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THE ROLE OF REPRESENTATIONS IN LEARNING THE DERIVATIVE

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ABSTRACT

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The role of representations in learning the derivative.
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It has been proposed that there are two ways how learning a mathematical concept may develop. A new object may be abstracted from actions performed on already existing objects. For example, the derivative concept may be abstracted from calculating values of the derivative. Another way is to act with the concept to be learnt and perceive it as an object. For instance, some properties of the derivative may be learnt by perceiving the derivative of a function from the graph of the function. These two ways correspond to learning in the symbolic and in the embodied worlds in the theory of the three worlds of mathematics. Several studies have suggested that the learning results of the derivative are enhanced if teaching takes into account, for example, working with several representations including the graphical ones, considering the limiting process inherent in the derivative thoroughly, supporting the process-object development and, in general, emphasizing problem solving.

However, there is still need for a detailed analysis on how students are thinking about the derivative in approaches which takes into account the above-mentioned suggestions. The aim of this study is to find out how students may use different kinds of representations for thinking about the derivative in a specific approach. To achieve this, I designed and implemented a five-hour teaching-learning sequence introducing the derivative concept in a Finnish high school (grade 11). The above-mentioned aspects of learning were taken into account in the design. After the teaching-learning sequence, I selected five students into carefully designed task-based interviews. From the interviews I analyzed what kind of representations the students used for thinking about the derivative and for which purpose and how they used these. Especially, using limiting processes and perceiving the derivative from the graph of a function were taken into account in the design of the interviews as well as in the analysis.

I found that the embodied world offered powerful thinking tools for the students. They used the increase, steepness, horizontalness and tangent of the graph for thinking about the derivative qualitatively without calculating anything. These representations were accompanied by gestures which were an essential part of thinking. At this very early stage of learning the derivative the students seemed to consider the derivative as an object, which has some properties, in the embodied world. Using the above-mentioned representations, they, for example, considered when the derivative is positive/negative, and what the maximum/minimum point of the derivative is. Therefore, this study supports the claims that learning may begin by considering the derivative as an object. The study also suggests that in the embodied world students may learn as the

representations become more and more transparent allowing seeing the derivative.

The students used various kinds of limiting processes and connected them in different ways to the limit of the difference quotient. Some of the students changed from one of these two representations to the other, and some of them explained one with the other. I named the two connections associative and reflective connections, respectively. One of the students, who made the reflective connection, had major difficulties in using the limit of the difference quotient. This suggests that a student may have conceptual knowledge of this notion without being able to use it for calculating the derivative.

On the basis of the analysis of the students' use of representations, I constructed a hypothetical learning path to the derivative. According to the learning path, the representations of the tangent, increase, steepness and horizontalness of the graph as well local straightness, moving a hand along the graph and placing a pencil as a tangent may be used to perceive the rate of change in the embodied world. In the symbolic world, students may calculate the average rate of change over different intervals. This way, students may build knowledge of the derivative even before being introduced to its definition and they have readiness to investigate the problem of the value of the instant rate of change.

Key words: conceptual knowledge, connection, derivative, embodied world, procedural knowledge, process-object, representation, symbolic world, task-based interview.

TIIVISTELMÄ

Markus Hähkiöniemi
Representaatioiden merkitys derivaatan oppimisessa.
Jyväskylän yliopisto. Matematiikan ja tilastotieteen laitos. Raportti 104.

Opiskelija voi alkaa muodostaa matemaattista käsitettä kahdella eri tavalla. Hän voi suorittaa toimintoja jo olemassa oleville objekteille ja abstrahoida näistä toiminnoista uuden objektin. Opiskelija voi esimerkiksi abstrahoida derivaatan arvojen laskemisen toiminnoista derivaatta käsitteen. Toisaalta hän voi toimia opittavan käsitteen parissa ja havaita sen objektina. Hän voi esimerkiksi oppia tiettyjä derivaatan ominaisuuksia havaitsemalla funktion kuvaajasta sen derivaatan. Nämä kaksi käsitteenmuodostuksen reittiä vastaavat oppimista matematiikan kolmen maailman teorian symboli- ja havaintomaailmassa. Useat tutkimukset viittaavat siihen, että derivaatan oppimiseen voidaan vaikuttaa myönteisesti ottamalla opetuksessa huomioon esimerkiksi erilaisten representaatioiden tuomat mahdollisuudet, käsittelemällä derivaattaan liittyvää rajankäyntiä perusteellisesti, tukemalla opiskelijoiden käsitysten kehittymistä prosessinomaisesta objektinomaiseksi ja painottamalla avointa ongelmanratkaisua.

Tällä tutkimuksella täydennän olemassa olevia tutkimuksia analysoimalla hyvin yksityiskohtaisesti opiskelijoiden derivaattaan liittyviä ajatteluprosesseja, kun opetuksen suunnittelussa on huomioitu edellä esitetyt seikat. Tutkin erityisesti, miten opiskelijat käyttävät erilaisia derivaatan representaatioita ajattelun työvälineinä ratkaistessaan ongelmia. Tätä tarkoitusta varten suunnittelin ja toteutin viisi tuntia kestävän derivaattaan johdattavan opetus–oppimisjakson lukion pitkän matematiikan Differentiaalilaskenta 1 -kurssilla. Suunnittelussa otin huomioon edellä esitetyt tekijät, jotka voivat vaikuttaa positiivisesti oppimiseen. Opetus–oppimisjakson jälkeen haastattelin viittä opiskelijaa heidän ratkaistessaan ongelmia tehtäväpohjaisessa haastattelussa. Haastatteluiden analyysissa kiinnitin huomiota siihen, millaisia representaatioita opiskelijat käyttivät sekä mihin tarkoitukseen ja miten he niitä käyttivät. Erityisesti analysoin miten he käyttivät erilaisia rajankäyntiprosesseja ja miten he havaitsivat derivaatan funktion kuvaajasta. Suunnittelin haastatteluissa käytetyt tehtävät erityisesti tätä tarkoitusta varten.

Tutkimuksen tulokset osoittavat, että havaintomaailma tarjosi haastatelluille opiskelijoille käyttökelpoisia representaatioita, joiden välityksellä derivaattaa voi käsitellä. Opiskelijat käyttivät funktion kuvaajan kasvamista, jyrkkyyttä, vaakasuoruutta ja tangenttia käsitelläkseen derivaattaa kvalitatiivisesti ilman laskutoimituksia. Näiden representaatioiden yhteydessä he käyttivät myös eleitä olennaisena osana ajatteluaan. Oppimisprosessin näinkin varhaisessa vaiheessa opiskelijat näkyivät käsittelevän derivaattaa objektina, jolla on tiettyjä ominaisuuksia havaintomaailmassa. Käyttämällä edellä mainittuja representaatioita he esimerkiksi havaitsivat derivaatan merkin sekä maksimi- ja minimipisteet funktion kuvaajasta. Siten tämä tutkimus tukee väitteitä, joiden mukaan

opiskelija voi aloittaa derivaatan oppimisen käsittelemällä derivaattaa objektina. Tutkimuksen perusteella näyttää myös siltä, että havaintomaailmassa opiskelijan käsitys kehittyy, kun representaatiot tulevat läpinäkyväksi ja hän näkee derivaatan niiden kautta.

Haastatellut opiskelijat käyttivät useita erilaisia derivaattaan liittyviä rajankäyntiprosesseja ja yhdistivät ne eri tavoin erotusosamäärän raja-arvoon. Eräät opiskelijat vaihtoivat toisesta representaatioista toiseen. Jotkut opiskelijat taas käyttivät toista representaatiota selittäessään toista. Nimesin nämä havaitut kaksi erilaista kytkentää assosiatiiviseksi ja reflektiiviseksi kytkennäksi. Eräs opiskelija muodosti reflektiivisen kytkennän, mutta hänellä oli suuria vaikeuksia erotusosamäärän raja-arvon käyttämisessä. Tämän perusteella vaikuttaa siltä, että opiskelijalla voi olla konseptuaalista tietoa erotusosamäärän raja-arvosta vaikka hän ei kykenekään käyttämään sitä laskeakseen derivaatan arvon.

Opiskelijoiden representaatioiden käytön analyysin pohjalta konstruoin hypoteettisen oppimispolun derivaattaan. Oppimispolun mukaan opiskelijat voivat havaita funktion hetkellisen kasvunopeuden käyttäen seuraavia havaintomaailman representaatioita: funktion kuvaajan tangentti, kasvaminen, jyrkkyys, vaakasuoruus, paikallinen suoruus sekä käden liikuttaminen kuvaajaa pitkin ja kynän asettaminen tangentiksi. Lisäksi opiskelijat voivat laskea symbolimaailmassa keskimääräisiä kasvunopeuksia eri väleillä. Oppimispolun mukaan opiskelijat voivat aloittaa derivaatan oppimisen ennen määritelmän esittämistä. Tällä tavoin opiskelijat saavat valmiuksia tutkiakseen, miten hetkellisen nopeuden arvon voisi määrittää.

Asiasanat: derivaatta, konseptuaalinen tieto, kytkennät, matematiikan kolme maailmaa, proseduraalinen tieto, prosessi–objekti, representaatio, tehtäväpohjainen haastattelu.

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Jyväskylä, November 2006
Markus Hähkiöniemi

CONTENTS

LIST OF INCLUDED ARTICLES

The included articles are referred to as [*i*], $i \in \{1, 2, 3, 4, 5\}$. Other references are presented as usually.

[1] Hähkiöniemi, M. 2006. Associative and reflective connections between the limit of the difference quotient and limiting process. Journal of Mathematical Behavior, 25(2), 170-184.

[2] Hähkiöniemi, M. 2006. Is there a limit in the derivative? – Exploring students' understanding of the limit of the difference quotient. In M. Bosch (Ed.) Proceedings of the fourth congress of the European society for research in mathematics education (CERME 4), Sant Feliu de Guíxols, Spain – 17 - 21 February 2005, 1758-1767. [http://ermeweb.free.fr/CERME4/].

[3] Hähkiöniemi, M. 2006. Perceiving the derivative: the case of Susanna. Nordic Studies in Mathematics Education, 11(1), 51-73.

[4] Hähkiöniemi, M. Submitted. How the derivative becomes visible: the case of Daniel. Submitted to Teaching Mathematics and Computer Science.

An earlier version of this paper is published as

[4]* Hähkiöniemi, M. 2006. Ajattelun apuvälineet – tapaustutkimus derivaatan representaatioista. [Tools of thinking – A case study of representations of the derivative.] University of Jyväskylä. Department of Teacher Education. Research report 82. 79 p.

[5] Hähkiöniemi, M. Submitted. Hypothetical learning path to the derivative. Submitted to Mathematical Thinking and Learning.

An earlier version of this paper is published as

[5]* Hähkiöniemi, M. 2005. The role of different representations in teaching and learning of the derivative through open approach. In E. Pehkonen (Ed.) Problem solving in mathematics education. Proceedings of the ProMath meeting June 30 – July 2, 2004 in Lahti. University of Helsinki, Department of Applied Sciences of Education. Research Report 261, 71-82.

1 INTRODUCTION

The aim of this study is to construct knowledge on high school students' mathematical thinking in the subject of the derivative. In this short introduction I clarify what aspects of the derivative concept are under focus and what characteristics of mathematical thinking are studied. This gives an introduction to the theoretical framework of the study. Furthermore, I will specify what kind of knowledge is sought and how that is acquired. These considerations give an introduction to the methodology of the study.

There seems to be a consensus in the mathematics education community that learning mathematics only as symbolic manipulations according to given rules is not meaningful. For example, according to Schoenfeld (1992), students should construct mathematical knowledge by solving problems and not just memorizing procedures, by investigating patterns and not just memorizing formulas, and by forming conjectures and not just doing exercises. This means that both procedural and conceptual knowledge should be emphasized in teaching (Haapasalo & Kadijevich, 2000). It is also proposed that teachers should focus on students' ideas instead of the teacher's and mathematicians' ideas (Davis & Maher, 1997). Thus, it is not that students should try to understand the teacher's ideas. Instead, the teacher should try to understand the students' ideas and support the further construction of these ideas. One teaching approach based on this view is open problem solving, whose aim is that students develop a variety of solutions to problems on the basis of their own ideas that they bring to the problem situations (Francisco & Maher, 2005; Nohda, 2000; Pehkonen, 1997).

Ideas of student centeredness and open problem solving inspired this study as I designed it to acquire information on how students think. The derivative is an interesting concept as it is a central notion in high school mathematics. It is also one of those concepts which includes limiting processes and is thus a turning point to more abstract mathematics (Tall, 1992). For example, Orton (1983) has reported students' difficulties with the derivative concept. Especially, the limiting process inherent in the derivative is reported to be difficult to understand (Orton, 1983; Heid, 1988; Repo, 1996; Zandieh, 2000). However, studies of Heid (1988), Repo (1996) and Asiala et al. (1997) have shown that with teaching based on the modern ideas, mentioned in the previous paragraph, it is possible to achieve better learning results of the derivative. This study fulfils these results by investigating qualitatively how students think about the derivative. The focus is not on how much they have acquired knowledge but on how they use their knowledge.

According to Goldin (1998) and Davis and Maher (1997), a useful way to investigate students' ideas is to examine how they use different representations. In this study, a representation is considered as a tool to think of something. Recently, more and more attention is focused on representations that are not symbolic but, for example, graphic or kinesthetic, or that are unconventional (Davis & Maher, 1997; Goldin, 1998; Gray & Tall, 2001; Tall, 2004a). In the case of the derivative and other calculus concepts it is proposed that emphasis on perceptual activity, for example, perceiving the derivative from the graph of a function, is beneficial for learning (Tall, 2003, 2004a, 2004b, 2005; Berry & Nyman, 2003; Heid, 1988; Speiser & al., 2003; Repo, 1996). Tall (2003, 2004a, 2004b, 2005) is developing a theory of three worlds of mathematics, which takes into account the role of different representations in learning. The three worlds are the embodied world of visuo-spatial images, the symbolic world where symbols act dually as processes and concepts, and the formal world of properties (ibid.). In the same way as in the APOS theory (Asiala & al., 1996) and in Sfard's (1991) reification theory, learning in the symbolic world begins by performing some procedure which is then encapsulated to an object (Tall, 2003, 2004a, 2004b, 2005). For example, a number may be encapsulated from the counting procedure. The other way, according to Tall (2003, 2004a, 2004b, 2005) and Gray and Tall (2001), is to start by considering the concept as an embodied object which becomes more abstract through reflection. For example, according to Gray and Tall (2001), the derivative may be considered as an object in a graphical context before any symbolic calculations.

Corresponding to Gray and Tall's (2001) theoretical perspective on two starting points for learning, I designed and implemented a short teaching-learning sequence introducing the derivative in a Finnish high school. In the teaching-learning sequence, open problem solving and working with different representations, especially graphs and the limit of the difference quotient, were emphasized. After the teaching-learning sequence I invited five students to task-based interviews. I designed and analyzed the interviews from the point of view of how the students use different representations. Particularly, I focused on representations related to the limit of the difference quotient with other limiting processes and perceiving the derivative from the graph of a function. Similarly to Speiser et al. (2003), the study concentrates on students' potentials instead of their misconceptions. I found that all the five students have good potential for learning the derivative and related concepts if they are allowed to use their own tools to think with, especially those representations which belong to the embodied world. On the basis of the analysis of the five students' use of different representations, I hypothesized how these representations could be used in the learning of the derivative. These considerations produced a hypothetical learning path to the derivative.

As already implicitly suggested, the intended results of this study are rich descriptions of the particular students' use of different representations. I do not intend to make general claims about the way that all students use these representations. Instead, these students' uses of the representations serve as an illustrative example on how it is possible to use these representations. This research increases our understanding about possibilities of students' personal thinking processes. In Ernest's (1997) classification of different research paradigms, these views are consistent with the qualitative research paradigm. According to him, this paradigm is also known as interpretative, naturalistic (cf. Lincoln & Guba, 1985) and alternative paradigms. In this paradigm, knowledge is viewed as constructed. Therefore, an absolute truth does not exist. However, in some occasions it is convenient to use expressions such as "correct" or "incorrect". In such cases the correctness refers to the compatibility of students' production with my interpretation of institutionalized mathematical knowledge.

According to Ernest, the constructivist perspective on learning is the central component of the qualitative research paradigm. My personal view of learning is a mixture of constructivist and sociocultural aspects. I view learning, on one hand, as an active individual construction and, on the other hand, as becoming a part of culture. In line with Cobb (1994), I argue that both views have to be taken into account to understand the complex phenomenon of learning. In this particular study, the focus is on individual students' use of representations. This implies the constructivist position. Some researchers (e.g., Simon, 1995) specify this kind of research applying a cognitive constructivist perspective. However, in this study the notion of cognition is widened to distributed cognition (Salomon, 1993). This means that in the analysis I take into account that the cognitions are distributed to sociocultural tools. The tools have an essential role in thinking; they are not just mere aids. Representations are not only tools for expressing our thoughts. They are tools to think with. The object of thinking is constructed through using different representations. A representation consists of an invisible internal side and of a visible external side. Unlike the classical internal versus external distinction, this view does not differentiate between two representations but considers them as the different sides of the same representation. For example, gestures are analyzed as an essential part of representations.

