UNIVERSITY OF JYVÄSKYLÄ
DEPARTMENT OF MATHEMATICS
AND STATISTICS

REPORT 116

UNIVERSITÄT JYVÄSKYLÄ
INSTITUT FÜR MATHEMATIK
UND STATISTIK

BERICHT 116

WEIGHTED HARDY INEQUALITIES AND THE BOUNDARY SIZE

JUHA LEHRBÄCK

JYVÄSKYLÄ
2008

UNIVERSITY OF JYVÄSKYLÄ
DEPARTMENT OF MATHEMATICS
AND STATISTICS

REPORT 116

UNIVERSITÄT JYVÄSKYLÄ
INSTITUT FÜR MATHEMATIK
UND STATISTIK

BERICHT 116

WEIGHTED HARDY INEQUALITIES AND THE BOUNDARY SIZE

JUHA LEHRBÄCK

To be presented, with the permission of the Faculty of Mathematics and Science of the University of Jyväskylä, for public criticism in Auditorium Paulaharju, Villa Rana, on November 29th, 2008, at 12 o'clock noon.

JYVÄSKYLÄ
2008

Editor: Pekka Koskela
Department of Mathematics and Statistics
P.O. Box 35 (MaD)
FI–40014 University of Jyväskylä
Finland

ISBN 978-951-39-3347-0
ISSN 1457-8905

University Printing House
Jyväskylä 2008

Acknowledgements

I want to express my deepest gratitude to my advisor, Professor Pekka Koskela, for his excellent guidance and support throughout my studies, and, above all, for sharing and doing mathematics with me. I would also like to thank the people at the Department of Mathematics and Statistics of the University of Jyväskylä for creating such a pleasant environment for studying and working.

For discussions, comments and other help related to this work I am grateful to Juha Kinnunen, Ville Suomala, Juhani Takkinen and Xiao Zhong. For financial support I am indebted to the *Center of Excellence of Geometric Analysis and Mathematical Physics* of the Academy of Finland.

My family, friends and relatives deserve a huge thank for their continued support and encouragement.

Finally, I thank my wife Piia for being the love of my life and my home; this work is dedicated to you and to our beloved children Juuso, Jere and Iida.

Jyväskylä, October 2008

Juha Lehrbäck

List of included articles

This dissertation consists of an introductory part and the following publications:

[KL] P. Koskela and J. Lehrbäck, 'Weighted pointwise Hardy inequalities', Preprint 332, Department of Mathematics and Statistics, University of Jyväskylä, 2007.

[L1] J. Lehrbäck, 'Pointwise Hardy inequalities and uniformly fat sets', *Proc. Amer. Math. Soc.* 136 (2008), no. 6, 2193–2200.

[L2] J. Lehrbäck, 'Self-improving properties of weighted Hardy inequalities', *Adv. Calc. Var.* 1 (2008), no. 2, 193–203.

[L3] J. Lehrbäck, 'Necessary conditions for weighted pointwise Hardy inequalities', Preprint 361, Department of Mathematics and Statistics, University of Jyväskylä, 2008.

[L4] J. Lehrbäck, 'Weighted Hardy inequalities and the size of the boundary', *Manuscripta Math.* 127 (2008), no. 2, 249–273.

1 Introduction

This thesis is devoted to different aspects of the so-called *Hardy inequalities.* We say that a domain (i.e. an open and connected set) $\Omega \subset \mathbb{R}^n$ *admits* the (p,β)-Hardy inequality, for $1 < p < \infty$ and $\beta \in \mathbb{R}$, if there exists a constant $C > 0$ such that

$$(1) \qquad \int_\Omega |u(x)|^p \, d_\Omega(x)^{\beta-p} \, dx \le C \int_\Omega |\nabla u(x)|^p \, d_\Omega(x)^\beta \, dx$$

holds for every smooth test-function $u \in C_0^\infty(\Omega)$. Here $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$. Our main interest is in finding both necessary and sufficient conditions for a domain $\Omega \subset \mathbb{R}^n$ to admit the (p,β)-Hardy inequality.

1.1 History and the basic setting

The origins of inequality (1) trace back to the early 20th century. The first appearance of a Hardy inequality was, in a rather weak form and without a proof, in the 1920 note [7] by G. H. Hardy; the main objects of interest in this note were the corresponding inequalities for infinite series. However, in the famous 1925 paper [8], Hardy proved that the inequality

$$(2) \qquad \int_0^\infty \left(\frac{1}{x} \int_0^x f(t) \, dt \right)^p dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p \, dx,$$

where $1 < p < \infty$, holds whenever $f \ge 0$ is measurable and, moreover, that the constant on the right-hand side is the best possible. An excellent account on the progress leading to inequality (2), illustrating the sometimes incoherent nature of mathematical discovery, is given in [17].

Inequality (2), often simply called *the Hardy inequality*, turned out to be of its own independent interest, and it, together with the many generalizations, has been studied extensively ever since. The proof of (2) is rather simple, the only tools needed are integration by parts and Hölder's inequality. Exactly the same method can be applied in order to prove one-dimensional *weighted Hardy inequalities* (cf. [9, §330]), which can be formulated as follows (essentially [16, Thm. 5.2]):

Theorem A. *Let* $1 < p < \infty$ *and* $\beta \neq p - 1$. *If* $u\colon (0,\infty) \to \mathbb{R}$ *is absolutely continuous with* $\lim_{x\to 0} u(x) = 0 = \lim_{x\to\infty} u(x)$, *then* u *satisfies the inequality*

$$(3) \qquad \int_0^\infty |u(x)|^p \, x^{\beta-p} \, dx \le \left(\frac{p}{|p-1-\beta|} \right)^p \int_0^\infty |u'(x)|^p \, x^\beta \, dx,$$

where the constant on the right-hand side is the best possible.

Notice that (3) is equivalent to (2) in the unweighted case $\beta = 0$, and also, related to (1), that here $x = d_\Omega(x)$ if we take $\Omega = (0, \infty)$. See the monograph [18] and the references therein for further developments concerning (3) and more general one-dimensional inequalities of Hardy type.

By Theorem A, it is justified to say that the domain $(0, \infty) \subset \mathbb{R}$ admits the Hardy inequality (1) whenever $1 < p < \infty$ and $\beta \neq p - 1$. On the other hand, it is quite easy to verify that a bounded interval $(a, b) \subset \mathbb{R}$, where $-\infty < a < b < \infty$, admits the (p, β)-Hardy inequality only if $1 < p < \infty$ and $\beta < p - 1$. These facts completely settle the question of finding conditions for one-dimensional domains to admit the (p, β)-Hardy inequality (1).

