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DEPARTMENT OF MATHEMATICS
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ON FRACTIONAL SMOOTHNESS AND APPROXIMATIONS OF STOCHASTIC INTEGRALS

ANNI TOIVOLA

JYVÄSKYLÄ
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Jyväskylä, November 2009

Anni Toivola

Abstract

Connections are revealed between fractional smoothness, strong convergence, and weak convergence of approximations of stochastic integrals with respect to the Brownian motion and the geometric Brownian motion. Strong convergence is considered with respect to the L_p norm for $p \geq 2$, and fractional smoothness is measured in terms of Besov spaces.

Tiivistelmä

Stokastisten integraalien approksimoinnista ja fraktionaalisesta sileydestä

Väitöskirjassa osoitetaan yhteyksiä vahvan suppenemisen, heikon suppenemisen ja fraktionaalisen sileyden välillä arvioitaessa stokastisia integraaleja Brownin liikkeen ja geometrisen Brownin liikkeen suhteen. Vahvaa suppenemista tarkastellaan L_p-normin suhteen, kun $p \geq 2$, ja fraktionaalisen sileyden kuvaamiseen käytetään Besov-avaruuksia.

List of included articles

This dissertation consists of an introductory part and the following publications:

[GT] S. Geiss and A. Toivola. Weak convergence of error processes in discretizations of stochastic integrals and Besov spaces. To appear in Bernoulli.

[T] A. Toivola. Interpolation and approximation in L_p. Preprint 380, Department of Mathematics and Statistics, University of Jyväskylä, 2009.

The author of this dissertation has actively taken part in the research of the joint paper [GT].

1 Introduction

In approximation theory, convergence results answer the question "How much more resources are required for a certain improvement in accuracy?" For random variables or processes, this "accuracy" can be measured in several ways. The strong convergence rate, most often considered with respect to the L_2 norm, describes the speed with which the norm of the approximation error converges to zero. Weak convergence, i.e. convergence in distribution, requires often less assumptions, but provides also less information. Both types of results yield also estimates for the probabilities of large errors, called tail estimates: denoting the approximation error by C, we would like the probability $\mathbb{P}(|C| > \lambda)$ to decrease rapidly as $\lambda > 0$ increases. The higher integrability of a weak limit or convergence with respect to a stronger norm provides better tail estimates. This is one motivation to study L_p convergence with $p > 2$.

Convergence properties are not always easy to observe from the stochastic integral itself. We assume that the integral is generated by a functional of Brownian motion, $f(W_1) \in L_2$, and look for connections between properties of f and the convergence of

$$\int_0^1 \Phi(t, W_t) dW_t - \sum_{i=1}^{n} \Phi(t_{i-1}, W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}),$$

where $\int_0^1 \Phi(t, W_t) dW_t = f(W_1) - \mathbb{E} f(W_1)$. The main contribution of this work is the close connection between strong convergence, weak convergence, and the integrability of a weak limit. The connecting link is the fractional smoothness of f, which for many functions can be observed by direct computations.

These kinds of convergence results can be applied to the hedging error in stochastic finance, where f is the pay-off function of an option, W models the pricing process, and the error appears when a theoretical, continuously rebalanced portfolio is replaced by another one with only finitely many trading times. Another field of application is simulation: a construction as in (2) below to simulate a stochastic integral retains the martingale property whereas a spline approximation does not.

To form a clear and consistent picture of the obtained results, in this overview we consider only stochastic integrals driven by the Brownian motion. For applications, other integrator processes are also of interest. Particularly in stochastic finance, the price processes should be positive, such as the geometric Brownian motion or a diffusion obtained by a suitable stochastic differential equation. In [GT], a more general setting is employed to include the case of geometric Brownian motion, and in many references, even a wider class of processes is considered. It would be of interest to see whether the results of [T] could be extended to more general diffusions; for the present, they are developed only for the Brownian motion.

We begin with the basic notation and definitions. After introducing the convergence problem and fractional smoothness, we discuss some previous results that form the background of this work. Section 2 compresses the main results of [GT] and [T] into three theorems, which are illustrated with some typical examples. Section 3 concludes the introductory part with possible extensions of this work and ideas for further research topics.

Throughout, γ denotes the standard Gaussian measure on the real line. To denote a process - usually a Brownian motion - independent of the underlying stochastic basis, we use a tilde above the process, for example $\tilde{W}$.