To summarize, the aim of the study is now specified as follows. In studying students' mathematical thinking, I consider how they use different kinds of representations in the embodied and in the symbolic worlds, what kind of procedural and conceptual knowledge they use, and how they consider the derivative as a process and as an object. The properties of the derivative which are in focus are the limit of the difference quotient, various limiting processes inherent in the derivative, and relations between a graph of a function and its derivative. The knowledge produced by the research are micro-level descriptions of how particular individual students use different representation for thinking about the derivative. This knowledge is achieved by investigating five students' reasoning as they solve problems in task-based interviews. Because of the detailed analysis of the qualities of the students' thinking processes, these cases are assumed to serve as illustrative examples of how also other students could use these representations.

2 OVERVIEW OF THE ARTICLES

The aim of the study is to investigate how students use different representations of the derivative after a short teaching-learning sequence introducing the derivative. On the basis of a literature review, theoretical background, and analysis of the derivative concept, I decided to emphasize the perceptual activity and limiting processes in the teaching-learning sequence. Also the task-based interviews were designed around these issues. Correspondingly, the students' uses of the limit of the difference quotient and various limiting processes are discussed in the articles [1] and [2]. The students' perceptual activity is discussed in the articles [3] and [4]. In the article [5] the common features of the students' use of representations is analyzed, and a hypothetical learning path to the derivative is constructed. In this section, slightly modified abstracts of the articles are presented to give an overview of the study.

[1] Hähkiöniemi, M. 2006. Associative and reflective connections between the limit of the difference quotient and limiting process. Journal of Mathematical Behavior, 25(2), 170-184.

This article reports a study of how students may connect the limiting process inherent in the derivative to the limit of the difference quotient when solving problems. It was found that the students used various limiting processes and they connected them in different ways to the limit of the difference quotient. Some of them changed from one of these two representations to the other, and some students explained one with the other. The two connections were named associative and reflective connections, respectively. One of the students, who made the associative connection, used the limit of the difference quotient skilfully. On the contrary, another student, who made the reflective connection, had major difficulties using the limit of the difference quotient. Therefore, students may, at the early stage of their learning process of the derivative, use different kinds of procedural and conceptual knowledge of the limit of the difference quotient.

[2] Hähkiöniemi, M. 2006. Is there a limit in the derivative? – Exploring students' understanding of the limit of the difference quotient. In M. Bosch (Ed.) Proceedings of the fourth congress of the European society for research in mathematics education (CERME 4), Sant Feliu de Guíxols, Spain – 17 - 21 February 2005, 1758-1767. [http://ermeweb.free.fr/CERME4/].

This paper continues examining the students' understanding of the limit of the difference quotient. The contribution of this paper is that it uses a different theoretical framework from the article [1]. This paper was written before the article [1], and the reader may see how the associative and reflective connections are still being developed. The students' procedural knowledge was analyzed using the APOS theory and conceptual knowledge by examining what kind of representations they had of the limiting process and how these were connected to the limit of the difference quotient. I found out that students had two kinds of connections: they changed from one representation to the other or they explained one representation with the other. Among the students, all combinations of good or poor procedural and conceptual knowledge of the limit of the difference quotient were found.

[3] Hähkiöniemi, M. 2006. Perceiving the derivative: the case of Susanna. Nordic Studies in Mathematics Education, 11(1), 51-73.

This article reports a study on how a less successful student (Susanna) perceives the derivative from the graph of a function. I analyzed the interview of Susanna to find out how she perceives the derivative from a graph of a function and what kind of representations she uses for this. The results show how she used representations of the increase, the steepness, and the horizontalness of the graph to perceive the derivative. Gestures were an integral part of her thinking. This case shows that with appropriate representations students can perceive essential aspects of the derivative from the graph of the function, and that students can consider the derivative as an object at the very beginning of the acquisition process.

[4] Hähkiöniemi, M. Submitted. How the derivative becomes visible: the case of Daniel. Submitted to Teaching Mathematics and Computer Science.

This paper continues with the topic of the article [3], but this time the interview of an advanced student (Daniel) is analyzed. Therefore, these two papers give important information on how a less successful and a very successful student reason in the embodied world. Daniel made very impressive perceptions using, for example, the steepness and the increase of a graph as well as the slope of a tangent as representations of the derivative. He followed the graphs sequentially and, for example, perceived where the derivative is increasing/decreasing. Gestures were an essential part of his thinking. Daniel's perceptions were reflected against those of Susanna (the article [3]). Unlike Susanna, Daniel seemed to use the representations transparently and could see the graph as a representation of the derivative.

[5] Hähkiöniemi, M. Submitted. Hypothetical learning path to the derivative. Submitted to Mathematical Thinking and Learning.

This paper builds on the other four papers and examines the role of different representations of the embodied and symbolic worlds in problem solving and in the learning of the derivative. I analyzed what kind of representations the five students used at the interviews and how they used them. On the basis of this analysis, I constructed a hypothetical learning path to the derivative which describes how students could use different representations in learning the derivative. According to the learning path, the representations of tangent, increase, steepness and horizontalness of the graph as well local straightness, moving a hand along the graph and placing a pencil as a tangent may be used to perceive the rate of change in the embodied world. In the symbolic world students may calculate the average rate of change over different intervals. This way, the students build knowledge of the derivative even before its definition and they have readiness to investigate the problem of the value of the instant rate of change. Solving this problem gives a reason to define the derivative.

3 INTRODUCTION TO THE DERIVATIVE CONCEPT

The purpose of this section is to present my view of the derivative. This view has affected the construction of the theoretical framework and the design of the empirical study. This section also orientates the reader to the themes of the derivative that are emphasised in this study: the underlying limiting process and perceptual activity.

The derivative is a mathematical concept which we usually meet at the advanced stage of our studies in mathematics. However, we have been dealing with the underlying structure of this concept and built related constructs all our lives. Velocity, for example, is a special case of the derivative. The velocity of an object means the rate by which the displacement of the object changes in relation to time. There are also other rates of change, for example, reading speed and the rate of change of population that we may have experienced. We can notice a lot of things from velocity and other rates of change. For example, when riding a bike, we can notice when velocity is great, when it is small, when it increases, when it decreases, when it is constant, when it is zero, and when it is at its greatest. We can also calculate an average velocity over some time interval. For example, we can calculate our average velocity on a school journey by dividing the length of the journey by the time that we spent on biking. Furthermore, we may notice that the velocity at a certain point during the journey is not necessarily the same as the average velocity. Nevertheless, we cannot know the exact value of the velocity at the point. The speedometer of the bike does not give the instant velocity but an average velocity on the basis of measuring how many times the wheel spins at a certain time interval. How good an estimate this is for the instant velocity depends on how much the velocity changed during the time interval of the measurement. A radar would give a better estimate because the time interval is smaller.

We may also model the motion using mathematics. For example, the displacement (s) of an object along the path travelled by the object may be represented as a function of time (t). We may know that the displacement of the object at a certain time is 5 times the time minus the time square. The algebraic expression for the function s would be $s(t) = -t^2 + 5t$. The function may be represented also graphically as in Figure 1.

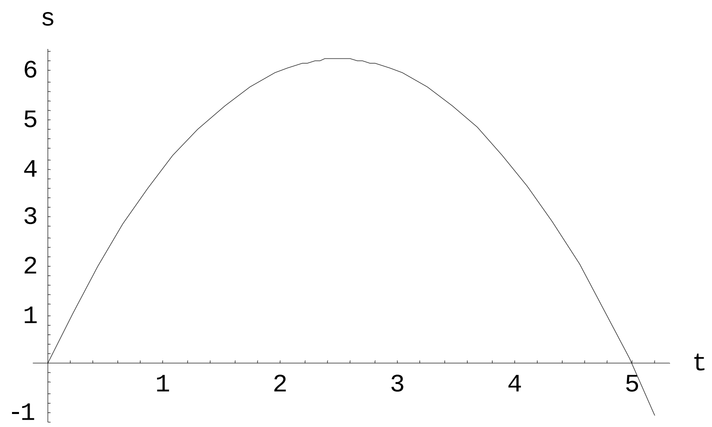

Figure 1. *The graph of the function* $s(t) = -t^2 + 5t$.

By investigating the graph of the function *s*, we may notice that the values of the function (the displacement of the object) are increasing until the point t = 2.5 and that after this the values are decreasing. Thus, the rate of change of the function (the velocity of the object) is positive until the point t = 2.5 and negative after that. Moving a hand along the graph from left to right may help to consider how the values are changing: when the hand raises, the values increase, and when the hand goes down, the values decreases. Furthermore, the steeper the hand raises the faster the values increase and the steeper the hand goes down the faster the values decrease. In perceiving the steepness of the graph, it may help if we imagine grasping the graph with a thumb and a forefinger as if the graph were a rope and place a pencil between the fingers[1]. We can also slide the hand from left to right keeping the grasp. When moving the hand, we can see the steepness of the position of the pencil at different points and feel whether the hand is raising or going down[2]. After these considerations we may notice that the values of the function *s* increase fastest at the point t = 0. After this the rate of change decreases and becomes zero at the point t = 2.5. After the point t = 2.5 the rate of change is negative and decreasing, which means that the values of the function are decreasing faster and faster. Thus, the velocity of the object is greatest at the beginning, positive and decreasing until the point 2.5, zero at the time 2.5 and negative and decreasing after that.

Let us now examine the rate of change of the function *s* (the velocity of the object) at the point t = 2. On the basis of the previous observations, we can notice that the rate of change at this point is positive but close to zero. How could we estimate the rate of change more accurately and how could we determine it exactly? Previously, we perceived the rate of change from the steepness of the

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[1] Note that this may also be used as an intuitive "definition" of a tangent as the tangent is known as a difficult concept to define without the limit concept. Definitions as "the tangent touches the graph at one point" are misleading. With the pencil-embodiment it becomes clear that, for example, the tangent may also intersect with the graph.

[2] This dynamic representation corresponds to dragging a tangent along the graph in computer algebra software.

position of the pencil. Thus, one estimation method could be calculating the steepness of the position of the pencil (Fig. 2). However, this estimate is not very accurate as it depends on how the pencil is placed on the graph.

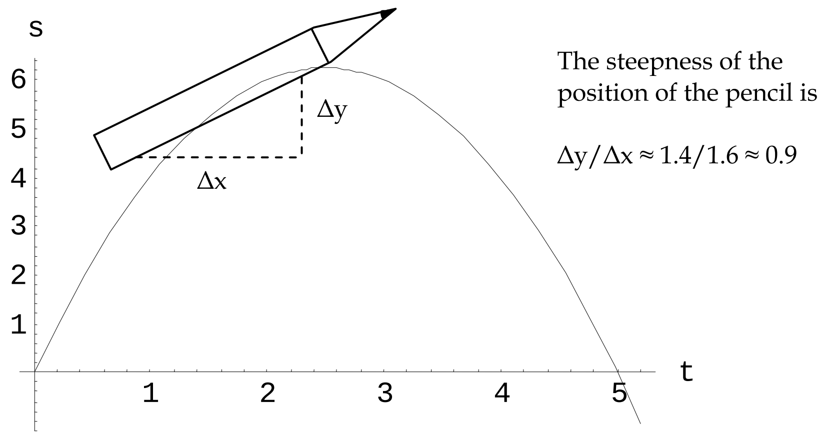

Figure 2. *The steepness of the position of the pencil.*

For another estimate, we can calculate the average rate of change (the average velocity of the object) over the interval [2, 4]. This is calculated by dividing the change in values by the length of the interval: $\frac{s(4)-s(2)}{4-2}=\frac{4-6}{4-2}=\frac{-2}{2}=-1$. In the graph this corresponds to the slope of the line (secant) intersecting the graph at the points 2 and 4 (Fig. 3). Obviously, this estimate is not good since we had previously noted that the rate of change at the point 2 should be positive. A better estimate could be achieved if the average rate of change was calculated over a smaller interval, for example, over the interval [2, 2.5]: $\frac{s(2.5)-s(2)}{2.5-2}=\frac{6.25-6}{2.5-2}=\frac{0.25}{0.5}=0.5$. This corresponds to the slope of a secant intersecting the graph at the points 2 and 2.5 (Fig. 3).

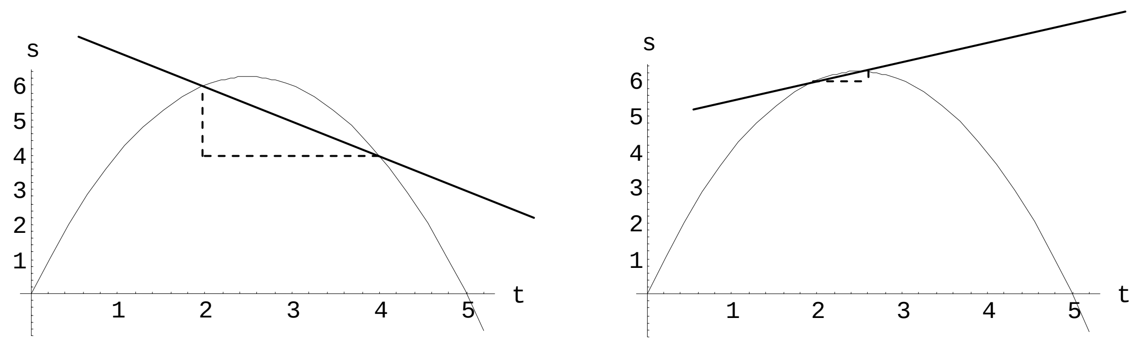

Figure 3. *The average rates of change over the intervals [2, 4] and [2, 2.5] and the corresponding secants.*

This estimate can be improved by calculating the average rate of change over the interval [2, 2.1]: $\frac{s(2.1)-s(2)}{2.1-2}=\frac{6.09-6}{2.1-2}=0.9$. An even better estimate is achieved by using the interval [2, 2.01]: $\frac{s(2.01)-s(2)}{2.01-2}=\frac{6.0099-6}{2.01-2}=0.99$. For the same purpose, the average rates of change can be also calculated, for example, over the intervals [1.5, 2], [1.9, 2], and [1.99, 2]. The smaller the interval is the better the estimate is. To get the exact value of the instant rate of change we can investigate what number the average rates of change approach when the length of the interval tends to zero. In this case they seem to approach number 1. This can be confirmed by investigating what number the average rate of change over the interval [2, *t*] approaches when *t* approaches 2. The average rate of change over the interval [2, *t*] is $\frac{s(t)-s(2)}{t-2}=\frac{-t^2+5t-6}{t-2}$. This expression can be manipulated to the form $\frac{(t-2)(3-t)}{t-2}=3-t$. From this we can see that when *t* approaches 2, the average rate of change (3 – *t*) approaches 1. This limit can be denoted as $\lim_{t\to 2}\frac{s(t)-s(2)}{t-2}=\lim_{t\to 2}\frac{-t^2+5t-6}{t-2}=\lim_{t\to 2}\frac{(t-2)(3-t)}{t-2}=\lim_{t\to 2}(3-t)=3-2=1$.

So the rate of change of the function *s* at the point 2 is 1. The limiting process was a crucial factor in determining the rate of change. The average rates of change correspond to slopes of secants. Thus, the limiting process can be interpreted to mean secants approaching a tangent (Fig. 4). Figures 2 and 4 show that the estimations on the basis of calculating the steepness of the position of the pencil and average rates of change mean the same thing as the average rates of change approach the steepness.

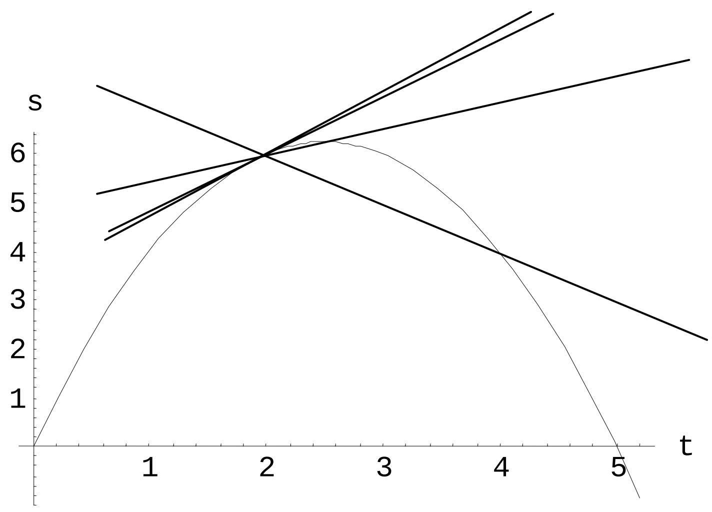

Figure 4. *Secants corresponding to average rates of change approaching the tangent.*

In general, the rate of change of the function *s* at a point *a* is determined as the limit $\lim_{t\to a}\frac{s(t)-s(a)}{t-a}$. The rate of change corresponds to the mathematical concept of the derivative. The derivative of a function f at a point a equals $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ if the limit is defined. According to the above considerations, the derivative of a function at a point means the rate of change of the function at the point, the slope of a tangent set on the point, and the steepness of the graph of the function at the point. In addition, the sign of the derivative indicates whether the function is increasing or decreasing at the point. The quotient $\frac{f(x) - f(a)}{x - a}$ is called the difference quotient and the limit $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ the limit of the difference quotient. The limit of the difference quotient can also be expressed, for example, as $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

For example, Grossman (1981, p. 93) gives the following definition to the derivative of a real valued function at a point:

> Let f be defined on an open interval containing the point x_0 and suppose that
>
> $$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$
>
> exists and is finite. Then f is said to be *differentiable* at x_0 and the *derivative of f* at x_0, denoted $f'(x_0)$, is given by
>
> $$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

The most important applications of the derivative in high school are determining the slope of a tangent, determining the rate of change of a function, determining velocity and acceleration, examining the behaviour of a function and determining the extreme values of a function.