The case of domains in $\mathbb{R}^n$, for $n \geq 2$, the one we are dealing with, is naturally much more involved. The first occurrence of Hardy inequalities in this higher dimensional setting was due to J. Nečas [23] in 1962, when he proved, perhaps a bit surprisingly, that for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ the results are exactly the same as for bounded intervals in $\mathbb{R}$.

Theorem B (Nečas)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then Ω admits the (p, β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < p - 1$.*

See also Kufner [16] for related results and applications of Hardy inequalities. We recall that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain if for every $w \in \partial\Omega$ there exists $r > 0$ such that $\partial\Omega \cap B(w, r)$ can be represented as (a part of) the graph of a Lipschitz-continuous function $\varphi_w \colon \mathbb{R}^{n-1} \to \mathbb{R}$. The proof of Theorem B relies heavily on the one-dimensional Hardy inequalities of Theorem A.

1.2 Uniform fatness and Hardy inequalities

The smoothness of the boundary, as in Theorem B, is however by no means necessary for a domain $\Omega \subset \mathbb{R}^n$ to admit Hardy inequalities. For more general domains an important sufficient requirement turns out to be that the complement of Ω is somehow “fat” enough — as is always the case with a smooth boundary.

Before being able to state such conditions we need to introduce concrete ways to measure the size (“fatness”) of a set. One possibility is given by the concept of capacity. When $\Omega \subset \mathbb{R}^n$ is a domain and $E \subset \Omega$ is a compact subset, we define the *(variational) p-capacity* of E (relative to Ω) as

$$\mathrm{cap}_p(E, \Omega) = \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in C_0^\infty(\Omega),\ u \geq 1 \text{ on } E \right\}.$$

For the basic properties of the p-capacity we refer to [12]. A closed set $E \subset \mathbb{R}^n$ is now said to be *uniformly p-fat* if there exists a constant $C > 0$ such that

$$\mathrm{cap}_p \big(E \cap \overline{B}(x, r), B(x, 2r) \big) \geq C \, \mathrm{cap}_p \big(\overline{B}(x, r), B(x, 2r) \big)$$

for every $x \in E$ and all $r > 0$. Actually, here $\mathrm{cap}_p(\overline{B}(x,r), B(x,2r)) = C(n,p)r^{n-p}$ for each ball $B(x,r) \subset \mathbb{R}^n$ (see e.g. [12]).

There are many natural examples of uniformly far sets. For instance, the complement of a Lipschitz domain is uniformly p-fat for all $p > 1$, and if $E \subset \mathbb{R}^n$ is an affine subspace of dimension m, then E is uniformly p-fat whenever $p > n - m$. Moreover, if $E \subset \mathbb{R}^n$ is a non-empty closed set, then E is uniformly p-fat for every $p > n$.

It is easy to see that if a set $E \subset \mathbb{R}^n$ is uniformly p-fat and $p' > p$, then E is also uniformly p'-fat. However, it is *not* easy to see that uniform fatness possesses the following self-improving property: If a set $E \subset \mathbb{R}^n$ is uniformly p-fat for some $1 < p < \infty$, then there exists $1 < q < p$ such that E is uniformly q-fat. This deep result is due to J. Lewis [19, Thm. 1]; see also [22] for another proof.

The notion of uniform p-fatness is closely related to Hardy inequalities, as can be seen from the following theorem by A. Ancona [2] (the case $p = 2$), J. Lewis [19], and A. Wannebo [25], dating to the late 1980's.

Theorem C (Ancona, Lewis, Wannebo)**.** *Let $1 < p < \infty$ and assume that the complement of a domain $\Omega \subset \mathbb{R}^n$ is uniformly p-fat. Then Ω admits the p-Hardy inequality (i.e.* (1) *with $\beta = 0$).*

Wannebo [25] proved in fact even more, namely that the p-fatness of Ω^c implies (p,β)-Hardy inequalities in Ω for all $\beta < \beta_0$, where $\beta_0 = \beta_0(p,n,\Omega)$ is a (small and implicit) positive number. Actually, it is rather easy to see that if Ω admits the p-Hardy inequality, then there exists some $\beta_0 > 0$ so that Ω admits (p,β)-Hardy inequalities whenever $|\beta| < \beta_0$ (cf. [5]), but it is not in general true that the p-Hardy inequality would imply (p,β)-Hardy inequalities for all $\beta < 0$. Taking this into account, we see that the full result of Wannebo is strictly stronger than the claim Theorem C. It is worth mentioning that Lewis et. al. defined uniform fatness in terms of different capacities, but by the well-known equivalence results (see e.g. [26]) all these definitions of fatness turn out to be essentially the same. There is also a converse to Theorem C in the case $p = n$, i.e., if $\Omega \subset \mathbb{R}^n$ admits the n-Hardy inequality, then Ω^c is uniformly n-fat, see [2] ($p = n = 2$) and [19, Thm. 3]. Results related to this are discussed in the next section.

A more geometric interpretation of fatness can be achieved by means of *Hausdorff contents.* Recall that the λ-Hausdorff content of a set $E \subset \mathbb{R}^n$ is defined as

$$\mathcal{H}^\lambda_\infty(E) = \inf\left\{ \sum_{i=1}^\infty r_i^\lambda : E \subset \bigcup_{i=1}^\infty B(z_i, r_i)\right\},$$

and that the *Hausdorff dimension* of $E \subset \mathbb{R}^n$ is then given by

$$\dim_{\mathcal{H}}(E) = \inf\left\{\lambda > 0 : \mathcal{H}^\lambda_\infty(E) = 0\right\}.$$

We say that a compact set E is of p-capacity zero, denoted $\mathrm{cap}_p(E) = 0$, if $\mathrm{cap}_p(E,\Omega) = 0$ for some domain Ω with $E \subset \Omega$. A closed set E

is of p-capacity zero if $\operatorname{cap}_p(E \cap \overline{B}(0,k)) = 0$ for all $k \in \mathbb{N}$. Now the standard result between capacity and Hausdorff dimension of $E \subset \mathbb{R}^n$ is that $\operatorname{cap}_p(E) = 0$ implies $\dim_{\mathcal{H}}(E) \leq n - p$, and conversely, if $\dim_{\mathcal{H}}(E) < n - p$, then $\operatorname{cap}_p(E) = 0$ (see e.g. [12]). However, there is also a quantitative version of this connection (cf. [12]), which leads to the following characterization: A closed set $E \subset \mathbb{R}^n$ is uniformly p-fat, for $1 < p < \infty$, if and only if there exists some exponent $\lambda > n - p$ and a constant $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(E \cap B(w,r)\big) \geq Cr^{\lambda} \quad \text{for every } w \in E \text{ and all } r > 0. \tag{4}$$

For our purposes such conditions in terms of Hausdorff contents turn out to be fruitful; some indication of this will be given already in the next section.

