On Linear Parabolic Partial Differential Equations

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Contents

Chapter 1

Introduction

In this thesis we'll consider the following linear parabolic partial differential equation (PDE):

$$u_t - \operatorname{div}(\mathbb{A}(x,t)Du) = 0, \tag{1.1}$$

where $u = u(x,t) : \Omega_T \to \mathbb{R}$ is a function. Here we have defined a cylinder $\Omega_T := \Omega \times T \subset \mathbb{R}^n \times [0,\infty)$ where Ω is an open domain in $\mathbb{R}^n$ and T is an open interval. The divergence $\operatorname{div}(\cdot)$ and the gradient Du are understood to be taken with respect to variable x only and furthermore we denote by $\mathbb{A} = \mathbb{A}(x,t)$ the symmetric $n \times n-$ matrix whose coefficients a^{ij} are measurable L^∞-functions that obey the ellipticity condition

$$\lambda|\xi|^2 \leq (\xi\,,\,\mathbb{A}\xi) \leq \Lambda|\xi|^2 \tag{1.2}$$

for all $\xi \in \mathbb{R}^n$ and some constants $\lambda, \Lambda > 0$.

We'll introduce some basic definitions and theorems that are needed later on including Sobolev spaces $W^{1,p}(\Omega_T)$ and parabolic Sobolev spaces $V^p(\Omega_T)$ in the second chapter. In the third we consider the heat equation (3.1) that is equation (1.1) with $\mathbb{A}$ being the identity matrix. We study the fundamental solution and prove the mean value theorem.

In the fourth chapter, we prove the weak and the strong maximum principles and Harnack's inequality assuming that the coefficients a^{ij} of equation (1.1) are smooth and depend only on the variable x.

The weak solutions of equation (1.1) are defined as follows.

Definition 1.0.1. We say that a function $u(x,t) \in V^2(\Omega_T)$ is a weak solution of equation

(1.1) if equation

$$\iint\limits_{\Omega_T} \phi_t u - (D\phi\,,\, \mathbb{A}Du)\, dt\, dx = 0 \tag{1.3}$$

holds for every $\phi \in C_0^\infty(\Omega_T)$.

In the fifth and final chapter we first prove the existence of weak solutions and then Harnack's inequality for equation (1.1), under the general assumption on measurable coefficients a^{ij} satisfying the ellipticity condition (1.2).

To state Harnack's inequality mentioned above, we make the following definitions. Let $D \subset \Omega$ be compact and connected and let $T^- := (t_1, t_2)$ and $T^+ := (t_3, t_4)$ be two subintervals of the time interval $T := (T_1, T_2)$ such that

$$T_1 < t_1 < t_2 < t_3 < t_4 \leq T_2.$$

We denote by $D^- := D \times T^-$ and $D^+ := D \times T^+$ the two corresponding subcylinders of the domain $\Omega \times T$. Now we are in position to state the main theorem of this thesis:

Theorem 1.0.2 (Harnack's inequality)**.** *Any non-negative weak solution of equation* (1.1) *satisfies the following inequality:*

$$\sup_{D^-} u \leq C^{\Lambda + \lambda^{-1}} \inf_{D^+} u, \tag{1.4}$$

where constant $C = C(D, D^-, D^+) > 1$.

Remarkable in this theorem is that C depends neither on u nor $\mathbb{A}$. The proof of this theorem is due to Moser [6], [7]. With this theorem one can prove the Hölder continuity of the solutions of (1.1) [6, p. 108]. A good introduction to the history of this problem can be found from [1].

Chapter 2

Preliminaries

2.1 L^p spaces

First define some norm spaces that we need to consider equation (1.1).

Definition 2.1.1. Let $1 \leq p < \infty$. Then the space $L^p(\Omega)$ includes every Lebesgue measurable function f such that

$$||f||_{L^p(\Omega)} := \left(\int_\Omega |f|^p \right)^{\frac{1}{p}} < \infty \,.$$

Moreover the space L^∞ contains every Lebesgue measurable function f such that

$$||f||_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{x \in \Omega} |f(x)| < \infty \,.$$

Remark 2.1.2. The definition above is a little bit vague in the sense that for L^p to be a normed space, it is required that there exists a unique function $f \in L^p$ such that $||f||_{L^p(\Omega)} = 0$. This is a problem since we know that $||f||_{L^p(\Omega)} = ||g||_{L^p(\Omega)}$ whenever f and g differ only in a set of a Lebesgue measure zero. However, we can overcome this problem by introducing an equivalence relation, $f \sim g$ when f and g differ only in a set of Lebesgue measure zero and defining $[f] =: f$.

Perhaps the most useful tool to study L^p spaces is the Hölder inequality. We omit the proof that can be found for example from [4, Theorem 3.2.1].

Theorem 2.1.3. *Let* $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. *If* $f \in L^p(\Omega), g \in L^q(\Omega)$, *then* $fg \in L^1(\Omega)$ *and*

$$||fg||_{L^1(\Omega)} \leq ||f||_{L^p(\Omega)} ||g||_{L^q(\Omega)} \,. \tag{2.1}$$

Next we would like to define a parabolic Sobolev space V^p from where we can find the solutions of the equation (1.1). For this purpose we first define L^p spaces involving time and the usual Sobolev-space $W^{1,p}$.

2.2 L^p spaces involving time

Definition 2.2.1 (Definitions)**.** Let X be a real Banach's space.

i) Function $s : [0,T] \to X$ is called a *simple* function, if it's of the form

$$s(t) = \sum_{i=1}^{m} \chi_{E_i}(t) u_i \quad (0 \leq t \leq T),$$

where every E_i is a Lebesgue measurable subset of $[0,T]$ and $u_i \in X$ for all $i = 1, \dots, m$.

ii) Function $f : [0,T] \to X$ is *strongly measurable*, if there exist simple functions $s_k : [0,T] \to X$ such that

$$s_k(t) \to f(t) \quad \text{for almost every } 0 \leq t \leq T.$$

Definition 2.2.2. Space $L^p(0,T;X)$ consists of every strongly measurable function $u : [0,T] \to X$ such that

$$||u||_{L^p(0,T;X)} := \left(\int_0^T ||u(t)||^p \, dt \right)^{\frac{1}{p}} < \infty$$

for $1 \leq p < \infty$ and

$$||u||_{L^\infty(0,T;X)} := \operatorname{ess\,sup}_{0 \leq t \leq T} ||u(t)|| < \infty,$$

where $|| \cdot ||$ denotes the norm of X.

2.3 Sobolev spaces $W^{1,p}$ and $W_0^{1,p}$

Next we define the Sobolev spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$.

Definition 2.3.1. Sobolev space $W^{1,p}(\Omega)$ is the completion of $C^\infty(\Omega) \cap L^p(\Omega)$ under the norm

$$||u||_{W^{1,p}(\Omega)} := ||u||_{L^p(\Omega)} + ||Du||_{L^p(\Omega)}, \ u \in C^\infty(\Omega) \cap L^p(\Omega).$$

Similarly we say that space $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the norm

$$||u||_{W_0^{1,p}(\Omega)} := ||Du||_{L^p(\Omega)}, \ u \in C_0^\infty(\Omega).$$

We need the following definition to state the next theorem.

Definition 2.3.2. For $1 \le p < n$, define

$$p^* := \frac{np}{n-p} .$$

Then p^* is called the Sobolev exponent of p. Note that $1/p^* = 1/p - 1/n$.

It is well known that $W^{1,p}$ is a Banach space [2, p. 249, thm 2]. Moreover functions $u \in W^{1,p}$ satisfy the following Sobolev inequality [3, p. 138].

Theorem 2.3.3 (Gagliardo-Nierenberg-Sobolev inequality)**.** *Assume $1 \le p < n$. Then there exists a constant C, depending only on p and n such that*

$$||u||_{L^{p*}(\mathbb{R}^n)} \le C||Du||_{L^p(\mathbb{R}^n)}, \tag{2.2}$$

for all $u \in W^{1,p}(\mathbb{R}^n)$.

We also need the following theorem

Theorem 2.3.4. *[2, p. 265] Assume Ω is a bounded, open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(\Omega)$ for some $1 \le p < n$. Then we have the estimate*

$$||u||_{L^q(\Omega)} \le C||Du||_{L^p(\Omega)}$$

for each $q \in [1, p^*]$, the constant C depending only on p, q, n and Ω. In particular, for all $1 \le p \le \infty$,

$$||u||_{L^p(\Omega)} \le C||Du||_{L^p(\Omega)} \ .$$

2.4 Sobolev spaces V^p and V_0^p

Next we define Sobolev spaces V^p and V_0^p for $1 \leq p \leq \infty$.

Definition 2.4.1. The parabolic Sobolev space $V^p(\Omega_T)$ is defined as

$$V^p(\Omega_T) := L^\infty(0,T;L^p(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega))$$

and similarly we define the space $V_0^p(\Omega_T)$ to be

$$V_0^p(\Omega_T) := L^\infty(0,T;L^p(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$$

both equipped with the norm

$$||u||_{V^p(\Omega_T)} := ||u||_{L^\infty(0,T;L^p(\Omega))} + ||u||_{L^p(0,T;W^{1,p}(\Omega))}$$

for $u \in V^p(\Omega_T)$.

Both $V^p(\Omega_T)$ and $V_0^p(\Omega_T)$ are Banach spaces and are embedded in $L^q(\Omega)$ for some $q > p$.

2.5 Inequalities

We need these inequalities later on.

Theorem 2.5.1 (Cauchy's inequality)**.**

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2} \quad (a, b > 0).$$

Proof. $0 \leq (a-b)^2 = a^2 - 2ab - b^2$. □

Theorem 2.5.2 (Cauchy's inequality with ε)**.**

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon} \quad (a, b > 0, \varepsilon > 0).$$

Proof. Write

$$ab = ((2\varepsilon)^{1/2} a) \left(\frac{b}{(2\varepsilon)^{1/2}} \right)$$

and apply Cauchy's inequality. □

Theorem 2.5.3 (Young's inequality)**.** *Let* $1 < p, q < \infty$*,* $\frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$ab \leq \frac{a^p}{p} + \frac{a^q}{q} \quad (a, b > 0).$$

Proof. The mapping $x \to e^x$ is convex, and consequently

$$ab = e^{\log a + \log b} = e^{\frac{1}{p}\log a^p + \frac{1}{q}\log b^q} \leq \frac{1}{p}e^{\log a^p} + \frac{1}{q}e^{\log b^q} = \frac{a^p}{p} + \frac{a^q}{q}.$$

□

Theorem 2.5.4 (Young's inequality with ε)**.**

$$ab \leq \varepsilon a^p + C(\varepsilon) b^q \quad (a, b > 0).$$

Proof. Write $ab = ((\varepsilon)^{1/p} a)\left(\frac{b}{(\varepsilon p)^{1/p}}\right)$ and apply Young's inequality. □

Theorem 2.5.5 (Cauchy-Schwartz inequality)**.** *[2, p. 624] Let A be a symmetric $n \times n$-matrix. Suppose that condition* (1.2) *holds. Then for all* $\xi, \eta \in \mathbb{R}^n$*,*

$$|(A\xi, \eta)| \leq (A\xi, \xi)^{1/2}(A\eta, \eta)^{1/2} \ .$$

Chapter 3

The heat equation

In the following we consider the heat equation

$$u_t - \Delta u = 0\,, \tag{3.1}$$

where $t > 0$ and $x \in \Omega$. Heat equation corresponds to equation (1.1) when $\mathbb{A} = Id$. Here we have used the following notation:

$$\Delta u(x,t) = \operatorname{div}(Du(x,t)) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \left(u(x,t)\right) \,.$$

3.1 Solving the heat equation

In this section we find a special solution to equation (3.1) and then define the fundamental solution to the heat equation. To find the solution, we first notice that if $u(x,t)$ is a solution to this equation, then also the function $w(x,t) := u(\lambda x, \lambda^2 t)$ is a solution for every $\lambda \in \mathbb{R}$ as one can easily verify that

$$w_t(x,t) - \Delta w(x,t) = \lambda^2 \left(u_t(\lambda x, \lambda^2 t) - (\Delta u)(\lambda x, \lambda^2 t)\right) \,.$$

If we choose

$$u(x,t) =: v\left(\frac{|x|^2}{t}\right) ,$$

then

$$u(\lambda x, \lambda^2 t) = v\left(\frac{|\lambda x|^2}{\lambda^2 t}\right) = v\left(\frac{|x|^2}{t}\right) = u(x,t)\,.$$

From now on we consider only the case $n = 1$, since the spherical symmetry of the solution v suggests that the solution provided by $n = 1$ applies also to the general case. So we seek for the solution $u(x,t)$ of the following equation:

$$u_t = u_{xx}\ , \tag{3.2}$$

where $x \in \mathbb{R}$. Now we would like to change equation (3.2) to an ordinary differential equation (ODE) of v. This is done by substituting $u = v\left(\frac{x^2}{t}\right)$ to the partial derivatives of u:

$$\begin{aligned} u_t &= \left(v\left(\frac{x^2}{t}\right)\right)_t = -\frac{x^2}{t^2}v'\ , \\ u_x &= \left(v\left(\frac{x^2}{t}\right)\right)_x = \frac{2x}{t}v'\ , \\ u_{xx} &= \left(\frac{2x}{t}v'\right)_x = \frac{2}{t}v' + \frac{4x^2}{t^2}v''\,. \end{aligned}$$

Then equation (3.2) implies that

$$-\frac{x^2}{t^2}v' = \frac{2}{t}v' + \frac{4x^2}{t^2}v''.$$

Defining $\frac{x^2}{t} =: y$ then gives

$$4yv''(y) + (2+y)v'(y) = 0\ ,$$

and writing $v'(y) =: w(y)$ implies

$$\frac{w'(y)}{w(y)} = -\frac{2+y}{4y}\,.$$

For this equation there is a (unique) solution [5, p. 10, 1.2 Lause]

$$w(y) = v'(y) = e^{-\frac{1}{2}\ln y - \frac{1}{4}y + C} = \widetilde{C}y^{-\frac{1}{2}}e^{-\frac{1}{4}y} \tag{3.3}$$

where $C, \widetilde{C}$ are constants. This gives

$$v(y) = \widetilde{C} \int_0^y z^{-\frac{1}{2}} e^{-\frac{1}{4}z} \, dz \, .$$

Now it seems that we have the solution we want, but as we shall soon see, we actually still need to take a partial derivative with respect to x of this function:

$$\begin{aligned} v_x &= y_x \, \widetilde{C} \, y^{-\frac{1}{2}} e^{-\frac{1}{4}y} \\ &= \widetilde{C} \, \frac{2x}{t} \, \frac{t^{\frac{1}{2}}}{x} e^{-\frac{x^2}{4t}} \\ &= 2 \, \widetilde{C} \, t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} \, . \end{aligned}$$

We can now check that $v_x =: \Psi(x,t)$ is a solution to (3.2). First we calculate

$$\begin{aligned} \Psi_t &= -2 \, \frac{\widetilde{C}}{2} \, t^{-\frac{3}{2}} e^{-\frac{x^2}{4t}} + 2 \, \widetilde{C} \, t^{-\frac{1}{2}} \left(-\frac{x^2}{4} \right) \left(-\frac{1}{t^2} \right) e^{-\frac{x^2}{4t}} \\ &= \left(\frac{1}{2} t^{-\frac{5}{2}} x^2 - t^{-\frac{3}{2}} \right) \widetilde{C} \, e^{-\frac{x^2}{4t}} \, . \end{aligned}$$

After that we see that

$$\Psi_x = -2\widetilde{C} \, \frac{2x}{4t} \, t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} = -\widetilde{C} \frac{x}{t^{\frac{3}{2}}} e^{-\frac{x^2}{4t}}$$

and thus

$$\begin{aligned} \Psi_{xx} &= -\widetilde{C} \frac{1}{t^{\frac{3}{2}}} e^{-\frac{x^2}{4t}} - \widetilde{C} \frac{x}{t^{\frac{3}{2}}} \left(-\frac{2x}{4t} \right) e^{-\frac{x^2}{4t}} \\ &= \left(\frac{1}{2} t^{-\frac{5}{2}} x^2 - t^{-\frac{3}{2}} \right) \widetilde{C} \, e^{-\frac{x^2}{4t}} \\ &= \Psi_t . \end{aligned}$$

From these considerations and the fact that v for arbitrary n was spherically symmetric with respect to x, we have the following definition.

