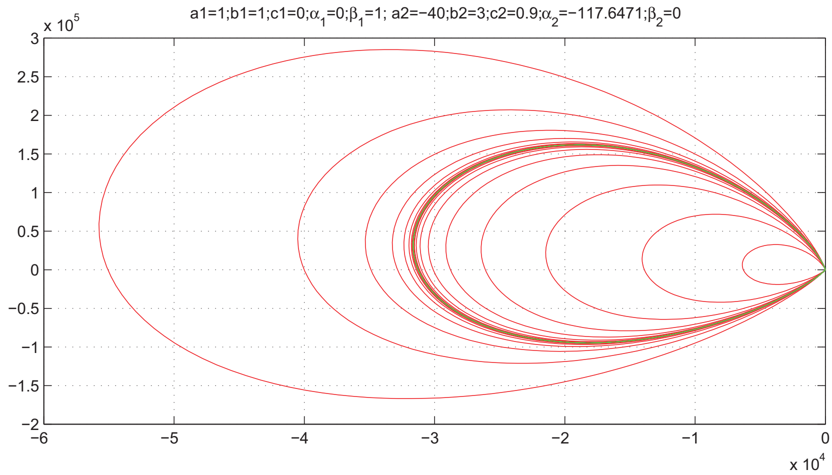

FIGURE 11 Visualization of one large limit cycle. $c_2 = 0.9$.

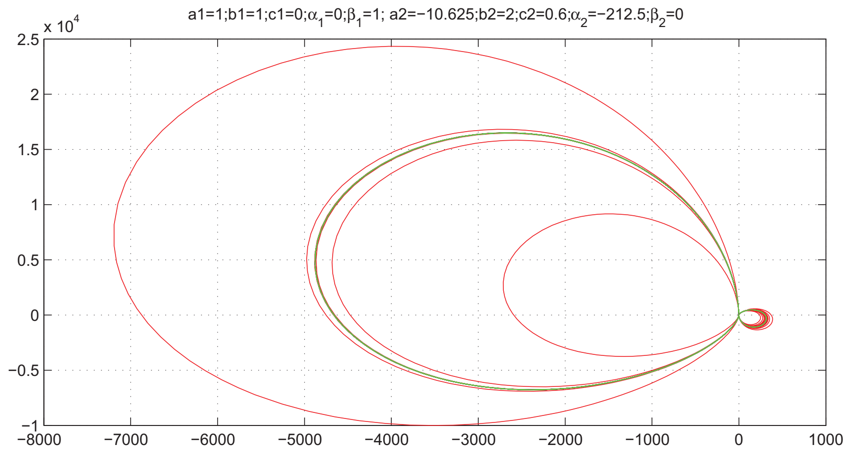

FIGURE 12 Visualization of two large limit cycles. $b_2 = 2$.

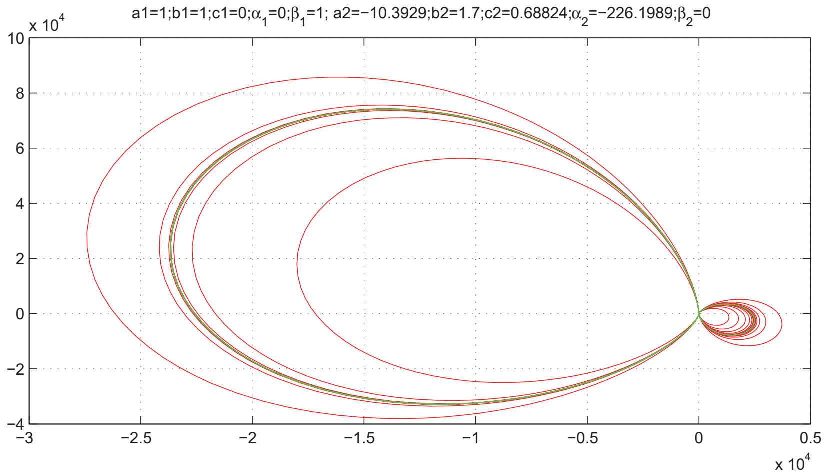

FIGURE 13 Visualization of two large limit cycles. $b_2 = 1.7$.

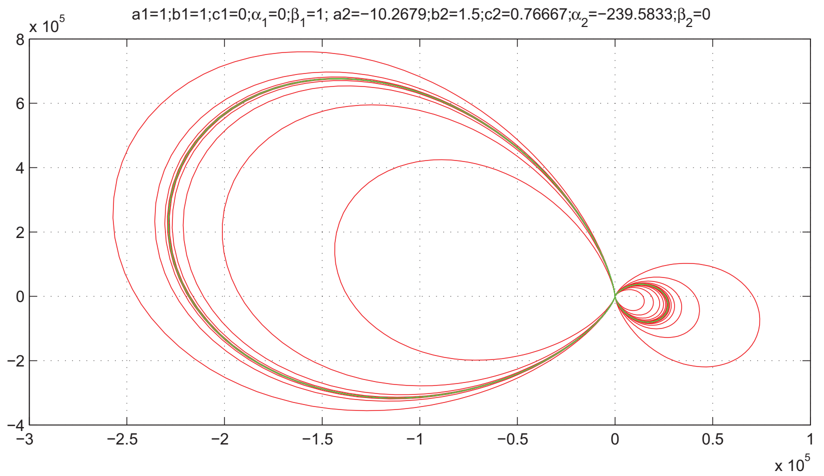

FIGURE 14 Visualization of two large limit cycles. $b_2 = 1.5$.

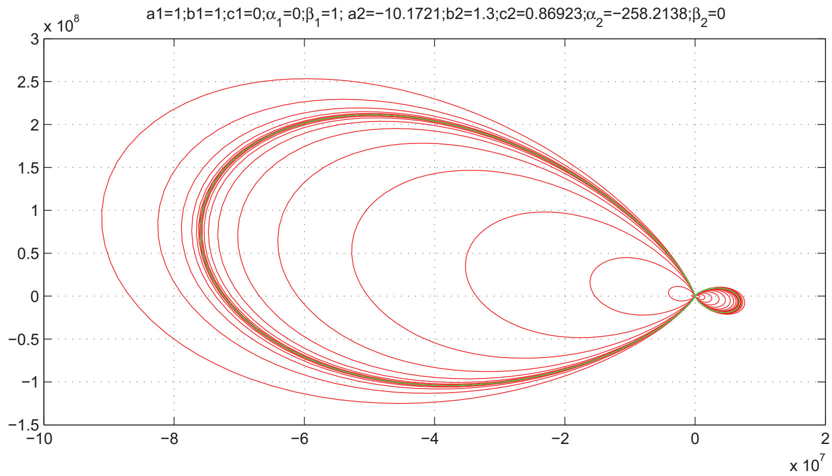

FIGURE 15 Visualization of two large limit cycles. $b_2 = 1.3$.

Here in Fig. 12, 13, 14, 15 are shown, by green color, 1 stable (on the right) and 1 unstable (on the left) "large" limit cycles. Besides, 2 "small" cycles around zero point can be obtained by small disturbances of parameters of the system. The figures show that in the case when the parameter b_2 approaches 3 the "large" cycle on the right gets smaller and becomes "small" limit cycle.

By Theorem 1 the domain of undisturbed coefficients, corresponding to the existence of four limit cycles, can be represented as $\{b_2 \in (1,3), \quad c_2 \in (\frac{1}{b_2}, 1), \quad a_2 < \frac{(b_2-1)^2}{4(c_2-1)}\}$. The projection of this domain on the plane (a_2, c_2) (see Fig. 16) can be described by the following inequalities $\{a_2 < \frac{(c_2-1)}{4c_2^2}, c_2 \in (1/3, 1)\}$. It means that for each pair of the coefficients (a_2, c_2) from this domain, one can construct 4 limit cycles (2 "large" and 2 "small") for certain b_2, namely b_2, satisfying the following conditions

$$\frac{1}{c_2} < b_2 < 2\sqrt{m} + 1, \quad m = \min(1, a_2(c_2 - 1)).$$

2.3 Three and four "large" limit cycles in quadratic system

The development of the method of asymptotic integration and the series of numerical experiments permitted one to obtain the domain of coefficients of quadratic system for which 3 "large" and one "small" limit cycles may exist. The different illustrations of "large" cycles, obtained in numerical experiments, correspond to different sets of coefficients from the obtained domain and will be given below.

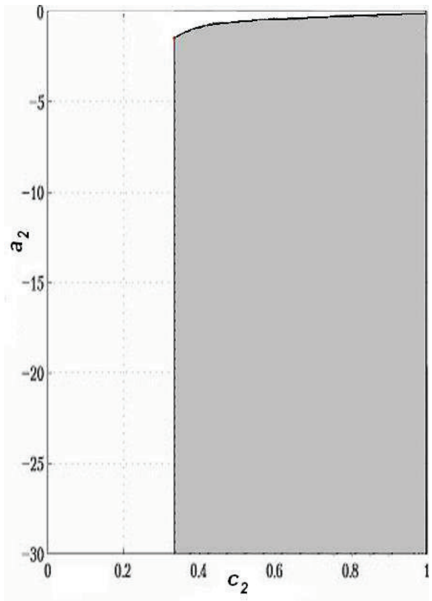

FIGURE 16 Projection of three-dimensional domain of the existence of two "large" limit cycles.

The disturbance of one more coefficient of system permits one to obtain the configuration of four "large" limit cycles. The illustrations, obtained in Matlab, corresponding to this configuration of cycles, demonstrate so-called "cycles dance". They are shown below.

2.3.1 Existence of three "large" limit cycles in quadratic system

Proceed to consider system (36). Criterion of existence of 2 "small" and 2 "large" limit cycles was formulated above (see Theorem 1).

In the work [Leonov, 2010[1]] there was formulated the following criterion of the existence of 1 "small" and 3 "large" limit cycles:

System (36) has 3 "large" limit cycles if it is satisfied the following conditions
$\alpha_2 = -\varepsilon^{-1}, \quad \beta_2 = 0, \quad c_2 \in (\frac{1}{3}, \frac{1}{2}), \quad b_2 > a_2 + c_2, \quad 2c_2 \leq b_2 + 1,$
$4a_2(c_2 - 1) > (b_2 - 1)^2, \quad b_2c_2 < 1,$ *where* $1 \gg \varepsilon \geq 0$.