Let us mention here that uniform fatness is a stronger condition than the famous Wiener criterion, which guarantees continuity up to the boundary for $\mathcal{A}$-harmonic functions, that is, solutions of the $\mathcal{A}$-harmonic equation $-\operatorname{div} \mathcal{A}(x, \nabla u) = 0$, where $\mathcal{A}\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies certain structural conditions (cf. [12]). Indeed, uniform p-fatness of Ω^c implies even Hölder-continuity up to the boundary for $\mathcal{A}$-harmonic functions in Ω (provided that the boundary values are Hölder-continuous); see [12, Chapter 6] for precise formulations and further information. In the following we give a new characterization of domains whose complements are uniformly p-fat, and so our work might be relevant in terms of such applications as well.

We would also like to remark that general capacity-type characterizations of V. G. Maz'ja (see for instance [21, Chapter 2.3]) yield necessary and sufficient conditions for a domain to admit Hardy inequalities. Altough very useful, these results are of quite a different nature than the above considerations based on the fatness of the complement, and hence they will not be discussed here any further.

1.3 Pointwise Hardy inequalities

A new chapter in the development of Hardy inequalities was opened in the late 90's, when P. Hajłasz [5] and J. Kinnunen and O. Martio [13] noticed, independently, that the uniform p-fatness of Ω^c not only implies the usual p-Hardy inequality, but also a stronger *pointwise p-Hardy inequality.*

Theorem D (Hajłasz, Kinnunen-Martio)**.** *Let $1 < p < \infty$ and assume that the complement of a domain $\Omega \subset \mathbb{R}^n$ is uniformly p-fat. Then there exist some exponent $1 < q < p$ and a constant $C > 0$ such that the inequality*

$$|u(x)| \leq C d_{\Omega}(x)\big(M_{2d_{\Omega}(x)}\big(|\nabla u|^q\big)(x)\big)^{1/q} \tag{5}$$

holds for all $u \in C_0^{\infty}(\Omega)$ at every $x \in \Omega$.

If the conclusion of Theorem D holds in a domain Ω we say that Ω admits the pointwise p-Hardy inequality. In (5) M_R is the usual restricted

Hardy-Littlewood maximal operator, defined by

$$M_R f(x) = \sup_{0<r\le R} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We remark that the self-improving property of uniform p-fatness plays a crucial role in the proof of Theorem D.

It is very easy to show, using the maximal theorem (cf. [24]), that if the pointwise p-Hardy inequality (5) holds for a function $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$ with a constant $C_1 > 0$, then u satisfies the usual p-Hardy inequality with a constant $C = C(C_1, p, q, n) > 0$. Hence we conclude that a domain admitting the pointwise p-Hardy inequality also admits the usual p-Hardy inequality. However, the pointwise p-Hardy inequality is not equivalent to the usual p-Hardy inequality, since there are domains which admit the latter for some p, but where the corresponding pointwise inequality fails to hold. One example of such a domain is the punctured unit ball $B(0,1)\setminus\{0\} \subset \mathbb{R}^n$, which admits the pointwise p-Hardy inequality only in the trivial case $p > n$, but where the usual p-Hardy inequality also holds when $1 < p < n$ (see [KL] for more details). This same example also shows that, apart from the case $p = n$, uniform p-fatness of the complement is not necessary for a domain to admit the p-Hardy inequality, as the complement of $B(0,1)\setminus\{0\} \subset \mathbb{R}^n$ is not uniformly p-fat for any $p \le n$; the failure happens of course at the origin.

Nevertheless, it turns out that uniform p-fatness of Ω^c is not only sufficient, but also *necessary* for Ω to admit the *pointwise* p-Hardy inequality. There is also another very natural sufficient and necessary condition for Ω to admit the pointwise p-Hardy inequality. Namely, that there exists a constat $C > 0$ and *some* exponent $\lambda > n - p$ so that

$$\mathcal{H}^\lambda_\infty\big(B(x, 2d_\Omega(x)) \cap \partial\Omega\big) \ge C d_\Omega(x)^\lambda \quad \text{for every } x \in \Omega. \tag{6}$$

We refer to requirements of the type (6) as *inner boundary density conditions.* All in all, let us combine the above equivalent conditions in a theorem, which is essentially [L1, Thm. 1].

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $1 < p < \infty$. Then the following assertions are quantitatively equivalent:*

(*a*) *The complement Ω^c is uniformly p-fat;*

(*b*) *Ω admits the pointwise p-Hardy inequality;*

(*c*) *There exists some $n - p < \lambda \le n$ so that Ω satisfies the inner boundary density condition* (6) *with the exponent λ;*

(*d*) *There exists some $n - p < \lambda \le n$ and $C > 0$ so that*

$$\mathcal{H}^\lambda_\infty\big(\Omega^c \cap B(w,r)\big) \ge C r^\lambda \quad \textit{for every } w \in \Omega^c \textit{ and all } r > 0.$$

As was already discussed, the implications $(a)\Rightarrow(b)$ and $(d)\Leftrightarrow(a)$ were previously known, whereas $(b)\Rightarrow(c)$ and $(c)\Rightarrow(d)$ were established in [L1]; a more detailed treatment concerning the constants in assertions (a)–(d) can be found in [L3].