Definition 3.1.1. We define the fundamental solution $\Phi(x,t)$ of the heat equation as follows:

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & \text{for } x \in \mathbb{R}^n, t > 0 \\ 0 & \text{for } x \in \mathbb{R}^n, t < 0\,. \end{cases} \tag{3.4}$$

This equals to our Ψ that we defined earlier if we take $n = 1$ and choose $\widetilde{C}$ suitably.

The constant $\widetilde{C}$ has been chosen such that $||\Phi||_{L(\mathbb{R}^n)} = 1$ and we show this next by a direct calculation:

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(x,t)\,dx &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\,dx \\ &= \frac{n\alpha(n)}{(4\pi t)^{n/2}} \int_0^\infty s^{n-1} e^{-\frac{s^2}{4t}}\,ds \\ &= \frac{n\alpha(n)}{(4\pi t)^{n/2}} \int_0^\infty \left(\sqrt{4t}s\right)^{n-1} e^{-s^2} \sqrt{4t}ds \\ &= \frac{n\alpha(n)}{\pi^{n/2}} \int_0^\infty s^{n-1} e^{-s^2}\,ds \\ &= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|x|^2}\,dx \\ &= \frac{1}{\pi^{n/2}} \prod_{i=1}^n \int_{-\infty}^\infty e^{-x_i^2}\,dx_i \\ &= \frac{1}{\pi^{n/2}} \prod_{i=1}^n \pi^{1/2} \\ &= 1\,. \end{aligned}$$

So we found a solution for the heat equation and chose a specific normalized one to be the fundamental solution.

3.2 Mean value formula for the heat equation

Next we prove the Mean Value Property (MVP) for solutions of this equation. This theorem resembles the MVP of the Laplace's equation, but the proof in our case is a bit more tricky as we shall see in the next section. We follow [2, p. 53-54] in this proof.
In the case of the Laplace's equation, the mean value of a solution is taken over the usual euclidean balls that are much more natural and simple compared to our choice of the

heat ball in the next definition. However there is a nice connection between these two: in both cases the boundaries of these balls happen to be level sets of the corresponding fundamental solutions.

Definition 3.2.1. We define for $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $r > 0$ the heat ball

$$B_h(x,t;r) = \left\{ (y,s) \in \mathbb{R}^{n+1} \,\middle|\, s \le t, \Phi(x-y,t-s) \ge \frac{1}{r^n} \right\}. \tag{3.5}$$

We also define space $C^{2,1}(\Omega_T) := \{u : \Omega_T \to \mathbb{R} \,|\, u, Du, D^2u, u_t \in C(\Omega_T)\}$, where Du and D^2u are understood to be taken with respect to variable x only.

Theorem 3.2.2 (The Mean value formula)**.** *Let $u \in C_1^2(\Omega_T)$ solve the heat equation. Then*

$$u(x,t) = \frac{1}{4r^n} \iint_{B_h(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dy\, ds \tag{3.6}$$

for each $B_h(x,t;r)$.

Proof. We can shift the point (x,t) to $(0,0)$ since transformation $(x,t) \to (x+a, t+b)$ preserves the solutions of (3.1) for every constant $a \in \mathbb{R}^n, b \in \mathbb{R}$. This is why we'll use the set $B_h(r) := B(0,0;r)$ instead of $B(x,t;r)$. We may also assume that the solution u is smooth.

First we calculate the integration boundaries of $B_h(r)$ in the usual euclidean spherical coordinates. Let $r > 0$ and $(y,s) \in B_h(r)$. By definition of $B(x,t;r)$ we have

$$B_h(r) = \left\{ (y,s) \in \mathbb{R}^{n+1} \,\middle|\, s \le 0,\ \Phi(-y,-s) \le \frac{1}{r^n} \right\}.$$

Since $s \le 0$, we have

$$\frac{|y|^2}{4s} \le 0\,. \tag{3.7}$$

Thus

$$e^{\frac{|y|^2}{4s}} \le 1\,,$$

which implies that

$$\frac{1}{(-4\pi x)^{n/2}} e^{\frac{|y|^2}{4s}} \leq \frac{1}{(-4\pi x)^{n/2}}.$$

Next we notice that the left side of this inequality equals to $\Phi(-y,-s)$ and since $(y,s) \in B_h(r)$ we have

$$\frac{1}{r^n} \leq \Phi(-y,-s) \leq \frac{1}{(-4\pi x)^{n/2}}\,.$$

Thus

$$(-4\pi x)^{n/2} \leq r^{2(n/2)}$$

and therefore

$$-4\pi s \leq r^2$$

which finally implies that

$$-\frac{r^2}{4\pi} \leq s \leq 0\,.$$

To get the boundaries for $|y|$ we start from the definition of B_h which gives

$$\Phi(-y,-s) = \frac{1}{(4\pi s)^{(n/2)}} e^{-\frac{|y|^2}{-4s}} \geq \frac{1}{r^{2(n/2)}}\,.$$

Thus

$$\frac{|y|^2}{4s} \geq \log\left(\frac{-4\pi s}{r^2}\right)^{\frac{n}{2}} \tag{3.8}$$

and this implies

$$|y| \leq \sqrt{2ns \log\left(\frac{-4\pi s}{r^2}\right)}\,.$$

Now the main point in the proof is to show that function

$$\phi(r) = \frac{1}{r^n} \iint_{B_h(r)} u(y,s) \frac{|y|^2}{s^2}\, dyds$$

is a constant function of r. This is done by showing that its derivative vanishes everywhere. Now, by the changing of variables from (y,s) to $(ry, r^2 s)$, we get that

$$\begin{aligned}
\phi(r) &= \frac{1}{r^n} \iint_{B_h(r)} u(y,s) \frac{|y|^2}{s^2}\, dy\, ds \\
&= \frac{1}{r^n} \iint_{B_h(1)} u(ry, r^2 s) \frac{r^2 |y|^2}{r^4 s^2}\, r^n r^2\, dy\, ds \\
&= \iint_{B_h(1)} u(ry, r^2 s) \frac{|y|^2}{s^2}\, dy\, ds
\end{aligned}$$

where the change of integration boundary from $B_h(r)$ to $B_h(1)$ can be done since

$$|ry| = \sqrt{2nr^2 s \log\left(\frac{-4\pi r^2 s}{r^2}\right)} = r\sqrt{2ns \log\left(\frac{-2\pi s}{1}\right)}$$

implies that

$$|y| \leq \sqrt{2sn\log(-4\pi s)}$$

and from

$$-\frac{r^2}{4\pi} \leq r^2 s$$

it follows that

$$-\frac{1}{4\pi} \leq s\ .$$

Next we want to define a new function $\psi(r)$ that vanishes at the boundary of $B_h(r)$ and has a nice form for our calculations further on. From (3.8) we know that

$$\frac{|y|^2}{4s} \geq \log\left(\frac{-4\pi s}{r^2}\right)^{\frac{n}{2}} = \frac{n}{2}\log(-4\pi s) - n\log r$$

or equivalently

$$0 \leq \frac{|y|^2}{4s} - \frac{n}{2}\log(-4\pi s) + n\log r \ ,$$

where the equality holds on the boundary of $B_h(r)$. This leads us to define

$$\psi(r) := \frac{|y|^2}{4s} - \frac{n}{2}\log(-4\pi s) + n\log r \ . \tag{3.9}$$

Then we need to calculate the partial derivatives of ψ because we need those when we show that $\phi'(r) = 0$. Simple calculations give

$$\psi_{y_i} = \frac{2y_i}{4s} = \frac{y_i}{2s} \quad \text{for} \qquad i = 1, \dots, n$$

and

$$\psi_s = -\frac{n}{2} \cdot \frac{-4\pi}{-4\pi s} - \frac{|y|^2}{4s^2} = -\frac{n}{2s} - \frac{|y|^2}{4s^2}.$$

Earlier we have showed that

$$\phi(r) = \iint_{B_h(1)} u(ry, r^2 s)\frac{|y|^2}{s^2}\, dy\, ds$$

and this gives

$$\begin{aligned}
\phi'(r) &= \iint_{B_h(1)} Du \cdot D(ry, r^2 s)\frac{|y|^2}{s^2}\, dy\, ds \\
&= \iint_{B_h(1)} \left(\sum_{i=i}^{n} u_{y_i} y_i + u_s \cdot 2rs \right) \cdot \frac{|y|^2}{s^2}\, dy\, ds \\
&= \iint_{B_h(1)} \sum_{i=i}^{n} y_i u_{y_i} \frac{|y|^2}{s^2}\, dy\, ds + \iint_{B_h(1)} 2r u_s \frac{|y|^2}{s}\, dy\, ds \\
&= \iint_{B_h(r)} \sum_{i=i}^{n} \frac{y_i}{r} u_{y_i} \frac{|(\frac{y}{r})|^2}{(\frac{s}{r^2})^2} \frac{1}{r^n}\frac{1}{r^2}\, dy\, ds + \iint_{B_h(r)} 2\frac{r}{r} u_s \frac{|(\frac{y}{r})|^2}{(\frac{s}{r^2})^2} \frac{1}{r^n}\frac{1}{r^2}\, dy\, ds \\
&= \frac{1}{r^{n+1}} \iint_{B_h(r)} (y, Dy)\frac{|y|^2}{s^2}\, dy\, ds + \frac{1}{r^{n+1}} \iint_{B_h(r)} 2u_s \frac{|y|^2}{s}\, dy\, ds \\
&=: A + B
\end{aligned}$$

where in the fourth equality we made another change of variables. Now we continue to calculate the term B:

$$
\begin{aligned}
B &= \frac{1}{r^{n+1}} \iint_{B_h(r)} 2u_s \frac{|y|^2}{s} \, dy \, ds \\
&= \frac{1}{r^{n+1}} \iint_{B_h(r)} 2u_s \sum_{i=1}^{n} \frac{y_i^2}{s} \, dy \, ds \\
&= \frac{1}{r^{n+1}} \iint_{B_h(r)} \sum_{i=1}^{n} 4u_s y_i \frac{y_i}{4s} \, dy \, ds \\
&= \frac{1}{r^{n+1}} \iint_{B_h(r)} \sum_{i=1}^{n} 4u_s y_i \psi_{y_i} \, dy \, ds.
\end{aligned}
$$

Then integrating by parts first with respect to y and then s and substituting ψ_s gives

$$
\begin{aligned}
B &= -\frac{1}{r^{n+1}} \iint_{B_h(r)} \sum_{i=1}^{n} 4y_i u_{sy_i} \psi \, dy \, ds - \frac{1}{r^{n+1}} \iint_{B_h(r)} 4nu_s \psi \, dy \, ds \\
&= \frac{1}{r^{n+1}} \iint_{B_h(r)} 4 \sum_{i=1}^{n} u_{y_i} y_i \psi_s \, dy \, ds - \frac{1}{r^{n+1}} \iint_{B_h(r)} 4nu_s \psi \, dy \, ds \\
&= \frac{1}{r^{n+1}} \iint_{B_h(r)} 4 \sum_{i=1}^{n} u_{y_i} y_i \left(-\frac{n}{2s}\right) - 4nu_s \psi \, dy \, ds + \frac{1}{r^{n+1}} \iint_{B_h(r)} 4 \sum_{i=1}^{n} y_i u_{y_i} \left(-\frac{|y|^2}{4s^2}\right) dy \, ds \\
&= -\frac{1}{r^{n+1}} \iint_{B_h(r)} 2 \sum_{i=1}^{n} \left(y_i u_{y_i} \frac{n}{s}\right) + 4nu_s \psi \, dy \, ds \; - A\,.
\end{aligned}
$$

Then moving A in the last term to the left hand side of the equation gives

$$
\begin{aligned}
\phi'(r) &= A + B \\
&= -\frac{1}{r^{n+1}} \iint_{B_h(r)} 2\sum_{i=1}^{n} \left(y_i u_{y_i} \frac{n}{s}\right) + 4n u_s \psi \, dy \, ds \\
&= -\frac{1}{r^{n+1}} \iint_{B_h(r)} 2\sum_{i=1}^{n} \left(y_i u_{y_i} \frac{n}{s}\right) + 4n \Delta u \psi \, dy \, ds \\
&= \frac{1}{r^{n+1}} \iint_{B_h(r)} -2\sum_{i=1}^{n} \left(y_i u_{y_i} \frac{n}{s}\right) + 4n \sum_{i=1}^{n} u_{y_i} \psi_{y_i} \, dy \, ds \\
&= \sum_{i=1}^{n} \frac{1}{r^{n+1}} \iint_{B_h(r)} -2\left(y_i u_{y_i} \frac{n}{s}\right) + 4n u_{y_i} \frac{y_i}{2s} \, dy \, ds \\
&= 0 \ ,
\end{aligned}
$$

where we used the fact that u is a solution to the heat equation, partial derivatives of ψ and integration by parts. So we have shown that $\phi'(r) = 0$ and this implies that $\phi(r) = C$ for some constant $C \in \mathbb{R}$.

Next we will determine this constant. To this end we will calculate

$$
\begin{aligned}
\phi(0) &= \lim_{r \to 0} \phi(r) \\
&= \lim_{r \to 0} \iint_{B_h(1)} u(ry, r^2 s) \frac{|y|^2}{s^2} \, dy \, ds \\
&= \iint_{B_h(1)} \lim_{r \to 0} u(ry, r^2 s) \frac{|y|^2}{s^2} \, dy \, ds \\
&= u(0,0) \iint_{B_h(1)} \lim_{r \to 0} \frac{|y|^2}{s^2} \, dy \, ds \\
&= u(0,0) \lim_{r \to 0} \iint_{B_h(1)} \frac{|y|^2}{s^2} \, dy \, ds \, .
\end{aligned}
$$

Now if we can show that the integral on the last line equals to 4, then we finish the proof of this theorem. We use two suitable changes of variables and the definition and the properties of the gamma function $\Gamma(x)$ in this calculation:

$$
\begin{aligned}
\iint_{B_h(1)} \frac{|y|^2}{s^2}\, dyds &= \int_{-\frac{1}{4\pi}}^{0} \frac{1}{s^2} \int_0^{\sqrt{2sn\log(-4\pi s)}} t^2 \cdot t^{n-1} n\alpha(n)\, dt\, ds \\
&= 4\pi n\alpha(n) \int_0^{\infty} e^{\hat{s}} \int_0^{\sqrt{\frac{2\hat{s}n}{4\pi} e^{-\hat{s}}}} t^{n+1}\, dt\, d\hat{s} \\
&= 4\pi n\alpha(n) \int_0^{\infty} \frac{e^{\hat{s}}}{n+2} \left(\frac{2\hat{s}n}{4\pi} e^{-\hat{s}} \right)^{\frac{n+2}{2}} d\hat{s} \\
&= \frac{2n^{\frac{n}{2}+2}\alpha(n)}{(n+2)(2\pi)^{\frac{n}{2}}} \int_0^{\infty} \hat{s}^{\frac{n}{2}+1} e^{-\frac{\hat{s}n}{2}}\, d\hat{s} \\
&= \frac{8\alpha(n)}{(n+2)\pi^{\frac{n}{2}}} \int_0^{\infty} \bar{s}^{\frac{n}{2}+1} e^{-\bar{s}}\, d\bar{s} \\
&= \frac{8\alpha(n)}{(n+2)\pi^{\frac{n}{2}}} \Gamma\left(\frac{n}{2}+2\right) \\
&= \frac{8\alpha(n)}{(n+2)\pi^{\frac{n}{2}}} \left(\frac{n}{2}+1\right) \Gamma\left(\frac{n}{2}+1\right) \\
&= \frac{4\alpha(n)}{\pi^{\frac{n}{2}}} \Gamma\left(\frac{n}{2}+1\right) \\
&= 4
\end{aligned}
$$

and the theorem is proved, since $\alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$ by [2, p. 615]. □

Chapter 4

Equations with smooth coefficients

In this section we assume that the coefficients of the equation (1.1) are $C^\infty(\Omega)$ functions and that they don't depend on time. In [2, Ch. 7.1.3] it is proved that under these assumptions the solutions of (1.1) are actually also in $C^\infty(\Omega)$. It is also possible to prove this result even if the coefficients do depend on the time, but we don't consider this here.