Besides, 1 "small" limit cycle can be obtained around the origin of coordinates by a disturbance of β_2.

Note that the condition $b_2c_2 < 1$, corresponding to that the inequality $L_1 < 0$ is satisfied, is not required.

Recall that $L_1(0) = \frac{-\pi}{4(\alpha_2)^{\frac{5}{2}}}(\alpha_2(b_2c_2 - 1) - a_2(b_2 + 2))$. Here the condition $L_1 < 0$ is satisfied for either $b_2c_2 < 1, \quad \alpha_2 < 0$, either $b_2c_2 > 1, \quad \alpha_2 < 0, \quad \alpha_2 > \frac{a_2(2+b_2)}{b_2c_2-1}$.

Note also that the condition $b_2 > a_2 + c_2$ is inessential requirement since it results from other conditions.

Further in the process of numerical experiments the question was considered on the possible enlargement of the domain of coefficients $(b_2,\ c_2)$, which correspond to the existence of 3 "large" and 1 "small" limit cycles. According to these experiments, the conditions $2c_2 \le b_2 + 1$ and $c_2 > \frac{1}{3}$ are necessary. While the condition $c_2 > \frac{1}{2}$ is not necessary. In this work it is experimentally obtained that for $c_2 < 1$ and $b_2 < 3$ three "large" and 1 "small" limit cycles can be found. The illustrations of cycles, corresponding to the obtained domain, see below.

Thus, new conditions of the existence of 4 limit cycles (3 "large" and 1 "small"), obtained by means of numerical-analytical approach, can be formulated in the following way:

System (36) has 3 "large" limit cycles if the following conditions are satisfied: $\beta_2 = 0,\quad c_2 \in (\frac{1}{3}, 1), c_2 \neq \frac{1}{2},\quad b_2 < 3,\quad 2c_2 \le b_2 + 1, 4a_2(c_2 - 1) > (b_2 - 1)^2$, *and either* $b_2c_2 < 1$, *or* $b_2c_2 > 1$ *and* $\frac{a_2(2+b_2)}{b_2c_2-1} < \alpha_2$.
Here $\alpha_2 < 0$ *and* $|\alpha_2|$ *is sufficiently large. Besides, 1 "small" limit cycle can be obtained around the origin of coordinates in the case of a disturbance of parameter* β_2.

The projection of the above described domain of coefficients on the plane $(b_2,\ c_2)$ is shown in Fig. 17.

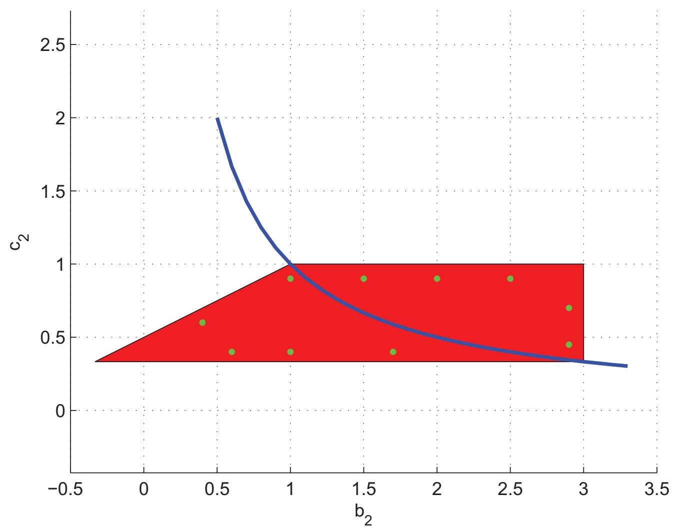

FIGURE 17 Projection of four-dimensional domain of the existence of three "large" limit cycles.

Below are given the illustrations (obtained in the numerical experiments), corresponding to two "large" cycles around zero point and one "large" cycle to the left of -1. The illustrations are given for different pairs of coefficients $(b_2,\ c_2)$, corresponding to the points lying near the boundaries of domain. The points are shown in Fig. 17 in green color.

See Fig. 18, 19, 20, 21, 22, 23. In this case the coefficients (a_2, α_2) were chosen with provision for all the rest of conditions, and β_2 is equal to zero.

2.3.2 Visualization of four "large" limit cycles and "cycles dance"

The numerical experiments result in the additional disturbance of β_2 permitting one to obtain the third "large" cycle (in place of "small" cycle) around zero. For example [Kuznetsov & Kuznetsova, Leonov, 2011], in Fig. 24, where for the set of coefficients $b_2 = 2.2$, $c_2 = 0.7$ it is chosen the coefficients $a_2 = -10$, $\alpha_2 = \dfrac{a_2(2+b_2)}{(b_2c_2-1)} + 5 = -72.777...$, $\beta_2 = 0$, one can observe 3 "large" limit cycles (2 "large" cycles around zero and 1 "large" cycle to the left of -1).

In Fig. 25 for the same set of coefficients choosing $\beta_2 = 0.0015$, one can observe 3 "large" cycles around zero and 1 "large" cycle to the left of -1.

In the present work different scenarios of so-called "appearance" and "degeneracy" of limit cycles were also studied. For this purpose it was considered "cycles dance", i.e. the varying of configuration of cycles in system for sequential increasing each of 5 given coefficients. A set of coefficients $a_2 = -10, b_2 = 2, c_2 = 0.4, \alpha_2 = -1950, \beta_2 = 0.13$ were chosen for the original set, which, in accordance to numerical-analytical investigation, corresponds to the configuration of four "large" cycles. The following results were obtained: for different coefficients, the radiuses of possible variation of the value, for which it is observed configuration of four cycles, are distinct from each other. For example, for c_2 it is approximately 0.002, for a_2 approximately 1, for α_2 approximately 200. And for the coefficients b_2 and β_2, the attempts to increase the value have practically not met with success and possible values of the coefficients can decrease only: for b_2 it is approximately 0.03, for β_2 approximately 0.05.

Remark. By reason of necessity to fit the time and the initial data of trajectories, the fulfilling of numerical experiments with visualization of the cycles is a highly laborious process. At present, by using even sufficiently large step (of order 0.1), the time, which is necessary for the check of the presence of cycles in five-dimensional space, is many millions years! This arrives to conclusion that for investigation of all five-dimensional space of coefficients of quadratic systems it is necessary to develop analytical methods.

Below are given illustrations which show the varying of limit cycles configuration, in which one of the 5 coefficients is increased and the remainder remain fixed.

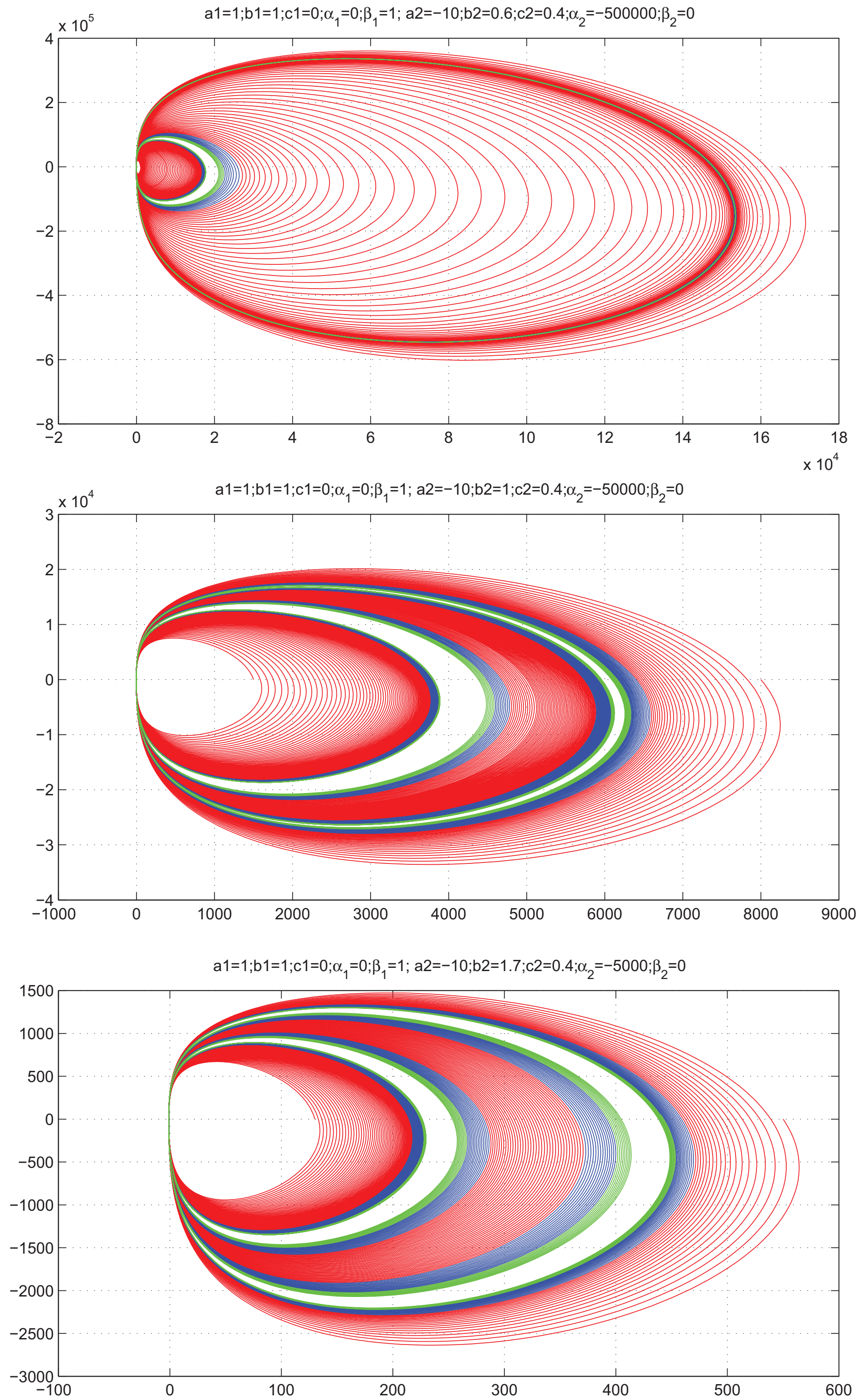

FIGURE 18 Visualization of two "large" limit cycles around zero equilibrium. b_2=0.6, c_2=0.4; b_2=1, c_2=0.4; b_2=1.7, c_2=0.4.