We mention here that, in the spirit of the pointwise p-Hardy inequality of Hajłasz and Kinnunen-Martio, the following pointwise version of the weighted Hardy inequality (1) was introduced in [KL]:

$$(7) \qquad |u(x)| \leq C d_\Omega(x)^{1-\frac{\beta}{p}} \Big(M_{2d_\Omega(x)} \big(|\nabla u|^q d_\Omega{}^{\frac{\beta}{p}q} \big)(x) \Big)^{1/q},$$

where again $1 < q < p$. We now say that a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise (p,β)-Hardy inequality if there exist some $1 < q < p$ and a constant $C > 0$ so that the inequality (7) holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$ with these q and C. As in the unweighted case, the pointwise (p,β)-Hardy inequality can be easily seen to imply the usual weighted (p,β)-Hardy inequality.

2 Model examples

New sufficient conditions for Hardy inequalities were established in [KL] and [L4]. Before going into the general results, let us briefly discuss what can be expected to hold in some particular domains, which will then work as model cases for our conditions in the following. By means of these examples we also hope to make the somewhat technical formulations of our theorems a bit more accessible and transparent. We let the exponent $1 < p < \infty$ be fixed throughout this section.

First of all, let us record a well-known case for reference:

Example 2.1 (Unit ball). Let $\Omega = B(0,1) \subset \mathbb{R}^n$ be the unit ball. It then follows from Theorem B that Ω admits the (p,β)-Hardy inequality whenever $\beta < p-1 = p-n+(n-1)$. Here $n-1 = \dim_{\mathcal{H}}(\partial\Omega)$. On the other hand, it is easy to see that Ω does not admit the (p,β)-Hardy inequality if $\beta \geq p-1$. For instace, we may consider test-functions $u_j \in C_0^\infty(\Omega)$ with $u_j(x) = 1$ if $d_\Omega(x) \geq 2^{-j+1}$, $u_j(x) = 0$ if $d_\Omega(x) \leq 2^{-j}$, and $|\nabla u_j(x)| \leq 2^{j+2}$ when $2^{-j} \leq d_\Omega(x) \leq 2^{-j+1}$. Then

$$(8) \qquad \frac{\int_\Omega |u_j|^p d_\Omega{}^{\beta-p}}{\int_\Omega |\nabla u_j|^p d_\Omega{}^{\beta}} \xrightarrow{j\to\infty} \infty$$

if $\beta \geq p-1$.

We remark that the above functions u_j implement a general rule which says, roughly, that the sequence $(|\nabla u_j|)$ concentrates at the boundary of Ω whenever the sequence (u_j) is "critical" for the p-Hardy inequality, as is the case in (8). See [20] for the precise statements.

Now one could ask what happens if we, instead of a domain with a smooth boundary, like the above unit ball, consider a domain with a highly irregular boundary, say a *fractal* like the *snowflake*.

Example 2.2 (von Koch snowflake)**.** The construction of the usual von Koch -snowflake curve K is indicated in Figure 1. More precisely, K is the self-similar invariant set of the iterated function system consisting of the four similitudes which map the segment $[(0,0),(1,0)] \subset \mathbb{R}^2$ to the leftmost curve in Fig. 1 — here the "end-points" of all the curves are assumed to be $(0,0)$ and $(1,0)$. Then, using the standard results of fractal geometry (see e.g. [4]), we easily obtain that $\dim_{\mathcal{H}}(K) = \lambda = \log 4/\log 3$.

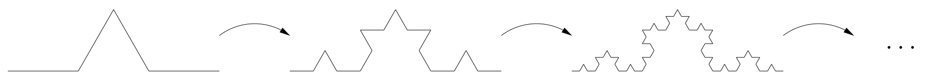

Figure 1: The construction of the von Koch -snowflake curve

The snowflake domain $\Omega \subset \mathbb{R}^2$ is constructed by joining three copies of K at their end-points in a suitable way; Ω is then the domain bounded by these curves, see Figure 3 for reference. We may now consider test-functions $u_j \in C_0^\infty(\Omega)$ which have exactly the same properties as the corresponding functions of the unit ball in Example 2.1 above, but now (8) holds only if $\beta \geq p - 2 + \lambda$ (notice that here $p - 2 + \lambda > p - 1$), meaning that the (p, β)-Hardy inequality *might* hold for a larger set of weight exponents β as in the case of $B(0,1) \subset \mathbb{R}^2$, and, indeed, we claim that Ω admits the (p, β)-Hardy inequality whenever $\beta < p - 2 + \lambda$.

It is not hard to see that if we remove a point, say the origin, from the ball $B(0,1) \subset \mathbb{R}^n$, then the domain $B(0,1) \setminus \{0\}$ admits the (p, β)-Hardy for all $\beta < p - 1$ but for $\beta = p - n = p - n + 0$; here $0 = \dim_{\mathcal{H}}(\{0\})$. Likewise, if we remove a point from the snowflake domain Ω of Example 2.2, the new domain should admit the (p, β)-Hardy inequality for all $\beta < p - 2 + \lambda$ but for $\beta = p - 2$. Behavior similar to this is expected even if we remove more complicated (compact) sets from the domain:

Example 2.3 (snowflake with dust)**.** Consider the snowflake domain Ω of the previous example, and let $I = [a, b] \subset \Omega$ be a compact line segment between points $a, b \in \Omega$. Now remove from Ω a standard $\frac{1}{3}$-Cantor set (or *Cantor's dust*), denoted C, obtained from the segment I by first removing the middle-third interval of I, and then repeating the same progress in the smaller intervals *ad infinitum*. Then C is a compact self-similar set of dimension $\mu = \log 2/\log 3$, and we claim, in accordance with the punctured unit ball and Example 2.2, that the domain $\Omega \setminus C$ admits the (p, β)-Hardy inequality for all $\beta < p - 2 + \lambda$ but for $\beta = p - 2 + \mu$.

As all the claims in the previous examples can indeed be verified, there seems to be a clear pattern: We remove a compact subset $E \subset \Omega$ of dimension $\mu < \lambda$ from a domain Ω which admits the (p,β)-Hardy inequality for all $\beta < p - n + \lambda$, and we obtain a new domain $\Omega \setminus E$ which admits the (p,β)-Hardy inequality for all $\beta < p - n + \lambda$ but for $\beta = p - n + \mu$. Let us now take a look at yet another example.

Example 2.4 (snowflake with an antenna)**.** Let $0 < \alpha < 1/2$. We construct a fractal called *the antenna set* with an iteration process as in Figure 2, where α is the "height" of the antenna set A and the "width" of the antenna is 1. Then A is a self-similar set of dimension μ, where μ is the solution of the equation $2 \cdot 2^{-\mu} + 2\alpha^{\mu} = 1$, in particular $\mu > 1$ (see [KL] for more details).