In the following we will prove the weak maximum principle, a version of Harnack's inequality and the strong maximum principle under these assumptions.

4.1 Weak maximum principle

Next we prove the weak maximum principle for the solutions of the equation

$$u_t = \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} =: Lu \tag{4.1}$$

where the coefficients a^{ij} are now continuous but not necessarily smooth. We also assume that they are symmetric. We also assume that these coefficients satisfy the ellipticity condition (1.2) for $\lambda, \Lambda > 0$. For this chapter and section 5.1 we define $\Omega_T := \Omega \times (0, T]$, where $(0, T] =: T$. Then the parabolic boundary Γ_T of Ω_T is defined by $\Gamma_T := \overline{\Omega}_T - \Omega_T$.

Theorem 4.1.1 (Weak maximum principle)**.** *Assume that $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$.*

i) If $u_t \le Lu$ in Ω_T, then

$$\max_{\overline{\Omega}_T} u = \max_{\Gamma_T} u\,. \tag{4.2}$$

ii) If $u_t \geq Lu$ in Ω_T, then

$$\min_{\overline{\Omega}_T} u = \min_{\Gamma_T} u \,. \tag{4.3}$$

Proof. We will prove, that under the assumption that u attains it's maximum in Ω_T, we can show that u can't be a supersolution.

We assume that u satisfies the strict inequality

$$u_t < Lu \quad \text{in } \Omega_T \tag{4.4}$$

and that we can find a point $(x_0, t_0) \in \overline{\Omega}_T$ such that

$$u(x_0, t_0) = \max_{\overline{\Omega}_T} u \,.$$

Then if we have $0 < t_0 < T$, we get that $u_t = 0$ at (x_0, t_0). We also have $Lu \leq 0$ as we will next show.

Now since (x_0, t_0) belongs to interior of Ω_T it follows that

$$\begin{aligned} Du(x_0, t_0) &= 0 \quad \text{and} \\ D^2u(x_0, t_0) &\leq 0 \,. \end{aligned}$$

Moreover from linear algebra we know that since a^{ij} is symmetric and positive definite, we find an orthogonal matrix $\mathbb{O}$ such that

$$\mathbb{O}\mathbb{A}\mathbb{O}^T = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

and $d_k > 0$ for every $k = 1, \ldots, n$. Now define y such that

$$y = x_0 + \mathbb{O}(x - x_0) \,.$$

This implies that

$$x - x_0 = \mathbb{O}^T(y - x_0) \,.$$

Therefore

$$\begin{aligned} u_{x_i x_j} &= \sum_{k=1}^{n} (u_{y_k} y_{x_i})_{x_j} \\ &= \sum_{k=1}^{n} o^{ki} u_{y_k x_j} \\ &= \sum_{k,l=1}^{n} o^{ki} \left(u_{y_k y_l} y_{x_j} \right) \\ &= \sum_{k,l=1}^{n} u_{y_k y_l} o^{ki} o^{lj} \end{aligned}$$

for all $i, j = 1, \dots, n$ and thus

$$\begin{aligned} Lu &= \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} \\ &= \sum_{i,j,k,l=1}^{n} o^{ki} a^{ij} o^{lj} u_{y_k y_l} \\ &= \sum_{k=1}^{n} u_{y_k y_k} \leq 0 \quad \text{at } (x_0, t_0) \end{aligned}$$

since $o^{ki} a^{ij} o^{lj} = d_k \delta_l^k$, $d_k > 0$ and $D^2 u \leq 0$. This implies that

$$u_t \geq Lu \quad \text{at } (x_0, t_0)$$

which is a contradiction to (4.4). Thus u can not have it's maximum in the interior of Ω_T.

Next we consider the case $t_0 = T$. Since u has it's maximum over Ω_T at (x_0, t_0) we have

$$u_t \geq 0 \quad \text{at } (x_0, t_0).$$

Since we still have $Lu \leq 0$ we get

$$u_t - Lu \geq 0 \quad \text{at } (x_0, t_0)$$

again and this is a contradiction to (4.4).

Then assume that

$$u_t - Lu \geq 0 \quad \text{in } \Omega_T \tag{4.5}$$

and define

$$u^\varepsilon := u(x,t) - \varepsilon t$$

where $\varepsilon > 0$ was fixed before. Then

$$u_t^\varepsilon - Lu^\varepsilon = u_t - Lu - \varepsilon < 0$$

in Ω_T. Thus

$$\max_{\overline{\Omega}_T} u^\varepsilon = \max_{\Gamma_T} u^\varepsilon$$

and letting $\varepsilon \to 0$ implies that

$$\max_{\overline{\Omega}_T} u = \max_{\Gamma_T} u \,.$$

This proves i), and ii) is proved by setting $\widetilde{u} := -u$ and then applying i). □

4.2 Harnack's inequality

Next we will prove our first version of the Harnack's inequality under the assumption that the coefficients a^{ij} of L are smooth. In this proof the constant C of equation (4.6) will depend on the coefficients.

Theorem 4.2.1 (Harnack's inequality, first version). *Assume that $u \in C^{2,1}(\Omega_T)$ solves* (4.1)*, with smooth coefficients, in Ω_T and that*

$$u \geq 0$$

in Ω_T. Suppose then that $D \subset\subset \Omega$ is connected. Then for each $0 < t_1 < t_2 < T$ there exists a constant $C = C(D, t_1, t_2, a^{ij})$ such that

$$\sup_{x \in D} u(x,t_1) \leq C \inf_{x \in D} u(x,t_2) \,. \tag{4.6}$$

Proof. First write

$$v := \log u$$

in Ω_T. Then we can just calculate

$$\begin{aligned} v_t &= (\log u)_t \\ &= \frac{1}{u} \cdot u_t \\ &= \frac{1}{u} \cdot \left(\sum_{i,j=1}^{n} a^{ij} u_{x_i\, x_j} \right) \\ &= \sum_{i,j=1}^{n} a^{ij} (v_{x_i\, x_j} + v_{x_i} v_{x_j}) \\ &= \sum_{i,j=1}^{n} a^{ij} v_{x_i\, x_j} + \sum_{i,j=1}^{n} a^{ij} v_{x_i} v_{x_j} \\ &=: w + \widetilde{w} \ , \end{aligned}$$

where w and $\widetilde{w}$ denote the two sums respectively and the fourth equality follows from

$$\begin{aligned} u_{x_i\, x_j} &= (e^v)_{x_i\, x_j} \\ &= \left(v_{x_i} e^v \right)_{x_j} \\ &= v_{x_i\, x_j} e^v + v_{x_i} v_{x_j} e^v \\ &= u \cdot \left(v_{x_i\, x_j} + v_{x_i} v_{x_j} \right) \ . \end{aligned}$$

Then for $k, l = 1, \ldots, n$, we have

$$v_{x_k\, x_l\, t} = w_{x_k\, x_l} + \widetilde{w}_{x_k\, x_l} \ . \tag{4.7}$$

For the last term on the right hand side of the equation we get

$$
\begin{aligned}
\widetilde{w}_{x_k x_l} &= \left(\sum_{i,j=1}^{n} a^{ij} v_{x_i} v_{x_j} \right)_{x_k x_l} \\
&= \sum_{i,j=1}^{n} \left(a^{ij}_{x_k} v_{x_i} v_{x_j} + a^{ij} v_{x_i x_k} v_{x_j} + a^{ij} v_{x_i} v_{x_j x_k} \right)_{x_l} \\
&= \sum_{i,j=1}^{n} \big(a^{ij}_{x_k x_l} v_{x_i} v_{x_j} + a^{ij}_{x_k} v_{x_i x_l} v_{x_j} + a^{ij}_{x_k} v_{x_i} v_{x_j x_l} \\
&+ a^{ij}_{x_l} v_{x_i x_k} v_{x_j} + a^{ij} v_{x_i x_k x_l} v_{x_j} + a^{ij} v_{x_i x_k} v_{x_j x_l} \\
&+ a^{ij}_{x_l} v_{x_i} v_{x_j x_k} + a^{ij} v_{x_i x_l} v_{x_j x_k} + a^{ij} v_{x_i} v_{x_j x_k x_l} \big) \\
&= \sum_{i,j=1}^{n} \left(2a^{ij} v_{x_i x_k x_l} v_{x_j} + 2a^{ij} v_{x_i x_k} v_{x_j x_l} \right) + R
\end{aligned}
$$

where

$$
\begin{aligned}
|R| &\leq \sum_{i,j=1}^{n} \left(2 \left| a^{ij}_{x_k} \right| \left| v_{x_i x_l} \right| \left| v_{x_j} \right| + 2 \left| a^{ij}_{x_l} \right| \left| v_{x_i x_k} \right| \left| v_{x_j} \right| + \left| a^{ij}_{x_k x_l} \right| \left| v_{x_i} \right| \left| v_{x_j} \right| \right) \\
&\leq n^2 \left(4 \left| D\mathbb{A} \right| \left| D^2 v \right| \left| Dv \right| + \left| D^2 \mathbb{A} \right| \left| Dv \right|^2 \right) \qquad (4.8) \\
&\leq \varepsilon \left| D^2 v \right|^2 + C(\varepsilon) \left| Dv \right|^2 .
\end{aligned}
$$

The last inequality follows from Theorem 2.5.2 that is the Cauchy's inequality with ε. From these calculations we get

$$
v_{x_k x_l t} = w_{x_k x_l} + \sum_{i,j=1}^{n} \left(2a^{ij} v_{x_i x_k x_l} v_{x_j} + 2a^{ij} v_{x_i x_k} v_{x_j x_l} \right) + R \, .
$$

This implies that

$$
\begin{aligned}
w_t &= \sum_{i,j=1}^{n} \left(a^{ij} v_{x_i x_j}\right)_t \\
&= \sum_{i,j=1}^{n} a^{ij}_t v_{x_i x_j} + \sum_{i,j=1}^{n} a^{ij} v_{x_i x_j t} \\
&= \sum_{i,j=1}^{n} a^{ij} w_{x_i x_j} + 2 \sum_{i,j,k,l=1}^{n} a^{kl} a^{ij} v_{x_l x_i x_j} v_{x_k} + 2 \sum_{i,j,k,l=1}^{n} a^{ij} a^{kl} v_{x_k x_i} v_{x_l x_j} + \widetilde{R} \\
&= \sum_{i,j=1}^{n} a^{ij} w_{x_i x_j} + 2 \sum_{k,l=1}^{n} a^{kl} v_{x_k} \left(\sum_{i,j=1}^{n} (a^{ij} v_{x_i x_j})_{x_l} - a^{ij}_{x_l} v_{x_i x_j} \right) \\
&+ 2 \sum_{i,j,k,l=1}^{n} a^{ij} a^{kl} v_{x_k x_i} v_{x_l x_j} + \widetilde{R} \\
&= \sum_{i,j=1}^{n} a^{ij} w_{x_i x_j} + 2 \sum_{k,l=1}^{n} a^{kl} w_{x_l} v_{x_k} + 2 \sum_{i,j,k,l=1}^{n} a^{ij} a^{kl} v_{x_k x_i} v_{x_l x_j} + \widetilde{\widetilde{R}} ,
\end{aligned}
$$

where we have absorbed the terms containing derivatives of a^{ij} to $\widetilde{R}$ and $\widetilde{\widetilde{R}}$ since they satisfy (4.8).

Next we choose ε to be small enough and then using the ellipticity condition (1.2) gives

$$
w_t - \sum_{i,j=1}^{n} a^{ij} w_{x_i x_j} + \sum_{i=1}^{n} b_i w_{x_i} \geq \theta^2 |D^2 v|^2 - C|Dv|^2 - C \tag{4.9}
$$

in Ω_T where $b_i := \sum_{j=1}^{n} -a^{ij} v_{x_j}$.

Our next task is to find a similar estimate for $\widetilde{w}$. To this end we first find out that

$$\begin{aligned}
\widetilde{w}_t - \sum_{k,l=1}^{n} a^{kl}\widetilde{w}_{x_k x_l} &= \sum_{k,l=1}^{n} a^{kl}(v_{x_k}v_{x_l})_t + \sum_{k,l=1}^{n} a^{kl}\left(w_{x_k x_l} - v_{x_k x_l}\right) + R \\
&= 2\sum_{i,j=1}^{n} a^{ij}v_{x_i t}v_{x_j} + \sum_{k,l=1}^{n}\left(-2\sum_{i,j=1}^{n} a^{ij}v_{x_i x_k x_l}v_{x_j}\right. \\
&\quad \left. - 2\sum_{i,j=1}^{n} a^{ij}v_{x_i x_k}v_{x_j x_l}\right) + R - \widetilde{\widetilde{R}} \\
&= 2\sum_{i,j=1}^{n} a^{ij}v_{x_j}\left(v_{x_i t} - \sum_{k,l=1}^{n} a^{kl}v_{x_i x_k x_l}\right) - 2\sum_{i,j,k,l=1}^{n} a^{ij}a^{kl}v_{x_i x_k}v_{x_j x_l} + \overline{R} \\
&= 2\sum_{i=1}^{n}(-b^i)w_{x_i} - 2\sum_{i,j,k,l=1}^{n} a^{ij}a^{kl}v_{x_i x_k}v_{x_j x_l} + \overline{R}
\end{aligned}$$

and this implies with condition (1.2) that

$$\widetilde{w}_t - \sum_{i,j=1}^{n} a^{ij}\widetilde{w}_{x_i x_j} + \sum_{i=1}^{n} b^i\widetilde{w}_{x_i} \geq -C|D^2v|^2 - C|Dv|^2 - C \tag{4.10}$$

in Ω_T.