Let $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,1]}\right)$ be a stochastic basis, and let $W = (W_t)_{t\in[0,1]}$ be a standard Brownian motion, with all paths continuous and $W_0 = 0$ for all $\omega \in \Omega$. Assume that $(\mathcal{F}_t)_{t\in[0,1]}$ is the augmentation of the filtration generated by W and that $\mathcal{F} = \mathcal{F}_1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function satisfying $f(W_1) \in L_2$ and define the function $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ by setting

$$F(t,x) := \mathbb{E}\left(f(W_1) \mid W_t = x\right) = \mathbb{E}f(x + W_{1-t}).$$

Then $F \in C^\infty([0,1[\times \mathbb{R})$ (see e.g. [6, Lemma A.2] or [4, p. 4]), and it satisfies

$$\begin{cases} \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} & = & 0, & 0 \le t < 1, x \in \mathbb{R} \\ F(1,x) & = & f(x), & x \in \mathbb{R} \end{cases} \tag{1}$$

and by Itô's formula, $f(W_1) = F(1, W_1) = \mathbb{E}f(W_1) + \int_0^1 \frac{\partial F}{\partial x}(s, W_s)dW_s$ a.s.

We discretize the integral on the interval $[0,t]$ with $t \le 1$ using a deterministic time net $\tau^{(n)} := (t_i^n)_{i=0}^n$ with $0 = t_0^n < t_1^n < \ldots < t_n^n = 1$, and get the approximation error process

$$C_t(f, \tau^{(n)}) := \int_0^t \frac{\partial F}{\partial x}(s, W_s)dW_s - \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}^n, W_{t_{i-1}^n})\left(W_{t_i^n \wedge t} - W_{t_{i-1}^n \wedge t}\right). \tag{2}$$

Moreover, we denote the size of the time net $\tau^{(n)}$ by

$$\left|\left|\tau^{(n)}\right|\right|_\infty := \max_{1\le i\le n} |t_i^n - t_{i-1}^n|.$$

For convenience, we denote by τ_n the equidistant time net of $n+1$ time points, i.e. $\tau_n = \left(\frac{i}{n}\right)_{i=0}^n$ and $||\tau_n||_\infty = \frac{1}{n}$.

For many functions f, the integrand $\frac{\partial F}{\partial x}$ has a singularity at time 1, and this makes approximation more difficult near 1. However, this can be counteracted by a suitable adaptation of the time nets: we place more time points near 1, so that the time nets are asymptotically more dense near 1, but at the other end point 0, the distance between time points is multiplied only by a constant factor.

This is our adaptation: for any $0 < \theta \leq 1$, we define the time nets $\tau_n^\theta = \left(t_i^{n,\theta}\right)_{i=0}^n$ by setting

$$t_i^{n,\theta} := 1 - \left(1 - \frac{i}{n}\right)^{\frac{1}{\theta}} \tag{3}$$

for $i = 0, 1, \ldots, n$. For $\theta = 1$, this definition yields an equidistant time net, and for smaller θ, an adjusted time net where the time knots are closer to each other near 1 (see Figure 1). We call θ the refinement parameter.

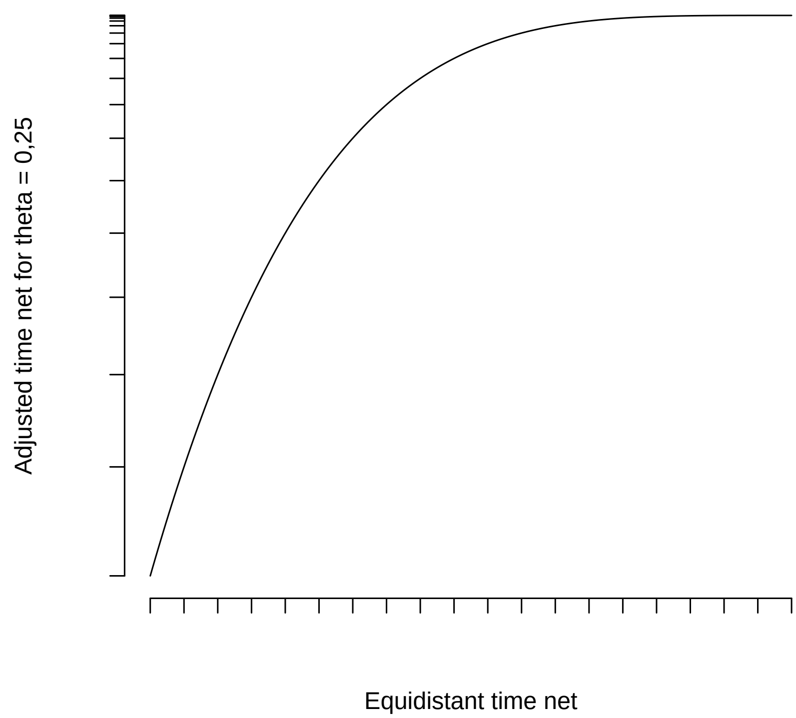

Figure 1: An illustration of a time net with refinement parameter $\theta = 0,25$.