4 STUDIES ON LEARNING THE DERIVATIVE

In this study, the literature is used in two different ways. On one hand, it is used to construct a framework through which the students' activity is analyzed. This kind of literature is discussed in section 5. On the other hand, I gained insights to what kind of conceptions students may have of the derivative and how they can learn it meaningfully from a number of existing studies considering the derivative. Knowing these studies was important for noticing new and essential findings of this study. The results of this study are related to previous ones by extending them, supporting their conclusions, proposing new directions or challenging the generality of previous results. This kind of literature is reviewed in this section. The existing studies are reviewed corresponding to themes of the findings of this study. Therefore, from the reviewed studies especially insights into limiting processes and working with graphs related to the derivative are discussed. In addition to studying students' reasoning and conceptions of the derivative, some studies have also proposed, tested or designed teaching-learning sequences on the derivative. The arguments for the relevance of these sequences vary from quantitative testing to design research. Also these studies are reviewed here as they are relevant for the construction of a hypothetical learning path to the derivative.

4.1 Teaching-learning sequences on the derivative

Several studies have suggested a particular research-based course, teaching-learning sequence[1] or learning activities through which students could learn the derivative meaningfully. From a researcher's perspective a critical question is how the proposed sequence is validated. What is the basis for the meaningfulness of the sequence? One approach is conducting a teaching experiment, testing the students' learning outcomes and comparing these to the outcomes of a control group. Another approach is to collect data that inform how an initial teaching-learning sequence could be improved.

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[1] Teaching–learning sequence is used in the sense of Méheut and Psillos (2004) to refer to topic-oriented instructional sequences instead of long-term curricula.

Testing the learning outcomes after a teaching-learning sequence

Asiala et al. (1997) constructed in the framework of the APOS theory (see section 5.4) a genetic decomposition of the derivative which described what kind of constructions students might make for learning the concept. First, researchers constructed an initial version of a genetic decomposition on the basis of theoretical analysis. Accordingly, they designed and implemented instruction for university students in a calculus course. Then empirical data were collected by interviewing 17 students after the reform instruction and 24 students after the traditional instruction. The data were analyzed in terms of the APOS theory. On the basis of the analysis, the genetic decomposition was revised. The initial and revised genetic decompositions included graphical and analytical paths to the derivative. The graphical path consisted of calculating the slopes of secants when the two points on the graph get closer and closer, and producing the slope of the tangent as a result of limiting. The analytical path consisted of calculating the average rate of change over a smaller and smaller interval, and producing the instant rate of change as a result of limiting. In the instruction the students constructed mathematical ideas with the computer using a programming language, investigated mathematical concepts with a symbolic computer system, and worked in cooperative learning groups in problem solving and in discussion.

According to Asiala et al., the calculus students whose course was based on the genetic decomposition had more success in constructing a graphical understanding of the function and the derivative than students from traditional courses. According to the analysis of the interviews, all the 17 students in the experimental group achieved a level of completely satisfactory understanding in the cases of the function and the derivative. Only 12 of the 24 students in the control group achieved this level in the case of the function and 10 in the case of the derivative. About a fourth of the students in the control group understood the concepts poorly. The rest were in between. Because of the striking differences between rates of success, Asiala et al. concluded that the experimental instructional treatment may be more effective than the traditional instruction.

Also Repo (1996)[1] conducted a study in the framework developed by Dubinsky (1991), which was later developed to the APOS theory. Repo developed a theoretical model of how the derivative could be learnt and designed a corresponding learning environment with learning activities. The 50-hour instruction was implemented in 1991 in a Finnish high school for 17 grade 11 students[2]. The computer software Derive was used intensively. The emphasis in the learning of the derivative was placed on the definition of the derivative and on the limiting process in the context of the average velocity/rate of change,

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[1] See also Lehtinen and Repo (1996).

[2] The mathematical contents of Repo's course seem to correspond roughly to the course in which the teaching-learning sequence of this study was implemented. However, in Repo's course there were 20 lessons more.

slope of secant and difference quotient. Also investigating a function using its derivative function and vice versa were emphasized. At every stage of the course different representations and their coordination were emphasized.

In Repo's study the definition of the derivative was introduced through examining the velocity of a ball rolling in an inclined plane. The students estimated the value of the instant velocity at a point using average velocity over a smaller and smaller interval. As a solution to this problem, the derivative was defined as the limit of the difference quotient and also the limiting process of secants approaching the tangent was discussed.

Repo compared the performance of the 17 students of the experimental group to the control group (N = 23) who studied the same course. She implemented pre-tests, post-tests and delayed post-tests (after six months) to both groups. She also supplemented the tests by conducting interviews. According to the results, the students of the experimental group constructed knowledge structures that were considerably higher in quality than those of the students of the control group. The difference between the groups was especially high in the case of students whose previous success in mathematics had been weak or average. The experimental group performed better in every item including those involving graphical representations and limiting processes. The overall difference between the experimental and the control groups was maintained in the delayed post-test both in the items of procedural knowledge and in the items of conceptual knowledge.

Also Heid (1988) conducted a study in which calculus students of two experimental groups (N = 39) showed more conceptual understanding than the students in the control group (N = 100). She designed an experimental applied calculus course for college students. In this course the traditional sequence of 'first skills then concepts' was reversed. Computer software was used in the course to perform symbolic calculations. The focus was on concepts, and various representations were emphasized. For example, working with graphs was used intensively. At the end of the course students practiced skills by performing the algorithms without computer. Heid based her conclusions on interviews and tests to the students.

Also Serhan's (2006) study suggests that experimental teaching, which utilizes graphical calculator and emphasizes connections between symbolic, visual and numerical representations, may influence positively on students' learning the derivative.

Collecting data to improve an initial teaching-learning sequence

The studies of Repo (1996), Heid (1988) and Asiala et al. (1997) suggest that in particular settings and with careful planning according to a strong theoretical framework, it is possible to improve students' learning achievements of the derivative. However, the pre-test/post-test procedures of these studies allow only suggesting very general answers to the question of what it was in the teaching-learning sequence that helped the students to learn the concept. In the study of Asiala et al., after analysing the data the researcher also revised the initial genetic decomposition describing the constructs that students should build to learn the derivative. Thus, their methodology had a cyclic nature aiming at refining iteratively the initial genetic decomposition. In recent years, methodological approaches have been developed whose main aim is to design teaching-learning sequences iteratively. Such approaches are, for example, design research (Edelson, 2002), design experiment (Cobb & al., 2003), didactical engineering (Artigue, 2005), and educational reconstruction (Duit & al., 1997; Kattmann & al., 1998). In these approaches, the effectiveness of a conducted teaching-learning sequence is not tested. Instead, data are collected to find out how the sequence could be improved. In the case of the derivative concept, Doorman (2005) and Artigue (2005) have carried out such a research.

In Doorman's (2005) design research, an integrated 10th-grade course of the principles of calculus and kinematics was developed. The course was designed as a guided reinvention course on modelling motion using computer tools. The initial design was based on theoretical considerations of previous studies on students' difficulties in calculus and kinematics, of the historical development of calculus and kinematics, of theories on symbolizing, and of a guided reinvention in the Realistic Mathematics Education –approach[1]. This design was further developed through collecting qualitative data mainly by videotaping the lessons of three implemented courses and reflecting on the design.

In Doorman's (2005) instructional sequence, the first step was to foster students' thinking about a change of position as a measure of motion. The students were guided to reason with displacements between successive positions and to make predictions of the motion. One-dimensional trace graphs, which signified successive displacements at equal time intervals, were introduced. Displaying the patterns in the displacements encouraged the students to invent two-dimensional discrete graphs of displacements between successive points and total displacements. Students started to use these graphs for reasoning about mathematical and kinematical notions and relationships. Velocity was represented as the length of displacements at successive time intervals in graphs of displacements and distance. The students came to understand that these graphs are related by taking sums and differences. They also noted that they did not have enough measurements to determine the instantaneous velocity. Then a transition to continuous models was made. Students used bar-graph approximations of continuous velocity-time graphs to signify displacements at corresponding time intervals. They also considered the difference between instantaneous and average velocities. The students used the difference quotient as a measure for the average rate of change: vertical displacements and corresponding time intervals were detected from continuous distance-time graphs. The approximation of instantaneous velocity built upon the linear continuation of a graph (a tangent-like continuation of the graph according to the movement that continues with the constant velocity). The students drew linear continuations to

[1] See also Gravemeijer and Doorman (1999) for the discussion about a guided reinvention course on calculus.

graphs and most students answered that the precise value of this velocity could not be determined. Only a few students invented an approximation process themselves. The last stage was a transition to reasoning with graphs of mathematical formulas. At this stage there were some problems. The students were familiar with graphs and their intervals but most of them had major difficulties in using the formula of a function in approximating the instantaneous change. Thus, Doorman concludes that in this transition, more attention is needed for approximation processes in data-based graphs.

Artigue (2005) reports a didactic engineering study in which a quite different approach to the derivative which utilizes a symbolic calculator is developed. In this approach the definition of the derivative is introduced by letting students conjecture graphically what the slope (and the equation) of the tangent would be at a particular point. Then this is proved or refined by examining the difference between values of the function and the tangent line at a point that approaches the point in question. Artigue found the design viable, but the analysis focused on the role of technology, and thus, it is not discussed more thoroughly here.

4.2 Limiting processes inherent in the derivative

In the above-reviewed innovative teaching-learning sequences to the derivative, a lot of emphasis was placed on the limiting process inherent in the derivative. This reflects the need for designs through which the limiting process might be learnt meaningfully. Several studies have reported that students have great difficulties understanding the limit concept (see, e.g., Cornu, 1991; Tall, 1992; Tall & Vinner, 1981; Merenluoto, 2001; Juter, 2006; Bergsten, 2006). This also causes difficulties in understanding other concepts which include limiting processes (Orton, 1983; Heid 1988; Tall, 1992; Tall, 1991; Tall & Vinner, 1981). In this section I review studies reporting these difficulties. Before this I clarify what I mean by limiting processes inherent in the derivative.

For example, Asiala et al. (1997) emphasized two kinds of limiting processes: the average rates of change approaching instant rate of change and the slope of secants approaching the slope of a tangent. Cottrill et al. (1996) have noted that the limit of a function consists of the coordination of two processes. Accordingly, also the limiting processes underlying the derivative consist of many processes. For example, the slopes of secants approaching the slope of a tangent include a point approaching the other point, secants approaching the tangent and the slope of the secants approaching the tangent. Other limiting processes that may represent the derivative include average velocity approaching instant velocity (cf. Repo, 1996; Doorman, 2005) and a numerical representation of the difference quotient approaching the derivative. Also local straightness (Tall, 2003) is a limiting process which means magnifying a graph so that it looks like a line[1]. Artigue (2005) discussed yet another limiting process, in which the difference between the values of the function and the assumed tangent line is considered. Usually, in these limiting processes underlying the derivative, the limit is viewed dynamically (cf. Cottrill & al., 1996; Cornu, 1991; Tall & Vinner, 1981) as something is approaching something else.

Zandieh (2000) developed a framework for understanding the derivative, in which special attention is given to the limiting processes inherent in the derivative. In addition to the limit of the difference quotient, Zandieh (2000) also considers limiting processes of the slope of the secant, rate of change and average velocity. In her framework the limit in these different representation contexts can be used as a process or as a (pseudo-)object. For example, for a student the process in the graphical context may be secants converting to tangents and the object may be the slope of the tangent line at a point. Applying the terms of Sfard (1992), Zandieh calls the object a pseudo-object because it does not necessarily include an internal structure of the limiting process for the student. In other words, as Zandieh and Knapp (2006) express it, a student may refer to the (pseudo-)object as a metonymy of the underlying processes or only of the intuitive idea. When applying the framework in case studies of nine calculus students, Zandieh (2000) found that the students could often describe the limit as a pseudo-object, but considering the limiting process was more difficult.

Orton (1983) has reported students' difficulties with the limiting process in a large study. Orton interviewed 60 students from the age range 16–18 years and 50 students from the age range 18–22 as they solved problems related to differentiation. He assessed students' success in these items numerically and studied what kind of errors they made. The students of the two age groups had similar success and failures in the items. According to the results, the students had major difficulties coping with the limiting process inherent in the derivative. The students scored weakest on items of "differentiation as a limit" and "use of δ-symbolism". In these and other items involving the limit, the students made lot of structural errors, which mean misunderstanding relationships involved in a problem or principles in a solution. On the contrary, students succeeded very well on carrying out differentiation. In these items they made only executive errors, which mean failing to carry out manipulations.

Also in Repo's (1996) study (see the section 4.1), neither the experimental nor the control group performed well (in relation to other items) in the items involving the limit of difference quotient and limiting processes. The studies of Repo, Orton and Zandieh, highlight the issue that students may perform well with other aspects of the derivative concept, but considering the limiting process may still be difficult. Also the study of Habre and Abboud (2006) gave indications of this.

If the limiting process is not properly understood, shortcomings in the use of the limit of the difference quotient may occur. Viholainen (2006, submitted a)

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[1] For example, Tall (1986) presented empirical evidence for the benefits of using the local straightness approach in calculus.

reports a case in which a function defined in two domains was not continuous, but a student (Mark) came to the conclusion that it has to be differentiable. The reason for this was that Mark used the limit of the difference quotient separately for both domains, and they happened to take the same value. Mark was interviewed after having studied mathematics as his main subject for five years at university and was an almost qualified mathematics teacher. Furthermore, Viholainen reports that more than one fourth of the 146 prospective teachers at the same phase as Mark at a written exam claimed that a discontinuous function is differentiable. Another student (Theresa), who made such a conclusion in the written exam, explained at the interview that a function which had a jump at one point (and was thus discontinuous) was differentiable. The reason, according to Theresa, was that the tangent could be drawn as if there were not a hole in the graph. Viholainen also noted that both of the students indicated a tendency to avoid using the limit of the difference quotient in interview tasks. This tendency was noted also among other 18 interviewed students in a particular challenging problem, as only three of them used the limit of the difference quotient as their primary method (Viholainen, submitted b). The conclusion from these research results should not be that, for example, the tangent representation should be avoided in teaching and the definition preferred. On the contrary, the tangent and other graphical representations should be used more and support students to connect them to formal representations. This means helping students to see the limit of the difference quotient as a limit instead of symbol manipulation (cf. Mark) and the tangent as a limit instead of a pseudo-object (cf. Theresa).

4.3 Graphical representations

The already reviewed studies of Repo (1996), Heid (1988) and Asiala et al. (1997) have shown that instruction which emphasizes, among other things, working with graphs may have positive impact on students' learning of the derivative. Also the research-based teaching-learning sequences designed by Doorman (2005) and Artigue (2005) emphasized working with graphs.

In addition to these studies, Berry and Nyman (2003) investigated how first-year university students think about connections between the graph of a function and the graph of the derivative function when solving a particular problem. The problem given to two groups of four students was to sketch the graph of the function from the graph of the derivative function. Furthermore, the students were asked to consider these as displacement-time and velocity-time graphs and to create the corresponding movement and compare that to the graph given by the motion detector. Berry and Nyman found that the students moved from an instrumental understanding of the calculus towards a relational understanding. They recommend that, before entering the formal symbolic calculus, students should build understanding of the underlying concepts. This can be enhanced with problems like the one used in their study. Also Speiser et al. (2003) report how a group of third-grade high school students made sense of the motion of a cat while they worked with several graphical representations.

There are also studies that have shown how even younger students may construct calculus-related ideas. Schorr (2003) reports a research project in which 8-11 students in grades 7 and 8 participated an after-school program in which they investigated motion especially with graphic representations and computer software. According to the results, the students built powerful ideas of related concepts. Thus, Schorr concludes that meaningful mathematical experiences in the mathematics of motion are possible even in the middle grades. Furthermore, in Wright's (2001) research, even a third/fourth-grade student was able to build mathematical ideas of motion when she was allowed to use her kinesthetic experience.

Besides the potentials of working in the graphical context in learning the derivative, there are also some documented difficulties. According to Nemirovsky and Rubin (1992), students' tendency to assume resemblance in change of a function and change of its derivative is quite a general phenomenon. This is also well documented in the context of kinematics (McDermott & al., 1987; Trowbridge & McDermott, 1980; Beichner, 1994).[1] According to Nemirovsky and Rubin (1992), students may overcome the assumption of resemblance by focusing on how one function describes the local variation in the other. Students may use different mathematical notions, such as steepness and slope, for this (ibid.). In Hauger's (1997) study, four pre-calculus students made a similar error to that described by Nemirovsky and Rubin. The students drew distance-time graphs which represented constant speed instead of varying speed. The students used graphical slope, steepness, shape of the graph and changes over intervals to correct their errors. Thus, Hauger concludes that these are powerful ways for pre-calculus students in thinking about the rate of change.