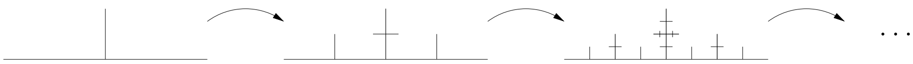

Figure 2: The construction of the antenna set with $\alpha = 1/4$

Now remove a suitably dilated copy of the antenna set of dimension $\mu < \lambda = \log 4/\log 3$, still denoted A, from the snowflake domain Ω of Example 2.2, see Fig. 3. By the previous examples it is expected that the domain $\Omega \setminus A$ admits the (p,β)-Hardy inequality for all $\beta < p - 2 + \lambda$, with the sole exception of $\beta = p - 2 + \mu$. However, by choosing appropriate test-functions $u_j \in C_0^\infty(\Omega \setminus A)$ supported in a small square below the antenna (cf. [KL] for similar considerations), it can be easily shown that Ω does not admit the $(p, p-1)$-Hardy inequality either; here $p - 1 < p - 2 + \mu$. We also remark that the inner boundary density condition (6) holds in Ω with the exponent μ, but still Ω does not admit the (p,β)-Hardy inequality for every $\beta < p - 2 + \mu$.

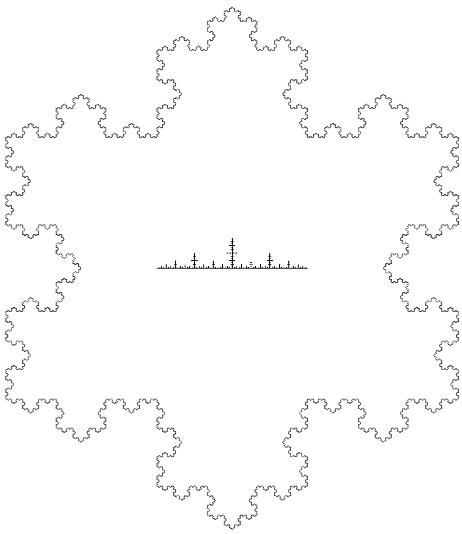

Figure 3: von Koch -snowfake domain with an antenna set removed

3 Sufficient conditions for Hardy inequalities

3.1 Hardy inequalities on domains with a uniformly dense boundary

The above Example 2.4 shows, among other things, that the density of the boundary alone is not sufficient for weighted Hardy inequalities. The problem in Example 2.4 is that even though the boundary of $\Omega \setminus A$ is dense in the sense of (6), with an exponent $\mu > 1$, it *appears* to be 1-dimensional from the points below (and near) the antenna, whence we say that the thick part of the boundary is not *visible* from such points.

One might then suggest that we should, instead of $B(x, 2d_\Omega(x)) \cap \partial\Omega$, consider in (6) only the x-component of $B(x, 2d_\Omega(x)) \cap \Omega$, denoted $D(x)$, i.e., require that $\mathcal{H}^\lambda_\infty(\partial D(x) \cap \partial\Omega) \geq C d_\Omega(x)^\lambda$ for every $x \in \Omega$. However, a slightly more complicated example from [KL] shows that even such a condition is not sufficient for Ω to admit the (p, β)-Hardy inequality for every $\beta < n - p + \lambda$. In the domain of Example 2.4 a similar phenomenon could be achieved by making small enough "holes" to the antenna.

It turns out that the desired visibility can be formulated in terms of John curves as follows: We say that $w \in \partial\Omega$ is in the *c-visible boundary near* $x \in \Omega$, denoted $v_x(c)$–$\partial\Omega$, if w is *accessible* from x by a *c-John curve*, that is, there exists a curve $\gamma = \gamma_{w,x} \colon [0, l] \to \Omega$, parametrized by arc length, with $\gamma(0) = w$, $\gamma(l) = x$, and satisfying $d(\gamma(t), \partial\Omega) \geq t/c$ for every $t \in [0, l]$. If there now is enough of the c-visual boundary near each $x \in \Omega$, for some fixed $c \geq 1$, the domain Ω admits even pointwise Hardy inequalities. This is the main theorem of [KL].

Theorem 2. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that there exist $0 \leq \lambda \leq n$, $c \geq 1$, and $C_0 > 0$ so that*

$$\mathcal{H}^\lambda_\infty\big(v_x(c)\text{–}\partial\Omega\big) \geq C_0 d_\Omega(x)^\lambda \qquad \text{for every } x \in \Omega. \tag{9}$$

Then Ω admits the pointwise (p, β)-Hardy inequality whenever $\beta < p - n + \lambda$.

To give a bit more concrete insight into Theorem 2 we mention that if $\Omega \subset \mathbb{R}^n$ is a *uniform* domain satisfying the inner boundary density condition (6) with an exponent λ, then Ω satisfies the visible boundary density condition (9) with the same exponent λ, and hence admits the (p, β)-Hardy inequality for all $\beta < p - n + \lambda$. See [KL] for the definitions and precise statements. As the snowflake domain of Example 2.2 is indeed a uniform domain and (6) holds in Ω with $\lambda = \log 4/\log 3$, we conclude that Ω admits (p, β)-Hardy inequalities for all $1 < p < \infty$ and $\beta < p - 2 + \lambda$. Likewise, it is easy to see that (9) holds in the domain $\Omega \setminus C$ of Example 2.3 with the exponent $\log 2/\log 3$, and in the domain $\Omega \setminus A$ of Example 2.4 with $\lambda = 1$, leading to pointwise Hardy inequalities for $\beta < p - 2 + \log 2/\log 3$ and $\beta < p - 1$, respectively.

Thorem 2 can in some sense be considered as a partial extension of both Theorem B (from smooth to non-smooth domains) and Theorem C (with an additional accessibility condition). For instance, if $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a uniform domain such that Ω^c is uniformly q-fat, for some $1 < q < \infty$, we obtain, by combining the knowledge from Theorem 1 with the previous remark on uniform domains, that Ω admits the (p, β)-Hardy inequality whenever $\beta \le p - q$. This is well in accordance with Theorems B and C.

The main ingredients in the proof of Theorem 2 are a rather standard chaining argument using the Poincaré inequality on cubes, and estimates for the *shadows of Whitney cubes* on the visual boundary with respect to John-curves; we measure the sizes of such shadows by a *Frostman measure* μ supported on the visual boundary.