To measure the smoothness properties of f, we define in Section 2.1 of [T] the Malliavin Sobolev space $\mathbb{D}_{1,2}(\gamma)$ and the derivative f' via the Hermite polynomials, and $\mathbb{D}_{1,p}(\gamma)$, $p > 2$ as a subset of $\mathbb{D}_{1,2}(\gamma)$ where both f and f' are L_p integrable.

Fractional smoothness is described by an index $\theta \in [0,1]$, where $\theta = 0$ implies no smoothness, and $\theta = 1$ differentiability in the Malliavin sense. More precisely, we use the Besov spaces

$$B^{\theta}_{p,q}(\gamma) = (L_p(\gamma), \mathbb{D}_{1,p}(\gamma))_{\theta,q} \tag{4}$$

which are intermediate spaces between $L_p(\gamma)$ and $\mathbb{D}_{1,p}(\gamma)$, $p \geq 2$, obtained by the real interpolation (see e.g. [2] and [T, Section 2.3]), with the interpolation parameters $0 < \theta < 1$ and $1 \leq q \leq \infty$. These spaces have a lexicographical order: for all $p \geq 2$,

$$B^{\theta_2}_{p,q_1}(\gamma) \subset B^{\theta_1}_{p,q_2}(\gamma)$$

for any $0 < \theta_1 < \theta_2 < 1$ and any $1 \leq q_1, q_2 \leq \infty$, and

$$B^{\theta}_{p,q_1}(\gamma) \subset B^{\theta}_{p,q_2}(\gamma)$$

for any $0 < \theta < 1$ and any $1 \leq q_1 \leq q_2 \leq \infty$. Figures 2 and 3 illustrate the inclusions.

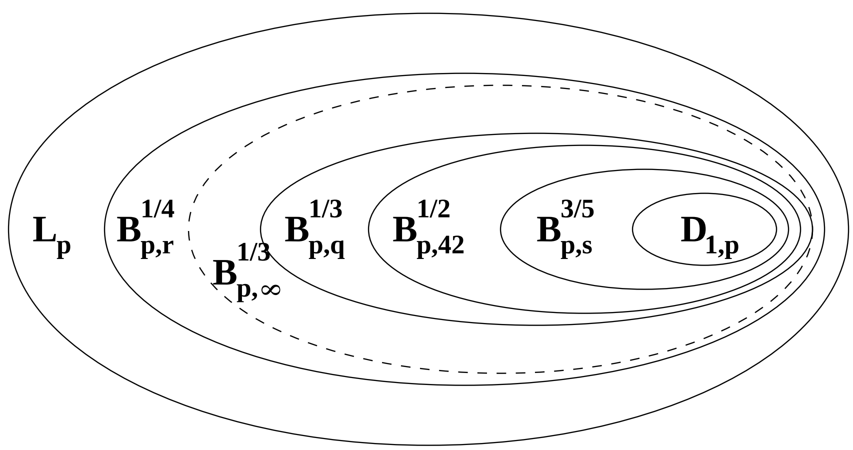

Figure 2: The spaces $B^{\theta}_{p,q}(\gamma)$ decrease as θ increases, regardless of q.

The purpose of this work is to investigate the convergence properties of the error defined in (2) as the size of the time net tends to zero, in connection with the smoothness properties of the function f.

There are several criteria to choose from when considering the convergence. It is known that the final error $C_1(f, \tau^{(n)})$ converges to zero in probability (see e.g. [11, Proposition 2.13 of Chapter IV]). The L_2 norm also