Tall (2003) argues that the local straightness would be a fruitful starting point for learning the derivative. The magnification window can also be dragged along the graph to see and feel the changing slope of the graph (ibid.). According to Gray and Tall (2001), instead of computer programs, also drawing the graph and moving a hand along the graph may be used for the same purpose. According to Tall and Watson's (2001) study, students (N = 13) whose teacher emphasized this kind of activity performed better in graphing tasks of a written test than other teachers' students (N = 27). Despite the benefits of perceptual activity, Tall (2003) and Gray and Tall (2001) note that the perceptual activity must be connected to the symbolism. According to Giraldo, Carvalho and Tall (2003) and Giraldo, Tall and Carvalho (2003) there are propositions in the literature that numerical representations may restrict the concept image. However, according to them, if the conflicts between numerical and formal representations are emphasized in teaching, students' concept images may become

[1] See Leinhardt et al. (1990) for the discussion on this and other misconceptions related to graphs.

richer. In this way, the limitations of individual representation are not generalized to the concept (ibid.).

The studies referred to in the section 4 suggest that many different representations have to be taken into account in teaching the derivative. As the studies of Kendal and Stacey (2000) and Bingolbali (2005) indicate, teachers' privileging of representations of the derivative affects students' competence with these representations. Therefore, the representations emphasized in teaching have to be carefully chosen.

5 THEORETICAL FRAMEWORK

In this section, the theoretical framework that guided the study is described. Although the theoretical framework was not used to make an advanced coding scheme for the analysis, it cannot be said that the theoretical framework did not influence the study. The previous experiences of a researcher cannot be excluded. Therefore, it is important to make the theoretical framework explicit. The theoretical framework is like spectacles through which students' activity is seen. The spectacles of this study focus, for example, on an individual's use of representations but not, for example, on classroom norms. The literature discussed in this section focuses more on theoretical perspectives which are more general than the particular research results discussed in the previous section. First, Tall's (2004a) evolving theory of three worlds of mathematics is described. This theory provides the general framework of the study. Other perspectives are used to discuss students' working in the different worlds. These include procedural and conceptual knowledge, representations, connections between representations, process-object nature of mathematical concepts, perceptual activity in mathematics and open problem solving.

5.1 Three worlds of mathematics

Similar to Piaget's pseudo-empirical and empirical abstractions, there are at least two ways how concept acquisition may begin in mathematics. These are explored in more detail in sections 5.4 and 5.5. One way is to perform a symbolic *action on* an object and from this action to construct a new concept. For example, in the APOS (Action-Process-Object-Schema) theory an action is interiorized as a process which is encapsulated as an object (Asiala & al. 1996; Dubinsky, 1994). Similarly, in Sfard's (1991) theory, a person may reify an operation to a static entity, which means a shift from an operational to a structural conception. The other way, according to Gray and Tall (2001), is that concept acquisition begins from the perception of an object and *acting with* the object. Gray and Tall call this kind of perceived object an *embodied object*. Embodied objects are mental constructs of the perceived reality, and through reflection and discourse they can become more abstract constructs, which do not anymore refer to specific objects in the real world (ibid.). These two ways how concept acquisition may begin are presented in Figure 5 taken from Gray and Tall (2001, p. 71).

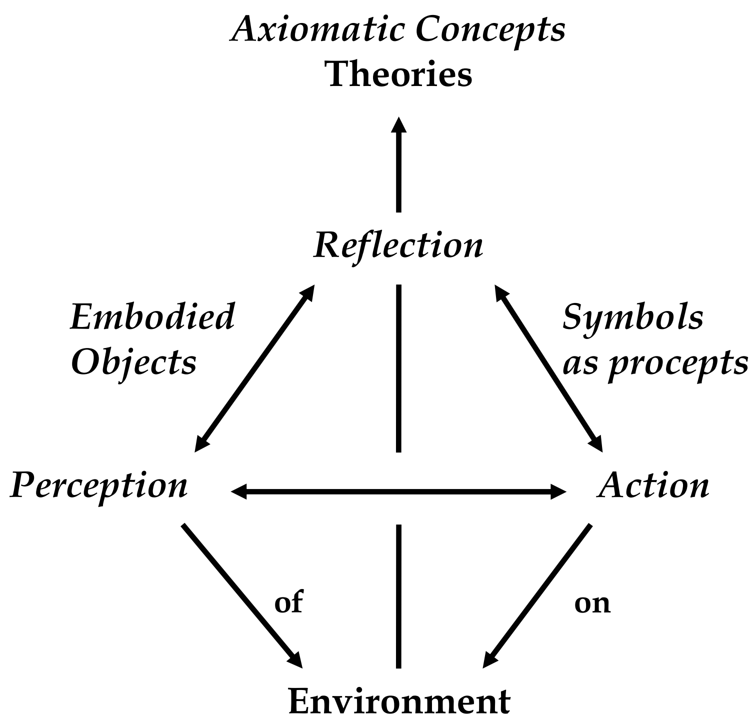

Figure 5. *Different kinds of mental entities arising through perception, action and reflection (Gray & Tall 2001, p. 71).*

Tall (2003, 2004a, 2004b, 2005) has further developed these ideas into the evolving theory of three worlds of mathematics.[1] Each of these worlds has its own characteristics for the development of sophistication and own warrants for the truth.

> *The (conceptual-)embodied world* (Tall, 2003, 2004a, 2004b, 2005) consists of thinking about things that can be perceived and sensed in the physical and the mental world. For example, mental conceptions that involve visual-spatial imagery belong to this world. The truth warrant in the embodied world is based on thought experiments and on "seeing" things to be true. For example, $2 + 3 = 3 + 2$ because it gives the same amount of blocks whether you add three blocks to two or two blocks to three (Tall, 2004b). Students' conceptions develop in this world through reflection and using more sophisticated language. The development of thinking is discussed in section 5.5.
>
> *The (proceptual-)symbolic world* (Tall, 2003, 2004a, 2004b, 2005) consists of using symbols for calculation and for thinking about concepts. Ideally,

[1] See also Tall's web page (www.davidtall.com) for information on the forthcoming book on this theory.

symbols act both as processes and as concepts (*pro*cess + con*cept* = *procept*), and a person can fluently change between these. For example, 2 + 3 may be viewed as an addition but it may also be viewed as a sum (Tall, 2005). In the symbolic world the truth warrant is based on calculations and manipulations of symbols. For example, 3 + 2 = 2 + 3 because both sides of the equation give the same result when counted (Tall, 2004b). Students build knowledge in this world by encapsulating new concepts from actions on old objects. The development is discussed in section 5.4.

The formal(-axiomatic) world (Tall, 2003, 2004a, 2004b, 2005) is based on axioms, definitions, theorems and deductive reasoning. In the formal world something is true if it is an axiom, a definition or it can be proved formally. For example, 3 + 2 = 2 + 3 because it follows from the axiom of commutativity (Tall, 2004b). This world is not described as much as the other worlds because this study focuses on the early stage of the learning of the derivative. This means that although deductive reasoning is used also in the embodied and symbolic worlds, it is not used to build the axiomatic structure. For example, the same definition of the derivative is used in the symbolic and in the formal worlds in a different way. In the symbolic world, its conclusive role in defining the derivative is not emphasized. Neither is it emphasized that theorems are part of the axiomatic system. Instead, theorems are more like mathematical laws that can be justified from known facts. Also some intuitiveness is allowed in the other worlds as, for example, the limit concept used in the definition of the derivative need not be defined formally.

The theory of three worlds of mathematics can be understood in two ways. First, it may describe students' journey in mathematics longitudinally (journey through geometry, arithmetic, algebra, calculus, analysis, etc.) as they work in the embodied and in the symbolic world, and some of them even continue to the formal world. Secondly, the theory may also focus on a specific concept and describe how students travel through different worlds when building that concept. In this study, Tall's theory is used in the latter sense by investigating how students work in the embodied and in the symbolic world at the beginning of the learning the derivative.

5.2 Procedural and conceptual knowledge

In the three worlds of mathematics, students may use both procedural and conceptual knowledge. There is a long-time debate on the importance of such knowledge as these (Hiebert & Lefevre, 1986; Hiebert & Carpenter, 1992; Haapasalo, 1997; Haapasalo & Kadijevich, 2000; Rittle-Johnson & al., 2001; Haapasalo, 2003; Kadijevich, 2003; Haapasalo, 2004). Several similar types of knowledge distinctions exist in the literature and are discussed in Haapasalo and Kadijevich (2000).

According to Hiebert and Lefevre's (1986, p. 6) classical definition, procedural knowledge consists of the formal language of mathematics, and of rules, algorithms, and procedures used to solve mathematical tasks. Conceptual knowledge, on the other hand, is knowledge that is connected to other pieces of knowledge and the holder of the knowledge also recognizes the connection (Hiebert & Lefevre, 1986, p. 3-4).

According to Haapasalo and Kadijevich (2000), the above-mentioned definitions may lead to the conception that procedural knowledge is dynamic and conceptual knowledge static. To highlight the dynamic nature of conceptual knowledge, they give the following definitions:

> *Procedural knowledge* denotes dynamic and successful utilization of particular rules, algorithms or procedures within relevant representation form(s). This usually requires not only knowledge of the objects being utilized, but also the knowledge of format and syntax for the representational system(s) expressing them.
>
> *Conceptual knowledge* denotes knowledge of and a skilful drive along particular networks, the elements of which can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or a rule) given in various representation forms. (Haapasalo & Kadijevich, 2000, p. 141.)

Haapasalo and Kadijevich (2000) emphasize that in some cases the two knowledge types can be distinguished only by the level of consciousness of the applied actions. These definitions clearly indicate that conceptual knowledge is not only "knowledge of" but also "using that knowledge". Therefore, students' conceptual knowledge cannot be investigated by simply asking them what they know. Instead, we have to study how they use this knowledge in solving problems. Often the procedural knowledge refers to the use of some representation and conceptual knowledge to making connections from this representation to other representations. In the articles [1] and [2] the students' procedural and conceptual knowledge of the limit of the difference quotient are analyzed. Both kinds of knowledge are conceived as dynamic in nature as the students use procedures and "drive along knowledge network" to solve problems. The degree of utilized procedural knowledge of the limit of the difference quotient means knowledge of and fluency in carrying out the mathematical algorithm of calculating the value of the derivative at a point using the limit of the difference quotient. The degree of utilized conceptual knowledge refers to students making connections between the limit of the difference quotient and other limiting processes.

According to Haapasalo and Kadijevich (2000) and Rittle-Johnson et al. (2001), there is a long-term debate on which type of knowledge is more important and in which order they should be learnt. They state that instead of arguing on this, it is more important to investigate how these two types of knowledge are connected. According to Haapasalo and Kadijevich (2000), all the four relations between procedural and conceptual knowledge have been supported by empirical studies: (1) procedural and conceptual knowledge are not related, (2) procedural knowledge is a necessary and sufficient condition for conceptual knowledge ($P \Leftrightarrow C$), (3) conceptual knowledge is a necessary but not sufficient condition for procedural knowledge ($P \Rightarrow C$), and (4) procedural knowledge is a necessary but not sufficient condition for conceptual knowledge ($C \Rightarrow P$). Also Rittle-Johnson et al. (2001) state that, it cannot be generalized which kind of knowledge should precede the other as it is only a matter of which one starts the learning process. They found empirical evidence that these knowledge types develop iteratively so that development in one type of knowledge leads to the development in the other.

5.3 Representations and connections between them

In the three worlds of mathematics students may use different representations. As stated in the previous section, procedural knowledge often denotes the use of a representation and conceptual knowledge denotes making of connections to other representations. Several classifications of representations can be found in the literature. For example, according to Goldin (1998), internal representational systems can be a) verbal/syntactic, b) imagistic, c) formal notational, d) strategic and heuristic, and e) affective. Imagistic representations are the main ingredient in the embodied world and formal notational representations in the symbolic world[1].

In this section, I clarify in what meaning the representation concept is used in this study. Traditionally, a representation is conceived as something which stands for something else, and representations are divided into internal and external ones (cf. Janvier, 1987). An internal representation refers to the mental construction and an external representation to the physical construction. For example, a graph on a paper may be a representation of a function for a person. The mental image of the graph is an internal representation. This view of the representation has been criticized recently. For example, there is a danger that representations may be thought to be mere representations of some objects and separated from meaning (Sfard, 2000). According to Sfard (2000), this position implies that objects and meanings are more important than representations, and these should be learnt before signs. For example, a picture of a triangle is not a triangle but just a representation of it. Then, what is the triangle? The triangle is something that is represented by the picture. Therefore, the meaning cannot be separated from the representation. How could one first learn what the triangle is and only after that represent it?

The traditional view of a representation implies that representations are only used to store information and that the role of signs and symbolic tools is only to support and aid students (Sfard & McClain, 2002; Radford, 2000; Meira, 1998). Thus, this view does not use all the potential power of representations

[1] In other literature, representations have been classified also as enactive, iconic, graphical, formal, symbolic, algebraic, numerical, verbal, etc.

and other tools. For example, the image of the triangle is not necessarily just storing some aspects of the triangle and used as an aid in thinking. Instead, the image may be an essential part of thinking and without it thinking would be really different.

Also, the dichotomy of internal versus external representations has been found artificial. According to Radford (2000) and Sfard and McClain (2002), traditional views often take a standpoint that external representations reflect the mental structures of an individual and that learning is the growth of mental structures. This would suggest that, for example, paper-and-pencil pictures of a triangle are reflections of a person's mental image. Sometimes it may be the other way around; the person constructs a mental copy of a physical picture. Clearly, this cannot be always the case as both the mental and the physical picture has some properties that cannot be replaced by the other.

Meira (1998) and Cobb et al. (1992) point out that even when the decisive role of a student is acknowledged, representations are often analyzed from an expert's point of view as if external representations included meanings. Thus, these analyses do not address the use or construction of the representations. Meira (1998) has emphasized that the focus of studies in representations should move towards students' use and construction of representations. This focus can be noticed in the studies of Davis and Maher (1997) as they describe how students use representations as "tools to think with". According to them, the key attribute of effective tools is that they can be used to carry out thought experiments and to test hypothetical scenarios. Research has to focus on students' ideas and not just on testing their compatibility with experts' ideas (ibid.). In line with this, Speiser et al. (2003) emphasize capabilities of students rather than their errors. This move towards conceiving representations as tools was made also by Radford (2000). In his study there is "a theoretical shift from what signs *represent* to what they *enable* us to do" (p. 241). Compatible with the view of a representation as a tool, Sfard (2000) has argued that representations are not born as such but they *may* become to stand for something else later. Several authors have also emphasized that meanings are constructed through the use of signs (e.g., Sfard, 2000; Radford, 2000).

Building on the above-reviewed criticism and new views, the representation is characterized in this study as follows:

> *A representation is a tool to think of something which is constructed through the use of the tool. A representation has the potential to stand for something else but this is not necessary. A representation consists of external and internal sides which are equally important and do not necessarily stand for each other but are inseparable. The external side is visible to other humans through the senses but the internal side is not.*

For example, a student may use the steepness of a graph of a function as a representation of a derivative of the function. This means that the steepness tool allows the student to perceive some aspects of the derivative, for example, the maximum point of the derivative. The student's conception of the derivative may have been constructed (and is being constructed) through the use of the steepness and other representation tools. There may be an external side of the steepness, for example, the mere graph on the paper, speech or some gestures. Obviously, there must be some internal side, because for some people the graph would not allow to perceive the derivative. It is not the case that the external side only reflects the internal side, but it is the interplay between them that allows the student to use this tool efficiently. External sides are important for research because all the interpretations are based on these.

Following Goldin (1998), it is pointed out that one representation (or representational system) may be thought to consist (and usually it does) of other representations (or representational systems), and it is a matter of convention if we want to think of a single representation or its constituents. Often a graph of a function is considered as one representation. Instead, in this study the focus is on more specific representations (e.g., steepness) which are used within the graph.

Connections, links and relations between representations or pieces of knowledge as well as students' ability to move, change, or translate among representations are important characteristics of learning and problem solving (Hiebert & Lefevre, 1986; Haapasalo & Kadijevich, 2000; Tall & Vinner, 1981; Goldin, 1998; Dreyfus, 1991; Hiebert & Carpenter, 1992; Zimmermann, 2005). Many studies discuss the fact that students are able to make particular connections but few explain the distinctions between different types of connections. For example, the classical book of representations edited by Janvier (1987) does not present any kind of analysis on the nature of different kinds of translations. One of the aims of this study was to characterize connections that are different in nature. Two connections, named associative and reflective connections, were characterized after extensive data analysis. Then, the data were (re)analyzed according to these connections. In the article [1] these connections were characterized as follows:

> *A person makes an associative connection between two representations if he or she changes from one representation to another.*
>
> *A person makes a reflective connection between two representations if he or she uses one representation to explain another.*

These characterizations highlight the nature of representations as tools for thinking. To solve a problem it is often important to change between representations and very often it is necessary to argue or reason by explaining one representation with another. During the data analysis it was found that the explanations in students' discourse are reasonably easy to observe. Therefore, the classification of connections was based on the explanations. It was also discovered that in some cases it was difficult to observe which one of the reflectively connected representations was used to explain and which one was explained. Moreover, the representations often seemed to be mutually explaining each other. Thus, the direction of the connections is not mentioned in the above characterization but is expressed in the results where appropriate. It is noteworthy that the connections are not assumed to exist in a person's mental structures but in his or her activity, such as in problem solving and in discourse. Thus, a person does not have a connection but he or she makes a connection.