3.2 Hardy inequalities on John domains

We say that a domain $\Omega \subset \mathbb{R}^n$ is a *c-John domain*, if there exists some $x_0 \in \Omega$ so that each $x \in \Omega$ can be joined to x_0 by a c-John curve, as defined in the previous section. Even though the visual boundary condition in Theorem 2 is obviously closely related to John domains, it is not true that Theorem 2 holds for all John domains with a uniformly big boundary, in the sense of (6); the domain of Example 2.4 works as a counterexample.

However, we were able to prove in [KL] that each simply connected John domain in the plane admits pointwise (p, β)-Hardy inequalities for all $\beta < p - 1$. This improves on Theorem B, since each bounded Lipschitz domain is in fact a John domain. More generally, if $\Omega \subset \mathbb{R}^n$, $n \ge 3$, is a John domain and if in addition Ω is *quasiconformally equivalent* to the unit ball of $\mathbb{R}^n$, then Ω admits pointwise (p, β)-Hardy inequalities for all $\beta < p - 1$, see [KL]. In spite of these results on John domains, we presume that in the case $\beta < p - 1$ the visibility of the boundary plays no essential role, but this question still remains to be more extensively studied.

3.3 Hardy inequalities on domains with a thin part in the boundary

To deal with the cases where $\partial\Omega$ is not dense enough to satisfy the assumption of Theorem 2 we need another concept of dimension, introduced by H. Aikawa (cf. [1]). When $E \subset \mathbb{R}^n$ is a closed set with an empty interior, we let $G(E)$ denote the set of those $s > 0$ for which there exists a constant $C_s > 0$ such that

$$\int_{B(x,r)} d(y, E)^{s-n}\, dy \le C_s r^s \tag{10}$$

for every $x \in E$ and all $r > 0$. Then the *Aikawa dimension* of E is defined by $\dim_{\mathcal{A}}(E) = \inf G(E)$. If a set E has a non-empty interior, we set

$\dim_{\mathcal{A}}(E) = n$. When $E \subset \mathbb{R}^n$ is closed, we always have that $\dim_{\mathcal{H}}(E) \leq \dim_{\mathcal{A}}(E) \leq n$, and there are many examples of sets with $\dim_{\mathcal{H}}$ strictly less than $\dim_{\mathcal{A}}$. However, if E is sufficiently regular, e.g. a compact submanifold of $\mathbb{R}^n$ or a nice self-similar fractal, then $\dim_{\mathcal{H}}(E) = \dim_{\mathcal{A}}(E)$. See [L4] for details and examples concerning the relations between these different notions of dimension.

In Theorem 2 the (p,β)-Hardy inequality holds, loosely speaking, because $\partial\Omega$ is fat enough, in particular $\dim_{\mathcal{H}}(\partial\Omega) > n - p + \beta$. But the Hardy inequality may also hold if $\partial\Omega$ is small, especially we need that $\dim_{\mathcal{A}}(\partial\Omega) < n - p + \beta$, as is the case in the following theorems. The first one, by P. Koskela and X. Zhong, was implicitly contained in [15].

Theorem E. *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $1 < p < \infty$. If $\dim_{\mathcal{A}}(\partial\Omega) < n - p$, then Ω admits the p-Hardy inequality.*

A weighted version of Theorem E was then obtained in [L4]:

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be an unbounded John domain, and let $1 < p < \infty$ and $\beta \in \mathbb{R}$. If $\dim_{\mathcal{A}}(\partial\Omega) < n - p + \beta$, then Ω admits the (p,β)-Hardy inequality.*

Moreover, it was shown in [L4] that, contrary to the unweighted case of Theorem E, the requirement that Ω is a John domain can not be removed from the assumptions in the weighted case of Theorem 3. See [L4] for the definition of unbounded John domains.

The situation with domains having boundary parts of different size, as in Examples 2.3 and 2.4, is naturally more complicated, and depends, in addition to the sizes and dimensions of the boundary parts, on the geometry of the domain as well. Sufficient conditions for Hardy inequalities in such domains are given in [L4]. Reflecting these conditions to the domain of Example 2.3, the idea is as follows: Suppose that a domain $\Omega \subset \mathbb{R}^n$ can be divided into two parts: (i) a "good" part Ω_g (the part near the snowflake curves in Ex. 2.3), where the visual boundary condition (9) holds with an exponent λ for all $x \in \Omega_g$ ($\lambda = \log 4/\log 3$ in the example), whence a slight modification of Theorem 2 yields the pointwise (p,β)-Hardy inequality at the points $x \in \Omega_g$ for $\beta < p - n + \lambda$; and (ii) a "bad" part Ω_b (the part near the dust in 2.3), with $\dim_{\mathcal{A}}(\partial\Omega_b \cap \partial\Omega) = \mu < \lambda$ ($\mu = \log 2/\log 3$ in the example). Then the geometric requirement in our condition is that there exists a constant $c \geq 1$ and some $x_0 \in \Omega_b$ satisfying the visual boundary condition (9) with the exponent λ, and such that for each point $x \in \Omega_b$ we can find a c-John curve γ_x joining x to x_0 in Ω_b. Under these assumptions we are able to conclude that Ω admits the (p,β)-Hardy inequality whenever $p - n + \mu < \beta < p - n + \lambda$. Notice the connection between this result and Theorem 3.

The case of Example 2.4 is even more sophisticated, as there the domain $\Omega \setminus A$ can be divided into three different parts based on the behavior with respect to Hardy inequalities. Namely, we have Ω_1, the good part near the snowflake boundary, where the pointwise (p,β)-Hardy inequality holds for all $\beta < p-2+\lambda$, with $\lambda = \log 4/\log 3$; and Ω_2, the part above the antenna A, with $\dim_{\mathcal{A}}(\partial\Omega_2 \cap \partial\Omega) = \mu$, the dimension of the antenna; and finally Ω_3, the part below the antenna, with $\dim_{\mathcal{A}}(\partial\Omega_3 \cap \partial\Omega) = 1$. Since it is obvious that the accessibility conditions (as in the above considerations related to Example 2.3) hold for both Ω_2 and Ω_3, we infer that Ω admits the (p,β)-Hardy inequality in the cases $p-1 < \beta < p-2+\mu$ and $p-2+\mu < \beta < p-2+\lambda$.