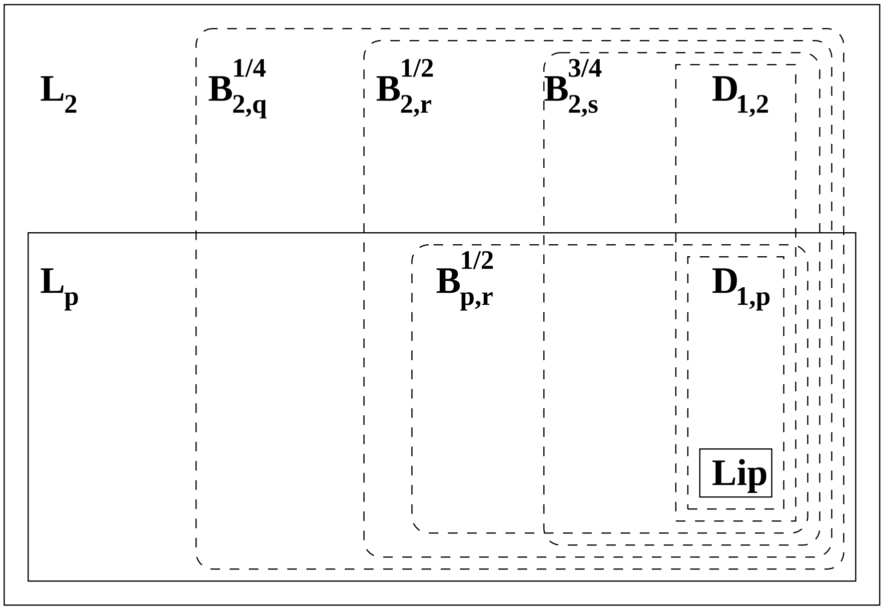

Figure 3: Some examples of inclusions for function spaces. Here $p > 2$.

converges to zero, and with equidistant time nets, the rate of L_2 convergence is $\frac{1}{\sqrt{n}}$ if and only if $f \in \mathbb{D}_{1,2}(\gamma)$, see [3, Theorem 2.6] and [8, Theorem 3.2]. This rate is optimal, provided that f is not an affine function, in which case the error is zero (see [4]). In [9], an L_2 convergence rate $n^{-\frac{1}{4}}$ was proved for $f(x) = \chi_{[K,\infty[}(x)$, and the rate $\left(\frac{1}{\sqrt{n}}\right)^{\alpha+\frac{1}{2}}$ for $f(x) = (x-K)_+^\alpha$, $K \in \mathbb{R}$, $0 < \alpha < \frac{1}{2}$, when using equidistant time nets. This showed that the regularity of f has some effect on the L_2 convergence rate, at least in these examples. In general, for a function with fractional smoothness $\theta \in]0,1[$, the L_2 convergence rate is $\left(\frac{1}{\sqrt{n}}\right)^\theta$ with equidistant time nets, or $\frac{1}{\sqrt{n}}$ when using appropriately adjusted time nets - the ones defined in (3). With correct choices of function spaces, these results are sharp. See [3] and [8] for the results in L_2, and [3] for a more extensive overview to the literature on related results in L_2.

A criterion even stronger than L_2 is considered in [7]: for Lipschitz continuous functions f, the error $C_1(f,\tau^{(n)})$ has bounded mean oscillation, and the convergence rate of $\left|\left|C(f,\tau^{(n)})\right|\right|_{BMO}$ is $\left|\left|\tau^{(n)}\right|\right|_\infty^{\frac{1}{2}}$. For equidistant nets, this rate equals $\frac{1}{\sqrt{n}}$.

These results form the starting point for [T]. Below they are listed omitting some technicalities, and the quantity $c > 0$ may vary from line to line:

$$f \in \mathbb{D}_{1,2}(\gamma) \iff ||C_1(f,\tau)||_{L_2} \leq c\,||\tau||_\infty^{\frac{1}{2}} \text{ for any time net } \tau \tag{5}$$

$$f \in B^\theta_{2,\infty}(\gamma) \iff ||C_1(f,\tau_n)||_{L_2} \leq cn^{-\frac{\theta}{2}} \tag{6}$$

$$f \in B^\theta_{2,2}(\gamma) \iff \left|\left|C_1\left(f,\tau_n^\theta\right)\right|\right|_{L_2} \leq cn^{-\frac{1}{2}} \tag{7}$$

$$f \in Lip \iff ||C(f,\tau)||_{BMO} \leq c\,||\tau||_\infty^{\frac{1}{2}} \text{ for any time net } \tau$$

Between L_2 and BMO there are L_p spaces with $2 < p < \infty$, and it is natural to expect some similar results for the L_p norm under suitable conditions. The techniques used in [7] for the BMO norm are still L_2 techniques, which rely heavily on orthogonality - or, more precisely, apply the very L_2 techniques locally (see [T, Appendix]). Interpolation techniques yield some estimates for the intermediate spaces, but for any sharp results, it was necessary to develope new techniques. Theorems 1.1 and 1.2 of [T] generalize the results (5) and (6) to L_p, $p > 2$.