Then we define function

$$\hat{w} := w + \kappa\widetilde{w}$$

where the constant $\kappa > 0$ is to be selected later on. Now combining (4.9) and (4.10) gives

$$\hat{w}_t - \sum_{i,j=1}^{n} a^{ij}\hat{w}_{x_i x_j} + \sum_{i=1}^{n} b^i\hat{w}_{x_i} \geq \frac{\theta^2}{2}|D^2v|^2 - C|Dv|^2 - C \tag{4.11}$$

provided $0 < \kappa \leq \frac{1}{2}$ is fixed to be small enough. Then we take an open ball $B \subset\subset \Omega$ and $0 < t_1 < t_2 \leq T$ and choose a cut-off function $\phi \in C^\infty(\Omega_T)$ such that

$$\begin{cases} 0 \leq \phi \leq 1, \\ \phi = 0 \quad \text{on} \quad \Gamma_T, \\ \phi = 1 \quad \text{on} \quad B \times [t_1, t_2]. \end{cases}$$

Then let μ be a positive constant that we determine later on and suppose that

$$\phi^4\hat{w} + \mu t$$

has a negative minimum at some point $(x_0, t_0) \in \Omega_T$. Then we have

$$0 = (\phi^4 \hat{w} + \mu t)_{x_k} = 4\phi^3 \phi_{x_k} + \phi^4 \hat{w}_{x_k}$$

for all $k = 1, \dots, n$. This implies that

$$4\phi_{x_k} \widetilde{w} + \phi \hat{w}_{x_k} = 0$$

for all $k = 1, \dots, n$ at (x_0, t_0). Moreover the calculations in the proof of the weak maximum principle show that

$$\sum_{i,j=1}^{n} a^{ij} (\phi^4 \hat{w} + \mu t)_{x_i x_j} \geq 0$$

and thus we get

$$0 \geq (\phi^4 \hat{w} + \mu t)_t - \sum_{i,j=1}^{n} a^{ij} (\phi^4 \hat{w} + \mu t)_{x_i x_j} \tag{4.12}$$

On the other hand

$$(\phi^4 \hat{w} + \mu t)_t = \mu + (\phi^4)_t \hat{w} + \phi^4 \hat{w}_t$$

where

$$|(\phi^4)_t \hat{w}| = 4|\phi^3 \phi_t \hat{w}| \leq M\phi^2 |\hat{w}|$$

for some $M > 0$. Then

$$\big(\phi^4 \hat{w} + \mu t\big)_{x_i x_j} = \big((\phi^4)_{x_i} \hat{w} + \phi^4 \hat{w}_{x_i}\big)_{x_j} = (\phi^4)_{x_i x_j} \hat{w} + (\phi^4)_{x_i} \hat{w}_{x_j} + (\phi^4)_{x_j} + \hat{w}_{x_i} + \phi^4 \hat{w}_{x_i x_j}$$

and thus we get for the second term of (4.12):

$$-\sum_{i,j=1}^{n} a^{ij} (\phi \hat{w} + \mu t)_{x_i x_j} = -\sum_{i,j=1}^{n} \big(a^{ij} \phi^4 \hat{w}_{x_i x_j} - 2a^{ij} (\phi^4)_{x_i} \hat{w}_{x_j} - a^{ij} (\phi^4)_{x_i x_j} \hat{w}\big)$$

where

$$\left| -\sum_{i,j=1}^{n} a^{ij} (\phi^4)_{x_i x_j} \hat{w} \right| \leq \widetilde{M} \phi^2 |\hat{w}|$$

for some $\widetilde{M} > 0$. Then if we write

$$R = (\phi^4)_t \hat{w} - \sum_{i,j=1}^{n} a^{ij} (\phi^4)_{x_i x_j} \hat{w} \,,$$

we get

$$|R| \leq C\phi^2 |\hat{w}|$$

for some $C > 0$. But in this case

$$0 \geq \mu + \phi^4 \left(\hat{w}_t - \sum_{i,j=1}^{n} a^{ij} \hat{w}_{x_i x_j} \right) - 2 \sum_{i,j=1}^{n} a^{ij} (\phi^4)_{x_i} \hat{w}_{x_j} + R \,. \tag{4.13}$$

Now we get from (4.11) and (4.12)

$$0 \geq \mu + \phi^4 \left(\frac{\theta^2}{2} |Dv|^2 - C|Dv|^2 - C - \sum_{i=1}^{n} b^i \hat{w}_{x_i} \right) + R \,, \tag{4.14}$$

where R is again a term satisfying

$$|R| \leq C\phi^2 |\hat{w}|$$

for some $C > 0$. Then from (4.12) and the definition of b^i it follows that

$$0 \geq \mu + \phi^4 \left(\frac{\theta^2}{2} |D^2 v|^2 - C|Dv|^2 - C \right) + \widetilde{R} \,,$$

where

$$|\widetilde{R}| \leq C\phi^2 |\hat{w}| + C\phi^3 |Dv| |\hat{w}| \,. \tag{4.15}$$

Now at (x_0, t_0) we have

$$\hat{w} = w + \kappa \widetilde{w} < 0 \,.$$

Since it also follows from the definition of w and $\widetilde{w}$ that

$$|Dv|^2 \leq C|D^2 v| \,,$$

we get

$$|\hat{w}| \leq C|D^2 v|$$

at (x_0, t_0). Then (4.15) gives

$$\begin{aligned} |\widetilde{R}| &\leq C\phi^2|D^2 v| + C\phi^3|D^2 v|^{3/2} \\ &\leq \varepsilon\phi^4|D^2 v|^2 + C(\varepsilon) \ , \end{aligned}$$

where the second inequality follows by Theorem 2.5.4.
So we see that there can't be a negative minimum of $\phi^4\hat{w} + \mu t$ at (x_0, t_0) if μ is large enough. This implies that

$$\phi^4\hat{w} + \mu t \geq 0$$

at Ω_T and in particular

$$\hat{w} + \mu t \geq 0$$

in $B \times [t_1, t_2]$. Then since $v_t = w + \widetilde{w}$ we have

$$w + \kappa\widetilde{w} + \mu t \geq 0 \ ,$$

from which follows

$$0 \leq v_t + (\kappa - 1)\widetilde{w} + \mu t \leq v_t - \frac{1}{2}\widetilde{w} + \mu t \ .$$

Thus

$$v_t \geq \frac{1}{2}\widetilde{w} - \mu t \alpha|Dv|^2 - \beta \geq \frac{\theta}{2}|Dv|^2 - \mu t$$

which finally implies

$$v_t \geq \alpha|Dv|^2 - \beta \tag{4.16}$$

in $B \times [t_1, t_2]$ for constants $\alpha > 0$ and $\beta > 0$.
To finish this proof we fix $x_1, x_2 \in B$, $t_2 > t_1$ and then we have

$$
\begin{aligned}
v(x_2,t_2) - v(x_1,t_1) &= \int_0^1 \frac{d}{ds} v\left(sx_2 + (1-s)x_1, st_2 + (1-s)t_1\right)\, ds \\
&= \int_0^1 Dv \cdot (x_2 - x_1) + v_t(t_2 - t_1)\, ds \\
&\geq \int_0^1 -|Dv||x_2 - x_1| + (t_2 - t_1)(\alpha|Dv|^2 - \beta)\, ds \\
&\geq -\gamma(\alpha, \beta, |x_1 - x_2|, |t_1 - t_2|)
\end{aligned}
$$

and finally, since $v(x,t) = \log u(x,t)$, we get

$$
\log u(x_2,t_2) \geq \log u(x_1,t_1) - \gamma\,.
$$

Thus

$$
u(x_2,t_2) \geq e^{-\gamma} u(x_1,t_1)\,.
$$

This result holds for any ball in Ω and thus we can extend the result to hold in the whole Ω by covering it with balls and applying the result above to them. □

4.3 Strong maximum principle

The last thing we will prove for this smooth equation is the strong maximum principle. To prove this we will use the weak maximum principle and the first version of the Harnack's equality.

Theorem 4.3.1 (The strong maximum principle)**.** *Assume that $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$, the coefficients a^{ij} of L are smooth and that Ω is connected. Then if*

i)

$$
u_t - Lu \leq 0
$$

in Ω_T and u attains it's maximum over $\overline{\Omega}_T$ at a point $(x_0, t_0) \in \Omega_T$, u is a constant function on $\Omega \times \{t = t_0\}$.

ii)

$$
u_t - Lu \geq 0
$$

in Ω_T and u attains it's minimum over $\overline{\Omega}_T$ at a point $(x_0, t_0) \in \Omega_T$, u is a constant function on $\Omega \times \{t = t_0\}$.

Proof. Assume $u_t - Lu \leq 0$ in Ω_T and u attains its maximum at some point $(x_0, t_0) \in \Omega_T$. Then we choose an open smooth set $\Omega' \subset\subset \Omega$ with $x_0 \in \Omega'$.
Then let v solve

$$\begin{cases} v_t - Lv = 0 & \text{in} \quad \Omega'_T \\ v = u & \text{on} \quad \overline{\Omega'}_T \backslash \Omega_T \,. \end{cases}$$

Then by the weak maximum principle we get that

$$u \leq v \leq M$$

for $M := \max_{\overline{\Omega}_T} u$. In this case we see that

$$v = M \quad \text{at } (x_0, t_0).$$

Next we write $\widetilde{v} := M - v$ and then we have

$$\begin{cases} \widetilde{v}_t - L\widetilde{v} = 0 \\ \widetilde{v} \geq 0 \end{cases}$$

in Ω'_T. Now choose any $\Omega'' \subset\subset \Omega'$ with $x_0 \in \Omega''$, Ω'' connected and let $0 < t < t_0$. Then Harnack's inequality gives

$$\max_{\Omega''} \widetilde{v}(x,t) \leq C \inf_{\Omega''}(x, t_0) \,. \tag{4.17}$$

But since $\widetilde{v} \geq 0$ and we have

$$\inf_{\Omega''} \widetilde{v}(x, t_0) \leq \widetilde{v}(x_0, t_0) = 0 \,,$$

equation (4.17) implies that $\widetilde{v} = 0$ on $\Omega'' \times \{t\}$ for each $0 < t < t_0$. This holds for every Ω'' in Ω and so

$$\widetilde{v} \equiv 0$$

on $\Omega' \times \{t = t_0\}$. Then going back to v gives

$$v \equiv M$$

in $\Omega \times \{t = t_0\}$. But now we have $v = u$ on $\overline{\Omega} \backslash \Omega'$ and thus

$$u \equiv M$$

on $\partial\Omega' \times \{t = t_0\}$. But finally we notice that since this holds for all sets Ω' it follows that

$$u \equiv M$$

on $\Omega \times \{t = t_0\}$ and the theorem is proved. $\square$

Chapter 5

Equations with bounded and measurable coefficients

In this chapter we assume that the coefficients $a^{ij}(x,t)$ of equation (1.1) are measurable L^∞-functions that satisfy the ellipticity condition (1.2).

5.1 Existence of weak solutions

Now we prove that there exists a weak solution to equation (1.1).

Theorem 5.1.1 (Existence of a weak solution)**.** *Suppose that $a_{ij}(x,t) \in L^\infty(\Omega_T)$ for all $i,j = 1,\dots,n$ satisfy ellipticity condition* (1.2) *for constants $\lambda, \Lambda > 0$. Then for any function $g \in L^2(\Omega)$, there exists a weak solution $u \in V^2(\Omega_T)$ of equation* (1.1) *such that $u(x,0) = g(x)$.*

Proof. We will prove this theorem by using Galerkin's method. That is, we first find solutions $u^m(x,t)$ of certain finite dimensional approximations to equation (1.3) and then show that $u^m \to u \in V^2(\Omega_T)$ such that the convergence is uniform with respect to the t variable and that $u(x,t)$ is indeed a weak solution to (1.1). We follow [8, Ch. III] and [2, ch. 7.1.2] in this proof.

First we take an orthogonal set of functions $\psi_k(x) \in W_0^{1,2}(\Omega)$ for all $k = 0,1,\dots$ such that they are an orthonormal in $L^2(\Omega)$. Then the solutions that we are looking for are

$$u^m(x,t) = \sum_{k=1}^{m} c_k^m(t)\psi_k(x) \tag{5.1}$$

where $(\cdot,\cdot)_{L^2(\Omega)}$ denotes the inner product in $L^2(\Omega)$ and $(\cdot,\cdot)$ is the inner product of $\mathbb{R}^n$. Now the functions $c_k^m(t)$ are determined from the conditions

$$\frac{d}{dt}(u^m,\psi_k)_{L^2(\Omega)} + \int_\Omega (\mathbb{A}Du^m, D\psi_k)\,dx = 0\,, \tag{5.2}$$

and

$$c_k^m(0) = (g,\psi_k)\ ,$$

for all $k = 1,\dots,m$. Now substituting u^m from (5.1) to (5.2) gives

$$c_k^{m\,\prime} + \sum_{l=1}^{m} A^{kl} c_l^m = 0 \tag{5.3}$$

where

$$A^{kl} := \int_\Omega (\mathbb{A}D\psi_k, D\psi_l)\,dx$$

are integrable functions of t for all $k = 1,\dots,m$. The well known theory for the ordinary differential equations provides us with a unique and absolutely continuous solution to (5.3) [2, p. 354]. This means that the solutions $c_k^m(t)$ are differentiable for a.e. $t \in T$ and that $c_k^{m\,\prime}$ is integrable on T for all $k = 1,\dots,m$.

Our next task is then to show that actually

$$||u^m||_{V^2(\Omega_t)} \le C \tag{5.4}$$

for all $m \in \mathbb{N}$ and where the constant C doesn't depend on m. To achieve this we first multiply equation in (5.2) by the corresponding C_k^m:

$$c_k^m \frac{d}{dt}(u^m,\psi_k) + \int_\Omega (\mathbb{A}D\psi_k, D(\psi_l c_k^m))\,dx = 0 \text{ (for a.e. } t \in T)\ .$$

Now from the product rule we get an equivalent equality:

$$\frac{d}{dt}\,(U^m,\psi_k c_k^m)_{L^2(\Omega)} - c_k^{m\,\prime}\cdot \int_\Omega u^m\psi_k\,dx + \int_\Omega (\mathbb{A}D\psi_k, D(\psi_l c_k^m))\,dx = 0\,. \tag{5.5}$$

But in this case we can calculate the second term as follows:

$$
\begin{aligned}
-c_k^{m\,\prime} \cdot \int_\Omega u^m \psi_k \, dx &= \sum_{l=1}^{m} A^{kl} c_l^m \int_\Omega u^m \psi_k \, dx \\
&= \sum_{l=1}^{m} \int_\Omega (\mathbb{A} D\psi_k, D\psi_l c_l^m(t)) \, dx \int_\Omega u^m \psi_k \, dx \\
&= \sum_{s=1}^{m} \int_\Omega (\mathbb{A} D\psi_k c_s^m, Du^m) \, dx \int_\Omega \psi_s \psi_k \, dx \\
&= \sum_{s=1}^{m} \int_\Omega (\mathbb{A} D\psi_k c_s^m, Du^m) \, dx \, \delta_s^k \\
&= \int_\Omega (\mathbb{A} D\psi_k c_k^m, Du^m) \, dx
\end{aligned}
$$

Then applying this to (5.5) and summing up those equations from $k = 1, \ldots, m$ gives

$$\frac{d}{dt} ||u^m||^2_{L^2(\Omega)} + 2 \int_\Omega (\mathbb{A} Du^m, Du^m) \, dx = 0 \, .$$

Now the ellipticity condition (1.2) implies that

$$\frac{d}{dt} ||u^m||^2_{L^2(\Omega)} + 2\lambda ||Du^m||^2_{L^2(\Omega)} \leq 0 \tag{5.6}$$

and thus

$$\frac{d}{dt} ||u^m||^2_{L^2(\Omega)} \leq 0.$$

Now integrating from 0 to t gives

$$||u^m(x,t)||^2_{L^2(\Omega)} - ||u^m(x,0)||^2_{L^2(\Omega)} \leq 0$$

that implies

$$||u^m(x,t)||^2_{L^2(\Omega)} \leq ||g||^2_{L^2(\Omega)} = M < \infty$$

with the constant M not depending on m. Thus we have

$$\sup_{t \in [0,T]} ||u^m(x,t)||^2_{L^2(\Omega)} \leq C_1 < \infty \tag{5.7}$$

where C_1 doesn't depend on m. We have now proved the first part of (5.4) and next we proceed to the second one. On the other hand (5.6) and Theorem 2.3.4 imply that

$$\frac{d}{dt}||u^m||^2_{L^2(\Omega)} + \lambda\left(C||u^m||^2_{L^2(\Omega)} + ||Du^m||^2_{L^2(\Omega)}\right) \leq 0 \tag{5.8}$$

for constant C. Thus for $\widetilde{C} =: \min(1, C)$ we have

$$\frac{d}{dt}||u^m||^2_{L^2(\Omega)} + \lambda\widetilde{C}||u^m||^2_{W^{1,2}(\Omega)} \leq 0$$

Then integrating this over $[0, T]$ gives

$$||u^m||_{L^p(\Omega)} + \int_0^T ||u^m||^2_{W^{1,2}(\Omega)}\, dt \leq ||g||_{L^2(\Omega)}$$

and thus

$$\int_0^T ||u^m||^2_{W^{1,2}(\Omega)}\, dt \leq ||g||_{L^2(\Omega)} < \infty. \tag{5.9}$$