See [L4] for the precise conditions and results in more general domains. The proofs of these results are again based on a chaining argument and estimates for the shadows of Whitney cubes with respect to John-curves. In these estimates the fact that $\dim_{\mathcal{A}}$ is small is crucial.

4 The dimension of the complement

4.1 Pointwise case

There are not only sufficient, but also necessary conditions for the size of the complement (or the boundary) of a domain Ω admitting Hardy inequalities. Examples of such conditions were already given in Theorem 1, where the p-fatness of the complement and inner density (6) of the boundary, with an exponent $\lambda > n-p$, were shown to be necessary for the pointwise p-Hardy inequality.

A similar density condition is necessary for weighted pointwise Hardy inequalities as well. The following theorem was proven in [L3]. Recall that when $x \in \Omega$, we let $D(x)$ denote the x-component of $B(x, 2d_\Omega(x)) \cap \Omega$.

Theorem 4. *Suppose that a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise (p,β)-Hardy inequality for $1 < p < \infty$ and $\beta \in \mathbb{R}$. Then there exist a constant $C > 0$ and some $\lambda > n-p+\beta$ such that*

$$\mathcal{H}^\lambda_\infty\big(\partial D(x) \cap \partial\Omega\big) \geq C d_\Omega(x)^\lambda \tag{11}$$

for every $x \in \Omega$.

In particular, it follows from Theorem 4 that if Ω admits the pointwise (p,β)-Hardy inequality, then Ω^c satisfies the uniform density condition (4) with an exponent $\lambda > n-p+\beta$ (see [L1] and [L3]), and so the Hausdorff dimension of Ω^c is, even locally, strictly greater than $n-p+\beta$. However, in the weighted case it is not possible to achieve such a nice equivalence as in Theorem 1 for the pointwise p-Hardy inequality, since the density alone is not sufficient for weighted Hardy inequalities, as was shown in Example 2.4.

Theorem 4 follows with a rather straight-forward construction if $\beta \geq 0$, but in the case $\beta < 0$ the same idea only leads to a weaker (*a priori*) density condition in terms of a Minkowski-type content, where all of the covering balls are required to be of the same radius. Nevertheless, it was shown in [L3] that such a Minkowski-type condition, for some $\lambda_0 > 0$, yields a similar condition in terms of the λ-Hausdorff content for all $\lambda < \lambda_0$, and so (11) holds in the case $\beta < 0$ as well.

4.2 Self-improvement

The following self-improving property of Hardy inequalities is needed as an essential tool in the considerations related to the size of the boundary in the general case. This result, established in [L2], is of course also of its own independent interest.

Theorem 5. *Let $1 < p < \infty$ and $\beta_0 \in \mathbb{R}$, and suppose that a domain $\Omega \subset \mathbb{R}^n$ admits the (p, β_0)-Hardy inequality. Then there exists $\varepsilon > 0$ such that Ω admits the (q, β)-Hardy inequality whenever $p - \varepsilon < q < p + \varepsilon$ and $\beta_0 - \varepsilon < \beta < \beta_0 + \varepsilon$. Moreover, the constant in all these inequalities can be chosen to be independent of the particular q and β.*

Theorem 5 generalizes earlier results from the unweighted case $\beta_0 = 0$ by Koskela and Zhong [15] and Hajłasz [5].

There is also another observation in [L2] which is needed for the necessary conditions of [L4]. Namely, if Ω admits the (p, β)-Hardy inequality, then Ω admits the $(p + s, \beta + s)$-Hardy inequality for all $s > 0$. In particular, if $\beta < 0$, we may in many occasions replace the original inequality with another inequality where the weight exponent is positive, so that the distance term on the right-hand side of (1) is bounded near the boundary, but, importantly, the quantity $n - p + \beta$ remains unaltered.

4.3 General case

Theorem 4 gives a rather strict necessary condition for domains which admit the pointwise (p, β)-Hardy inequality. For the usual Hardy inequality (1) the possibilities are much more varied, but there are nonetheless some requirements for the size of the complement — which actually are realized as requirements for the dimension of the boundary in the case where (a part of) the complement has an empty interior. Again, the following theorems from [L4] are generalizations of unweighted results from [15]. To begin with, let us state a global dichotomy result for the dimension of the complement.

Theorem 6. *Let $1 < p < \infty$ and $\beta \neq p$, and suppose that a domain $\Omega \subset \mathbb{R}^n$ admits the (p, β)-Hardy inequality. Then there exists $\delta > 0$, depending only on the given data, such that either $\dim_{\mathcal{H}}(\Omega^c) > n - p + \beta + \delta$ or $\dim_{\mathcal{A}}(\Omega^c) < n - p + \beta - \delta$.*

Recall here that always $\dim_{\mathcal{H}}(\Omega^c) \leq \dim_{\mathcal{A}}(\Omega^c)$. Such a dichotomy for the dimension of the complement also holds locally, in the sense of the next theorem.

Theorem 7. *Let $1 < p < \infty$, $\beta \neq p$, and assume that a domain $\Omega \subset \mathbb{R}^n$ admits the (p, β)-Hardy inequality. Then there exists $\delta > 0$, depending only on the given data, such that for each ball $B \subset \mathbb{R}^n$ either*

$$\dim_{\mathcal{H}}(4B \cap \Omega^c) > n - p + \beta + \delta$$

or

$$\dim_{\mathcal{A}}(B \cap \Omega^c) < n - p + \beta - \delta.$$

The requirement $\beta \neq p$ is really needed in the preceding theorems, as an example from [L4] shows. Notice however that this requirement is only relevant in the case where the domain Ω is unbounded, since a bounded domain Ω can not admit the (p, β)-Hardy inequality for any $\beta \geq p$. To see this, it suffices to consider functions u_j as in Example 2.1.

We immediately obtain from Theorem 7 that if the complement Ω^c has an isolated part of (Hausdorff or Aikawa) dimension $\mu < n$, then Ω can not admit the $(p, p - n + \mu)$-Hardy inequality. Especially, the domain $\Omega \setminus C$ of Example 2.3 can not admit the $(p, p - 2 + \mu)$-Hardy inequality for $\mu = \dim_{\mathcal{H}}(C) = \dim_{\mathcal{A}}(C) = \log 2/\log 3$. As we already observed that $\Omega \setminus C$ admits the (p, β)-Hardy inequality for every $\beta < p-2+\mu$ and every $p-2+\mu < \beta < p - 2 + \lambda$, with $\lambda = \log 4/\log 3$, all the claims of this example are now justified. Likewise, the domain $\Omega \setminus A$ of Example 2.4 can not admit the $(p, p-2+\dim_{\mathcal{H}}(A))$-Hardy inequality, and so this domain admits the (p, β)-Hardy inequality for all $\beta < p-2+\log 4/\log 3$, but for $\beta = p-2+\dim_{\mathcal{H}}(A)$ and $\beta = p - 1$.