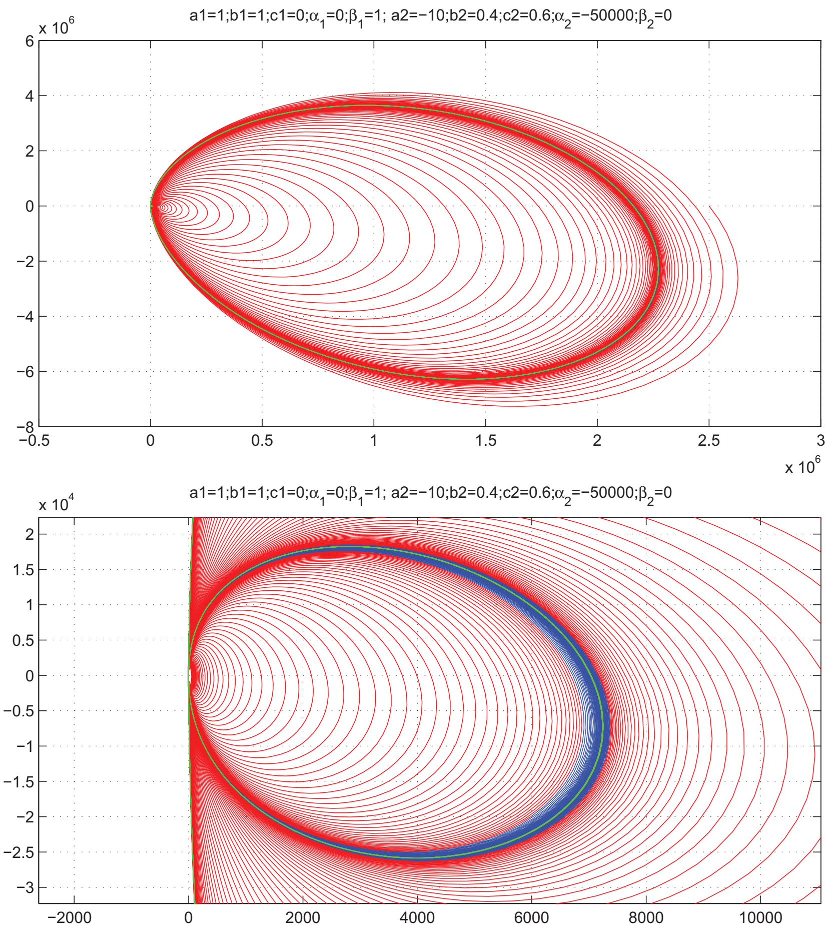

FIGURE 19 Visualization of two "large" limit cycles around zero equilibrium. b_2=0.4, c_2=0.6 (in lower figure is shown one of the cycles scaled-up).

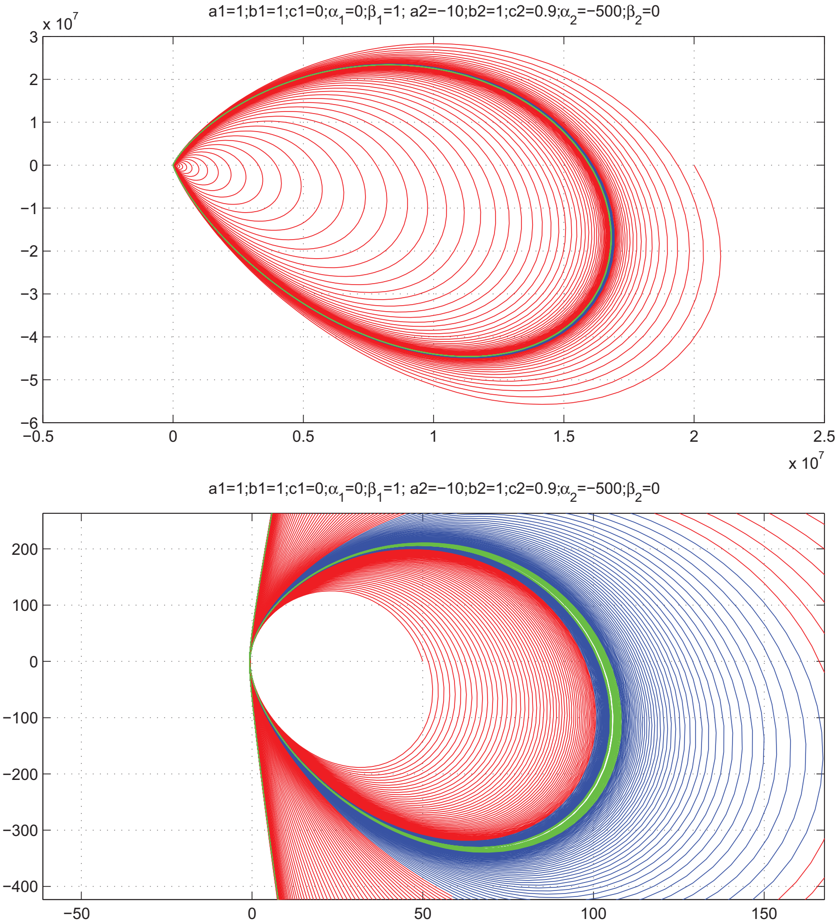

FIGURE 20 Visualization of two "large" limit cycles around zero equilibrium. b_2=1, c_2=0.9 (in lower figure is shown one of the cycles scaled-up).

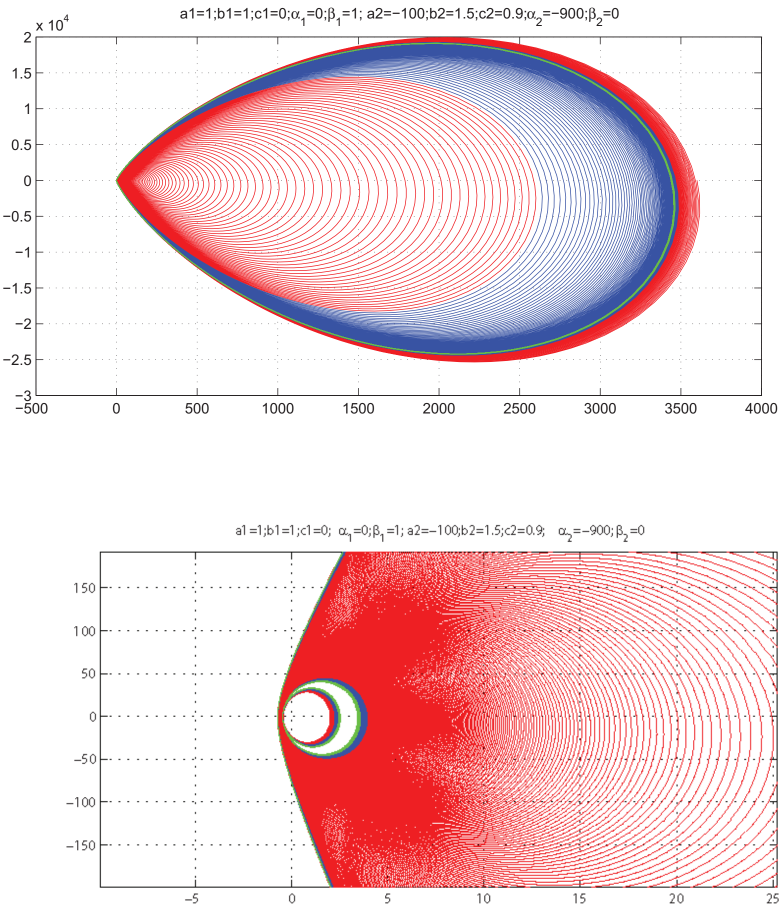

FIGURE 21 Visualization of two "large" limit cycles around zero equilibrium. b_2=1.5, c_2=0.9. (in lower figure is shown one of the cycles scaled-up).

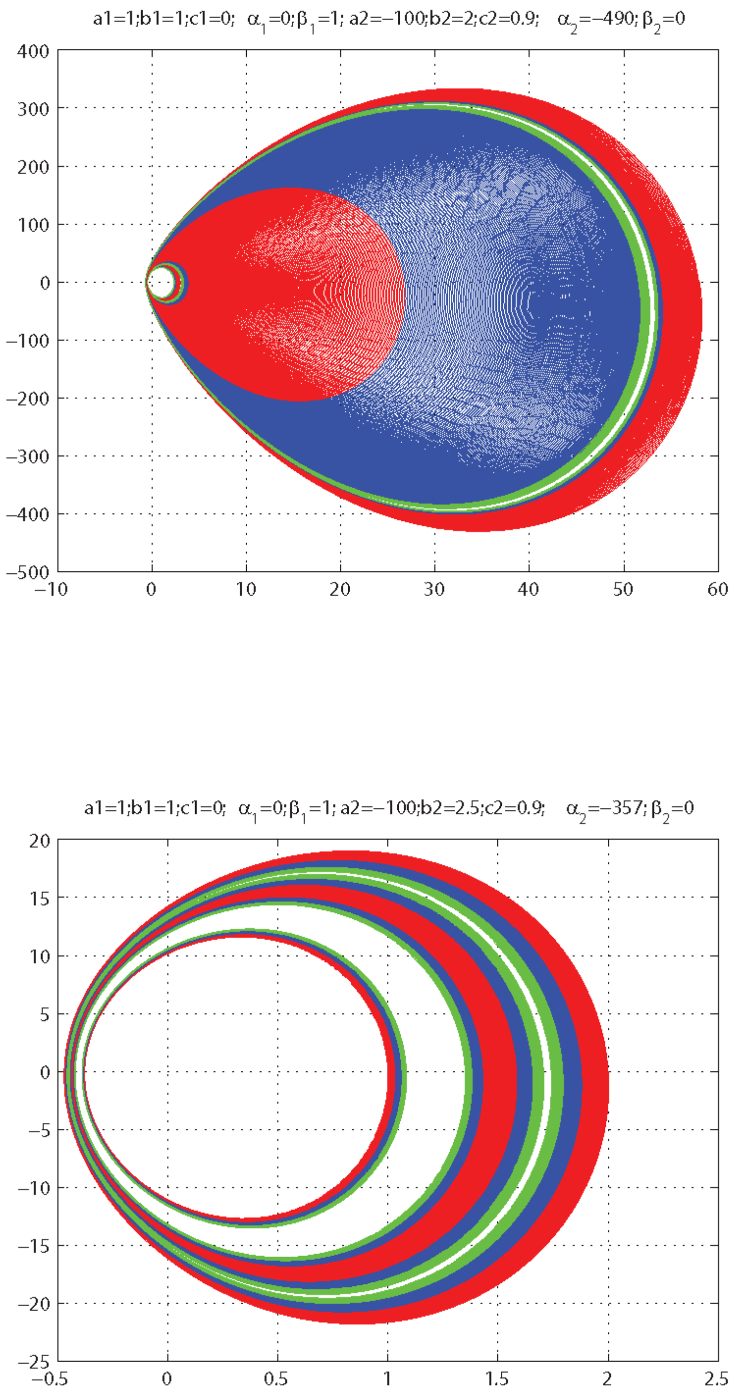

FIGURE 22 Visualization of two "large" limit cycles around zero equilibrium. b_2=2.0, c_2=0.9; b_2=2.5, c_2=0.9.