Now combining this inequality with (5.7) proves finally (5.4). This implies that functions

$$l_{m,k}(t) := \int_\Omega u^m \psi_k\, dx$$

are uniformly bounded. Our next aim is to show that $l_{m,k}$ are actually equicontinuous on $[0, T]$ for fixed k and arbitrary $m \geq k$. For this purpose we fix $\varepsilon > 0$. Then integrating (5.2) from 0 to t gives

$$l_{m,k}(t) - l_{m,k}(0) = -\int_0^t \int_\Omega (\mathbb{A}Du^m, \psi_k)\, dxdt$$

for all $0 < t < T$. From this equality we can estimate that

$$
\begin{aligned}
|l_{m,k}(t+\delta) - l_{m,k}(t)| &= \left| \int_t^{t+\delta} \int_\Omega (Du^m, \mathbb{A} D\psi_k)\, dxdt \right| \\
&\leq \int_t^{t+\delta} \int_\Omega |(Du^m, \mathbb{A} D\psi_k)|\, dxdt \\
&\leq \int_t^{t+\delta} \int_\Omega (Du^m, \mathbb{A} Du^m)^{\frac{1}{2}} (D\psi_k, \mathbb{A} D\psi_k)^{\frac{1}{2}}\, dxdt \\
&\leq \int_t^{t+\delta} \int_\Omega \Lambda |Du^m||D\psi_k|\, dxdt \\
&\leq \int_t^{t+\delta} \Lambda ||Du^m||_{L^2\Omega} ||D\psi_k||_{L^2(\Omega)}\, dt \\
&\leq \Lambda ||Du^m||_{L^2(t,t+\delta;L^2(\Omega))} ||D\psi_k||_{L^2(t,t+\delta;L^2(\Omega))} \\
&\leq \Lambda \left(\int_t^{t+\delta} ||u^m||^2_{W^{1,2}(\Omega)}\, dt \right)^{\frac{1}{2}} ||D\psi_k||_{L^2(t,t+\delta;L^2(\Omega))}\,,
\end{aligned}
$$

where we used Theorem 2.5.5. Then we notice that from (5.9) it follows that

$$
\int_0^{t+\delta} ||u^m||^2_{W^{1,2}(\Omega)}\, dt
$$

is bounded uniformly for every m and thus we find $\delta > 0$ that is independent of m and so small that we have

$$
\left(\int_t^{t+\delta} ||u^m||^2_{W^{1,2}(\Omega)}\, dt \right)^{\frac{1}{2}} < \varepsilon\,.
$$

but now since also $||D\psi_k||_{L^2(t,t+\delta;L^2(\Omega))}$ is bounded we get that

$$
|l_{m,k}(t+\delta) - l_{m,k}(t)| \leq \varepsilon\, C
$$

for some constant C that doesn't depend on m or k and this implies that $l_{m,k}$ is equicontinuous on $[0, T]$.

Now we are ready to go towards taking the limit $m \to \infty$. To this end we choose a subsequence $l_{m_i,k}$ that converges uniformly to a continuous function $l_k(t)$ for each $k = 1, 2, \ldots$. These functions l_k define a function

$$
u(x,t) = \sum_{k=1}^{\infty} l_k(t) \psi_k(x)\,.
$$

Next we show that

$$||u||_{V^2(\Omega_T)} < \infty\,.$$

To achieve this we first show that $u^{m_i} \to u$ weakly in $L^2(\Omega_T)$ such that the convergence is uniform with respect to $t \in [0,T]$.

Now let $\psi(x) \in L^2(\Omega)$. Then

$$\begin{aligned}
\int_\Omega (u^{m_i} - u)\psi\,dx &= \int_\Omega (u^{m_i} - u)\sum_{k=1}^{\infty}\left(\int_\Omega \psi_k\psi\,dx\right)\psi_k\,dx \\
&= \int_\Omega (u^{m_i} - u)\sum_{k=1}^{s}\left(\int_\Omega \psi_k\psi\,dx\right)\psi_k\,dx \\
&+ \int_\Omega (u^{m_i} - u)\sum_{k=s+1}^{\infty}\left(\int_\Omega \psi_k\psi\,dx\right)\psi_k\,dx \\
&= \sum_{k=1}^{s}\left(\int_\Omega \psi_k\psi\,dx\right)\int_\Omega (u^{m_i} - u)\,\psi_k\,dx \\
&+ \int_\Omega (u^{m_i} - u)\sum_{k=s+1}^{\infty}\left(\int_\Omega \psi_k\psi\,dx\right)\psi_k\,dx\,.
\end{aligned}$$

For the last term we get the following estimate:

$$\begin{aligned}
&\left|\int_\Omega (u^{m_i} - u)\sum_{k=s+1}^{\infty}\left(\int_\Omega \psi_k\psi\,dx\right)\psi_k\,dx\right| \\
&= \left|\sum_{k=s+1}^{\infty}\int_\Omega (u^{m_i} - u)\left(\int_\Omega \psi_k\psi\,dx\right)\psi_k\,dx\right| \\
&\le \sum_{k=s+1}^{\infty}\int_\Omega \left|(u^{m_i} - u)\left(\int_\Omega \psi_k\psi\,dx\right)\psi_k\right|\,dx \\
&\le \sum_{k=s+1}^{\infty} ||u^{m_i} - u||_{L^2(\Omega)}\left|\left|\left(\int_\Omega \psi_k\psi\,dx\right)\psi_k\right|\right|_{L^2(\Omega)} \\
&= \sum_{k=s+1}^{\infty}\left|\left(\int_\Omega \psi_k\psi\,dx\right)\right| ||u^{m_i} - u||_{L^2}||\psi_k||_{L^2} \\
&\le C\sum_{k=s+1}^{\infty}\left|\left(\int_\Omega \psi_k\psi\,dx\right)\right| \\
&\to 0 \quad \text{as}\, s \to \infty\,.
\end{aligned}$$

and where C is independent of m_i. Now we can find $S \in \mathbb{N}$ such that the above sum is less than $\varepsilon/2$. But since we have earlier proven that for fixed s the sum

$$\sum_{k=1}^{s} \left(\int_{\Omega} \psi_k \psi \, dx \right) \int_{\Omega} (u^{m_i} - u) \, \psi_k \, dx$$

will be less than $\varepsilon/2$ for all $t \in [0, T]$. This implies that $|(u^{m_i} - u, \psi)|$ can be made arbitrarily small for all $t \in [0, T]$. In this same way we can show that $u^{m_i} \to u$ weakly in $L^2(\Omega_T)$ and uniformly with respect to t.

Now from (5.4) and the sequential compactness of $V^2(\Omega_T)$ we see that we can choose such a subsequence $u^{m_{i_j}}$ of u^{m_i} that also $Du^{m_{i_j}} \to Du$ in $L^2(\Omega_T)$. Then by weak convergence we see that

$$||u||_{V^2(\Omega_T)} \leq C < \infty \tag{5.10}$$

and thus $u \in V^2(\Omega_T)$.

Then we finally show that u satisfies (1.3). To this end we multiply the equation (5.2) by a smooth function $d_k(t)$ that has compact support in T and sum over all k from 1 to $m' \leq m$ and integrate the result from 0 to T. This gives

$$\sum_{k=1}^{m'} \left(\int_0^T \frac{d}{dt} (u^m, \psi_k)_{L^2(\Omega)} \, d_k \, dt + \iint_{\Omega_T} (\mathbb{A} Du^m, D\psi_k) d_k dx dt \right) = 0 \,.$$

Then integrating by parts in the first term and defining

$$\phi^{m'}(x,t) = \sum_{k=1}^{m'} d_k(t) \psi_k(x)$$

we get

$$-\int_0^T \int_{\Omega} u^m \phi_t^{m'} \, dx \, d_k \, dt + \sum_{i,j=1}^{n} \iint_{\Omega_T} a^{ij} u^m_{x_i} \phi^{m'}_{x_j} \, dx dt = 0 \,.$$

But now taking the weak limit with respect to u^m gives

$$-\int_0^T (u, \phi_t^{m'}) d_k \, dt + \iint_{\Omega_T} \left(Du, \mathbb{A} D\phi^{m'} \right) \, dx dt = 0 \,.$$

Then finally we can find a sub sequence $\phi^{m'_i}$ of $\phi^{m'}$ such that $\phi^{m'_i} \to \phi$ weakly in $V_0^2(\Omega_T)$. This completes the proof of the theorem since $C_0^\infty(\Omega_T) \subset V_0^2(\Omega_T)$. □

5.2 Proof of the Harnack's inequality

We now assume that the coefficients of the equation are smooth and the weak solution of the equation is actually a classical one. We will establish Harnack's inequality with a constant that is independent of these smooth coefficients and the solution itself. This is why we can't use the methods of the last section and why we need to use totally new ideas. The proof is due to Moser [6], [7].

5.2.1 Main lemmas

The proof consists of three main lemmas. There are also four other lemmas that are needed in the proofs of the three main ones and these are proved later on. The first two of the main lemmas are needed to prove the third one that is then used in the proof of the Harnack's inequality. In the first lemma we show that the supremum of the solution of (1.1) is controlled by its L^p-norm with p near to zero. This corresponds to the geometrical mean of the solution.

We denote by $R(\rho)$ the cylinder

$$R(\rho) := \left\{(x,t) \in \mathbb{R}^{n+1} \,\middle|\, |x| < \rho,\ |t| < \rho^2\right\} ,$$

where $\rho > 0$. Similarly, let

$$R^+(\rho) := \left\{(x,t) \in R(\rho) \,\middle|\, |x| < \rho,\ 0 < t < \rho^2\right\}$$

and

$$R^-(\rho) := \left\{(x,t) \in R(\rho) \,\middle|\, |x| < \rho,\ -\rho^2 < t < 0\right\} .$$

Lemma 5.2.1. *Let $\frac{1}{2} \le \rho < r \le 1$ and $\mu = \lambda^{-1} + \Lambda$ and assume that u is a positive solution of (1.1). Then there exists a positive constant $C_1 = C_1(n)$ such that*

$$\sup_{R(\rho)} u^p \le \frac{C_1}{(r-\rho)^{n+2}} \iint_{R(r)} u^p \, dx\, dt \tag{5.11}$$

for all p such that $0 < p < \mu^{-1}$. Similarly

$$\sup_{R^-(\rho)} u^p \le \frac{C_1}{(r-\rho)^{n+2}} \iint_{R^-(r)} u^p \, dx\, dt \tag{5.12}$$

for all p such that $0 < -p < \mu^{-1}$.

Proof. Let $\mu = \lambda^{-1} + \Lambda$. We first prove (5.11) under the assumption that $\rho = \frac{1}{2}$ and $r = 1$. We can then easily extend the result to the more general case. At first let $\phi \in C^\infty(R(1))$ and have compact support in $|x| < 1$ for every t. Then we define $\psi \in C^\infty(R(1))$ for every $p \neq 0, 1$ by setting

$$\phi = u^{1-p}\psi^2$$

and we also define function

$$v = u^{p/2}.$$

Now we fix $\frac{1}{2} \leq \rho < r \leq 1$ and use the fact that u is a solution to (1.1) which implies that

$$\iint \phi u_t + (D\phi, \mathbb{A}Du)\, dx\, dt = 0\,. \tag{5.13}$$

We would like to write this with ψ and v defined above and for this purpose we calculate next ϕu_t and $(D\phi, \mathbb{A}Du)$. Now since

$$\begin{cases} \phi = u^{p-1}\psi^2 = \left(v^{2/p}\right)^{p-1}\psi^2 = v^{2(1-1/p)}\psi^2 \quad \text{and} \\ u_t = \left(v^{2/p}\right)_t = \left(v^2\right)_t \frac{1}{p} v^{2(1/p-1)} \end{cases}$$

we get

$$\phi u_t = \frac{v^{2(1/p-1)}}{v^{2(1/p-1)}} \cdot \frac{1}{p}\left(v^2\right)_t \psi^2 = \frac{1}{p}\left(v^2\right)_t \psi^2. \tag{5.14}$$

Moreover we have

$$\begin{aligned} D\phi &= D(v^{2(1-1/p)}) \\ &= 2(1-1/p)v^{2(1-1/p)-1}Dv\phi^2 + v^{2(1-1/p)} \cdot 2\psi D\psi \\ &= 2v^{2(1-1/p)}\left[(1-\frac{1}{p})v^{-1}\psi^2 Dv + \psi D\psi\right] \end{aligned}$$

and

$$Du = D(v^{(2/p)}) = \frac{2}{p}v^{2/p-1}Dv = \frac{2}{p}v^{2(1/p-1)+1}Dv$$

which gives us

$$
\begin{aligned}
(D\phi, \mathbb{A}Du) &= \frac{2}{p}\frac{v^{2(1/p-1)+1}}{v^{2(1/p-1)}} \cdot 2\left(\left(1-\frac{1}{p}\right)v^{-1}\psi^2 Dv + \psi D\psi, \mathbb{A}Dv\right) \\
&= \frac{4v}{p}\left(v^{-1}\left(1-\frac{1}{p}\right)\psi^2 Dv + \psi D\psi, \mathbb{A}Dv\right) \\
&= \frac{4}{p}\left(1-\frac{1}{p}\right)\psi^2 (Dv, \mathbb{A}Dv) + \frac{4v}{p}\psi(D\psi, \mathbb{A}Dv).
\end{aligned}
$$

Then substituting these to (5.13) we get

$$
\iint \frac{4}{4p}(v^2)_t\psi^2 + \frac{4}{p}\left(1-\frac{1}{p}\right)\psi^2(Dv, \mathbb{A}Dv) + \frac{4v}{p}\psi(D\psi, \mathbb{A}Dv)\, dx\, dt = 0
$$

that implies

$$
\begin{aligned}
&\frac{1}{4}\iint \psi^2(v^2)_t\, dx\, dt + \left(1-\frac{1}{p}\right)\iint \psi^2(Dv, \mathbb{A}Dv)\, dx\, dt \\
=&-\iint v\psi(D\psi, \mathbb{A}Dv)\, dx\, dt.
\end{aligned} \tag{5.15}
$$

Our next task is to estimate the last integrand $-v\psi(D\psi, \mathbb{A}Dv)$. To do this we first fix $\varepsilon > 0$. Then by Theorem 2.5.5,

$$
|v\psi(D\psi, \mathbb{A}Dv)| \le \frac{v^2}{4\varepsilon}(D\psi, \mathbb{A}D\psi) + \varepsilon\psi^2(Dv, \mathbb{A}Dv).
$$

Substituting this to equation (5.15) gives the two inequalities

$$
\begin{aligned}
&\frac{1}{4}\iint \psi^2(v^2)_t\, dx\, dt + \left(1-\frac{1}{p}\right)\iint \psi^2(Dv, \mathbb{A}Dv)\, dx\, dt \\
\le&\frac{1}{4\varepsilon}\iint v^2(D\psi, \mathbb{A}\psi)\, dx\, dt + \varepsilon\iint \psi^2(Dv, \mathbb{A}Dv)\, dx\, dt
\end{aligned} \tag{5.16}
$$

and

$$
\begin{aligned}
&-\frac{1}{4}\iint \psi^2(v^2)_t\, dx\, dt - \left(1-\frac{1}{p}\right)\iint \psi^2(Dv, \mathbb{A}Dv)\, dx\, dt \\
\le&\frac{1}{4\varepsilon}\iint v^2(D\psi, \mathbb{A}\psi)\, dx\, dt + \varepsilon\iint \psi^2(Dv, \mathbb{A}Dv)\, dx\, dt.
\end{aligned} \tag{5.17}
$$

For (5.16) we notice that from the product rule $\left(\psi^2 v^2\right)_t = \left(\psi^2\right)_t v^2 + \psi^2 \left(v^2\right)_t$ we get that $\psi^2 \left(v^2\right)_t = \left(\psi^2 v^2\right) - v^2 \left(\psi^2\right)_t$. This implies

$$\begin{aligned}
&\frac{1}{4}\iint \left(\psi^2 v^2\right)_t + \left[-\varepsilon + \left(1 - \frac{1}{p}\right)\right] \psi^2 (Dv, \mathbb{A}Dv)\, dx\, dt \\
\leq &\frac{1}{4}\iint v^2 \left(\frac{1}{\varepsilon}(D\psi, \mathbb{A}D\psi) + \left(\psi^2\right)_t\right)\, dx\, dt\ .
\end{aligned}$$