5 Final remarks

We have seen that the different forms of Hardy inequalities and the size and geometry of the boundary have an intimate connection for domains of the Euclidean space $\mathbb{R}^n$. In the light of our theorems and examples a full geometric characterization of domains admitting the (p, β)-Hardy inequality seems to be slightly beyond our reach, or at least such a characterization is destined to be of a rather complicated nature. Still, the necessary and sufficient conditions introduced in the previous sections complement each other in quite a nice way, and offer many possibilities to determine in concrete examples the values of p and β for which a domain admits Hardy inequalities, as the considerations related to Examples 2.2 – 2.4 show.

Even tough we have restricted ourselves to the case of domains in $\mathbb{R}^n$, our techniques do not rely on the special characteristics of Euclidean spaces. Indeed, all of our main tools, including Poincaré inequalities, chaining arguments using Whitney-type cubes, the different notions of dimension, and even the self-improving property of p-fatness (see [3]), are available also in the more general setting of a *metric measure space* (X, d, μ), under some assumptions on the geometry of X and the measure μ. More precisely, in order to be able to develop the basic machinery of analysis, one usually has to presume that the space (X, d, μ) supports a Poincaré inequality and that the measure μ is doubling. Natural examples of such metric spaces — and substantially different from the Euclidean spaces — are the Heisenberg groups. We refer to [6], [10], the excellent survey [11], and the references therein for more information about the analysis on metric spaces.

Hardy inequalities have already been studied in metric measure spaces to some extent. For instance, Theorem C (cf. [3]), a converse to Theorem C (cf. [14]), and the theorems of [15] hold in suitable metric spaces. As it now seems that at least most of our results can be transferred into metric measure spaces with a reasonable amount of effort, we hope that our work provides one small tool for the further development of analysis in this more general setting as well.

References

[1] H. Aikawa and M. Essén, 'Potential theory—selected topics', Lecture Notes in Mathematics, 1633, Springer-Verlag, Berlin, 1996.

[2] A. Ancona, 'On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^n$', *J. London Math. Soc.* (2) 34 (1986), no. 2, 274–290.

[3] J. Björn, P. MacManus and N. Shanmugalingam, 'Fat sets and pointwise boundary estimates for p-harmonic functions in metric spaces', *J. Anal. Math.* 85 (2001), 339–369.

[4] K. Falconer, 'Fractal Geometry. Mathematical foundations and applications', John Wiley & Sons, Ltd., Chichester, 1990.

[5] P. Hajłasz, 'Pointwise Hardy inequalities', *Proc. Amer. Math. Soc.* 127 (1999), no. 2, 417–423.

[6] P. Hajłasz and P. Koskela, 'Sobolev met Poincaré', *Mem. Amer. Math. Soc.* 145 (2000), no. 688.

[7] G. H. Hardy, 'Note on a theorem of Hilbert', *Math. Z.* 6 (1920), no. 3-4, 314–317.

[8] G. H. Hardy, 'Notes on some points in the integral calculus (LX)', *Messenger of Math.* 54 (1925), 150–156.

[9] G. H. Hardy, J. E. Littlewood and G. Pólya, 'Inequalities' (Second edition), Cambridge, at the University Press, 1952.

[10] J. Heinonen, 'Lectures on analysis on metric spaces', Universitext, Springer-Verlag, New York, 2001.

[11] J. Heinonen, 'Nonsmooth calculus', *Bull. Amer. Math. Soc.* (N.S.) 44 (2007), no. 2, 163–232.

[12] J. Heinonen, T. Kilpeläinen and O. Martio, 'Nonlinear potential theory of degenerate elliptic equations', Oxford University Press, 1993.

[13] J. Kinnunen and O. Martio, 'Hardy's inequalities for Sobolev functions', *Math. Res. Lett.* 4 (1997), no. 4, 489–500.

[14] R. Korte and N. Shanmugalingam, 'Equivalence and self-improvement of p-fatness and Hardy's inequality, and association with uniform perfectness', preprint, arXiv:0709.2013v2 [math.FA], 2007.

[15] P. Koskela and X. Zhong, 'Hardy's inequality and the boundary size', *Proc. Amer. Math. Soc.* 131 (2003), no. 4, 1151–1158.

[16] A. Kufner, 'Weighted Sobolev spaces', John Wiley & Sons, Inc., New York, 1985.

[17] A. Kufner, L. Maligranda and L. E. Persson, 'The prehistory of the Hardy inequality', *Amer. Math. Monthly* 113 (2006), no. 8, 715–732.

[18] A. Kufner and L. E. Persson, 'Weighted inequalities of Hardy type', World Scientific Publishing Co., New Jersey, 2003.

[19] J. L. Lewis, 'Uniformly fat sets', *Trans. Amer. Math. Soc.* 308 (1988), no. 1, 177–196.

[20] M. Marcus, V. J. Mizel and Y. Pinchover, 'On the best constant for Hardy's inequality in $\mathbb{R}^n$' *Trans. Amer. Math. Soc.* 350 (1998), no. 8, 3237–3255.

[21] V. G. Maz'ja, 'Sobolev Spaces', Springer, Berlin, 1985.

[22] P. Mikkonen, 'On the Wolff potential and quasilinear elliptic equations involving measures', Ann. Acad. Sci. Fenn. Math. Diss. No. 104 (1996)

[23] J. Nečas, 'Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle', *Ann. Scuola Norm. Sup. Pisa* (3) 16 (1962), 305–326.

[24] E. M. Stein, 'Singular integrals and differentiability properties of functions', Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J. 1970.

[25] A. WANNEBO, 'Hardy inequalities', *Proc. Amer. Math. Soc.* 109 (1990), 85–95.

[26] W.P. ZIEMER, 'Weakly Differentiable Functions', Graduate Texts in Mathematics 120, Springer, New York, 1989.

ISBN 978-951-39-3347-0
ISSN 1457-8905