Instead of examining strong convergence rates, we can look for the right scaling factor for the error so that the limiting distribution is non-trivial, and observe the properties of the achieved limit distribution. A natural guess for the factor would be the inverse of the strong convergence rate, but this leads to trouble. For example, in [9], with equidistant time nets the L_2 convergence rate for $f(x) = \chi_{[K,\infty[}(x)$ is found to be $n^{-\frac{1}{4}}$ (which was later also verified by (6)), but a limit distribution $\tilde{W}_{\frac{1}{2}\int_0^1\left[\frac{\partial^2 F}{\partial x^2}(t,W_t)\right]^2 dt}$ is achieved with the scaling factor $\sqrt{n}$ as formulated in (8). This mismatch between the convergence rate and the scaling factor is in [9] explained by the fact that the limit distribution is not L_2 integrable for the above mentioned example. In fact, the limit distribution with equidistant time nets is L_2 integrable if and only if $f \in \mathbb{D}_{1,2}(\gamma)$ (see [3, Theorems 2.3 and 2.6]). Denoting the weak convergence by $\Longrightarrow$, the results to begin with are thus

$$\sqrt{n}C_1(\chi_{[K,\infty[},\tau_n) \Longrightarrow \tilde{W}_{\frac{1}{2}\int_0^1\left[\frac{\partial^2 F}{\partial x^2}(t,W_t)\right]^2 dt} \text{ and} \tag{8}$$

$$f \in \mathbb{D}_{1,2}(\gamma) \iff \tilde{W}_{\frac{1}{2}\int_0^1\left[\frac{\partial^2 F}{\partial x^2}(t,W_t)\right]^2 dt} \in L_2. \tag{9}$$

In [GT] we use non-equidistant time nets to obtain a square integrable limit distribution for a wider class of functions, and the refinement index for the time net is the same as the fractional smoothness of f. Moreover, we obtain conditions under which the limit distribution is L_p integrable with $p > 2$.

2 Results

We begin with strong convergence. With the equidistant time nets, Theorem 1.2 of [T] generalizes the L_2 result seen in (6) to

$$f \in B^{\theta}_{p,\infty}(\gamma) \Longleftrightarrow ||C_1(f,\tau_n)||_{L_p} \leq cn^{-\frac{\theta}{2}} \tag{10}$$

for $p \geq 2$. It thus also provides an alternative proof for (6). For smooth functions, interpolation between the results in L_2 and BMO yields

$$f \in \mathbb{D}_{1,p}(\gamma) \Longrightarrow ||C_1(f,\tau)||_{L_p} \leq c\,||\tau||_{\infty}^{\frac{1}{2}} \text{ for any time net } \tau,$$

as seen in Theorem 1.1 of [T]. We do not know how to deduce the equivalence as in (5). However, the result is nearly sharp, as we see from (10).

For weak convergence, we employ the non-equidistant time nets defined in (3). With $0 < \theta < 1$ and $f \in L_2(\gamma)$, the weak limit of the error at time $t \in [0,1[$ is computed in [GT] as

$$\sqrt{n}C_t\left(f,\tau_n^{\theta}\right) \Longrightarrow \tilde{W}_{\frac{1}{2\theta}\int_0^t\left[(1-u)^{1-\theta}\frac{\partial^2 F}{\partial x^2}(u,W_u)\right]^2 du},$$

where $\tilde{W}$ is an independent Brownian motion. If $f \in B^{\theta}_{2,2}(\gamma)$, this limit extends to $t = 1$ and is L_2 integrable. Moreover, if $p > 2$ and $f \in B^{\theta+\epsilon}_{p,\infty}(\gamma)$ for some $0 < \epsilon < 1-\theta$, then the limit is L_p integrable.

We can formulate the main results of [T] and [GT] in the following three theorems, assisted with an L_2 result from [3] and [8]. These theorems consist of implications between strong convergence, weak convergence and fractional smoothness. In addition, results in [GT] connect the fractional smoothness of f to the L_p convergence of $\mathbb{E}\left(f(W_1) \mid \mathcal{F}_t\right)$ as $t \to 1$, and establish links to other concepts of fractional Sobolev spaces (see Proposition 3.5 and Remark 3.6 of [GT]).

Let us shorten the notation with the following abbreviation:

$\left(M_{L_p}\right)$: On some stochastic basis, there exists a continuous L_p integrable martingale $M = (M_t)_{t\in[0,1]}$, such that

This abbreviation will be used in context of weak convergence. We consider the weak convergence on the level of processes, in particular the error processes $C\left(f,\tau^{(n)}\right)$ associated with the time nets $\tau^{(n)}$, instead of the random variables $C_1\left(f,\tau^{(n)}\right)$. This we do to obtain the equivalence we are looking for; we do not know whether the fractional smoothness follows from the weak convergence of the final error. We denote the weak convergence in $C[0,1]$ by $\Longrightarrow_{C[0,1]}$.

For convenience, in the following theorem we use the notation $B^1_{2,2}(\gamma) := \mathbb{D}_{1,2}(\gamma)$.