In the article [1] the similarities of these connections to other proposed connections are discussed. In short, these connections resemble a kind of combination of Goldin and Kaput's (1996) weak and strong links and Sfard's (2000) discursive connections. According to Goldin and Kaput, two external representations may be linked internally in the mind of a person who produced or perceived them. The link would be weak if the individual is able to predict, identify, or produce the counterpart of a given external representation. The link would be strong when given an action to one of the external representations, the individual is able to predict, identify, or produce the result of the corresponding action on its external counterpart. According to Sfard (2000), one kind of relationship is "the awareness of some sort of 'kinship' between signs", which usually is prior to any mention of a referent. The other kind of relationship appears in isomorphic discourses which mean "a relations-preserving correspondence between the two discourses". According to Sfard (2000, p. 82), the latter relationship exists, for example, in a mapping between discourses about algebraic formulas ("$2x+1=26-3x$ when $x=5$") and discourses about coordinate graphs ("the straight lines with the slopes 2 and -3 and y-intercepts 1 and 26, respectively, cross each other at the point (5, 11)") when the logical relationships are preserved. In the article [2] also the intra, the inter and the trans stages of a schema development described by Clark et al. (1997) are used in the analysis. This framework is an extension of the APOS theory and is discussed in the next section.

5.4 Process-object theories

An important feature of mathematical thinking is that mathematical concepts can be viewed as processes or as objects (e.g., Sfard, 1991; Asiala & al., 1996; Gray & Tall, 2001). For example, 3 + 5 may be understood as adding 5 to 3. On the other hand, 3 + 5 may also be understood as a sum without a demand to perform some calculation. Similarly, 0.999… may be understood as a process in which the number 9 is "repeated" infinitely many times. This may suggest that 0.999… is infinitely close to 1. However, 0.999… may also be conceived as an object, as a number which is a result of the limiting process ("repeating" 9 infinitely many times). The latter point of view may suggest that 0.999… = 1. This illustrates how big the difference between these two views is. For example, Francisco and Hähkiöniemi (2006) presented a case in which three children arrived to a correct rule in "Guess My Rule" –game but did not write the rule as (x + 1)(x + 1). This might have been because for the children x + 1 was a process which had to be calculated first. Therefore, their rule was: add 1 and multiply the answer by itself.

The APOS theory (Asiala & al., 1996) and Sfard's (1991) reification theory are based on the mentioned dualism of the mathematical concepts and on the assumption that process conception precedes object conception. According to

Sfard (1991), also in the history of mathematics several concepts have first been conceived as processes and only afterwards as objects. A similar development has been noticed in individuals (ibid.). For example, the concept of number is learnt usually through counting. The number as an object is abstracted from the counting process. Sfard (1991) calls process conception *operational* outlook and object conception *structural* view. According to the *reification theory* (Sfard, 1991; Sfard, 1992; Sfard & Linchevsky, 1994) learning proceeds through three stages:

> At the first stage a learner *interiorize* a process that is performed on an already existing object. The learner becomes skilled in performing the process and can consider the process without actually performing it.
>
> At the second phase long sequences of operations are *condensed* into more manageable units. A person becomes more capable of thinking about the process as a whole and he/she does not have to go into details. The process is considered as an input-output relation rather than as operations.
>
> At the third stage a mathematical entity is *reified*. This means that the notion is conceived as an object and is detached from the process that produced it. The object gains its meaning from being a member of a certain category rather than from the process. At this stage processes may be performed on this new object.

Sfard presents one exception for this kind of development. This is called *pseudo-structural* (Sfard, 1992; Sfard & Linchevsky, 1994) or *quasi-structural* (Sfard, 1991) conception. This means that a student manipulates a concept according to certain rules as if it was an object, but the object does not have underlying operational structure (Sfard, 1991; Sfard, 1992; Sfard & Linchevsky, 1994). The representation of the concept is just standing for itself without any meaning (ibid.). According to Sfard and Linchevsky (1994), an example of this in the case of algebra is that changing the name of a variable leads to a totally different equation. At the beginning of this section an example of operational and structural views of number 0.999… was given. To continue this example, imagine that a person considers 0.333… as a number 1/3 and proves this by manipulating 0.333… according to a well-practiced algorithm so that $9 \cdot 0.333... = 10 \cdot 0.333... - 0.333... = 3.333... - 0.333... = 3$ and thus, concluding that $0.333... = 3/9 = 1/3$. The person may do this by considering the 0.333… as a pseudo-structural object which does not include any limiting process. In an unfamiliar context of 0.999… this person may not notice that 0.999… = 1 if the algorithm is not practiced also in this case. This is an important detail for this study as the derivative may be easily considered as a pseudo-structural object, for example, using differentiation rules.

In the *APOS theory* developed by Dubinsky et al. (Asiala & al., 1997; Cottrill & al., 1996; Asiala & al., 1996; Breidenbach & al., 1992; Dubinsky & McDonald, 2001; Dubinsky, 1991; Dubinsky, 1994; DeVries, 2001) especially for advanced mathematics, there are similar stages of Action, Process, Object and Schema:

> An *action* is a physical or a mental transformation of objects to obtain another object. The action is a reaction to external stimuli and it is carried out step by step without an individual's conscious control of the action.
>
> When the individual reflects on the action and gets a conscious control of it, the action is *interiorized* to a *process* and the individual can describe the steps in the transformation without necessarily doing them.
>
> The process becomes *encapsulated* as an *object* when the individual becomes aware of the totality of the process and is able to perform new actions to it.
>
> A *schema* is a coherent collection of processes, objects and other schemas. An object can also be created when a schema is thematized to an object.

Examples of the stages in the case of advanced mathematical concepts can be found in the mentioned literature on the APOS theory. For the case of the function concept, Asiala et al. (1996), Breidenbach et al. (1992), Dubinsky and McDonald (2001) and DeVries (2001) give the following examples. A student has an action conception if he/she is limited to calculating values of a function with a given formula. One is at the process level if he/she is thinking about the function as an input-output machine. At the object level a student is able to perform actions on functions. For example, he/she can think about a function as a sum of two functions or as an element of a set.

Clark et al. (1997) have extended the APOS theory by developing a three-staged framework for analyzing the schema development. According to this framework, at the *intra stage*, a student focuses on a single item isolated from other items, at the *inter stage* he or she recognizes relationships between different items, and at the *trans stage* the coherent structure of relationships is structured. For example, at the intra stage a student may have a collection of differentiation rules, at the inter stage he or she recognizes that in some way they are related, and at the trans stage he or she considers those rules as special cases of the chain rule (ibid.).

Roughly saying, the three first levels in the APOS theory correspond to the stages in Sfrad's theory. For a detailed comparison of these theories and other similar theories, see Meel (2003). The most important common thing is, however, the development from processes to objects. In this study, these theories are used mainly in this sense. It is examined whether students are thinking about the derivative operationally as a process or structurally as an object. In addition, in the article [2], the APOS theory is used in more detail. In the article [2], it is examined on which levels the students are with the procedure of calculating the limit of the difference quotient and with their limiting processes, and on which stage they are in their schema development.

In the theory of reification and the APOS theory, a stage cannot be reached before all the previous stages have been passed. However, when one stage is reached, the previous stage does not vanish. In the APOS theory this means that it is important to be able to de-encapsulate the object back to a process. Sfard (1991) uses a metaphor that operational and structural views are like different sides of the same coin. The same basic idea is also inherent in Gray and Tall's (2001) notion of *procept* which means that symbols act dually as processes and concepts. According to Tall (2003, 2004a, 2004b, 2005), learning in the symbolic world proceeds similarly to the APOS theory and Sfard's (1991) theory. Gray and Tall (2001) propose levels of sophistication which are *procedure* (step-by-step solution for routine problem), *process* (flexible solution with possible alternatives) and *procept* (ability to think about mathematics symbolically). However, Tall (1999) has criticized the APOS theory because learning does not always proceed according to the APOS levels. Therefore, Tall (2003, 2004a, 2004b, 2005) has included the embodied world to his evolving theory of three worlds of mathematics, in which perceptions have an important role. In the next section learning in the embodied world is discussed.

5.5 Perceptual activity in mathematics

Learning in the symbolic world proceeds by constructing objects from actions. However, in the embodied world it is not meaningful to describe the development in the same way. Gray and Tall (2001) claim that learning may also begin by perceiving the concept to be learnt as an object. In this way a student constructs the concept by acting with it. This means that before encapsulating the object from the process, the concept already exists as an *embodied object* (ibid.). According to Gray and Tall, for example, the derivative can be perceived from the graph of a function before any numerical calculation or symbolic manipulation. Gray and Tall proposed that perceptions may become more abstract constructs, which do not anymore refer to specific objects in the real world. For example, the conception of a line may develop from a line drawn by a ruler to a perfectly straight line that has no width and is arbitrarily extensible in either direction (ibid.). According to Gray and Tall (2001), this kind of development is similar to the development described by van Hiele in geometry[1]. This kind of development is assumed to happen in the embodied world through reflection and use of language (Tall, 2004a). According to Tall (2004a), this is quite different from the development in the symbolic world.

Later, Tall has developed the description of learning in the embodied world so that it has similarities with the development in the symbolic world (see Tall, 2005; Pegg & Tall, 2005). This was inspired by one student who noticed that one hand translation combined with another had the same effect as a single translation corresponding to the sum of the two vectors (Tall, 2005; Poynter, 2004; Watson & al. 2003). Thus, in the embodied world, students may learn by shifting their focus from *actions* to the *effects* of those actions (Tall, 2005; Poynter, 2004; Pegg & Tall, 2005). For example, in the case of the vector concept students

[1] See Silfverberg (1999) for an extensive discussion on van Hiele levels and comparison to other theories, including Sfard's (1991) theory.

may shift their focus from translations of a hand to the effects of the translations (Tall, 2005; Poynter, 2004). Another example in the case of fractions is given by Pegg and Tall (2005). According to them, the action of dividing a quantity into 6 equal parts and selecting 3 of them leads to the same effect as dividing the quantity into 4 equal parts and selecting 2. Similarly, in the symbolic world 3/6 = 2/4.

In the articles [3] and [4] I have proposed that the action-effect development in the embodied world is similar to the concept of *transparency*. The transparency of a tool means that the tool is visible for acquiring detailed information of the tool, but invisible for getting access to a phenomenon that can be seen through the tool (Meira, 1998; Roth, 2003; Lave & Wenger, 1991). Similarly, eyeglasses are visible to a person so that he/she may notice when it is time to clean the glasses. But the eyeglasses are invisible so that the person sees the world through the glasses, and trashes and frames do not disturb him/her[1]. Transparency is not a property of a tool but an emerging relation between the user and the tool (Meira, 1998; Roth, 2003). The tool does not mean only physical tools, but also representations are tools. For example, a graph may become transparent to the user so that he/she sees the phenomenon behind the graph and does not only focus on the physical appearance of the graph (Ainley, 2000; Roth, 2003). Roth (2003) points out that in his study the graph affected what scientists were able to see of the phenomenon. He uses the eyeglasses metaphor that the graph was like glasses which allowed seeing the world clearer when the user was accustomed to the glasses. In mathematics, learning to use a graph means beginning to see essential aspects of mathematical objects that are represented in the graph. In the above-mentioned action-effect example in the case of the vector concept, the transparency of the hand movement embodiment would mean that one sees the vector as an effect through the hand movement. If the hand movement is not transparent, one focuses on the action of moving the hand. But if the hand movement is transparent, one focuses on the effect of the action.

Noble et al. (2004) have investigated students' working with graphs and how students learn to see these in a meaningful way. They have proposed a mechanism through which a disciplined way of seeing may evolve. According to them the disciplined way of seeing may evolve from *not seeing a whole* to *recognizing in* and to *seeing as*. Not seeing a whole means that one may be able to see the parts of an image without being able to see the whole. However, one may recognize in the image something he/she is familiar with. The experiences

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[1] In section 5, I presented a metaphor that the theoretical framework of this study is like spectacles through which students' activity is seen. The theoretical framework has to be visible for introspection. On the other hand, the researcher has to be familiar with it so that he does not have to read what this and that means every time he makes inferences of students' activity. Thus, the theoretical framework has to be also invisible so that, for example, the technical terms do not prevent the researcher from seeing what happens. For evaluating the transparency of the theoretical framework the reader may judge from this section how visible it is and from the empirical part how invisible it is for the author.

of "recognizing in" may cause one to see the image as something that he/she was not able to see before. This framework may be translated to the context of this study in the following way. Not seeing a whole means that a student may see a graph as a representation of a function. However, he/she does not see the derivative of the function in the graph. Gradually, the student may recognize in, for example, where the derivative is positive and where it is negative by focusing on the graph going upward and downward. Finally, the student may see the graph as a representation of the derivative. In the articles [3] and [4], I have proposed that the representations of the steepness, increase and horizontalness of a graph are things that may be recognized in the graph. Depending on the transparency, these representations may allow to see the derivative in the graph.

Also gestures have an important role in the embodied world, as can be seen from the hand translation example. In this study, gestures are considered as external sides of representations. McNeill (1992) has argued that gestures together with speech are an essential part of thinking processes. According to him, *deictic gestures* indicate something, *iconic gestures* resemble something and *metaphoric gestures* represent abstract ideas. For example, pointing with a finger to something is a deictic gesture. When talking about a round object one makes an iconic gesture if he/she makes a circle in the air with a hand. An example of metaphoric gesture is moving a hand between oneself and listener when talking about communication. Roth and Welzel (2001) have demonstrated by their case studies that gestures have an important role in constructing explanations in physics. They argue that gestures allow constructing complex explanations even in the absence of the scientific language and coordinating phenomenal and conceptual layers of the content. According to their study, gestures seemed to make abstract entities visible. Similarly, Radford et al. (2003) report that gestures with words allowed a student to make sense of a distance-time graph of a moving object. Also, a study of Moschkovich (1996) highlights the importance of gestures, particularly, when describing graphical objects. According to her, ninth-grade students used coordinated gestures and talk to negotiate a meaning for steepness of a linear graph. Rasmussen et al. (2004) illustrate in their study how gestures are part of expressing, communicating and reorganizing one's thinking also in the advanced mathematics of differential equations.

5.6 Open problem solving

This section differs from the other sections in the theoretical framework. The difference is that the literature on open problem solving is not used in the analysis or discussing the results. Instead, open problem solving was an underlying perspective on what meaningful mathematics learning means. These ideas had very much influence on the design of the study, particularly on the design of the teaching-learning sequence.

In line with the constructivist perspective, Schoenfeld (1992) and Davis and Maher (1990) have proposed that to learn mathematics students should engage in doing mathematics and in solving problems. Particularly, students should also work with open problems (see, e.g., Pehkonen, 1997; Pehkonen, 2004; Nohda, 2000; Francisco & Maher, 2005). A problem is open if the starting situation or the goal situation is not exactly defined so that the solver has to make selections (Pehkonen, 1997). Nohda (2000) adds that also the process may be open so that the problem has multiple ways of solving them. Although all mathematical problems have multiple correct ways to solve them, Nohda points out that the openness of the process need to be emphasized instead of answer-driven problem solving. Especially, the openness of the process was emphasized in the teaching-learning sequence of this study. The openness of the process allows a solver to use or create different representations, and solutions utilizing different representations may be compared to each other. Students can also solve these problems in the different worlds of mathematics and work at their level of concept development. In the already referred studies of Repo (1996), Berry and Nyman (2003), Doorman (2005), and Speiser & al. (2003), students worked with open problems and openness seemed to be one factor influencing students' learning.

6 THE IMPLEMENTED TEACHING-LEARNING SEQUENCE

The studies and theories used in constructing the initial learning path and instruction are reviewed above. This was supplemented by the analysis of the derivative concept (see section 3). The main features of the derivative that would be included in the teaching-learning sequence were the velocity as a special case of the derivative, the average and instant rate of change, the slope of a tangent line, the steepness of the graph, the relation between a graph of a function and its derivative, various underlying limiting processes and the limit of the difference quotient. The aim of the instruction was that students build these constructs by solving open problems in which they can use several representations. The definition of the derivative was designed to be introduced to students as they engage in solving the problem of a value of an instant rate of change of a function. First, I shortly describe how calculus is generally taught in Finland. Then, the initial hypotheses of how students might learn the derivative are summarized. Finally, the implemented teaching-learning sequence is described. For the discussion on methodology involved in the construction of the teaching-learning sequence, see section 7.4.

6.1 Calculus in Finnish high school and university

Finnish students in the advanced syllabus of high school mathematics face the explicit notion of the limit for the first time usually during the second year (grade 11) in the Differential calculus I -course. After this course the derivative concept is dealt with in the courses Differential calculus II, Integral calculus, Number series and in the optional courses Analysis and Numerical mathematics (The Finnish National Board of Education, 1994). Courses have had some changes after the new framework curriculum 2003 (The Finnish National Board of Education, 2003), which was for the first time implemented in 2005. In the new framework curriculum the concepts of limit, continuity, and derivative are introduced in the course Derivative. After this the derivative is also handled in the courses Root functions and logarithm functions, Trigonometric functions and number series, Integral calculus as well as in the optional courses Numerical and algebraic methods and Extension course of differential and integral calculus.