Then putting in $(\psi^2)_t = 2\psi\psi_t \leq 2|\psi\psi_t|$ gives

$$\begin{aligned}
&\frac{1}{4}\iint \left(\psi^2 v^2\right)_t + \left[-\varepsilon + \left(1 - \frac{1}{p}\right)\right] \psi^2 (Dv, \mathbb{A}Dv)\, dx\, dt \\
\leq &\frac{1}{4}\iint v^2 \left(\frac{1}{\varepsilon}(D\psi, \mathbb{A}D\psi) + 2|\psi\psi_t|\right)\, dx\, dt\ .
\end{aligned} \tag{5.18}$$

Similar calculations for (5.17) imply that

$$\begin{aligned}
&\frac{1}{4}\iint -\left(\psi^2 v^2\right)_t + \left[-\varepsilon - \left(1 - \frac{1}{p}\right)\right] \psi^2 (Dv, \mathbb{A}Dv)\, dx\, dt \\
\leq &\frac{1}{4}\iint v^2 \left(\frac{1}{\varepsilon}(D\psi, \mathbb{A}D\psi) + 2|\psi\psi_t|\right)\, dx\, dt\ .
\end{aligned} \tag{5.19}$$

Now we would like to use the conditions of (1.2) to $(D\psi, \mathbb{A}D\psi)$ and $(Dv, \mathbb{A}Dv)$ but before this we have to assure that both $\left[-\varepsilon + \left(1 - \frac{1}{p}\right)\right]$ of (5.18) and $\left[-\varepsilon - \left(1 - \frac{1}{p}\right)\right]$ of (5.19) are nonnegative. Since this inequality holds for all $\varepsilon > 0$ we can now choose it to be

$$\varepsilon = \frac{1}{2}\left|1 - \frac{1}{p}\right|. \tag{5.20}$$

This is possible since we only consider the case where $p \neq 0, 1$. Now if $p > 1$ we have

$$\left(1 - \frac{1}{p}\right) = 2\varepsilon\,.$$

Substituting this to (5.18) gives

$$\begin{aligned}&\frac{1}{4}\iint \left(\psi^2 v^2\right)_t \, dx\, dt + \varepsilon \iint_{R(r)} \psi^2 (Dv, \mathbb{A} Dv)\, dx\, dt \\ \leq &\frac{1}{4}\iint v^2 \left(\frac{1}{\varepsilon}(D\psi, \mathbb{A} D\psi) + 2|\psi\psi_t|\right) dx\, dt \ .\end{aligned}$$

If on the other hand $p < 1$, we have

$$-\left(1 - \frac{1}{p}\right) = 2\varepsilon$$

and this with (5.18) implies that

$$\begin{aligned}-&\frac{1}{4}\iint \left(\psi^2 v^2\right)_t \, dx\, dt + \varepsilon \iint_{R(r)} \psi^2 (Dv, \mathbb{A} Dv)\, dx\, dt \\ \leq &\frac{1}{4}\iint v^2 \left(\frac{1}{\varepsilon}(D\psi, \mathbb{A} D\psi) + 2|\psi\psi_t|\right) dx\, dt \, .\end{aligned}$$

Now conditions (1.2) imply

$$\begin{aligned}&\frac{1}{4}\iint \left(\psi^2 v^2\right)_t \, dxdt + \varepsilon\lambda \iint \psi^2 |Dv|^2 \, dx\, dt \\ \leq &\frac{1}{4}\iint v^2 \left(\frac{\Lambda}{\varepsilon}|D\psi|^2 + 2|\psi\psi_t|\right) dx\, dt\end{aligned} \tag{5.21}$$

for $p > 1$ and

$$\begin{aligned}-&\frac{1}{4}\iint \left(\psi^2 v^2\right)_t \, dxdt + \varepsilon\lambda \iint \psi^2 |Dv|^2 \, dx\, dt \\ \leq &\frac{1}{4}\iint v^2 \left(\frac{\Lambda}{\varepsilon}|D\psi|^2 + 2|\psi\psi_t|\right) dx\, dt\end{aligned} \tag{5.22}$$

for $p < 1$. Next we will consider these inequalities in suitable regions of integration. First we take (5.22) and notice that since the second term in the above is nonnegative, we get

$$-\iint \left(\psi^2 v^2\right)_t \, dxdt \leq \iint v^2 \left(\frac{\Lambda}{\varepsilon}|D\psi|^2 + 2|\psi\psi_t|\right) dx\, dt \tag{5.23}$$

Then we choose ψ such that $\psi = 1$ in $R^+(\rho)$, $\psi = 0$ outside of $R^+(r)$, $|D\psi| \leq \sqrt{C_1}\frac{1}{r-\rho}$

and $|\psi_t| \leq C_1 \frac{1}{r^2-\rho^2}$ for a suitable constant C_1 that doesnt depend on any of the parameters. This gives

$$\left(\frac{\Lambda}{\varepsilon}|D\psi|^2 + 2|\psi\psi_t|\right) \leq C_1\left(\frac{\Lambda}{\varepsilon}\frac{1}{(r-\rho)^2} + \frac{1}{r^2-\rho^2}\right) \leq \frac{C_2}{(r-\rho)^2}\left(\frac{\Lambda}{\varepsilon}+1\right)$$

where C_2 is another absolute constant.

If we now fix the integration to be taken over the set $R^+_\sigma := \{(x,t) \in R^+(r) : 0 < \sigma < t < r^2, |x| < \rho\}$, where σ is chosen such that

$$\int_{|x|<\rho} v(x,\sigma)^2\,dx \geq \frac{1}{2}\sup_{0<t<r^2}\int_{|x|<\rho} v^2\,dx\,,$$

then it follows from (5.23) that

$$\frac{1}{2}\sup_{0<t<r^2}\int_{|x|<\rho} v^2\,dx \leq \frac{C_2}{(r-\rho)^2}\left(\frac{\Lambda}{\varepsilon}+1\right)\iint_{R^+_\sigma} v^2\,dx\,dt.$$

But now well known properties of the supremum and integration finally imply that

$$\sup_{0<t<\rho^2}\int_{|x|<\rho} v^2\,dx \leq \frac{2C_2}{(r-\rho)^2}\left(\frac{\Lambda}{\varepsilon}+1\right)\iint_{R^+(r)} v^2\,dx\,dt$$

for $p<1$. With the same argument we get

$$\sup_{0<-t<\rho^2}\int_{|x|<\rho} v^2\,dx \leq \frac{2C_2}{(r-\rho)^2}\left(\frac{\Lambda}{\varepsilon}+1\right)\iint_{R^-(r)} v^2\,dx\,dt$$

when $p>1$.

Then we can deduce that for $p>0, p\neq 1$ we have

$$\sup_{0<|t|<\rho^2}\int_{|x|<\rho} v^2\,dx \leq \frac{2C_2}{(r-\rho)^2}\left(\frac{\Lambda}{\varepsilon}+1\right)\iint_{R(r)} v^2\,dx\,dt \tag{5.24}$$

On the other hand we can consider (5.21), with the same choice of ψ, over the region $R(r)$ and then remove the first term to get

$$\iint_{R(r)} \psi^2|Dv|^2\,dxdt \leq \frac{1}{4\varepsilon\lambda}\iint_{R(r)} v^2\left(\frac{\Lambda}{\varepsilon}|D\psi|^2 + 2|\psi\psi_t|\right)dx\,dt$$

for $p>0, p\neq 1$. But then the rules of integration and the choice of ψ imply

$$\varepsilon\lambda\iint_{R(\rho)} |Dv|^2\,dx\,dt \leq \frac{C_2}{4\varepsilon\lambda(r-\rho)^2}\left(\frac{\Lambda}{\varepsilon}+1\right)\iint_{R(r)} v^2\,dx\,dt \tag{5.25}$$

From lemma 5.2.6 we know that

$$\iint_{R(\rho)} v^{2\kappa}\,dx\,dt \le C_3 \sup_{|t|<\rho^2} \left(\int_{|x|<\rho} v^2\,dx\right)^{\kappa-1} \iint_{R(\rho)} v^2 + |Dv|^2\,dx\,dt\,.$$

Now substituting (5.24) and (5.25) in this inequality gives

$$\begin{aligned}\iint_{R(\rho)} v^{2\kappa}\,dx\,dt &\le C_3 \left[\frac{2C_2}{(r-\rho)^2}\left(\frac{\Lambda}{\varepsilon}+1\right)\iint_{R(r)} v^2\,dx\,dt\right]^{\kappa-1} \\ &\quad\cdot \left[\iint_{R(\rho)} v^2\,dx\,dt + \frac{C_2}{4\varepsilon\lambda(r-\rho)^2}\left(\frac{\Lambda}{\varepsilon}+1\right)\iint_{R(r)} v^2\,dx\,dt\right] \\ &= C_4\left(\iint_{R(\rho)} v^2\,dx\,dt\right)^{\kappa}\end{aligned}$$

where

$$C_4 = C_3\left[\frac{2C_2}{(r-\rho)^2}\left(\frac{\Lambda}{\varepsilon}+1\right)\right]^{\kappa-1}\cdot\left[\frac{C_2}{4\varepsilon\lambda(r-\rho)^2}\left(\frac{\Lambda}{\varepsilon}+1\right)+1\right].$$

This constant may seem a bit nasty, but we can estimate it to get a shorter one: since $\mu = \Lambda + \lambda^{-1} > 1$ we get

$$\begin{aligned}C_4 &= C_3\left[\frac{C_2}{(r-\rho)^2}\left(\frac{\Lambda}{\varepsilon}+1\right)\right]^{\kappa}\frac{2^\kappa}{4\varepsilon\lambda} + C_3\left[\frac{C_2}{(r-\rho)^2}\left(\frac{\Lambda}{\varepsilon}+1\right)\right]^{\kappa-1} 2^{\kappa-1} \\ &= C_3\left(\frac{C_2}{(r-\rho)^2}\right)^{\kappa}\left[\left(\frac{\mu}{\varepsilon}+1\right)\frac{2^\kappa\mu}{4\varepsilon} + \frac{(r-\rho)^2}{C_2}\left(\frac{\mu}{\varepsilon}+1\right)^{\kappa-1}\right] \\ &\le C_5\left(\frac{C_2}{(r-\rho)^2}\right)^{\kappa}\left(\frac{\mu}{\varepsilon}+1\right)^{\kappa+1} \\ &= \frac{C_6}{(r-\rho)^{2\kappa}}\left(\frac{\mu}{\varepsilon}+1\right)^{\kappa+1} \\ &=: C\end{aligned}$$

where the constant C_6 doesn't depend on the solution or the factors a^{ij}. Then going back to u gives

$$\left(\iint_{R(\rho)} u^{p\kappa}\,dx\,dt\right)^{\frac{1}{\kappa}} \le C^{\frac{1}{\kappa}} \iint_{R(r)} u^p\,dx\,dt \tag{5.26}$$

for $p > 0, p \neq 1$. Our next task is to iterate this inequality to get (5.11) with $r = 1$ and $\rho = \frac{1}{2}$. To do this we define

$$\begin{cases} \rho_j = \frac{1}{2}\left(1 + 2^{-j}\right) \\ p_j = p\kappa^j \\ \varepsilon_j = \frac{1}{2}\left|1 - p_j^{-1}\right| \end{cases}$$

for $j = 0, 1, 2, \ldots$. Then we notice that we should have $p_j \neq 1$ for every j. This can be done by choosing

$$p = \frac{1}{2}\kappa^i(\kappa + 1)$$

for some $i \in \mathbb{Z}$. But we recall that

$$\left(\frac{1}{|R(r)|}\iint_{R(r)} u^p \, dx \, dt\right)^{\frac{1}{p}},$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$, is a monotone function of p. Thus for any $\widetilde{p}$ we can choose $i \in \mathbb{Z}$ such that $p \leq \widetilde{p} < p\kappa$. Then from lemma 5.2.7 we get that

$$|p_j - 1| \leq \frac{1}{2}(1 - \kappa^{-1})$$

for every $j \in \mathbb{N}$, $i \in \mathbb{Z}$. And thus for $0 < p < \mu^{-1}$ we can calculate

$$\begin{aligned} \frac{\mu}{\varepsilon_j} &= \frac{\mu}{\frac{1}{2}|1 - p_j^{-1}|} = \frac{2\mu}{p_j^{-1}|p_j - 1|} = \frac{2\mu p_j}{|p_j - 1|} \\ &\leq \frac{2\mu p\kappa^j}{\frac{1}{2}(1 - \kappa^{-1})} \leq \frac{4\kappa^j}{(1 - \kappa^{-1})} = \frac{4\kappa^j}{(1 - \kappa^{-1})} \,. \end{aligned}$$

Then we get for the iteration constants

$$
\begin{aligned}
C_j &= \frac{C_6}{(\rho_j - \rho_{j+1})^{2\kappa}} \left(\frac{\mu}{\varepsilon_j} + 1 \right)^{\kappa+1} \\
&\leq \frac{C_6}{\left(\frac{1}{2} + \frac{1}{2^{j+1}} - \frac{1}{2} - \frac{1}{2^{j+2}}\right)^{2\kappa}} \left(\frac{4\kappa^j}{(1-\kappa^{-1})} + 1 \right)^{\kappa+1} \\
&\leq \frac{C_6}{\left(\frac{1}{2^{j+1}}(1-\frac{1}{2})\right)^{2\kappa}} \left(\frac{4 \cdot 2^j}{(1-\kappa^{-1})} + 1 \right)^{\kappa+1} \\
&\leq C_6 \left(2^{j+2}\right)^{2\kappa} \left(\frac{2^{j+2}}{(1-\kappa^{-1})} \left(1 + \frac{1}{2^{j+2}}\right) \right)^{\kappa+1} \\
&\leq C_6 \left(2^{4\kappa}\right)^{j+1} \left(2^{2(\kappa+1)}\right)^{j+1} \left(1 + \frac{1}{2^{j+2}}\right)^{\kappa+1} \\
&=: C_6 \cdot C_7^{j+1} \cdot C_8 \leq C_9^{j+1}
\end{aligned}
$$

where C_9 is yet again an absolute constant. Then from (5.26) if follows that:

$$
\left(\iint_{R(\rho_j+1)} u^{p_{j+1}} \, dx \, dt \right)^{\frac{1}{\kappa^{j+1}}} \leq \left(C_j^{\frac{1}{\kappa}} \right)^{j+1} \left(\iint_{R(\rho_j)} u^{p_j} \, dx \, dt \right)^{\frac{1}{\kappa^j}} \tag{5.27}
$$

for all $j = 0, 1, \ldots$. Then iterating this inequality implies finally what we need:

$$
\sup_{R(\frac{1}{2})} u^p = \lim_{j \to \infty} \left(\iint_{R(\rho_j)} u^{p_j} \, dx \, dt \right)^{\frac{1}{\kappa^j}} \leq \prod_{j=0}^{\infty} \left(C_j^{\frac{1}{\kappa}} \right)^{j+1} \cdot \iint_{R(1)} u^p \, dx \, dt \, . \tag{5.28}
$$

This proves (5.11) for $r = 1$ and $\rho = \frac{1}{2}$ since we have the estimate

$$
\prod_{j=0}^{\infty} \left(C_j^{\frac{1}{\kappa}} \right)^{j+1} \leq \prod_{j=0}^{\infty} C_9^{\frac{j+1}{\kappa^{j+1}}} = C_9^{\sum_{j=1}^{\infty} \frac{j}{\kappa^j}} \leq C_9^{C_{10}} =: C_{11}
$$

with C_{11} being an absolute constant. Now we should prove that (5.11) holds generally. For this purpose we define a transformation

$$
(x, t) \to (\alpha x, \alpha^2 t) \, .
$$

Then we get a new solution to (1.1) with another absolute constant denoted also by C_{11}. Then we get from (5.28) that

$$
\sup_{R(\frac{\alpha}{2})} u^p \leq \frac{C_{11}}{\alpha^{n+2}} \iint_{R(\alpha)} u^p \, dx \, dt \, .
$$

Then letting $\frac{1}{2} \le \rho < r \le 1$ and setting $\alpha = r - \rho$ implies

$$\sup_{R(\frac{r-\rho}{2})} u^p \le \frac{C_{11}}{(r-\rho)^{n+2}} \iint_{R(\alpha)} u^p \, dx \, dt$$

and applying this inequality to all cylinders that are obtained from $R(r-\rho)$ by translations. Now the centers of these cylinders cover $R(\rho)$ which then gives finally (5.11). We can prove (5.12) in this same way, but in that case one doesn't take care of p going too close to 1 since it's negative. □

Next we prove another lemma that controls the measures of certain level sets of $\log u$.