Theorem 2.1. *Let $f \in \mathrm{L}_2(\gamma)$ and $0 < \theta \leq 1$. Then the following are equivalent:*

(i) $f \in B^\theta_{2,2}(\gamma)$,

(ii) (M_{L_2}) : $\sqrt{n}C\left(f, \tau^\theta_n\right) \Longrightarrow_{C[0,1]} M$, *and*

(iii) $\left|\left|C_1\left(f, \tau^\theta_n\right)\right|\right|_{\mathrm{L}_2} \leq c\frac{1}{\sqrt{n}}$ *for some* $c > 0$.

Proof. For the equivalence of (i) and (iii), see [3] and [8]. The equivalence of (i) and (ii) is included in Theorem 2.1 of [GT]. □

Along with (6), reproduced below, Theorem 2.1 provides a complete picture in the L_2 setting:

$$f \in B^\theta_{2,\infty}(\gamma) \Longleftrightarrow ||C_1(f, \tau_n)||_{L_2} \leq cn^{-\frac{\theta}{2}} \text{ for some } c > 0.$$

Theorem 2.2. *Let $f \in \mathrm{L}_2(\gamma)$ and $2 < p < \infty$, and consider the following conditions:*

(i) $f \in \mathbb{D}_{1,p}(\gamma)$,

(i') $f \in \bigcap_{0<\theta<1} B^\theta_{p,\infty}(\gamma)$,

(ii) $\left(M_{\mathrm{L}_p}\right)$: $\sqrt{n}C\left(f, \tau_n\right) \Longrightarrow_{C[0,1]} M$, *and*

(iii) $||C_1\left(f, \tau_n\right)||_{\mathrm{L}_p} \leq c\frac{1}{\sqrt{n}}$ *for some* $c > 0$.

Then the following implications hold:

$$\text{(i)} \Longleftrightarrow \text{(ii)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(i}'\text{)}.$$

Proof. Theorem 1.1 and Corollary 1.3 of [T] include the implications (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (i$'$). By the Burkholder-Davis-Gundy inequalities,

$$\sup_{0<t<1} \left|\left|\frac{\partial F}{\partial x}(t, W_t)\right|\right|_{L_p} < \infty \tag{11}$$

implies that $f \in L_p(\gamma)$. Thus (11) is equivalent to (i) by Lemma 4.8 of [T]. Furthermore, (11) is equivalent to

$$\sup_{0\leq t<1} \left|\left|\frac{\partial F}{\partial x}(t, W_t) - \frac{\partial F}{\partial x}(0,0)\right|\right|_{L_p} < \infty,$$

which coincides with the condition given in [GT, Corollary 3.3 (ii)] for Brownian motion and $\beta = 1$. □

Theorem 2.3. *Let $2 < p < \infty$, $f \in \mathrm{L}_p(\gamma)$ and $0 < \theta < 1$, and consider the following conditions:*

(i) $f \in B^{\theta}_{p,\infty}(\gamma)$,

(ii) $\left(M_{\mathrm{L}_p}\right)$: $\sqrt{n}C\left(f, \tau_n^{\theta}\right) \Longrightarrow_{C[0,1]} M$,

(ii') $\left(M_{\mathrm{L}_p}\right)$: $\sqrt{n}C\left(f, \tau_n^{\theta-\epsilon}\right) \Longrightarrow_{C[0,1]} M$ *for any* $0 < \epsilon < \theta$, *and*

(iii) $||C_1\left(f, \tau_n\right)||_{\mathrm{L}_p} \leq cn^{-\frac{\theta}{2}}$ *for some* $c > 0$.

Then the following implications hold:

$$\text{(ii)} \Longrightarrow \text{(i)} \Longleftrightarrow \text{(iii)} \Longrightarrow \text{(ii}'\text{)}.$$

Proof. The equivalence of (i) and (iii) is proven in Theorem 1.2 of [T]. By Lemma 4.7 of [T], condition (i) is equivalent to

$$\sup_{0<t<1} (1-t)^{\frac{1-\theta}{2}} \left|\left|\frac{\partial F}{\partial x}(t, W_t)\right|\right|_{L_p} < \infty.$$

Corollary 3.4 of [GT] for Brownian motion completes the proof. □

The equivalence of (i) and (iii) in Theorem 2.3 generalizes (6) to L_p for $p > 2$. The disparity between conditions (ii) and (ii') in Theorem 2.3 in comparison with Theorem 2.1 suggests that for an equivalence concerning weak convergence, one might want to consider $B^{\theta}_{p,q}(\gamma)$ for some $q < \infty$ instead of $B^{\theta}_{p,\infty}(\gamma)$. We formulate this as a conjecture.