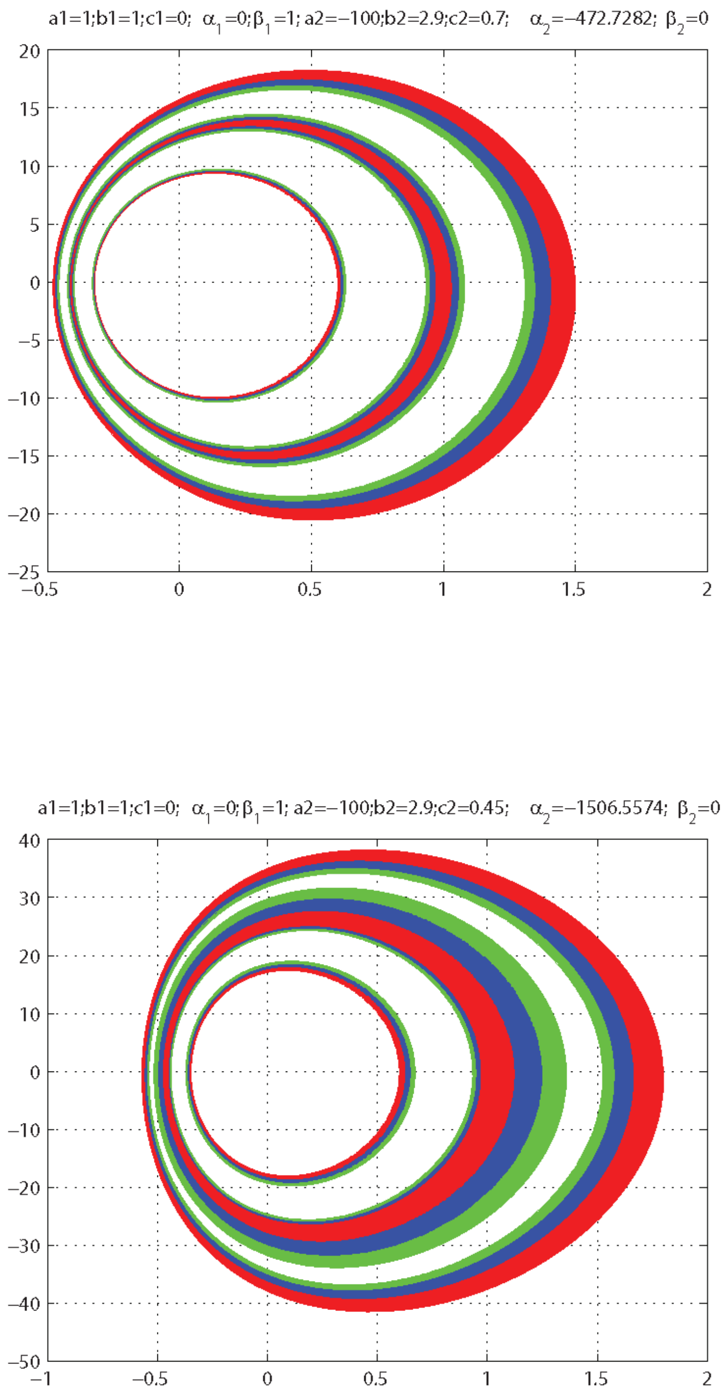

FIGURE 23 Visualization of two "large" limit cycles around zero equilibrium. b_2=2.9, c_2=0.7; b_2=2.9, c_2=0.45.

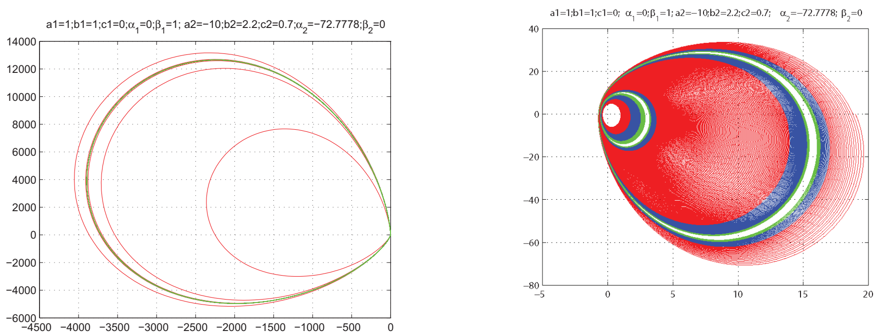

FIGURE 24 Visualization of three "large" cycles. b_2=2.2, c_2=0.7, $\beta_2 = 0$.

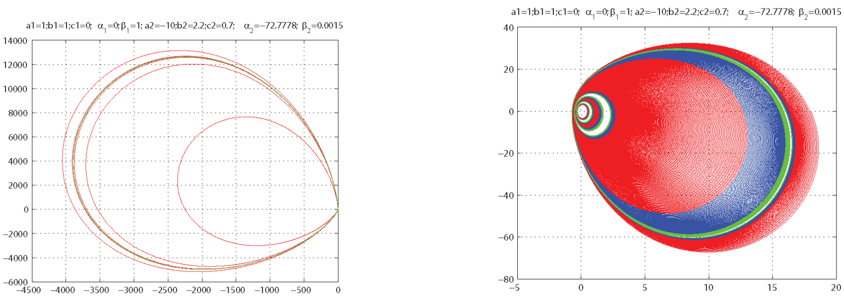

FIGURE 25 Visualization of four "large" cycles. b_2=2.2, c_2=0.7, $\beta_2 = 0.0015$.

Varying of the coefficient c_2.

The coefficient c_2 is varying from 0.395 to 0.405 under the condition that the rest of coefficients are fixed. For $c_2 = 0.395$ it can be constructed only 1 "large" limit cycle, for $c_2 = 0.398$ to inner cycle it is added 2 more cycles, which are yet preserved for $c_2 = 0.401$ (but they are already rather close to one another). For $c_2 = 0.405$ it remains only 1 "large" limit cycle (2 inner cycles disappear). See Fig. 26.

Varying of the coefficient α_2.

The coefficient α_2 is varying from -2300 to -1750 under the condition that the rest of coefficients are fixed. For $\alpha_2 = -1750$ it can be constructed only 1 "large" limit cycle, for $\alpha_2 = -1800$ to inner cycle it is added 2 more limit cycles (all 3 cycles are distinctly observed), for $\alpha_2 = -2200$ 3 cycles are yet observed, while for $\alpha_2 = -2300$ it remains only 1 "large" limit cycle (2 inner cycles disappear). See Fig. 27.

Varying of the coefficient a_2.

The coefficient a_2 is varying from -11.1 to -8.6 under the condition that the rest of coefficients are fixed. For $a_2 = -11.1$ it can be constructed only 1 "large" limit cycle. For $a_2 = -11$ three cycles (2 outer cycles are added) are observed. For $a_2 = -8.9$ it is yet preserved the configuration of three cycles. But for $a_2 = -8.5$ it is observed only 1 cycle, the largest of three. See Fig. 28.

Varying of the coefficient b_2.

The coefficient b_2 is varying from 1.96 to 2.02 under the condition that the rest of coefficients are fixed. For $b_2 = 1.96$ it is observed only 1 "large" limit cycle. For $b_2 = 1.97$ it is already observed configuration of three "large" cycles around zero (2 outer cycles are added) and all 3 cycle are distinctly observed. For $b_2 = 2.01$ it is yet observed configuration of three "large" cycles around zero, but for $b_2 = 2.02$ it is observed only 1 (outer) "large" limit cycle. See Fig. 29.

Varying of the coefficient β_2.

The coefficient β_2 is varying from 0.07 to 0.15 under the condition that the rest of coefficients are fixed. For $\beta_2 = 0.07$ it is observed only 1 "large" limit cycle. For $\beta_2 = 0.08$ it is already observed the configuration of three "large" cycles around zero (2 outer cycles are added) and all 3 cycles are distinctly observed. For $\beta_2 = 0.14$ it is yet observed configuration of three "large" cycles around zero (2 inner cycles are rather close to one another). But for $\beta_2 = 0.15$ it is already observed only 1 (outer) "large" limit cycle. See Fig. 30.