Lemma 5.2.2. *If $u > 0$ is a solution of* (1.1) *in $Q \times T$ where*

$$Q := \{x \in \mathbb{R}^n \, | |x| < 2\}$$

and $T := [t_1, t_2]$. Then there exists constants a, b such that

$$\left|\left\{(x,t) \in R^+(1) \, | \log u < -s + a\right\}\right| + \left|\left\{(x,t) \in R^-(1) \, | \log u > s + a\right\}\right| \le \frac{b\mu}{s} \tag{5.29}$$

for all $s > 0$ where a depends on the solution u and b depends only on n.

Proof. Let u be a supersolution of (1.1). Then for any $\phi \in C_0^\infty(Q)$, $\phi \ge 0$ we have

$$\iint \phi u_t + (D\phi, \mathbb{A}Du) \, dx \, dt \ge 0 \; .$$

If we now define $\phi := \frac{1}{u(x,t)}\psi(x)^2$ for some fixed $\psi \in C_0^\infty$ and $v := -\log u$, we get

$$\iint \left(\psi^2 \frac{1}{e^{-v}}\right) \cdot (-v_t e^{-v}) + \left(D\left(\psi^2 \frac{1}{e^{-v}}\right), \mathbb{A}D\left(e^{-v}\right)\right) dx \, dt \ge 0$$

and this implies that

$$-\iint \psi^2 v_t \, dx \, dt - \iint (D\psi^2, \mathbb{A}Dv) \, dx \, dt - \iint \psi^2 (Dv, \mathbb{A}Dv) \ge 0$$

Then fixing the integration with respect to t to be from t_1 to t_2 gives

$$\int_Q \psi^2 v\,dx\,dt\Big|_{t_1}^{t_2} + 2\int_{t_1}^{t_2}\int_Q \psi(D\psi,\mathbb{A}Dv)\,dx\,dt + \int_{t_1}^{t_2}\int_Q \psi^2(Dv,\mathbb{A}Dv)\,dx\,dt \le 0\,. \qquad (5.30)$$

Now by Theorem 2.5.5 we know that

$$(D\psi,\mathbb{A}Dv) \le (D\psi,\mathbb{A}D\psi)^{1/2}(Dv,\mathbb{A}Dv)^{1/2}$$

and then this and the Hölder's inequality of Theorem 2.1.3 implies that

$$\begin{aligned}
2\int_{t_1}^{t_2}\int_Q \psi(D\psi,\mathbb{A}Dv)\,dx\,dt &\le 2\int_{t_1}^{t_2}\int_Q (D\psi,\mathbb{A}D\psi)^{1/2}\psi(Dv,\mathbb{A}Dv)^{1/2}\,dx\,dt \\
&\le 2\left(\int_{t_1}^{t_2}\int_Q (D\psi,\mathbb{A}D\psi)\,dx\,dt\right)^{\frac{1}{2}}\cdot\left(\int_{t_1}^{t_2}\int_Q \psi^2(Dv,\mathbb{A}Dv)\,dx\,dt\right)^{\frac{1}{2}} \\
&\le \int_{t_1}^{t_2}\int_Q (D\psi,\mathbb{A}D\psi)\,dx\,dt + \int_{t_1}^{t_2}\int_Q \psi^2(Dv,\mathbb{A}Dv)\,dx\,dt\,.
\end{aligned}$$

Thus we have

$$2\int_{t_1}^{t_2}\int_Q \psi(D\psi,\mathbb{A}Dv)\,dx\,dt - \int_{t_1}^{t_2}\int_Q \psi^2(Dv,\mathbb{A}Dv)\,dx\,dt \le \int_{t_1}^{t_2}\int_Q (D\psi,\mathbb{A}D\psi)\,dx\,dt\,.$$

This implies with (5.30) that

$$\int_Q \psi^2 v\,dx\,dt\Big|_{t_1}^{t_2} + 2\int_{t_1}^{t_2}\int_Q \psi^2(Dv,\mathbb{A}Dv)\,dx\,dt \le \int_{t_1}^{t_2}\int_Q (D\psi,\mathbb{A}D\psi)\,dx\,dt\,.$$

Next we notice that from the ellipticity condition (1.2) it follows that

$$\int_Q \psi^2 v\,dx\,dt\Big|_{t_1}^{t_2} + 2\lambda\int_{t_1}^{t_2}\int_Q \psi^2|Dv|^2\,dx\,dt \le \Lambda\int_{t_1}^{t_2}\int_Q |D\psi|^2\,dx\,dt\,.$$

Then since $\lambda \geq \mu^{-1}$ and $\Lambda \leq \mu$, we get

$$\int_Q \psi^2 v \, dx \, dt \Bigg|_{t_1}^{t_2} + \frac{1}{C_1 \mu} \int_{t_1}^{t_2}\int_Q \psi^2 |Dv|^2 \, dx \, dt \leq \mu \int_{t_1}^{t_2}\int_Q |D\psi|^2 \, dx \, dt \, .$$

where $C_1 = \frac{1}{2}$. Then then we choose $\psi \in C_0^\infty(\Omega)$ such that

$$\begin{cases} \psi & = 1 \quad \text{in } |x| \leq 1, \\ \psi & = 0 \quad \text{in } |x| \geq 2, \\ |D\psi| & \leq C_2 \quad , \end{cases}$$

where C_2 is a suitable absolute constant and all the level surfaces of ψ are convex. This with Lemma 5.2.5 implies that

$$\int_Q \psi^2 v \, dx \, dt \Bigg|_{t_1}^{t_2} + \frac{1}{C_1 \mu} \int_{t_1}^{t_2}\int_Q (v(x,t) - V(t))^2 \psi^2 \, dx \, dt \leq \mu C_2^2 \int_{t_1}^{t_2}\int_Q dx \, dt = C_3 \mu |Q| (t_2 - t_1) \ ,$$

where

$$V(t) = \frac{\int_Q v(x,t) \psi^2 \, dx}{\int_Q \psi^2 \, dx}$$

and $C_3 := C_2^2$. Then dividing both sides of the equation with $Q(t_2 - t_1)$ implies that

$$\begin{aligned} &\frac{\int_Q \psi^2(x,t_2) v(x,t_2) \, dx - \int_Q \psi^2(x,t_1) v(x,t_1) \, dx}{\int_Q dx (t_2 - t_1)} \\ &+ \frac{1}{|Q| \mu C_1 (t_2 - t_1)} \int_{t_1}^{t_2}\int_Q (v(x,t) - V(t))^2 \psi^2 \, dx \, dt \\ &\leq C_3 \mu \, . \end{aligned}$$

If we now write the first term with $V(t)$, we get

$$\frac{V(t_2) - V(t_1)}{t_2 - t_1} + \frac{1}{|Q| \mu C_1 (t_2 - t_1)} \int_{t_1}^{t_2}\int_Q (v(x,t) - V(t))^2 \psi^2 \, dx \, dt \leq C_3 \mu \ . \tag{5.31}$$

Next define $U := \{(x,t) \in Q \times T \,||t| \leq 1|x| < 1\}$. Now since (5.31) doesn't change if we add a constant to v. In this case there is a constant a such that transformation $v \to v + a$ normalizes V such that $V(0) = 0$. Then denote $U^+ := \{(x,t) \in U \,|0 \leq t \leq 1\}$ and defined

$$\begin{cases} w(x,t) := v(x,t) - C_3\mu t\,, \\ W(t) := V(t) - C_3\mu t\,. \end{cases}$$

we know that W is certainly differentiable and then by taking the limit $t_2 \to t_1$ we get

$$\frac{dW}{dt} + \frac{1}{C_3\mu}\int_Q (w - W)^2 dx \leq 0 \tag{5.32}$$

Now set $Q_s(t) := \{x \in Q \,|w(x,t) > s\}$ for $0 < t < 1$. Then for $s > 0$ we have

$$0 < s - W \leq w - W$$

since $W(t) \leq 0$ in $0 < t < 1$. Then if we restrict (5.32) to $Q_s(t)$, we get

$$\frac{dW}{dt} + \frac{1}{C_3\mu}\frac{|Q_s|}{|Q|}(s - W)^2 \leq 0$$

and this implies that

$$C_3\mu(s - W)^2 \frac{d(s - W)}{dt} \geq \frac{|Q_s|}{|Q|}\,.$$

Then integrating this from 0 to 1 with respect to t gives

$$\frac{C_3\mu}{s} \geq \frac{1}{|Q|}\int_0^1 |Q_s| dt = \frac{1}{|Q|}\iint_{w>s} dx\,dt\,.$$

But writing the last integral as a measure then gives

$$\left|\left\{(x,t) \in U^+ \,|\log u < -s + a\right\}\right| \leq \frac{C_4(n)\mu}{s}$$

since

$$\left\{(x,t) \in U^+ \,|\log u < -s + a\right\} \subset \left\{(x,t) \in U^+ \,|\log u < -s + at\right\}$$

for all $0 < t < 1$. Similarly we get

$$\left|\left\{(x,t) \in U^- \,|\log u > s + a\right\}\right| \leq \frac{C_4(n)\mu}{s}$$

and this implies the claim. □

Now we have proved the first two lemmas and we are ready to proceed to the third one. This lemma is the main result of [7].

Lemma 5.2.3. *Let $m, \mu, C_0, \frac{1}{2} \leq \theta < 1$ be positive constants and let $w > 0$ be a continuous function defined in a neighbourhood of $Q(1)$ for which*

$$\sup_{Q(\rho)} w^p < \frac{C_0}{(r-\rho)^m |Q(1)|} \iint_{Q(r)} w^p \, dx \, dt \tag{5.33}$$

for all p, r, ρ satisfying

$$\begin{aligned} \frac{1}{2} \leq \theta \leq \rho < r \leq 1, \\ 0 < p < \mu^{-1} \, . \end{aligned}$$

In addition let

$$|\{(x,t) \in Q(1) \,|\log w > s\}| < \frac{C_0 \mu}{s} |Q(1)| \tag{5.34}$$

for all $s > 0$. Then there exists a constant function $\gamma = \gamma(\theta, m, C_0)$ such that

$$\sup_{Q(\theta)} w < \gamma^\mu \; . \tag{5.35}$$

Proof. We may assume that $\mu = 1$ and $Q(1) = 1$. Then define

$$\phi(\rho) := \sup_{Q(\rho)} \log w$$

for $\theta \leq \rho < 1$, which is a nondecreasing function. We want first to get an estimate

$$\phi(\rho) < \frac{3}{4}\phi(r) + \frac{\gamma_1}{(r-\rho)^2 m} \tag{5.36}$$

with $\theta \le \rho < r \le 1$ and some constant $\gamma_1 = \gamma(\theta, m, C_0)$. For this purpose we estimate

$$\iint_{Q(r)} w^p \, dx \, dt$$

by defining sets

$$\begin{aligned} Q_1(r) &:= \left\{ (x,t) \in Q(r) \,\middle|\, \log w > \frac{1}{2}\phi(r) \right\} \\ Q_2(r) &:= \left\{ (x,t) \in Q(r) \,\middle|\, \log w \le \frac{1}{2}\phi(r) \right\} \ . \end{aligned}$$

Then we have

$$\begin{aligned} \iint_{Q(r)} w^p \, dx \, dt &\le \iint_{Q_1(r)} w^p \, dx \, dt + \iint_{Q_2(r)} w^p \, dx \, dt \\ &\le \sup_{Q_1(r)} w^p \iint_{Q_1(r)} dx \, dt + \iint_{Q_2(r)} e^{\frac{\phi p}{2}} \, dx \, dt \\ &\le \sup_{Q(r)} w^p \frac{2C_0}{\phi(r)} + e^{\frac{\phi p}{2}} \iint_{Q(1)} dx \, dt \\ &\le e^{\phi(r) p} \frac{2C_0}{\phi(r)} + e^{\frac{\phi p}{2}} \\ &= 2e^{\frac{\phi p}{2}} \ , \end{aligned}$$

where the equality of the last line holds if we choose p such that

$$\frac{2C_0}{\phi(r)} = e^{-\frac{\phi p}{2}} \ ,$$

that is

$$p = \log\left(\frac{\phi(r)}{2C_0}\right) \cdot \frac{2}{\phi(r)} \ .$$

But since we require $0 < p < 1$ we have

$$\phi(r) > 2C_0 =: C_1 \, . \tag{5.37}$$

Then the first assumption of the lemma gives

$$\sup_{Q(\rho)} w^p < \frac{2C_0}{(r-\rho)^m} e^{\frac{p\phi}{2}}.$$

This implies that

$$\sup_{Q(\rho)} w < \left(\frac{2C_0}{(r-\rho)^m} e^{\frac{p\phi}{2}} \right)^{\frac{1}{p}}$$

and thus

$$\phi(\rho) < \frac{1}{p} \log \left(\frac{2C_0}{(r-\rho)^m} e^{\frac{p\phi}{2}} \right) = \frac{1}{p} \log \left(\frac{2C_0}{(r-\rho)^m} \right) + \frac{\phi}{2}$$

Next we substitute our choice of p to this inequality:

$$\phi(\rho) < \frac{\phi}{2} \frac{\log\left(\frac{2C_0}{(r-\rho)^m}\right)}{\log\left(\frac{\phi(r)}{2C_0}\right)} + \frac{\phi}{2} \tag{5.38}$$

and we consider the case when

$$\frac{\log\left(\frac{2C_0}{(r-\rho)^m}\right)}{\log\left(\frac{\phi(r)}{2C_0}\right)} < \frac{1}{2}.$$

To get some sense of this we solve ϕ from this inequality:

$$\frac{2C_0}{(r-\rho)^m} < \left(\frac{\phi(r)}{2C_0} \right)^{\frac{1}{2}}$$

and thus

$$\phi(r) > \frac{8C_0^3}{(r-\rho)^{2m}}. \tag{5.39}$$

So in this case we get from (5.38) that

$$\phi(\rho) < \frac{3}{4}\phi(r).$$

On the other hand if (5.37) or (5.39) is violated, then we get

$$\phi(r) \le \frac{\gamma_1}{(r-\rho)^{2m}}$$

for $\theta \le \rho < r < 1$, since from (5.37) it follows that

$$\phi(\rho) \le C_1 \frac{(r-\rho)^{2m}}{(r-\rho)^{2m}} \le \frac{(1-\theta)^{2m}}{(r-\rho)^{2m}}.$$

So in any case equation (5.36) holds. Now we can iterate this inequality to prove this lemma. To this end fix a sequence

$$0 \le r_0 < r_1 < \cdots < r_k \le 1.$$

Then we get from (5.36) that

$$\phi(r_0) < \left(\frac{3}{4}\right)^k \phi(r_k) + \gamma_1 \sum_{i=1}^{k} \left(\frac{3}{4}\right)^i \frac{1}{(r_{i+1}-r_i)^{2m}}.$$

Since k was arbitrary and

$$\phi(r_k) \le \phi(1) < \infty$$

we get for $k \to \infty$ that

$$\phi(\theta) \le \phi(r_0) \le \gamma_1 \sum_{i=1}^{\infty} \left(\frac{3}{4}\right)^i \frac{1}{(r_{i+1}-r_i)^{2m}}\,.$$