Conjecture 2.4. *Let $2 < p < \infty$, $f \in \mathrm{L}_p(\gamma)$ and $0 < \theta < 1$. Then there exists a parameter $1 \leq q < \infty$ such that the following condititions are equivalent:*

(I) $f \in B^{\theta}_{p,q}(\gamma)$,

(II) $\left(M_{\mathrm{L}_p}\right)$: $\sqrt{n}C\left(f, \tau_n^{\theta}\right) \Longrightarrow_{C[0,1]} M$, *and*

(III) $\left|\left|C_1\left(f, \tau_n^{\theta}\right)\right|\right|_{\mathrm{L}_p} \leq c\frac{1}{\sqrt{n}}$ *for some* $c > 0$.

A natural first guess for q, arising from Theorem 2.1, would be $q = 2$ or $q = p$. Theorem 2.3 shows a close connection, if not quite an equivalence, between (I) and (II), but for (III), we do not yet have a characterization. Theorem 5.3 of [T] shows that improvement in the L_p convergence rate is possible with the non-equidistant time nets defined in (3), but it does not provide a clear connection between the properties of f and the refining parameter necessary for the optimal convergence rate. This directions remains open for further research.

2.1 Examples

We illustrate the obtained results by an example.

Example 2.5. *Let* $-\frac{1}{2} < \alpha \leq 1$ *and define*

$$f_\alpha(x) := \begin{cases} 0, & x \leq 0 \\ x^\alpha, & x > 0 \end{cases} .$$

Then, for $p \geq 2$*:*

- *if* $\frac{1}{p} + \alpha > 1$*, then*
  - $f_\alpha \in \mathbb{D}_{1,p}(\gamma)$,
  - $||C_1(f_\alpha, \tau_n)||_{\mathrm{L}_p} \leq \frac{c}{\sqrt{n}}$ *for some* $c > 0$ *depending only on* p *and* α,
  - *for all* $\lambda > 0$,

    $$\mathbb{P}\left(|C_1(f_\alpha, \tau_n)| \geq \frac{\lambda}{n^{\frac{1}{2}}}\right) \leq c^p \lambda^{-p}$$

    for some $c > 0$ *depending only on* p *and* α*, and*
  - *for an independent Brownian motion* $\tilde{W}$,

    $$\sqrt{n} C_1\left(f_\alpha, \tau_n\right) \Longrightarrow \tilde{W}_{\frac{1}{2}\int_0^1 \left[\frac{\partial^2 F}{\partial x^2}(u, W_u)\right]^2 du} \in \mathrm{L}_p,$$

    where we ignore the set of measure zero where $\int_0^1 \left[\frac{\partial^2 F}{\partial x^2}(u, W_u)\right]^2 du$ *may be infinite;*

- *if* $0 < \frac{1}{p} + \alpha < 1$*, then*
  - $f_\alpha \in B_{p,\infty}^{\frac{1}{p}+\alpha}(\gamma)$,
  - $||C_1(f_\alpha, \tau_n)||_{\mathrm{L}_p} \leq c\left(\frac{1}{\sqrt{n}}\right)^{\frac{1}{p}+\alpha}$ *with some* $c > 0$ *depending only on* p *and* α,
  - *for all* $\lambda > 0$,

    $$\mathbb{P}\left(|C_1(f_\alpha, \tau_n)| \geq \frac{\lambda}{n^{\frac{1+\alpha p}{2p}}}\right) \leq c^p \lambda^{-p}$$

    for some $c > 0$ *depending only on* p *and* α*, and*

– *for any* $0 < \theta < \frac{1}{p} + \alpha$, *and an independent Brownian motion* $\tilde{W}$,

$$\sqrt{n} C_1\left(f_\alpha, \tau_n^\theta\right) \Longrightarrow \tilde{W}_{\frac{1}{2\theta}\int_0^1 \left[(1-u)^{1-\theta}\frac{\partial^2 F}{\partial x^2}(u, W_u)\right]^2 du} \in \mathrm{L}_p,$$

where we ignore the set of measure zero where $\int_0^1 \left[(1-u)^{1-\theta}\frac{\partial^2 F}{\partial x^2}(u, W_u)\right]^2 du$ *may be infinite.*

Computations to verify the fractional smoothness of f_α are similar to Examples 5.1 and 5.2 of [T]. The convergence properties follow from the theorems above, and the tail estimates from Chebysev's inequality.

Notice that $\frac{1}{p} + \alpha = 1$ does not imply that $f_\alpha \in \mathbb{D}_{1,p}(\gamma)$, but only that $f_\alpha \in B^\theta_{p,\infty}(\gamma)$ for all $0 < \theta < 1$. For f_0 we have the jump function, and for $\alpha < 0$, the function f_α has a singularity at 0. Functions with singularities are studied more carefully in Section 3 of [GT], pages 14-16.