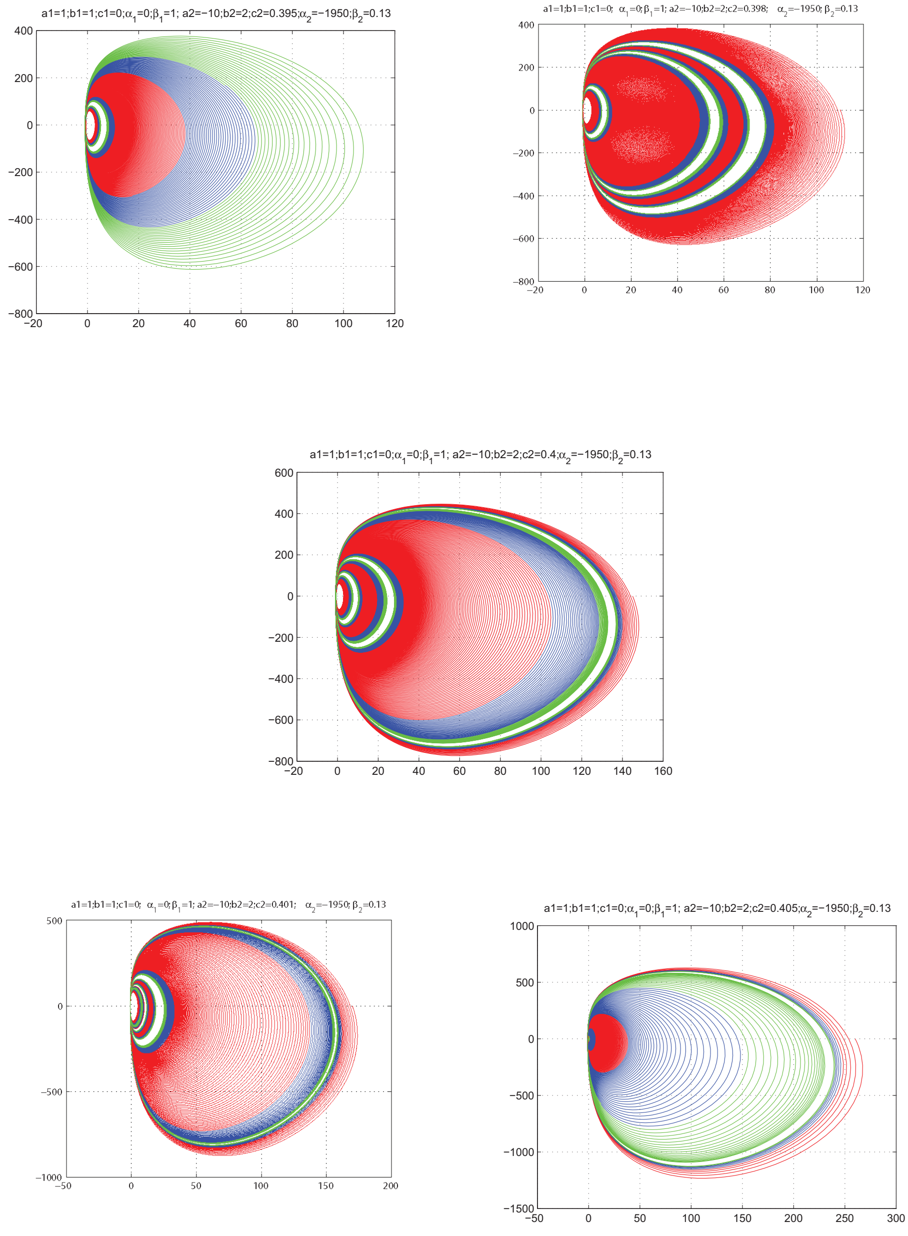

FIGURE 26 "Cycles dance" under the varying of coefficient c_2.

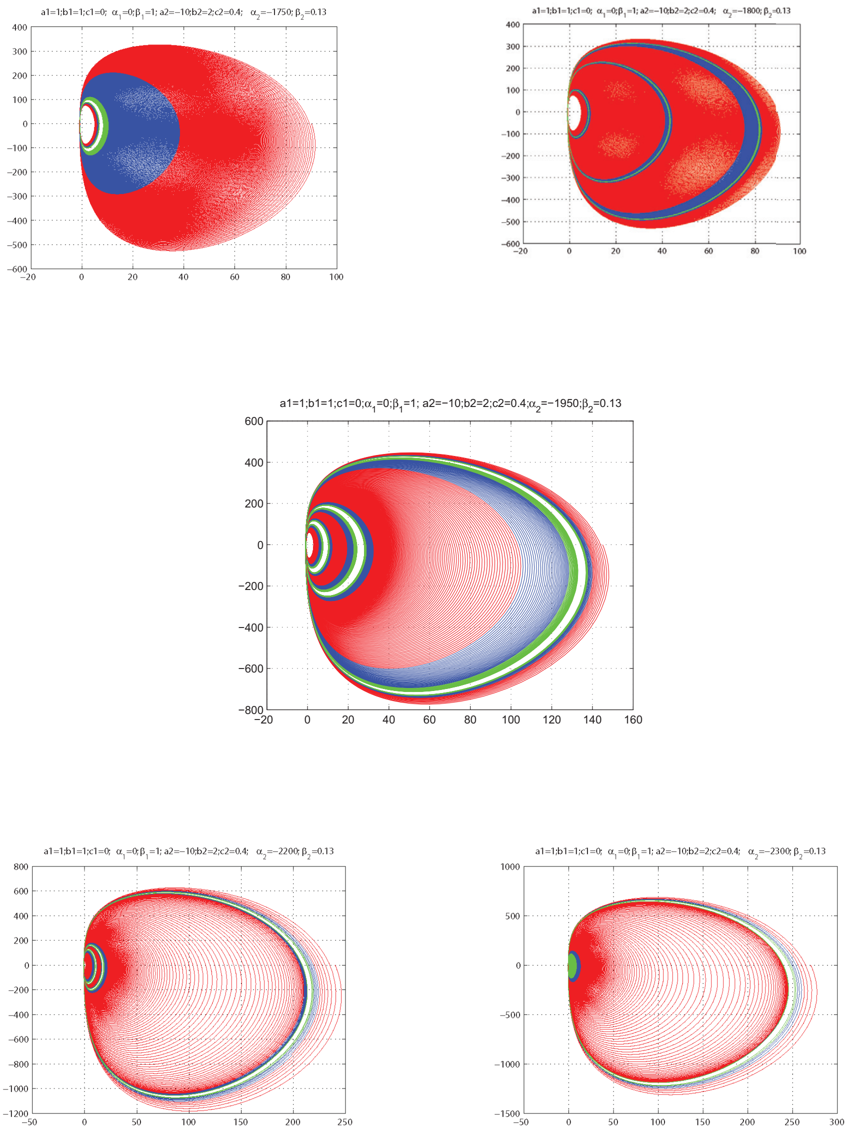

FIGURE 27 "Cycles dance" under the varying of coefficient α_2.

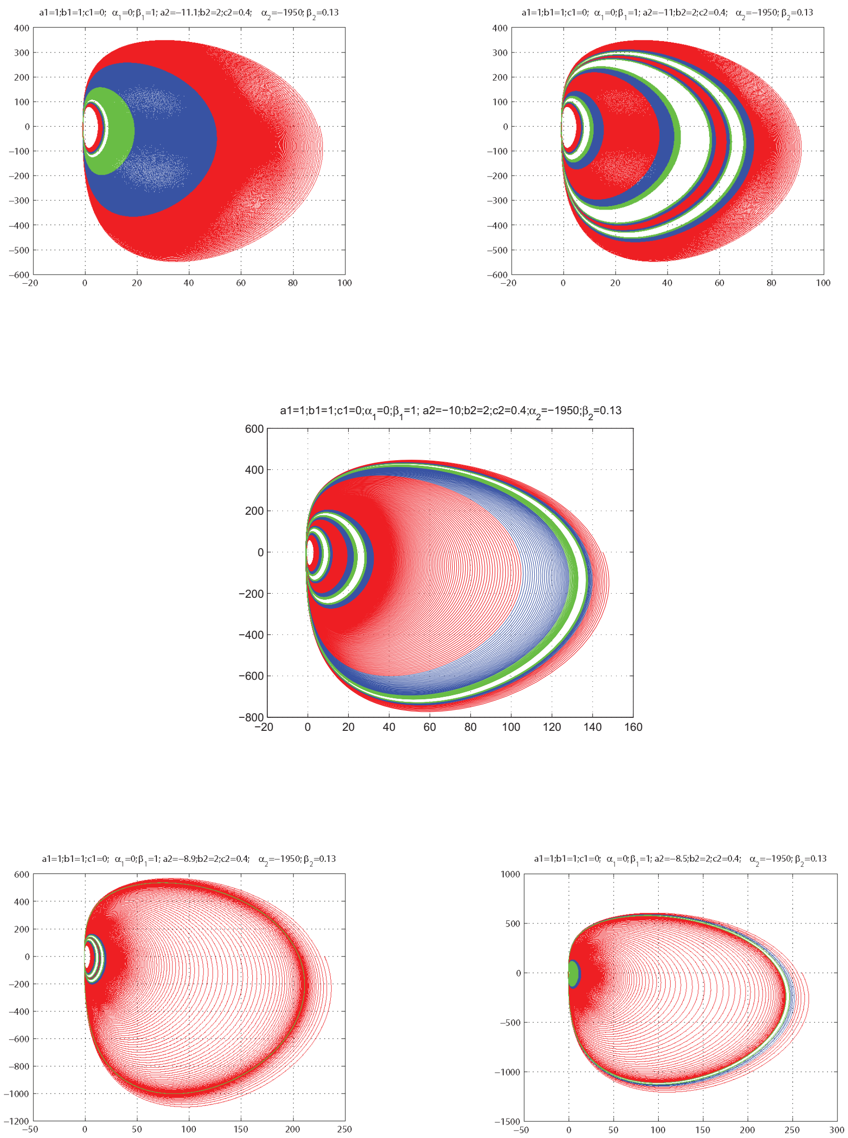

FIGURE 28 "Cycles dance" under the varying of coefficient a_2.