Typically, the limit is defined informally (without the $\varepsilon - \delta$-definition) including statements, such as getting the values of the function arbitrarily close to the limit value (Kontkanen & al., 1999; Merikoski & al., 2000; Kangasaho & al., 2002; Kangasaho & al., 2006). Thus, in high school the limit is viewed dynamically excluding the formal definition. Shortly, the rules for calculating algebraically limits of elementary functions are introduced. The continuity and derivative are then defined using the limit concept. When proceeding like this, there is a danger that students will conceive limiting only as computing limits algebraically (cf. Tall, 1991, p. 17-18). The derivative concept offers one possibility to avoid this if the (dynamic) limiting process in the definition of the derivative is emphasized in the instruction. The definition of the derivative is followed by differentiation rules. The derivative is used in applications to investigate functions and to find extreme values.

Those who continue to study mathematics at university will face the formal $\varepsilon - \delta$-definition of the limit concept in Analysis courses. Although calculus is usually considered as advanced mathematics, it is considered totally differently in high school and at university (see Hähkiöniemi & Viholainen, 2005). In high school, calculus is hardly dealt with in Tall's (2004a) formal-axiomatic world.

6.2 The initial hypothetical learning path to the derivative

The initial theoretical analysis of how students may learn the derivative meaningfully is summarized in Figure 6 (cf. Figure 5)[1]. Learning the derivative begins by perceiving functions and their properties in graphical representations and by performing actions on functions represented symbolically. From the graph of a function students can make perceptions about the rate of change even before the definition of the derivative. Moving a hand along the curve, placing a pencil as a tangent, looking how steep the graph is and the local straightness of the graph can be used for this.

Especially, the following innovation was developed to help the students. The graph of a function may be imagined to be a rope. One can grasp this rope by a thumb and a forefinger. Keeping the rope (the graph) between the thumb and the forefinger one may now slide the hand along the rope (the graph). A pencil may now be placed between the fingers so that it resembles a tangent. By zooming in to the graph it may be noticed that the pencil is actually a good approximation of how the graph behaves locally. When the pencil points upward/downward, the rate of change is positive/negative. The steeper the pencil is the greater the rate of change is. When one now slides the hand along the graph, he/she sees how the rate of change is changing.

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[1] This analysis was presented as a poster including Figure 6 at the symposium of the Finnish mathematics and science education research association (Hähkiöniemi, 2003).

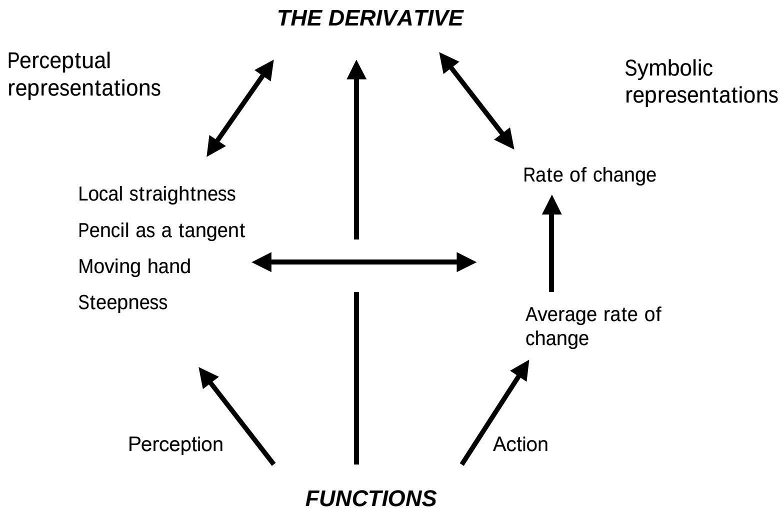


Figure 6. *The initial learning path to the derivative.*

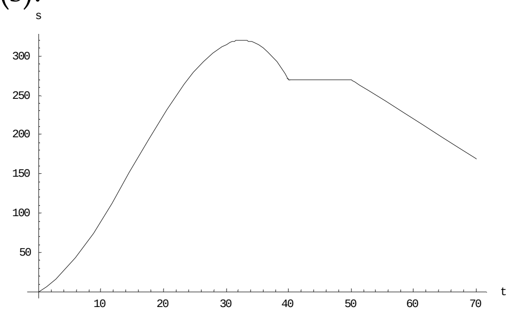
These perceptions can form different embodied objects for the derivative. On the other hand, students can perform different symbolic actions on functions, for example, they can define algebraically the average rate of change over different intervals. These actions may be interiorized to a process in which the average rate of change is calculated over a smaller and smaller interval. This process may be encapsulated to the instant rate of change. Also, at the same time the embodied objects of the derivative become more abstract. Thus the learner's conception of the derivative is not anymore limited to the real-world situation from which learning began. The main thing is that the learner links different representations of the derivative, so that he/she forms a rich concept image and can use different appropriate representations in problem solving.

6.3 The teaching-learning sequence

I taught the teaching-learning sequence in the autumn of 2003 as a part of a Finnish grade 11 (age 17) course Differential calculus I in the advanced syllabus of high school mathematics. The teaching-learning sequence consisted of the first five lessons (one single lesson and two double lessons) on the subject of the derivative. It was conducted in an ordinary classroom of 14 students without using the computer. However, most of the students had a graphical calculator. There were no other researchers in the classroom than the author as a teacher. I had been the students' teacher also before this course. The lessons were videotaped with one stationary camera. In the following, the teaching-learning sequence is described with a sample of important tasks. The description is based on the teaching journal, lesson plans and videotapes of the lessons. The tasks were especially designed for this course. In addition to these tasks, students also had a course book (Kangasaho & al., 2002). During the teaching-learning sequence the book was used only to give some extra tasks to students. Before the teaching-learning sequence the course dealt with topics of function, piecewise function, absolute value function, limit and continuity.

The teaching-learning sequence began by examining motion represented graphically. The word derivative was not mentioned. Among other tasks, the students were given Task 1.

1. The figure represents the distance travelled *s* (m) by a car as a function of time *t* (s).



   a) What is the furthest point where the car is and how far does it end up?
   b) How long did the travel take?
   c) What is the average velocity of the car during the whole trip?
   d) What can you say about the velocity of the car at different points?

In the class I discussed[1] with individual students or groups of students and with the whole class how to determine the average velocity and what the instant velocity means. Students considered, for example, the negative velocity and the maximum value of the velocity. Students also pointed out conflicts with the real life. After this we agreed that the distance travelled in this situation meant the displacement of the car from the starting point on a straight road[2]. Also moving a hand along the graph and placing a pencil tangent-like to a

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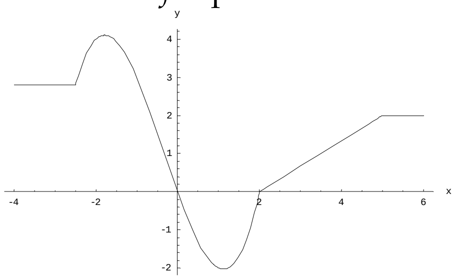
[1] A discussion with an individual or a small group of students means asking questions and sharing ideas during problem solving. I avoided giving students too much advice or directions. Instead, I asked them, for example, what they were doing and where they were aiming at. I also tried to encourage students to explore more problem situations and find alternative solution methods. A whole-class discussion means that I collected the ideas that students had invented in solving problems. At this kind of discussion I asked more specific questions about solution methods. Unlike at the discussion with students during problem solving, I also contributed to answering these questions myself.

[2] From the scientific point of view, the distance travelled and the displacement are different concepts. For example, the distance travelled can never decrease. For these tasks to be meaningful, the displacement has to be considered along the path travelled by the object. Dealing with these issues thoroughly in this situation would had taken the teaching-learning sequence to a long side route. Therefore, the distance travelled was (incorrectly) identified with the displacement.

graph and moving it along the graph were considered at a teacher-led discussion with the whole class. After this, the rate of change of a function was considered, for example, in Tasks 2 and 3.

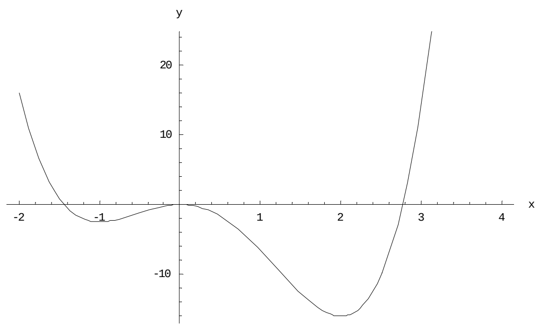
2. The graph of the function *f* is presented in the figure.



   a) What is the greatest value that the function *f* takes?
   b) At which point does the function *f* take its smallest value?
   c) When do the values of the function *f* increase and when decrease?
   d) Make observations of the rate of change of the function *f*.

At this point the teaching-learning sequence had lasted about one lesson (45 minutes). The classroom discussion concerned increasing and decreasing of the values of a function as well as the quality and magnitude of the rate of change. Also the local straightness was considered, and it was noticed that a pencil as a tangent illustrates the graph well locally. Especially, it was noticed that the graph indeed goes horizontally at the top of the graph, which means that the rate of change equals to zero.

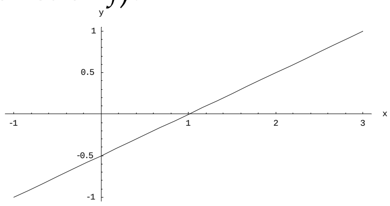
3. $f(x) = \frac{3}{2}x^4 - 2x^3 - 6x^2$



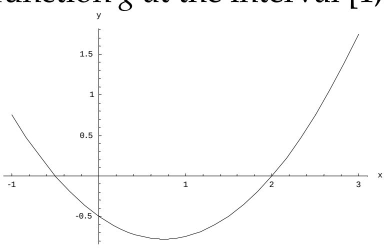
   a) Determine the zero points of the function *f*.
   b) What is the smallest value of the function *f*? What about the greatest?
   c) When is the rate of change of the function *f* positive and when negative?
   d) When is the rate of change of *f* zero?
   e) When is the rate of change of *f* greatest and when smallest?

Then the average rate of change was scrutinized in Task 4 among other tasks. During this task and after solving it, we discussed in the class how the average rate of change is determined by dividing the change in the values of the function by the change in the variable, by calculating the slope of the corresponding secant and by calculating the difference quotient $((f(a)-f(b))/(a-b))$. At this point I mentioned the difference quotient for the first time.

4. a) How many units does the value of the function f increase when x increases by one unit (that is, what is the rate of change of the function f)?



b) How many units does the value of the function $g(x)=\frac{1}{2}x^2-\frac{3}{4}x-\frac{1}{2}$ increase on average at the interval [1, 3] when x increases by one unit (that is, what is the average rate of change of the function g at the interval [1, 3])?



After dealing with the average rate of change, the students were given Task, 5 in which the value of the instant rate of change was asked.

5. A car starts at the time t = 0 and its distance s (m) from the starting point depends on the time t (s) according to the function $s(t)=t^2$.
   a) What is the distance of the car from the starting point at the time t = 5?
   b) When has the car travelled 100 meters?
   c) How does the velocity of the car change as time increases? How about the rate of change of the function s?
   d) Determine the average velocity of the car at the interval [5, 7].
   e) Determine the average rate of change of the function s at the interval [4, 5].
   f) What is the velocity of the car at the moment t = 5? Invent different ways to determine or estimate the velocity.

In connection with the solution of Task 5 I discussed with the class the estimations of the instant velocity by drawing a tangent and by decreasing the interval over which the average rate of change is calculated. Average rates of change were listed over different intervals. It was examined which number they seemed to approach when the interval diminished. Also secants corresponding to average rates of change were drawn (and this was related to the tangent). Finally, the exact value of the rate of change was determined by the limit of the difference quotient. Ideas for these solution methods came from students[1]. At this point the teaching-learning sequence had lasted three lessons.

After the solution methods for Task 5 were discussed, I defined the derivative of a function f at a point a as $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$. Other forms of the definition were not discussed in the class or presented in the course book. Then the students evaluated and determined exact values of the derivative, for example, in Tasks 6 and 7.

6. a) Estimate with some method what is the derivative (that is the rate of change) of the function $f(x) = 2x^2 - 3$ at the point $x = -1$.
   b) Determine the exact value of the derivative of the function f at the point $x = -1$.

7. $f(x) = |x|$.
   a) Determine the derivative of the function f at the point $x = 4$.
   b) Determine $f'(-2)$.
   c) Investigate with different methods what is the derivative of the function f at other points.
   d) Investigate what could be the derivative of f especially at the point $x = 0$.

During the discussions of Task 7 it was noticed from the graph of the function that the function does not have an unambiguous rate of change or a tangent at the point $x = 0$, and, therefore, the derivative does not exist. This was also proven by calculating both hand limits of the difference quotient. It was also graphically noted that a function does not have a derivative at a point

[1] All the students did not arrive at all these solutions. According to my teaching journal (the stationary camera did not manage to record individual students' solutions), I noticed that individual students had solved the problem, for example, by calculating the slope of the tangent and by calculating an average rate of change over smaller interval. One student also calculated the average of the average rates of change over intervals [5, 7] and [4, 5]. Another student calculated the exact value of the instant rate of change as the limit of the average rates of change before any interaction with me. After I noticed this, he explained how he did this. He also used secants approaching the tangent in his explanation.

when it is not continuous. The whole teaching-learning sequence lasted five lessons.

After this, another teacher continued the course with the following topics: derivative function, differentiation rules, investigations of polynomial functions, extreme values of a function, applications, differentiation rule for the product of functions, differentiation rule for the function $f(x)^n$, differentiation rule for division of functions, rational function, derivative of rational function and applications (cf. Kangasaho & al., 2002). In all, the course lasted 30 lessons. The 14 students were evaluated by their teacher on scale 4–10, in which 4 means failed. Two students got mark 10, four students got 9, four students got 8, three students got 7 and one student got 6. The average of the students' course marks was 8.2[1].

[1] The information about students' marks are only intended as general background information about the course. Any conclusions about the effectiveness of the teaching-learning sequence cannot be made from this.

7 METHODOLOGY

7.1 Research questions

The aim of the research is to find out how students may use different kinds of representations for thinking about the derivative at the early stage of learning the derivative in a specific approach. The following more specific research questions were set to guide the study. In stating the aim of the research, I use the phrase "how students may use" but in the two first research questions the expression "how the students use" is employed. This is to emphasize that the aim is to find illustrative information on what ways of reasoning students may have in general. This information is acquired by studying the five students in depth.

Research question 1: How do the students use the procedure of the limit of the difference quotient, how do they use other limiting processes, and how do they connect the limiting processes to the limit of the difference quotient?

The limit is an essential feature of the derivative, and, therefore, it deserves special attention. Also one of the main aims of the teaching-learning sequence was to introduce the definition of the derivative. I investigate how the students carry out the procedure of calculating the value of a derivative at a point using the limit of the difference quotient ($f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$). More important, I sought for detailed information on how the students reason about other limiting process inherent in the derivative and how they connect these to the limit of the difference quotient. As the limiting process inherent in the derivative is reported to be difficult to understand for students, answering this research question is an important contribution to the literature on what kind of ideas students may use of the limiting process. After noticing that there exist very few characterizations of different connections between representations, I also aimed at producing such a characterization. Producing this kind of characterization extends the theoretical framework of connections between representations. This research question is answered in the articles [1] and [2].

Research question 2: What kind of representations do the students use for thinking about the derivative? For which purpose and how do they use these representations?

In addition to the representations of limiting processes, also other representations are of interest. I try to find effective tools that students may use for thinking about the derivative, particularly, in a similar approach to the derivative to that reported here. Especially, perceptual activity was emphasized in the teaching-learning sequence, and therefore, I greatly focus on how the students perceive the derivative from the graph of a function. I investigate what representations the students use, how they use them, and what they enable the students to do. In the literature it is suggested that this kind of perceptual activity may be beneficial for learning calculus. Answering this question extends the previous studies by describing how the students use different representations to perceive the derivative. To present enough detailed analysis, the perceptual activity of only two students is reported in the articles. The case of one less successful student is reported in the article [3] and the case of one very successful student in the article [4]. Other students and also other representations are discussed in the article [5].

Research question 3: How could different representations be used in learning the derivative?

I investigated how the students use different representations at the early stage of their learning process of the derivative in a specific approach. On the basis of these characteristics, I hypothesized how these representations could be used by students to learn the derivative. This research question is different from the other two as this builds on them. The answer to this question needs building a hypothetical learning path to the derivative that is based on the analysis of the interviews and the literature. In the literature there exist some proposed learning paths (or similar constructs) or teaching-learning sequences to the derivative. This study takes into account some new aspects of students' ways to learn the derivative on the basis of the analysis of how different representations are used. This research question is answered in the article [5].

7.2 Data collection

After the teaching-learning sequence the five students were invited into task-based interviews. Tommi and Niina were interviewed right after the teaching-learning sequence. Samuel was interviewed one lesson after, Susanna three and Daniel five lessons after the teaching-learning sequence. During that time the teacher of the course continued with the concept of the derivative function and with differentiation rules. Students were selected on the basis of their differences in success in previous mathematics courses and their performance at a pre-test. Because I had taught these students also in previous courses, I could roughly classify their success on these previous courses and at the pre-test. Daniel had been very successful, Tommi and Samuel had been average successful, and Niina and Susanna had been less successful. It should be emphasized that it was a coincidence that the least successful students happened to be girls[1].