Next we choose sequence r_i such that the right hand side converges. This is indeed the case if we choose

$$r_i := 1 - \frac{1-\theta}{1+j}$$

as we will show in what follows. At first we may just calculate

$$\begin{aligned}
\gamma_1 \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j (r_{j+1}-r_j)^{-2m} &= \gamma_1 \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j \left(1 - \frac{1-\theta}{1+j+1} - \left(1 - \frac{1-\theta}{1+j}\right)\right)^{-2m} \\
&= \gamma_1 \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j \left(\frac{-(1+j)(1-\theta)+(2+j)(1-\theta)}{(j+2)(j+1)}\right)^{-2m} \\
&= \gamma_1 \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j \left(\frac{(1-\theta)}{(j+2)(j+1)}\right)^{-2m} \\
&\le \gamma_1 \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j \frac{(j+2)^{4m}}{(1-\theta)^{2m}}\,.
\end{aligned}$$

Now for $\widetilde{k} \in \mathbb{N}, \widetilde{k} > 4m$ we have

$$\begin{aligned}
\left(\frac{5}{4}\right)^x &= \left(\frac{5}{4}\right)^{(x+2)-2} \\
&= \left(\frac{4}{5}\right)^2 \cdot \left(\frac{5}{4}\right)^{x+2} \\
&= \left(\frac{4}{5}\right)^2 e^{(x+2)\log\left(\frac{5}{4}\right)} \\
&= \left(\frac{4}{5}\right)^2 \sum_{k=0}^{\infty} \frac{1}{k!} \left((x+2)\log\left(\frac{5}{4}\right)\right)^k \\
&\geq \left(\frac{4}{5}\right)^2 \frac{1}{(\widetilde{k}+1)!} \left((x+2)\log\frac{5}{4}\right)^{\widetilde{k}+1} \\
&> \left(\frac{4}{5}\right)^2 \frac{1}{(\widetilde{k}+1)!} \left(\log\frac{5}{4}\right)^{\widetilde{k}+1} (x+2)^{4m+1} \ .
\end{aligned}$$

Here we can replace x with j and the inequality above holds for every $j \in \mathbb{N}$. This implies that

$$\begin{aligned}
\left(\frac{5}{4}\right)^j &> \left(\frac{4}{5}\right)^2 \frac{1}{(\widetilde{k}+1)!} \left(\log\frac{5}{4}\right)^{\widetilde{k}+1} (j+2)(j+2)^{4m} \\
&\geq \frac{(j+2)^{4m}}{(1-\theta)^{2m}}
\end{aligned}$$

for every

$$j > -2 + \frac{1}{\left(\frac{4}{5}\right)^2 \frac{1}{(\widetilde{k}+1)!} \left(\log\frac{5}{4}\right)^{\widetilde{k}+1} (1-\theta)^{2m}} =: M \ .$$

Thus we get

$$\begin{aligned}
\gamma_1 \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j (r_{j+1}-r_j)^{-2m} &< \gamma_1 \sum_{j=0}^{M} \left(\frac{3}{4}\right)^j (r_{j+1}-r_j)^{-2m} + \gamma_1 \sum_{j=M+1}^{\infty} \left(\frac{3}{4}\right)^j \left(\frac{5}{4}\right)^j \\
&= \widetilde{M} + \gamma_1 \sum_{j=m+1}^{\infty} \left(\frac{15}{16}\right)^j \\
&< \infty
\end{aligned}$$

and this proves the lemma.

$\square$

5.2.2 Proof of Theorem 1.0.2.

Next we will prove Theorem (1.0.2) in specially chosen domains D, D^+ and D^-. Idea in the proof is to show that the assumptions (5.33) and (5.34) hold for $w := e^{-a}u$ where the constant a is the constant given by Lemma 5.2.2.

Proof. Let

$$\begin{aligned} D &= \{(x,t) \in \Omega \times T \,||x| < 2, |t| < 1\} \,, \\ D^+ &= \left\{(x,t) \in D \,\middle|\, |x| < \frac{1}{2}, \frac{3}{4} < t < 1\right\} \text{ and} \\ D^- &= \left\{(x,t) \in D \,\middle|\, |x| < \frac{1}{2}, -\frac{3}{4} < t < -\frac{1}{4}\right\} . \end{aligned}$$

Now assumption (5.33) clearly holds since for $Q(1) := R^-(1)$ we know that $|Q(1)| = |R^-(1)| = c(n)$. To show that the second assumption holds we have to make the following calculation: Lemma 5.2.2 gives constants c_2, a such that

$$\begin{aligned} \frac{c_2\mu}{s} &\geq \left|\left\{(x,t) \in R^-(1) \,|\, \log u > s + a\right\}\right| \\ &= \left|\left\{(x,t) \in R^-(1) \,|\, \log u - \log e^a > s\right\}\right| \\ &= \left|\left\{(x,t) \in R^-(1) \,\middle|\, \log e^{-a}u > s\right\}\right| \\ &= \left|\left\{(x,t) \in R^-(1) \,|\, w > s\right\}\right| . \end{aligned}$$

This implies that the second assumption also holds for w since $|Q(1)| = c(n)$. Then set

$$Q(\rho) = \left\{(x,t) \in D \,\middle|\, |x| < \frac{\rho}{\sqrt{2}}, \left|t + \frac{1}{2}\right| < \frac{1}{2}\rho^2\right\} .$$

for $\frac{1}{2} \leq \rho < 1$. We notice that $Q(\rho)$ can be obtained from $R(1)$ by an admissible transformation

$$\begin{cases} x \to \frac{1}{\sqrt{2}}x \\ t \to t + \frac{1-\rho^2}{2} \end{cases}$$

that preserves the solutions of (1.1). Now taking $\theta = \frac{1}{\sqrt{2}}$ gives $Q(\theta) = D^-$ and Lemma 5.2.3 implies

$$\sup_{D^-} e^{-a} < c_4^{\mu}\,, \tag{5.40}$$

where $c_4 = \gamma$ for $m = n + 2$. In the same way we can show that (5.33) and (5.34) hold for $w = e^a u$ setting

$$Q(\rho) = \left\{(x,t) \in D \,\middle|\, |x| < \rho, 0 < 1 - t < \rho^2\right\} .$$

Thus by Lemma 5.2.3 we get for $\theta = \frac{1}{2}$, $Q(\theta) = D^+$,

$$\sup_{D^+} e^{a} < c_5^{\mu}\,. \tag{5.41}$$

Multiplying inequalities (5.40) and (5.41) imply (1.4).

□

Our next purpose is to extend this result from these specially chosen domains to more general domains. To this end we first need the following lemma:

Lemma 5.2.4. *Let Ω be an open set in $\mathbb{R}^n$ and K a compact, connected subset of Ω. Then there exists a constant $C_6 = C_6(K, \Omega)$ with the following property: for any ρ in*

$$0 < 2\rho < \operatorname{dist}(K, \partial\Omega) =: \delta$$

and for any $\xi, \eta \in K$, there exists a chain of balls $B_j(2\rho)$, $j = 1, 2, \ldots, l$, in Ω such that the concentric balls $B_j(\rho)$ connect ξ, η in the sense that $\xi \in B_0(\rho)$, $\eta \in B_l(\rho)$, $B_{j-1}(\rho) \cap B_j(\rho) \neq \emptyset$, $j = 1, 2, \ldots, l$, and such that l doesn't exceed C_6/ρ.

Proof. A simple proof, based on a finite covering of K by balls and a polygonal path through some of the centers of these balls can be found in [7, p. 735]. □

One can also find the proof of 1.0.2 in general domains in [7, p. 735]. It uses Lemma 5.2.4 above to prove that for a arbitrary points $(\xi^+, t^+) \in D^+$ and $(\xi^-, t^-) \in D^-$ it holds that

$$u(\xi^-, t^-) \leq C^{N\mu} u(\xi^+, t^+),$$

where $N = N(D, D^+, D^-)$. This proves 1.0.2 since we assumed that u is smooth.

5.2.3 Lemmas

Here are the proofs for the lemmas used earlier in this chapter.

Lemma 5.2.5. *Let $p(x) \geq 0$ be a continuous function of compact support of breath B and such that the domains level surfaces $p(x) =$ constant are convex. Then for any* *function f such that f and $|Df|$ are in L^2 with respect to the measure given by $p(x)dx$ we have*

$$\min_k \int (f(x)-k)^2\, p(x)\, dx \leq CB^2 \int |Df|^2\, p(x)\, dx \ , \tag{5.42}$$

where

$$C = \frac{\max p(x)}{2\int p(x)\, dx} \int_{p>0} |Df|^2\, p(x)\, dx \ .$$

Then minimum is assumed for

$$k = \frac{\int f(x)\, p(x)\, dx}{\int p(x)\, dx} \ .$$

Proof. First we notice that

$$\iint (f(x)-f(y))^2 p(x)p(y)\, dx\, dy = 2\int p(x)\, dx \int (f-k)^2\, p(y)\, dy \tag{5.43}$$

with k as in the statement of the lemma. Then let γ be the staright line segment

$$\gamma := \{xt + (1-t)y\, |t \in [0,1]\} \ .$$

Then we can calculate as follows:

$$\begin{aligned}
(f(y)-f(x))^2 &= \left(\int_0^1 (Df, \gamma'(t))\, dt\right)^2 \\
&\leq \left(\int_0^1 \sqrt{p}|Df| \cdot |\gamma'(t)|^{1/2} \frac{1}{\sqrt{p}} |\gamma'(t)|^{1/2}\, dt\right)^2 \\
&\leq \left(\int_0^1 p|Df|^2 |\gamma'(t)|\, dt\right)\left(\int_0^1 \frac{|D\gamma'(t)|}{p}\, dt\right) \\
&= \left(\int_\gamma p|Df|^2\, d\gamma\right)\left(\int_\gamma \frac{1}{p}\, d\gamma\right) \ .
\end{aligned}$$

Now $p(tx+(1-t)y) \geq p(x)$ for all $t \in [0,1]$ and this implies that

$$\int_\gamma \frac{1}{p}\, d\gamma \leq \frac{1}{p(x)} \int_\gamma d\gamma \leq \frac{C}{p(x)}$$

for some $C \in \mathbb{R}$. Then

$$(f(x) - f(y))^2 \, p(x)p(y) \le \int_\gamma |Df|^2 p \, d\gamma C p(y)$$

and by integrating this for $z = y - x$ over x we get

$$\int (f(x) - f(x+z))^2 \, p(x)p(x+z) \, dx \le \int |Df|^2 p \, dx C^2 \max p \, .$$

Finally integration over z gives

$$\iint (f(x) - f(y))^2 \, p(x)p(y) \, dx \, dy \le \int |Df|^2 dx C^2 \max p \int_{p>0} dx.$$

This, when combined with (5.43), proves this lemma. □

Then we prove another lemma that is a Sobolev type inequality that we need in the proof of Lemma 5.2.1.

Lemma 5.2.6. *For any function $v \in V^{1,2}(\Omega_T)$ we have*

$$\iint_{R(\rho)} v^{2\kappa} \, dxdt \le C \sup_{|t|<\rho^2} \left(\int_{|x|<\rho} v^2 \, dx \right)^{\kappa - 1} \iint_{R(\rho)} v^2 + |Dv|^2 \, dxdt \tag{5.44}$$

with $C = C(n)$ and

$$\kappa = \begin{cases} 1 + \frac{2}{n} & \text{for } n > 2 \, , \\ \frac{5}{3} & \text{for } n = 1, 2 \, . \end{cases}$$

Proof. Let $n > 2$. Then Hölder's inequality implies that

$$\begin{aligned} \int v(x,t)^{2(1+2/n)} \, dx &= \int v(x,t)^2 v(x,t)^{4/n} dx \\ &\le \left(\int v(x,t)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \left(\int v(x,t)^{\frac{4}{n} \cdot \frac{n}{2}} \, dx \right)^{\frac{2}{n}} \\ &\le \left(\int v(x,t)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \left(\sup_t \int v(x,t)^2 \, dx \right)^{\frac{2}{n}} . \end{aligned}$$

also the Sobolev inequality from Theorem 2.3.3 in the preliminaries section asserts that

$$\left(\int v(x,t)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \leq C(n)\int v^2 + |Dv|^2\,dx\ .$$

Combining these inequalities then gives

$$\int v(x,t)^{2\left(1+\frac{2}{n}\right)}\,dx \leq C(n)\left(\sup_t \int v(x,t)^2\,dx\right)^{\frac{2}{n}} \int v^2 + |Dv|^2\,dx$$

and integrating over t gives (5.44) for $n > 2$ for $\kappa = 1 + \frac{2}{n}$. If we neglect one or two of the $x-$variables for the case $n = 3$, we get (5.44) for $n = 1, 2$ with $\kappa = 5/3$. □

Next lemma is also for the proof of Lemma 5.2.1.

Lemma 5.2.7. *Let $0 < p' < \mu^{-1}$, $i \in \mathbb{Z}$ such that $p \leq p' < \kappa p$ for $p = \frac{1}{2}\kappa^i(\kappa+1)$ and for κ as in lemma 5.2.6 and $p_j = p\kappa^j$ for every $j \in \mathbb{N}$. Then we have*

$$|p_j - 1| \geq \frac{1}{2}(1-\kappa^{-1}) \tag{5.45}$$

Proof. Let $j \in \mathbb{N}$. Then

$$\begin{aligned}
|p_j - 1| &= |p\kappa^j - 1| \\
&= |(1/2)\kappa^i(\kappa+1)\kappa^j - 1| \\
&= \kappa|(1/2)\kappa^i(1+1/\kappa)\kappa^j - 1/\kappa| \\
&> \frac{1}{2}|\kappa^{i+j} + \kappa^{i+j-1} - \kappa^{-1} - \kappa^{-1}| \\
&= \frac{1}{2}|(\kappa^{i+j} - \kappa^{-1}) + (\kappa^{i+j-1} - \kappa^{-1})| \\
&= \begin{cases} \frac{1}{2}|1-\kappa^{-1}| = \frac{1}{2}(1-\kappa^{-1}) & \text{if}\, i+j=0 \\ \frac{1}{2}|\kappa^{-2}-\kappa^{-1}| \geq \frac{1}{2}|\kappa^{-1}-1| = \frac{1}{2}(1-\kappa^{-1}) & \text{if}\, i+j=-1 \\ \frac{1}{2}(\kappa-\kappa^{-1}+1-\kappa^{-1}) \geq \frac{1}{2}(1-\kappa^{-1}) & \text{if}\, i+j=-1 \end{cases}.
\end{aligned}$$

But if we have $i + j < -1$ we get

$$\begin{aligned}
&\frac{1}{2}|(\kappa^{i+j} - \kappa^{-1}) + (\kappa^{i+j-1} - \kappa^{-1})| \\
=&\frac{1}{2}\left[(\kappa^{-1} - \kappa^{i+j}) + (\kappa^{-1} - \kappa^{i+j-1})\right] \\
\geq&\frac{2}{2}(\kappa^{-1} - \kappa^{-2}) \geq \frac{1}{2}(1-\kappa^{-1}).
\end{aligned}$$

Finally for $i + j > 1$ we get

$$\begin{aligned}
&\frac{1}{2}|(\kappa^{i+j} - \kappa^{-1}) + (\kappa^{i+j-1} - \kappa^{-1})| \\
=&\frac{1}{2}(\kappa^{i+j} - \kappa^{-1} + \kappa^{i+j-1} - \kappa^{-1}) \\
\geq&\frac{1}{2}(\kappa^{i+j-1} - \kappa^{-1}) \\
\geq&\frac{1}{2}(1 - \kappa^{-1})
\end{aligned}$$

and the lemma is proved.

□

Lemma 5.2.8. *Let $p(x) \geq 0$ be continuous and have compact support of breadth B such that the domains $p(x) \geq$ constant are convex for all positive constants C. Then for any function for which $f, Df \in L^2$ with respect to $p(x)dx$ we have*

$$\min_k \int (f(x) - k)^2 \, p(x)dx \leq CB^2 \int |Df|^2 \, p(x) \, dx \tag{5.46}$$

where

$$C = \frac{\max p(x)}{2 \int p(x)dx}$$

Proof. Proof can be found from [6, p. 132]. □

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