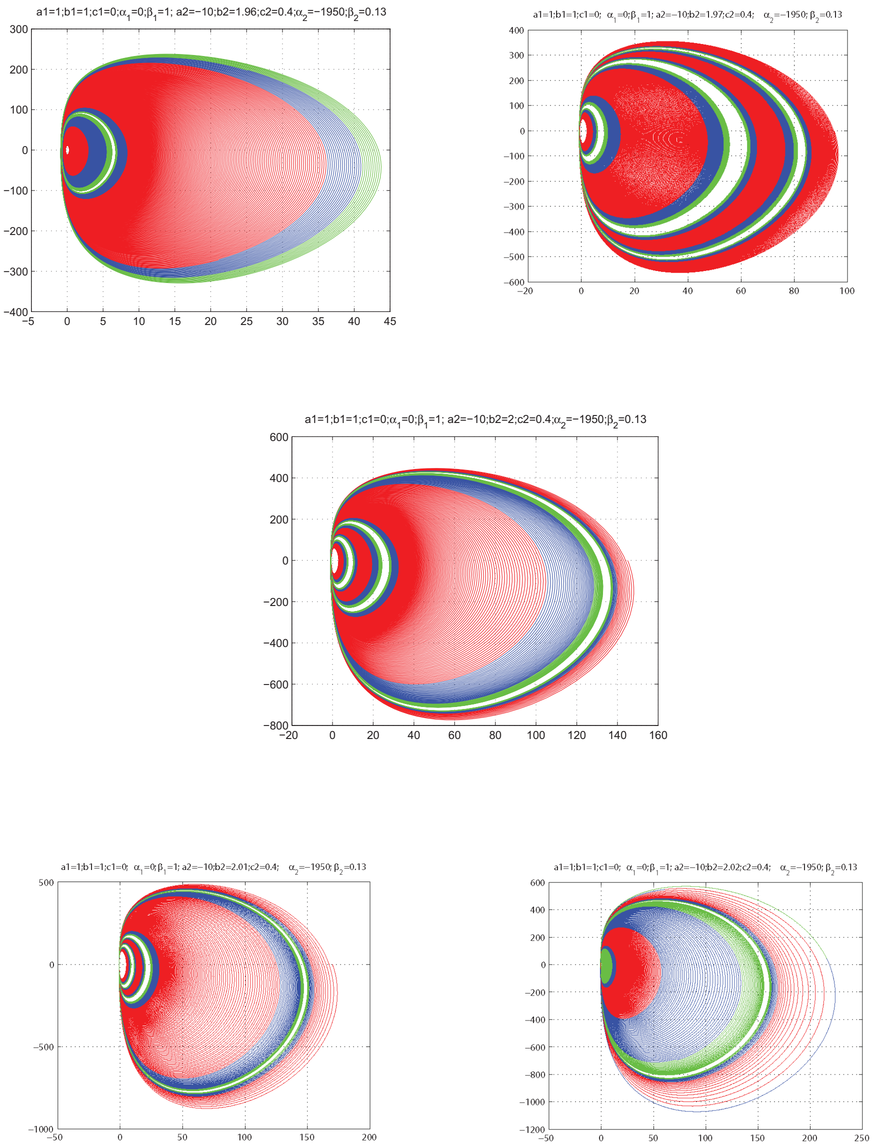

FIGURE 29 "Cycles dance" under the varying of coefficient b_2.

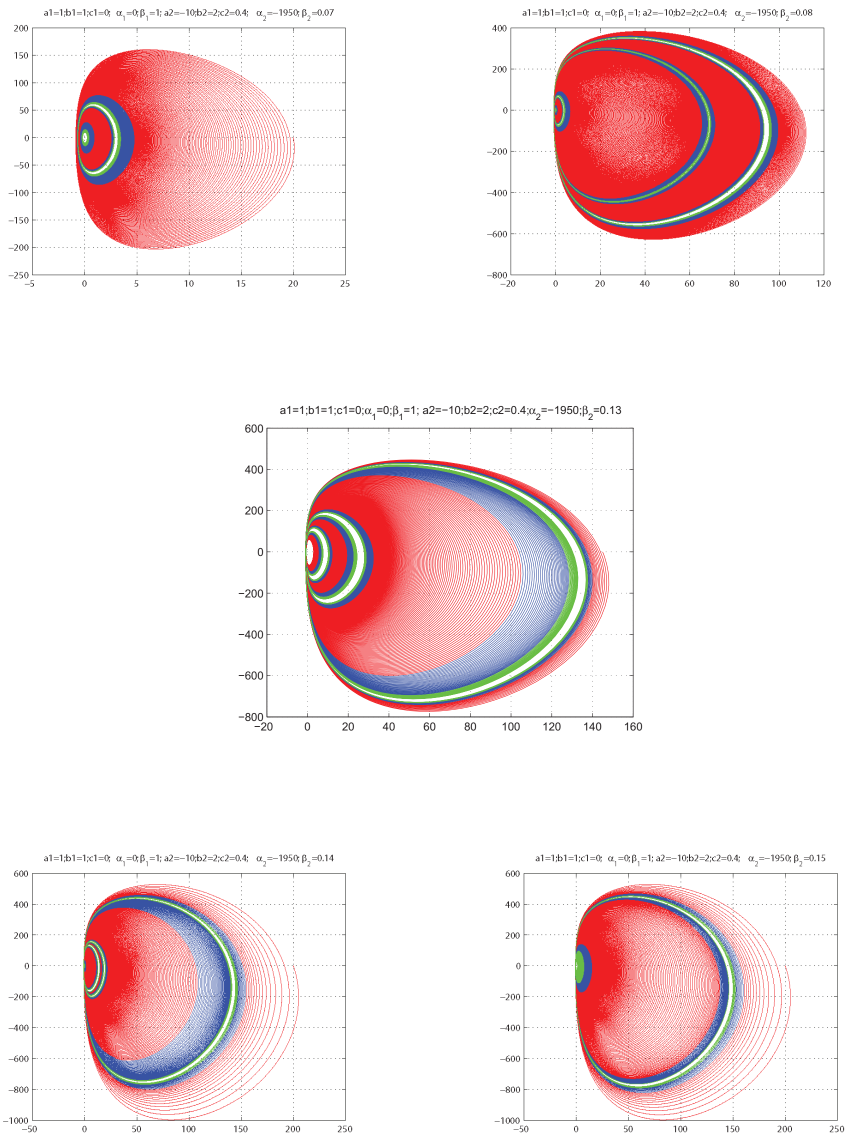

FIGURE 30 "Cycles dance" under the varying of coefficient β_2.

2.4 Matlab Programming code, used in the work for visualization of large limit cycles

For the work it was constructed a number of illustrations of "large" limit cycles. This became possible by virtue of the obtained analytical results and also the numerical algorithms, constructed on their basis in Matlab.

Below is given a simple function, which realizes the construction of three "large" limit cycles around zero point in computation software tool Matlab.

```
function f=fcyclesPlot(a1,b1,c1,al1,bt1,a2,b2,c2,al2,bt2, ->
    x0_1,x0_2,x0_3,x0_4,acc,len1,len2,len3,len4,len5,len6)
  syms x
         %create the title of the figure
  title_coefQs =['a1=',num2str(a1),';b1=',num2str(b1), ->
       ';c1=',num2str(c1),';\alpha_1=',num2str(al1), ->
       ';\beta_1=',num2str(bt1),';a2=',num2str(a2),';b2=',->
       num2str(b2),';c2= ',num2str(c2),';\alpha_2=', ->
       num2str(al2), ';\beta_2=',num2str(bt2)];

  RelTol = acc; AbsTol = acc; InitialStep = acc; %set accuracy
  options = odeset('RelTol',RelTol,'AbsTol',AbsTol, ->
      'InitialStep',InitialStep, 'NormControl','on');
       %draw trajectories, defining the location of 3 cycles
  x0=x0_1;y0=0;[T,XY]=ode45(@fQsys,[0 len1],[x0 y0],options);
       fPlotTrajectory(XY(:,1),XY(:,2),'red','blue','green');
       hold on; grid on;
  x0=x0_2;y0=0;[T,XY]=ode45(@fQsys,[0 len2],[x0 y0],options);
       fPlotTrajectory(XY(:,1),XY(:,2),'red','blue','green');
       hold on; grid on;

  x0=x0_2;y0=0;[T,XY]=ode45(@fQsys,[0 len3],[x0 y0],options);
       fPlotTrajectory(XY(:,1),XY(:,2),'red','blue','green');
       hold on; grid on;
  x0=x0_3;y0=0;[T,XY]=ode45(@fQsys,[0 len4],[x0 y0],options);
       fPlotTrajectory(XY(:,1),XY(:,2),'red','blue','green');
       hold on; grid on;

  x0=x0_3;y0=0;[T,XY]=ode45(@fQsys,[0 len5],[x0 y0],options);
       fPlotTrajectory(XY(:,1),XY(:,2),'red','blue','green' );
       hold on; grid on;
  x0=x0_4;y0=0;[T,XY]=ode45(@fQsys,[0 len6],[x0 y0],options);
       fPlotTrajectory(XY(:,1),XY(:,2),'red','blue','green' );
       hold on; grid on;

  title({title_coefQs})
```

In this code it is called the functions fQsys and fPlotTrajectory, which are the following:

```
function f = fPlotTrajectory(X,Y,Color1,Color2,Color3)
        %length of trajectories
  lenTr = length(X); lenTr3 = round(lenTr/9);
  lenColor1 = round(4*lenTr3); lenColor2 =  round(7*lenTr3);
```

```
            %draw trajectory in three colors
  plot(X(1 :lenColor1),Y(1 :lenColor1),Color1); hold on;
  plot(X(lenColor1:lenColor2),Y(lenColor1:lenColor2),Color2);
  hold on;
  plot(X(lenColor2:length(X)),Y(lenColor2:length(Y)),Color3);
```

```
function dz = fQsys(t,z) %form system of equations
  global a1 b1 c1 al1 bt1 a2 b2 c2 al2 bt2
  dz = zeros(2,1); % z = ( z(1),z(2) ) = (x,y)

  dz(1) = (a1*z(1)^2+b1*z(1)*z(2)+c1*z(2)^2+al1*z(1)+bt1*z(2));
  dz(2) =(a2*z(1)^2+b2*z(1)*z(2)+c2*z(2)^2+al2*z(1)+bt2*z(2));
```

For example, for the set of parameters

$a1 = 1$; $b1 = 1$; $c1 = 0$; $al1 = 0$; $bt1 = 1$; $a2 = -10$; $b2 = 2.7$; $c2 = 0.4$; $al2 = -437.5$; $bt2 = 0.003$; $x0\_1 = 1$; $x0\_2 = 3$; $x0\_3 = 5.5$; $x0\_4 = 10$; $acc = 1 * 10^{-6}$; $len1 = 180\pi$; $len2 = 380\pi$; $len3 = -150\pi$; $len4 = -150\pi$; $len5 = 130\pi$; $len6 = 80\pi$
the result of the work of function *fcyclesPlot* is given by the following illustration (see Fig. 31).

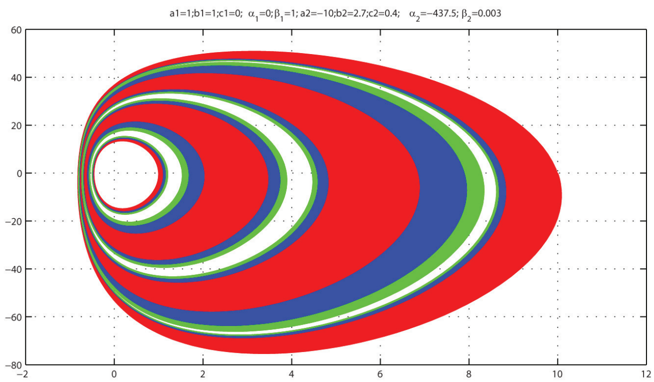

FIGURE 31 Example of the result of programming code work.