At the pre-test (Appendix 1) all the five students could determine an average velocity from a graph of a distance, but only Daniel estimated the instant velocity. They all, except Susanna, determined the sign of the velocity correctly. Niina and Susanna had some difficulties with functions and they could not draw a tangent. The other three students could also draw a tangent of slope zero and, except Samuel, determine the slope of a particular tangent. Only Daniel could interpret the difference quotient as the slope of a secant and estimate how it changes when the base interval decreases.

The *task-based interviews* were designed according to the principles given in Goldin (2003, 2000, 1997). Also more general interview techniques were taken into account (Hirsjärvi & Hurme, 2000; Kvale, 1996; Clement, 2000). Task-based interviews are interviews in which a subject and an interviewer interact in relation to one or more tasks designed in advance (Goldin, 2000). As applied in this study, the task-based interview seems to correspond to the *semi-structured thematic interview* (Hirsjärvi & Hurme, 2000; Kvale, 1996), where specific themes are carefully chosen for discussion and options for additional questions are preplanned. In this case a task corresponds to a theme; the additional questions depend on the students' behaviour. According to Goldin (2000), task-based interviews allow to focus more directly on students' thinking processes than the traditional tests which focus on patterns of incorrect and correct answers. Thus, the task-based interviews suit especially well for this study. Goldin (2000, p. 539-544) presents the following principles for designing task-based interviews: design task-based interviews to address advance research questions, choose tasks that are accessible to the students, choose tasks that embody rich representational structures, develop explicitly described interviews and establish criteria for major contingencies, encourage free problem solving, maximize interaction with the external learning environment, decide what will be recorded and record as much as possible, train the clinicians and pilot-test the interview, design to be alert to new or unforeseen possibilities, and compromise when appropriate. In the following it is described how these and some other things were taken into account in the design and implementation of the interviews.

During the interview the only people present were the student and I. I gave tasks and subtasks one at a time on separate sheets of paper and red them aloud. Students could use extra sheets, pencils, erasers, rulers and a graphic calculator. Each interview was recorded by one video camera focused on the sheets and hands of the interviewee. I selected this focus because it was more important to capture the hand gestures and the order of written notations than facial expressions. I also collected all the written documents.

The task-based interview method was combined with the *think aloud* method (Ericsson & Simon, 1993; Van Someren & al., 1994). I asked the students

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[1] I wanted to choose students from different levels and only students that had been in every lesson. In addition, one girl did not want to participate in the interview.

to think their solutions aloud. Thinking aloud was not demanded very strictly. If there was a long silence, I could, for example, ask what the student is thinking. I told the students before the interview that in addition to such questions as these, they would be asked for arguments without meaning that they were wrong. Usually, at the think aloud interviews questions are minimized (Ericsson & Simon, 1993; Van Someren & al., 1994). This guideline was not followed in this study, as the methods of task-based interviews were considered more relevant for the purposes of this study. Thus, the role of thinking aloud was to help to reveal the students' thought processes in addition to the task-based interviews methods. According to Ericsson and Simon (1993), a good instruction to thinking aloud includes a reference to a familiar procedure. In this study, a television game show "Hermopeli" was used as an example of thinking aloud and of the interviewer's questions[1]. In this game show the host continuously asks what the competitor is thinking to let the audience take part in his decision making processes. I told the students that the tasks are not intended to test their skills but rather to get information about different ways of thinking. I also told them that the interview does not affect the evaluation of their studies and that the tapes are dealt with confidentially. It was also noted that their face could not be recognized from the video because of the focus of the camera. The camera was positioned so that the student was not facing towards the camera. In this way, the camera (with a blinking red light) was not in the student's sight all the time. I hoped that this positioning would help the students not to be interrupted by the camera and to even "forget" it. After the interviews the students approved that the videos could be shown to other researchers. The interviewed students were given a small reward from the school's cafeteria.

At approximately 45-minute interviews the following tasks were given. Depending on the situation, some subtasks were skipped with some students.

Task 1. a) Tell in your own words what the derivative is.
b) The derivative of the function f at the point $x = -5$ is 3. What does this mean?

Task 2. The graph of a function f is given in the figure (see Appendix 2). What observations can you make about the derivative of the function f at different points?

Task 3. Estimate as accurately as possible the value of the derivative of the function $f(x) = 2^x$ at the point $x = 1$.

Task 4. a) Interpret from the figure (see Appendix 2) what the quotient $\left(f(1+h) - f(1)\right)/h$ means.
b) Interpret from the figure (see Appendix 2) what the limit $\lim_{h \to 0} \left(f(1+h) - f(1)\right)/h$ means.
c) Estimate the value of the limit $\lim_{h \to 0} \left(f(1+h) - f(1)\right)/h$.

[1] The first interviewee invented the metaphor of the game show. Later, it was also told to the other interviewees.

Task 5. A car starts at the time $t = 0$ from the starting point. The figure (see Appendix 2) represents the velocity v (m/s) of the car as a function of time t (s).
a) What is the velocity of the car at the point $t = 7$?
b) When does the distance travelled by the car increase and when does it decrease?
c) Sketch the graph of the distance travelled s (m) by the car as a function of time t (s) in the given (t, s)-coordinates.
d) What is the average acceleration of the car at the interval 2s - 7s?
e) What is the acceleration of the car at the point $t = 7$s?
f) Sketch the graph of the acceleration a (m/s^2) of the car as a function of time t (s) in the given (t, a)-coordinates.

The purpose of Task 1 was to be a warm-up question and to investigate what are the things that give meaning to the derivative for the students. Task 2 was designed to give information on how students can see the derivate from the graph of a function. The equation of the function was not given so that all the conclusions would be made from the graph.

Task 3 was chosen to get information on how students estimate the derivative of a function for which they do not know the differentiation rule, and using the limit of the difference quotient is too difficult. At this stage of their learning process the students do not know that they cannot use the limit of the difference quotient for this function. Thus, they will probably try to use it. This will reveal more of their reasoning because they may describe what they would do if they could. Among other methods to estimate the value of the derivative, they may use those including limiting processes. Estimations for the derivatives of exponent functions were not discussed at this stage in the course. A graph was not given to avoid restricting possible estimation methods to those involving a graph.

Task 4 was designed to explore what kind of limiting processes student may use and how they connect these with symbolism. The students had not yet faced this form of the difference quotient (neither in the class nor in the textbook). Thus, they could not only recall what they had seen but they had to reason.

Task 5 was planned to be similar to Task 2 but in a different context. The difference is that this task corresponds to the situation where the graph of a function (velocity) is given, and students are asked how the values of the integral function (distance) and derivative function (acceleration) are changing. In this task they were also asked to draw the graphs. Students' ways to determine the average and instant acceleration is also investigated in this task.

7.3 Analysis of the interviews

The analysis of the interviews followed general guidelines given to a qualitative analysis of interview data (Hirsjärvi & Hurme, 2000; Kvale, 1996). These general guidelines included transcribing the data, reading through the data and "living with it", describing the data, coding or categorizing, synthesizing and moving back and forth between the mentioned stages. The analysis has similarities with grounded theory and particularly with the constant comparison method (Glaser & Strauss, 1967; Strauss & Corbin, 1990). The analysis has also some similarities with the video data analysis procedures developed by Powell et al. (2003). In the following, the steps in the analysis are described in more detail than in the articles [1]-[5].

I began the analysis of the interviews (if the interpretations made before and during the interviews are not included) by viewing the videotapes. This is an important stage of the analysis because the analyst has to become familiar with the data and know it as a whole before making interpretations of individual incidents (Powell & al., 2003). The familiarization process continued as I transcribed all the tapes. Also the gestures and inscriptions made by the students were described in the transcript. From this on the analysis was based on the transcript and the original videos as well as on the written work of the students. The transcripts allowed going into details with several points of the interview simultaneously and to see long segments at a glance, which is not possible from the videotapes. The role of the original recordings was to show the situation in a more holistic way than captured in the transcript. Particularly, students' gestures and inscriptions could not be transcribed without losing something. When making interpretations, I checked the relevant points from the video. The students' written work was used together with the videotapes to see, for example, what they were writing and to what they were pointing. After transcribing and becoming familiar with the data the main steps in the analysis were:

1. *Describing.* I described how the students solved the problems. I focused on what the students did at the interviews. This process was started already during viewing the tapes and transcribing. For example, I listed what observations the students made in Task 2. This stage corresponds to the stage of "describing the video data" in the analysis framework of Powell et al. (2003). According to Powell et al. (2003), it is important that the researcher avoids making strong interpretations but describes what happens (of course, this is also an interpretation but not so strong).
2. *Locating representations.* I located indications of the use of some representation from each of the interviews. The analysis was inductive in a sense that no coding scheme was used in advance. Instead, any kinds of representations were sought. Of course, as Strauss and Corbin (1990) note, the theoretical background and previous experiences affect the analysis. Moreover, according to Strauss and Corbin (1990, 41-47), it is important for the researcher to acquire theoretical sensitivity. For example, without having theoretical (and practical) sensitivity, it could happen that the researcher would note only conventional representations. When I found a new indication, I searched

whether similar indications could be found in the other points of the interviews.

3. *Analyzing uses of representations.* From each of the located situation, I analyzed how and for which purpose a student used the representation and how he/she connected it to the other representations. At this stage the associative and reflective connections (the article [1]) were not yet characterized. Instead, these connections were being created as I noted which properties the connections seemed to have.
4. *Comparing one student's use of different representations.* Then, I compared all the situations where a student used some representation to other situations where the same student used another representation. This allowed noticing some common and some distinct features among the representations that the student used. It was meaningful to speak about specific representations, and I grouped together the points where these were used. This grouping of the uses of the same representations in different situations allowed deepening the analysis of the representations. At this stage I also collected together each student's connections between representations. I noted, for example, that some connections seemed to be "stronger" than others.
5. *Comparing the use of one representation among the five students.* After naming specific representations that each of the students used, I compared these representations among the students. Like above, some representations resembled each other so much that I grouped them in the same category. This again allowed deepening the analysis. When comparing one student's use of one representation to how other students used the same representation, I noted some new instances. For example, some feature in the use of a representation may be more visible in some student than in others. However, after noticing this feature it can also be seen in other students as the researcher has gained sensitivity to this feature. At this stage also the common and distinct characteristics in the students' connections between representations were clarified. I characterized the associative and the reflective connections.
6. *Comparing the students.* Finally, I compared the students' activities during the whole interview to each other's activities. At this stage I compared how students used different representations in the whole interviews. This was an important stage because after the new findings at the previous stages the interviews had to be viewed again in order not to lose the sense of the whole. At this stage I re-analyzed the connections between representations according to the characterizations of the associative and the reflective connections.

The comparisons between different representations of one student, between different students' use of one representation and between different students' use of all his/her representations correspond to the *constant comparison method* of the grounded theory (Glaser & Strauss, 1967; Strauss & Corbin, 1990).

As Goldin (2000) and Clement (2000) emphasize, in the analysis of interviews, there are two levels of interpretations made by the researcher: observations of students doing something and interpretations of these observations. The observations are more reliable than the interpretations. Quotes from interviews are presented, so that the reader may control the reliability of the observations and the interpretations. The quotes from transcripts in the articles [1]–[5] are translated from Finnish and some irrelevant expressions of the spoken language are left out because translating them would be my interpretation, and thus it would not increase reliability but, instead, confuse reader. Symbol "[...]" in the students' transcripts means that the text is snipped and "[ ]" that one word was not audible. In this study there is also a third level of interpretations. This consists of hypotheses of how different representations could be used in learning the derivative. These are based on interpretations at the second level. In the next section the rationale for making these hypotheses are discussed.

7.4 Constructing a hypothetical learning path to the derivative

As an answer to the third research question, I construct a hypothetical learning path to the derivative that describes how a student could learn the derivative concept by working with different representations (the article [5]). For constructing the learning path, I apply the rationale of *educational reconstruction* (Duit & al., 1997; Duit & Komorek, 1997; Kattmann & al., 1998) developed in science education. In this methodology, researchers relate the three components: *analysis of the scientific content, empirical investigations of students' perspectives* and *construction of instruction*. The content analysis aims at detecting the interconnected set of core ideas of the content from the perspective of key aims of education (Duit & al., 1997). Empirical investigations focus on the structure and quality of students' conceptions instead of quantities (Kattmann & al.,1998). According to Kattmann et al. (1998), in constructing the instruction the researchers should, for example, evaluate the most relevant elements of the students' conceptions to be respected and the opportunities that are opened by certain elements of students' conceptions.

Many factors in these components correspond to those of Simon's (1995) model of factors influencing a mathematics teacher's construction of a *hypothetical learning trajectory* which consists of the teacher's learning goal, plan for learning activities and hypotheses of learning processes. In Simon's model, the factors are the teacher's knowledge of mathematics, knowledge of mathematical activities and representations, hypothesis of students' knowledge, theories about mathematics learning and teaching, knowledge of students' learning of particular content and continuous assessment of students' knowledge during the instruction. Simon's model describes the teacher's making of hypotheses, but this model can also be used to guide research. Combining the model with educational reconstruction helps to account different factors for the construction of the hypothetical learning path. Moreover, constructing the hypothetical learning path through educational reconstruction should help teachers to construct hypothetical learning trajectories[1] which are based on the explicit consideration of the mentioned factors. Particularly, one aim in constructing the learning path is to encourage teachers to consider alternative trajectories to those supported by textbooks.

This study applies the ideas of educational reconstruction and a hypothetical learning trajectory for constructing the hypothetical learning path to the derivative in the following way.

1. *Theoretical analysis.* First, I analyzed the structure of the derivative concept (section 3), studies of learning the derivative (section 4), and theories about mathematical learning (section 5) in relation to each other.
2. *Designing and implementing the teaching-learning sequence.* I designed and implemented a five-hour teaching-learning sequence introducing the derivate according to the theoretical analysis (section 6).
3. *Collecting and analyzing data.* After the teaching-learning sequence, I collected empirical data by interviewing five students on how they used different representations in solving problems about the derivative (section 8).
4. *Constructing the hypothetical learning path.* I constructed the hypothetical learning path on the basis of the micro-level analysis of the data in relation to the theoretical analysis (the article [5]).

From a researcher's perspective, a critical question is how the proposed learning path is validated. In section 4.1, some approaches which have been used to validate hypothetical learning paths, teaching-learning sequences or some similar constructs were reviewed. One such approach is conducting a teaching experiment, testing the students' learning outcomes and comparing these to the outcomes of a control group (Repo, 1996; Asiala & al., 1997; Heid, 1988). Another approach is to collect and analyze data to find out how an initially designed teaching-learning sequence (or learning path) could be improved. Such approaches are, for example, *design research* (Edelson, 2002), *design experiment* (Cobb & al., 2003) and *didactical engineering* (Artigue, 2005). Also the rationale of educational reconstruction as applied in this study belongs to this category. The main similarity to design research is that the initial learning path, which is based on theoretical analysis, is revised on the basis of empirical data. According to Cobb et al. (2003) and Edelson (2002), developing a domain-specific theory is one aim of design research. Thus, another similarity is that I am constructing a local framework of how students could learn the derivative

[1] I use the phrase "hypothetical learning path" to signify the construct which resulted from this study. The phrase "hypothetical learning trajectory" is used in the sense of Simon (1995) to signify the teacher's hypotheses of the way how students might learn the concept.

using particular representations. Moreover, nor does this study aim to show the effectiveness of teaching[1]. Instead, it makes an effort for finding effective tools that students may use for thinking about the derivative.

A difference to design research studies (e.g., Doorman, 2005; Artigue, 2005) is that the learning path constructed in this study does not include practical instructional preferences but students' constructions. In this study, the aim is to find micro-level information on individual students' use of representations, and on the basis of this, to hypothesize how these representations could be used in learning the derivative. Thus, the hypothetical learning path does not include research-based information on how the activities in the classroom should be arranged. The design for the activities is left to teachers who want to support students' use of representations as suggested by the learning path. The aim at micro-level information on individuals was also a reason for collecting data through task-based interviews, as this aim was thought to be difficult to achieve by analyzing classroom episodes. Thus, this study has also similarities with *APOS studies* (Asiala & al., 1997; Cottrill & al., 1996; Asiala & al., 1996; Breidenbach & al., 1992; Dubinsky & McDonald, 2001), as the initial hypothetical learning path or genetic decomposition is revised on the basis of the analysis of the interview data. Another similarity is that the genetic decomposition as well as the hypothetical learning path do not include information on how the activities in the classroom should be arranged. A difference to the APOS studies is that the design of the hypothetical learning path and the analysis of data is not tied to the APOS theory. Unlike in the APOS studies, I do not intend to analyze whether students have made some proposed constructions. Instead, I try to analyze the characteristics of students' use of representations in an open way and, on the basis of this, hypothesize how these could be used in learning the derivative. Thus, the hypothetical learning path is also very different from frameworks for understanding the derivative (cf. Zandieh, 2000; Santos & Thomas, 2003; Kendal & Stacey, 2000) because it does not describe what the students should know. Instead, the learning path is a kind of a local framework of how students could learn the derivative using particular representations.