Figure 4 illustrates how the fractional smoothness θ decreases when p increases, for certain functions. The dash curves represent the positions of f_β, f_0, $f_{\frac{1}{2}}$ and f_α with $-\frac{1}{2} < \beta < -\frac{1}{p}$ and $\frac{1}{2} < \alpha < 1 - \frac{1}{p}$ in the Besov spaces $B^\theta_{p,\infty}$ for different $p \geq 2$ and different $0 < \theta < 1$. Recall that $f_{\frac{1}{2}} \in B^\theta_{2,\infty}(\gamma) \setminus \mathbb{D}_{1,2}(\gamma)$ for any $0 < \theta < 1$, and that $f_\beta \notin L_p(\gamma)$.

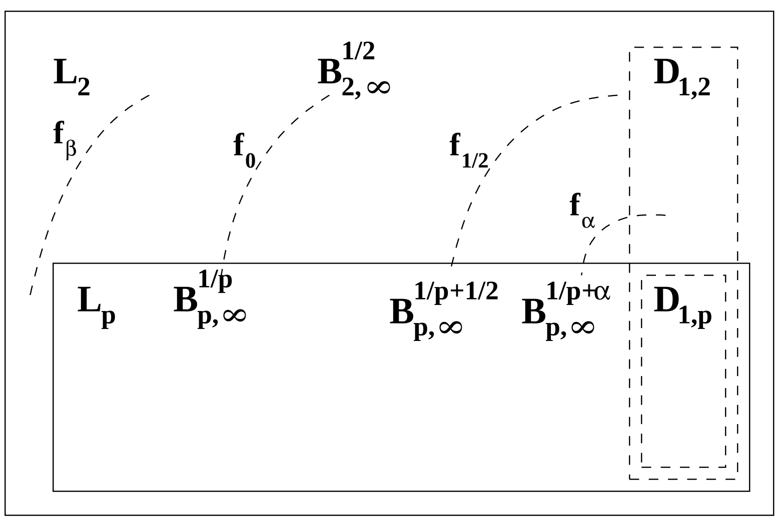

Figure 4: An illustration of f_β, f_0, $f_{\frac{1}{2}}$ and f_α when $-\frac{1}{2} < \beta < -\frac{1}{p}$ and $\frac{1}{2} < \alpha < 1 - \frac{1}{p}$.

For the exponential tail estimates of the weak limit, see Remark 3.13 of

[GT]. For an η-Hölder continuous function f, for example,

$$\mathbb{P}\left(\left|\tilde{W}_{\frac{1}{2\theta}\int_0^1\left[(1-u)^{1-\theta}\frac{\partial^2 F}{\partial x^2}(u,W_u)\right]^2 du}\right| > \lambda\right) \leq ce^{-\frac{\lambda^2}{c}}$$

for all $\lambda > 0$ with any $0 < \theta < \eta \leq 1$, where we ignore the set of measure zero where $\int_0^1 \left[(1-u)^{1-\theta}\frac{\partial^2 F}{\partial x^2}(u, W_u)\right]^2 du$ may be infinite.

3 Conclusions

In this thesis, we have obtained close connections between strong convergence, weak convergence and fractional smoothness when approximating stochastic integrals. These results are restricted to the case of Brownian integrals, partly including integrals with respect to the geometric Brownian motion.

Extentions to more general diffusions would be interesting for applications especially in stochastic finance. The results of [T] seem to be extendable to the geometric Brownian motion. In [3], the related L_2 results are obtained for diffusions satisfying $dY_t = \sigma(Y_t)dW_t$, $Y_0 = y_0$, with some conditions on σ, so that one might expect similar extensions for L_p, $p > 2$. For processes with jumps, the ongoing research of [5] extends some L_2 results of [8] to Lévy processes.

A natural next step within the Brownian motion setting would be a complete characterization of time nets required for optimal L_p convergence rate as seen in Conjecture 2.4.

In [GT], there are also connections to other concepts of fractional smoothness, for example the one mentioned in [1] related to approximation of stochastic differential equations. Further study in this direction would extend the setting beyond the Gaussian case.

The results presented here concern first-order approximations of stochastic integrals. Considering higher orders of fractional smoothness might reveal interesting convergence properties for higher order approximations.

For more irregular functions, existence of arbitrarily slow L_2 convergence rates was shown in [10] when optimized over all deterministic time nets. In [12], there is a characterization of functions for which the optimized L_2 convergence rate is $\frac{1}{\sqrt{n}}$, and an example where, for a given $0 < \beta < 1$, the optimized convergence rate is $n^{-\frac{\beta}{2}}$. This takes us outside the polynomial scale of the Besov spaces considered here. Further research on this logarithmic approach might reveal new connections to L_p convergence for $p > 2$ or even to weak convergence.

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