YHTEENVETO (FINNISH SUMMARY)

Tässä väitöskirjassa käsitellään tehokkaita menetelmiä kaksiulotteisten dynaamisten järjestelmien rajasyklien tutkimiseen.

Ensimmäisessä osassa tarkastellaan Lyapunovin arvojen laskemista ja pienten rajasyklien analysointia. Väitöskirjassa on johdettu ensimmäistä kertaa täydellinen muoto neljännelle Lyapunovin arvolle. Lisäksi on laskettu Lienardin yhtälön viidennen, kuudennen ja seitsemännen Lyapunovin arvojen symboliset muodot. Lienardin yhtälö kuvaa todellisia sekä sähköisiä että mekaanisia malleja, ja L.A. Cherkasin ja G.A. Leonovin mukaan sillä on tärkeä rooli neliöllisten järjestelmien rajasyklien sekä polynomisten järjestelmien rajasyklien lukumääräarvioiden tutkimisessa. Analyyttisten laskentamenetelmien kehittyminen, niiden perusteella luotujen algoritmien toteuttaminen sekä modernien symbolisen laskennan ohjelmistojen käyttö ovat mahdollistaneet näiden uusien Lyapunovin muotojen kehittämisen.

Kehitettyjä Lyapunovin arvojen laskenta- ja analysointimenetelmiä sovelletaan toisessa osassa käyttäen Bautinin tunnettua tekniikkaa. Erityisesti analysoidaan ja visualisoidaan niiden kertoimien alueita, jotka vastaavat neliöllisten järjestelmien rajasyklien olemassaolevia eri konfiguraatioita. Tähän käytetään akateemikko A.N. Kolmogorovin tehtävää, joka liittyy neliöllisten järjestelmien syklien lokalisointiin ja muodostamiseen. Neliöllisen järjestelmän muuntaminen Lienardin järjestelmäksi, jolla on epäjatkuva oikea puoli, sekä G.A. Leonovin ehdottaman asymptoottisen integrointimenetelmän käyttö mahdollistivat tehokkaat kriteerit ratojen laadulliselle käyttäytymiselle riittävän suurilla alkutiedoilla. Kehitetyillä ratojen globaalin käyttäytymisen analyyttisillä kriteereillä, Lyapunovin arvojen paikallisella analyysillä sekä laskentakokeilla tutkittiin niiden kertoimien alueita, jotka vastaavat neliöllisen järjestelmän neljän rajasyklin olemassaoloa. Lisäksi väitöskirjassa kehitetyt menetelmät mahdollistivat neliöllisten järjestelmien kolmen ja neljän rajasyklin visualisoinnin sekä "syklien tanssin"laskennallisen tutkimisen.

REFERENCES

[Arnold, 2005] Arnold V.I. [2005] Experimental Mathematics// Fazis, Moscow. [in Russian]

[Artes & Llibre, 1997] Artes J.C. and Llibre J. [1997] Quadratic Vector Fields with a Weak Focus of Third Order// Publications Mathematiques, 41, pp. 7–39.

[Artes *et al.,* 2006] Artes J.C., Llibre J., and Schlomiuk D.[2006] The geometry of quadratic differential systems with a weak focus of second order// International Journal of Bifurcation and Chaos, 16(11), pp. 3127–3194.

[Bamon, 1985] Bamon R. [1985] The solution of Dulac's problem for quadratic vector fields// Ann. Acad. Bris. Cienc., 57, pp. 111–142.

[Bautin, 1949] Bautin N.N. [1949] Behavior of Dynamical Systems Near the Boundaries of the Stability Region// Gostekhizdat, Moscow-Leningrad. [in Russian]

[Bautin, 1952] Bautin N.N. [1952] On the number of limit cycles arising with the variation of the coefficients from an equilibrium point of focus or center type// Mat. Sbornik, 30(72), pp. 181–196. [in Russian] [Transl. into English [1954] in reprint, AMS Transl., 1(5)[1962], pp. 396–413]

[Blows & Lloyd, 1984[1]] Blows T.R. and Lloyd N.G. [1984] The number of limit cycles of certain polynomial differential equations// Proc. Roy. Soc. Edinburgh Sect., A98, pp. 215–239.

[Blows & Lloyd, 1984[2]] Blows T.R. and Lloyd N.G. [1984] The number of small-amplitude limit cycles of Lienard equations// Math. Proc. Cambridge Philos. Soc., 95, pp. 359–366.

[Blows & Perko, 1994] Blows T.R. and Perko L.M. [1994] Bifurcation of limit cycles from centers and separatrix cycles of planar analytic systems// SIAM review, 36(3), pp. 341–376.

[Cesari, 1959] Cesari L. [1959] Asymptotic Behavior and Stability Problems in Ordinary Differential Equations// Springer, Berlin.

[Chance *et al.,* 1973] Chance B., Pye E.K., Ghosh A.K., and Hess B. [1973] Biological and Biochemical Oscillators// Academic Press, New York.

[Chavarriga & Grau, 2003] Chavarriga J. and Grau M. [2003] Some open problems related to 16th Hilbert problem// Sci. Ser. A Math. Sci. N.S., 9, pp. 1–26.

[Chen & Wang, 1979] Chen L.S. and Wang M.S. [1979] The relative position and number of limit cycles of the quadratic differential systems// Acta Math. Sinica, 22, pp. 751–758.

[Cherkas, 1976] Cherkas L.A. [1976] Number of limit cycles of an autonomous second-order system// Differential Equations, 5, pp. 666–668.

[Cherkas *et al.*, 2003] Cherkas L.A., Artes J. C., and Llibre J. [2003] Quadratic systems with limit cycles of normal size// Buletinul Academiei de Stiinte a Republicii Moldova. Matematica, 41(1), pp. 31–46.

[Chow & Hale, 1982] Chow S.N. and Hale J.K. [1982] Methods of Bifurcation Theory// Springer-Verlag, NY.

[Christopher & Li, 2007] Christopher C. and Li Ch. [2007] Limit cycles of differential equations. Advanced Courses in Mathematics// Birkhauser Verlag, CRM Barcelona, Basel.

[Christopher & Lloyd, 1996] Christopher C. and Lloyd N. [1996] Small-amplitude limit cycles in polynomial Lienard systems// NoDEA : Nonlinear Differential Equations and Applications, vol. 3, 2, pp. 183–190.

[Christopher & Lynch, 1999] Christopher C.J. and Lynch S. [1999] Small-amplitude limit cycles bifurcations for Lienard systems with quadratic or cubic damping or restoring forces //Nonlinearity, 12, pp. 1099–1112.

[Dulac, 1923] Dulac H. [1923] Sur les cycles limites// Bull. Soc. Math. France, 51, pp. 45–188.

[Dumortier & Li, 1996] Dumortier F. and Li C. [1996] On the uniqueness of limit cycles surrounding one or more singularities for Lienard equations// Nonlinearity, 9, pp. 1489–1500.

[Dumortier *et al.*, 2006] Dumortier F., Llibre J., and Artes J. [2006] Qualitative Theory of Planar Differential Systems// Springer.

[Ecalle, 1992] Ecalle J. [1992] Introduction aux Functions Analysables et Preuve Constructive de la Conjecture de Dulac// Hermann Publ., Paris.

[Epstein *et al.*, 1983] Epstein I.R., Kustin K., De Kepper P., and Orban M. [1983] Oscillating chemical reactions// Sci. Am., 248(3), p. 112–123.

[Fairen & Velarde, 1979] Fairen V. and Velarde M.G. [1979] Time-periodic oscillations in a model for the respiratory process of a bacterial culture// J. Math. Biol., 9, p. 147–157.

[Frommer, 1928] Frommer M. [1928] Die Integralkurven einer gewonlichen Differentialgleichungen erster Ordnung in der Umgebung rationaler Unbestimmtheitsstellen// Math Annalen. Bd., 99, pp. 222–272.

[Gasull *et al.*, 1997] Gasull A., Guillamon A., and Manosa V. [1997] An explicit expression of the first Liapunov and period constants with applications// J. Math. Anal. Appl, 211, pp. 190–212.

[Gasull & Prohens, 1997] Gasull A. and Prohens R. [1997] Effective computation of the first Liapunov quantities for a planar differential equation// Appl. Math, 24(3), pp. 243–250.

[Gine, 2007] Gine J. [2007] On some problems in planar differential systems and Hilbert's 16th problem// Chaos, Solutions and Fractals, vol. 31, pp. 1118-1134.

[Goldbeter, 1980] Goldbeter A. [1980] Models for oscillations and excitability in biochemical systems// In "Mathematical Models in Molecular and Cellular Biology", L.A. Segel (ed.), Cambridge University Press, Cambridge.

[Han, 1999] Han M. [1999] Liapunov constants and Hopf cyclicity of Lienard systems// Ann. Diff. Eqns, 15(2), pp. 113–126.

[Ilyashenko, 1985] Ilyashenko Yu.[1985] Dulac's memoir *On limit cycles* and related problems of the localtheory of differential equations// Russian Math. Surveys VHO, pp. 1–49.

[Ilyashenko, 1991] Ilyashenko Yu.S. [1991] Finiteness for Limit Cycles// Transl. Math. Monographs, vol. 94, Amer. Math. Soc., Providence, RI.

[Jones *et al.*, 2010] Jones D.S., Plank M.J., and Sleeman B.D. [2010] Differential Equations And Mathematical Biology// CRC Press, London.

[Kudryashova, 2009] Kudryashova E. [2009] Cycles in Continuous and Discrete Dynamical Systems: Computations, Computer-assisted Proofs, and Computer Experiments// Jyväskylä University Printing House.

[Kudryashova *et al.*, 2008] Kudryashova E.V., Kuznetsov N.V., Leonov G.A., and Neittaanmäki P. [2008] Large cycles in a series of computer experiments// Abstracts of Int. conference "Space, astronomy and programming", p. 81.

[Kuznetsov, 2008] Kuznetsov N.V. [2008] Stability and Oscillations of Dynamical Systems: Theory and Applications// Jyväskylä University Printing House.

[Kuznetsov & Leonov, 2008[1]] Kuznetsov N.V. and Leonov G.A. [2008] Computation of Lyapunov quantities// 6th EUROMECH Nonlinear Dynamics Conference, http://lib.physcon.ru/?item=1802.

[Kuznetsov & Leonov, 2008[2]] Kuznetsov N.V. and Leonov G.A. [2008] Lyapunov quantities, limit cycles and strange behavior of trajectories in two-dimensional quadratic systems// Journal of Vibroengineering, 10(4), pp. 460–467.

[Kuznetsov & Kuznetsova, Leonov, 2011] Kuznetsov N.V., Kuznetsova O.A. and Leonov G.A. [2011] Investigation of limit cycles in two-dimensional quadratic systems// Proceedings of 2nd International symposium Rare Attractors and Rare Phenomena in Nonlinear Dynamics (RA'11), pp. 120–123.

[Kuznetsova, 2010[1]] Kuznetsova O.A. [2010] Sixth and seventh Lyapunov quantities for Lienard system// Vestnik of St.Petersburg University, ser. 10 "Applied mathematics, informatics and control processes", iss. 4, pp. 25–29. [in Russian]

[Kuznetsova, 2010[2]] Kuznetsova O.A. [2010] Computation of Lyapunov Quantities and Limit Cycles Existence// XI International conference "Stability and Oscillations of Nonlinear Control Systems" (Pyatnitsky conference), Moscow, p. 229.

[Kuznetsova & Leonov, 2010[1]] Kuznetsova O.A. and Leonov G.A. [2010] Localization of limit cycles in two-dimensional dynamical systems// Proceedings of the Third International Conference on Dynamics, Vibration and Control, Hangzhou, pp. 249–250.

[Kuznetsova & Leonov, 2010[2]] Kuznetsova O.A. and Leonov G.A. [2010] Computation of Lyapunov quantities and limit cycles// Proceedings of IFAC Workshop "Periodic Control Systems", Antalya, p. 8.

[Lefschetz, 1957] Lefschetz S. [1957] Differential Equations: Geometric Theory// Interscience Publishers, New York.

[Lengyel *et al.*, 1990] Lengyel I., Rabai G., and Epstein I.R. [1990] Experimental and modeling study of oscillations in the chlorine dioxide-iodine-malonic acid reaction// J. Am. Chem. Soc., 112, pp. 9104–9110.

[Leonov, 1997] Leonov G.A. [1997] Two-Dimensional Quadratic Systems as a Lienard Equation// Differential Equations and Dynamical Systems, vol. 5, 3/4, pp. 289–297.

[Leonov, 2009] Leonov G.A. [2009] Limit cycles of Lienard system with discontinuous coefficients// Doklady Akademii Nauk, 426(1), pp. 47–50.

[Leonov, 2010[1]] Leonov G.A. [2010] The criteria of four cycles existence in quadratic systems// Applied Mathematics and Mechanics, 74(2), pp. 191–202.

[Leonov (2010[2])] Leonov G.A. [2010] Necessary and sufficient conditions of the boundedness of two-dimensional quadratic systems solutions in positively invariant half-plane// Doklady Akademii Nauk, 420(2), pp. 1–3.

[Leonov *et al.*, 2008] Leonov G.A., Seledzhi S.M., Fyodorov A., Kudryashova E., and Kuznetsova O. [2008] Analytical-Numerical analysis methods of control systems// Proceedings of the 23rd IAR workshop on advanced control and diagnosis, pp. 287–291.

[Leonov, Kuznetsov & Kudryashova, 2008] Leonov G.A., Kuznetsov N.V., and Kudryashova E.V. [2008] Limit cycles of two-dimensional systems. Calculations, proofs, experiments// Vestnik St.Petersburg University. Mathematics, 41(3), pp. 216–250.

[Leonov *et al.*, 2009] Leonov G.A., Seledzhi S.M., Kuznetsova O., Fyodorov A., and Kudryashova E. [2009] Periodical oscillations of control systems. Analytical and numerical approach// Proceedings "MATHMOD 09", Vienna, pp. 416–427.

[Leonov *et al.*, 2011] Leonov G.A., Kuznetsov N.V., Kudryashova E.V., and Kuznetsova O.A. [2011] Modern symbolic computation methods: Lyapunov quantities and 16th Hilbert problem// SPIIRAS Proceedings, iss. 16, pp. 5–36.

[Leonov & Kuznetsov, 2007] Leonov G.A. and Kuznetsov N.V. [2007] Time-Varying Linearization and Perron effects// International journal of bifurcation and chaos, vol. 17, iss. 4, pp. 1–29.

[Leonov & Kuznetsova, 2009] Leonov G.A. and Kuznetsova O.A. [2009] Evaluation of the First Five Lyapunov Exponents for the Lienard System// Doklady Physics, 54(3), pp. 131–133.

[Leonov & Kuznetsova, 2010] Leonov G.A. and Kuznetsova O.A. [2010] Lyapunov Quantities and Limit Cycles of Two-dimensional Dynamical Systems. Analytical Methods and Symbolic Computation// Regular and Chaotic Dynamics, vol. 15, 2–3, pp. 356–379.

[Li, 2003] Li J. [2003] Hilbert's 16th problem and bifurcation of planar polynomial vector fields// Int. J. Bifurcation Chaos, 13, pp. 47-106.

[Lienard, 1928] Lienard A. [1928] Etudes des Oscillations Entretenues// Rev. Gen. Electr., 23, pp. 901–912.

[Lloyd, 1988] Lloyd N.G. [1988] Limit cycles of polynomial systems – some recent developments// New Direction in Dynamical Systems, Cambridge University Press, pp. 192–234.

[Lotka, 1925] Lotka A.J. [1925] Elements of Physical Biology// Williams and Wilkins.

[Lyapunov, 1892] Lyapunov A.M. [1892] The General Problem of the Stability of Motion// Doctoral dissertation, Univ. Kharkov. [in Russian] [Transl. into English [1966] Stability of Motion, Academic Press, New York and London]

[Lynch, 2005] Lynch S. [2005] Symbolic computation of Lyapunov quantities and the second part of Hilbert's sixteenth problem// In "Differential Equations with Symbolic Computations", Wang Dongming, Zheng Zhiming (eds.), pp. 1–26.

[Odell, 1980] Odell G.M. [1980] Qualitative theory of systems of ordinary differential equations, including phase plane analysis and the use of the Hopf bifurcation theorem. Appendix A.3// In "Mathematical Models in Molecular and Cellular Biology", L.A. Segel (ed.), Cambridge University Press, Cambridge.

[Perko, 1990] Perko L.M. [1990] Global families of limit cycles of planar analytic systems// Amer. Math. Soc., vol. 322, pp. 627–656.

[Perko, 1991] Perko L.M. [1991] Differential Equations and Dynamical Systems// Springer-Verlag, NY.

[Petrovskii & Landis, 1957] Petrovskii I.G. and Landis E.M. [1957] About number of limit cycles of the equation dy/dx = P (x, y)/Q(x, y), where P and Q are polynomials// Matem. Sb., 43(85), pp. 149–168.

[Petrovskii & Landis, 1959] Petrovskii I.G. and Landis E.M. [1959] Corrections to the article "About number of limit cycles of the equation dy/dx = P (x, y)/Q(x, y), where P and Q are polynomials"// Matem. Sb., 48(90), pp. 255–263.

[Poincare, 1885] Poincare H. [1885] Memoire sur les courbes definies par lesequations diffeentielles// J. de. Mathematiques Pures et Appliquees, 4, vol. 1, pp. 167–244.

[Rayleigh, 1877] Rayleigh J.W.S. [1877, 1878] The theory of sound// Macmillan, London.

[Reyn, 1994] Reyn J.W. [1994] A bibliography of the qualitative theory of quadratic systems of differential equations in the plane. Third edition// Report 94-02, Delft University of Technology.

[Romanovskii, 1993] Romanovskii V.G. [1993] On a calculation of Liapunov focus quantities in the case of two imaginary roots// Diff. Eqns., 29, pp. 782–787.

[Romanovskii, 1996] Romanovskii V.G. [1996] Grobner Basis Theory for Monoidal Rings and 16-th Hilbert Problem// Mat.-Report, 1996-08, Depart. of Math. Technical Univ. of Denmark.

[Rousseau, 1993] Rousseau C. [1993] Bifurcation methods in polynomial systems// In "Bifurcations and Periodic Orbits of Vector Fields", Schlomiuk D. (eds.), NATO ASI Series C., vol. 408, pp. 383–428, Kluwer Academic, London.

[Sabatini & Chavarriga, 1999] Sabatini M. and Chavarriga J. [1999] A survey of isochronous centers// Qualitative Theory of Dynamical Systems, 1, pp. 1–70.

[Schuko, 1968] Schuko S.D. [1968] Lyapunov quantities computation with the help of EVZM// Trudy Gor'kovskogo inst. inzh. vodn. transp., 94, pp. 97–109. [in Russian]

[Serebryakova, 1959] Serebryakova N.[1959] Behavior of dynamical system with one degree of freedom near the points of boundaries of domain of stability in the case when a safe boundary goes to the dangerous one// Izvestiya AN SSSR. Division of technical sciences, mechanics and engineering, 2, pp. 178–182. [in Russian]

[Shi, 1980] Shi S.L. [1980] A concrete example of the existence of four limit cycles for plane quadratic systems// Sci. Sinica, 23, pp. 153–158.

[Strogatz, 1994] Strogatz H. Steven [1994] Nonlinear Dynamics and Chaos. With Applications to Physics, Biology, Chemistry, and Engineering// Westview Press.

[van der Pol, 1926] van der Pol B. [1926] On relaxation-oscillations // Philosophical Magazine and Journal of Science, vol. 7, no. 2, pp. 978-992.

[Volterra, 1926] Volterra V. [1926] Pages 409–448 in Chapman R. N. [1931] Animal ecology// McGraw-Hill, New York.

[Ye, 1986] Ye Yian-Qian [1986] Theory of Limit Cycles// Translations of. Math. Monograqphs, 66 Amer. Math. Soc., Providence.

[Yu, 1998] Yu P. [1998] Computation of normal forms via a perturbation technique// J. Sound Vibr., 211, pp. 19–38.

[Yu & Chen, 2008] Yu P. and Chen G. [2008] Computation of focus values with applications// Nonlinear Dynamics, 51(3), pp. 409–427.

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