[1] Some studies (e.g., Leppäaho, in press; Viiri, 2004) have combined the two approaches in that a teaching-learning sequence is constructed through design research or educational reconstruction but the effectiveness of the sequence is also tested using pre-test/post-test procedures.

8 SUMMARY OF THE RESULTS

In this section the results presented in the articles [1]–[5] are summarized. Only some quotes from the transcripts are presented for illustration. The reliability of the observations and interpretations should be evaluated on the basis of the original publications which are more detailed.

8.1 Limiting processes in the derivative

The analysis of the students' use of limiting processes inherent in the derivative focused on their use of the limit of the difference quotient, on their use of other limiting processes and on how they make connections between these. These results are presented in the articles [1] and [2]. Corresponding to the task design of the interviews, the most important tasks for this section are Tasks 3 and 4.

I found that only Samuel and Tommi used the limit of the difference quotient fluently to calculate the value of the derivative at a point. Niina and Daniel tried to remember the formula but could not even get started in using it. Susanna used the limit of the function to calculate the derivative when, apparently, she meant to use the limit of the difference quotient. In terms of the APOS theory it could be said that Niina, Susanna and Daniel were at the action level and Tommi and Samuel were at the process level. Accordingly, Tommi and Samuel demonstrated good procedural knowledge of the limit of the difference quotient, while Niina, Susanna and Daniel demonstrated poor procedural knowledge. For example, Samuel started to calculate the derivative in Task 3 as "$Df(x)=(f(x)-f(1))/(x-1)=(2^x-2)/(x-1)$" and described the procedure as:

> Now you can't substitute one here *(points to x at numerator and denominator)*, because it would be zero here. [...] You should find some common factor from there *(points to numerator).* [...] If you could find a common factor from here and the other factor would be *x* minus one, then you could cancel out. [...] Then you would substitute one to what's left.

Samuel's notation was insufficient, but he described the procedure correctly. Actually, in the successful use of the procedure the notation $\lim_{x\to 1}$ symbolizes only that 1 is substituted to x after the expression is manipulated to an appropriate form. Therefore, it is really not needed in the procedure. If the notation $\lim_{x \to 1}$ means some kind of limiting, then some other limiting process than the procedure is considered.

All the students used also some other limiting process besides the limit of the difference quotient but not all of them connected these. Niina used zooming in the graph to explain in Task 2 why the derivative is zero at the top of the graph of a function if there is not a sharp corner. In Task 5a she first thought that the vertical axis is the distance and suggested taking the average velocity over a small interval to estimate the instant velocity. Susanna used secants approaching the tangent in Task 3 to estimate the derivative. She did this graphically by adjusting secant lines to the graph and concluded that this would not be a more accurate estimation than the slope of a tangent line. Thus, she did not notice that the slope of the secant could have been determined exactly using the expression of the function. Tommi used the idea of the average rate of change over a diminishing interval in Task 3 and 4. However, in Task 3 he was not able to evaluate the derivative by this method. Also in Task 4 he had constructed an incorrect correspondence between the given quotient and the average rate of change. Daniel used secants approaching the tangent in Task 4a to interpret what the quotient means:

> Let h be also one. [...] Then here h would be the distance *(points to x-axis at [1, 2])*. [Pause.] From that value *(points to the graph at 2)* we subtract that value *(points to the graph at 1),* so it would be this interval, difference of these values *(points to y-axis at [1.2, 3.2])*. So that divided by the lower part *(points to x-axis at [1, 2])*. How does this go? I assumed that this would, of course, be connected to this kind of line *(sketches the corresponding secant in the air).* Oh yeah, is this then? What's the difference quotient? This could be quite close to the difference quotient *(sketches the secant).* This defines also the tangent. I'm not quite sure, don't remember if the formula of the difference quotient was just like this. If it was it, then it would be the slope of that line *(sketches to the secant)*. Yes, it comes from here, too. This distance *(points to the y-axis at [1.2, 3.2])* divided by this *(points to x-axis at [1, 2]). [...] (Draws the secant.)* So it would be the slope of that line, that's like, how to say it, average derivative at that interval.

In Task 4b he interpreted what the limit of the quotient means:

> What does the limit mean? When h tends to zero. Obviously the derivative at the point one *(points to the x-axis at 1)* is wanted here. Because h is this distance and if h tends to zero, so h would be zero here *(points to the graph at 1).* So it would be to this point, you would get the tangent here *(draws the tangent)*, which slope would come out from that formula *(points to the formula in the task).* So you would get the derivative at that point.

Samuel used difference quotients over diminishing intervals in Task 3 to estimate the value of the derivative and explained these with corresponding secants converting to the tangent (Fig. 7). He used these limiting representations also in Task 4 to interpret what the quotient and its limit mean. In terms of the APOS theory, only Daniel and Samuel seemed to have interiorized the limiting as a process. The other students seemed to be at the action level.

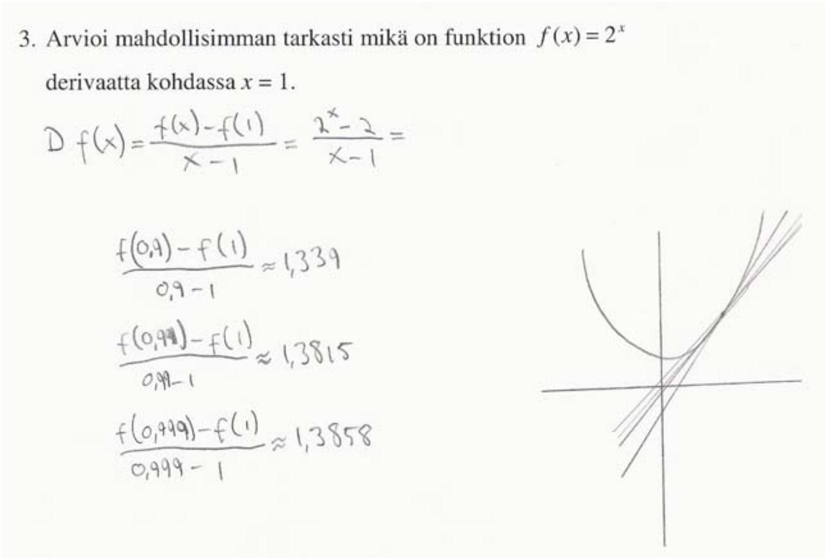


Figure 7. *Samuel's solution to Task 3.*

The students used various limiting processes but they also connected these in different ways to the limit of the difference quotient. Niina used the limiting process for thinking about the derivative but did this separately of the limit of the difference quotient. Tommi and Susanna changed between using the limit of the difference quotient and a limiting process. This connection was named an associative connection (the article [1]). Daniel and Samuel even used the limiting process and the limit of the difference quotient to explain each other. This connection was named a reflective connection (the article [1]). The associative and the reflective connections were taken as means to evaluate the conceptual knowledge of the limit of the difference quotient used by the students. Daniel and Samuel demonstrated good conceptual knowledge by making reflective connections. Instead, Tommi and Susanna did not demonstrate so good conceptual knowledge as they made associative connections. Also Niina was considered to demonstrate some conceptual knowledge as she used limiting processes although she did not make a connection from these to the limit of the difference quotient.

8.2 Perceiving the derivative

In this section results concerning the students' perceptual activity in the embodied world are presented. Corresponding to the task design of the interviews, the most important tasks for this section are Tasks 2, 3 and 5. First, I summarize the students' main findings in these tasks. Then, as a more important part of the study, I analyze how they used different representations in their reasoning. A more detailed presentation of the results of this section can be found in the articles [3]–[5].

The students' achievements in the embodied world

In Task 2 all the students perceived the intervals when the derivative is positive, negative and constant. They also perceived the zero points and the maximum point of the derivative. They all, except Susanna, argued that the derivative does not exist at the point 2. Also Susanna considered the point as problematic but continued to estimate the derivative with inappropriate methods. Susanna and Niina determined the minimum point of the derivative incorrectly. Instead, Daniel, Tommi and Samuel determined also the minimum point of the derivative correctly. Daniel even observed where the derivative increases and where it decreases. Susanna, on the contrary, confused the increase of the derivative to the increase of the function.

In Task 5 they all, except Susanna, drew appropriate time–distance-graphs of the car from the time–velocity-graph (Fig. 8). Samuel, Tommi and Daniel also drew appropriate time–acceleration-graphs (Fig. 9).

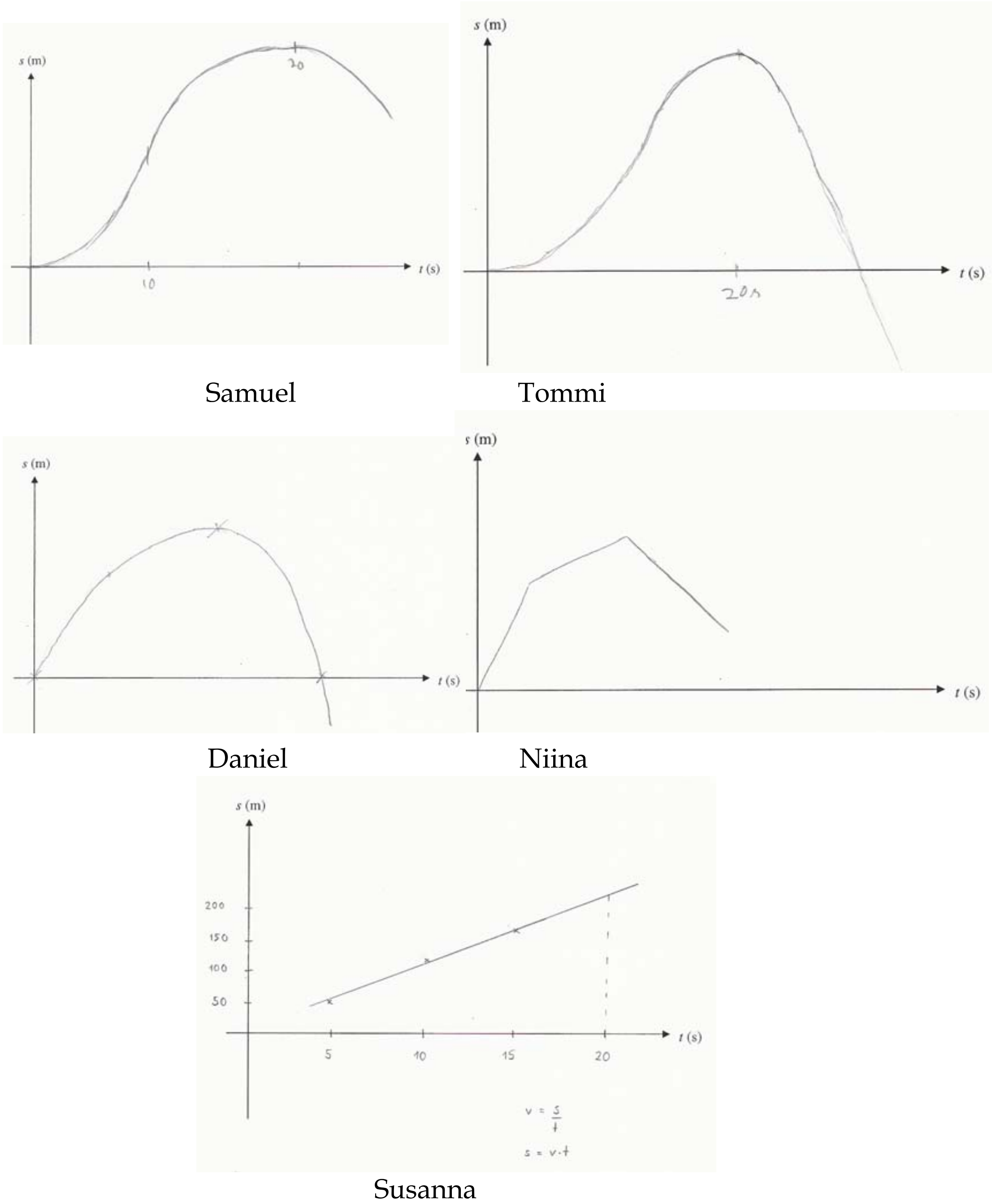
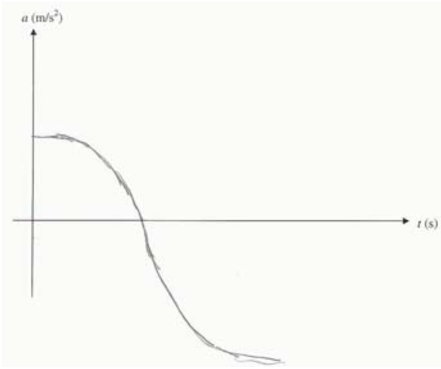
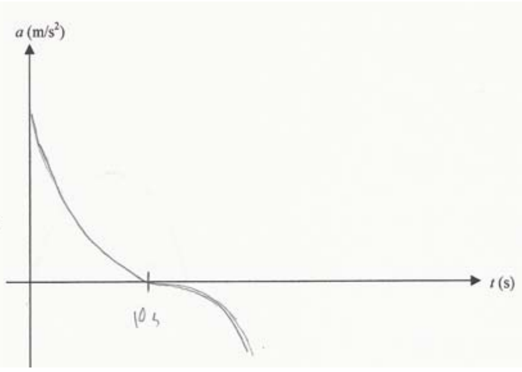


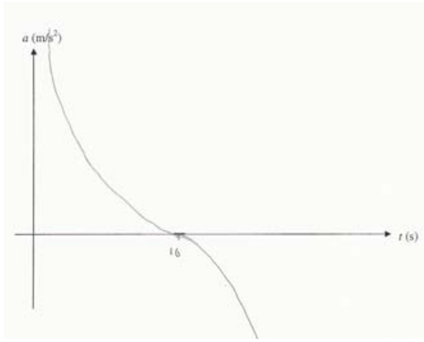
Figure 8. *The distance-time graphs produced by the students' in Task 5c.*



Samuel



Tommi



Daniel

Figure 9. *The acceleration-time graphs produced by the students' in Task 5f.*

Also in Task 3 the students used some estimation methods that included perceiving the derivative from the graph. Samuel's solution is already presented in Figure 7. He was the only student who sketched the graph only very roughly. The other students calculated some points of the graph and fitted the curve to these. Daniel did not draw anything on the paper but obviously sketched the graph mentally. After sketching the graph, Susanna and Tommi drew a tangent to the point in question and calculated its slope. Also Niina mentioned that the derivative could be estimated by calculating the slope of the tangent as she did in Task 1. However, she did not do that because she was uncertain. Daniel estimated the value of the derivative by imagining the shape of the graph, calculating the derivative over an interval and by imagining the tendency of decrease of the graph.

The representations used by the students

For the mentioned observations the students used representations of the increase, steepness, horizontalness, and tangent of the graph of a function. They also used some related gestures.

All the five students used the increase of the graph as a representation of the derivative. Using this representation means that a student uses increasing/decreasing or rising/going down of the graph of a function for thinking about the derivative. The increase seemed to be a tool to perceive how a function changes at some interval. For example, a sign of the derivative could be observed using the increase representation. Daniel, Tommi, Susanna and Niina used the increase to perceive the sign of the derivative from the graph of the function correctly in Task 2. In Task 5 all the students, including Samuel, examined the increase of the velocity to investigate the distance or used to the velocity to make inferences about increase of the distance. At many points the increase representation was accompanied by a gesture of tracing the graph or moving a hand in the air up or down. A thorough discussion on these gestures can be found in the article [3]. Susanna and Daniel used increase also in Task 3. Susanna used it to convince herself that she had calculated the derivative incorrectly. Daniel used the derivative over an interval (average rate of change), the shape of the imagined graph, and the tendency of decrease for his estimation:

> Because it is a kind of half of a parabola, if I imagined it correctly. The value at the point 1 was 2 and at the point 2 it would be 4, then at that interval the derivative would be something, it would be 2 at that interval. Yes, it would be like a half of a parabola. If we then assume that it would continue to decrease in the same way also after the 2 power 1, then it would be, it would be about slightly above 1.

The use of the steepness representation means that a student refers to the steepness of the graph of a function when considering the derivative of the function. All the students, except Tommi, were noticed to use the steepness of the graph for thinking about the magnitude of the change of the function. The students seemed to use steepness together with increase. This may be natural as increase seemed to represent how the values change and steepness represented the magnitude of the change. As increase is a property of an interval, steepness is a pointwise characteristic. Thus, it is a good tool, for example, to perceive the maximum and minimum points of the derivative. For example, Susanna used increase and steepness in Task 2:

Interviewer: When would the derivative be positive in general at the whole graph and when negative?

Susanna: It would be positive approximately from here to somewhere there *(points to the graph at -2.6 and -1.5)*, when the graph rises upward *(moves the pencil upward)*. And then from somewhere here to there *(points to the graph at 0.8 and 2)*. [...] Negative from somewhere here to about there *(points to the graph at -1.5 and 0.8)*. And from that on *(points to the x-axis from 2 to 4)*. [...]

Interviewer: Could it be said when the derivative is at its greatest and when at its smallest?

Susanna: At its greatest it is when the graph rises most steepest upward *(moves the pencil upward)*. It could be *(puts the pencil as a tangent to the graph at points -3 and 1.9, see Fig. 10)*. Hmm. Somewhere there *(points to the graph at [-3, -2.6])*. Or here *(points to the graph at 1.9)*.

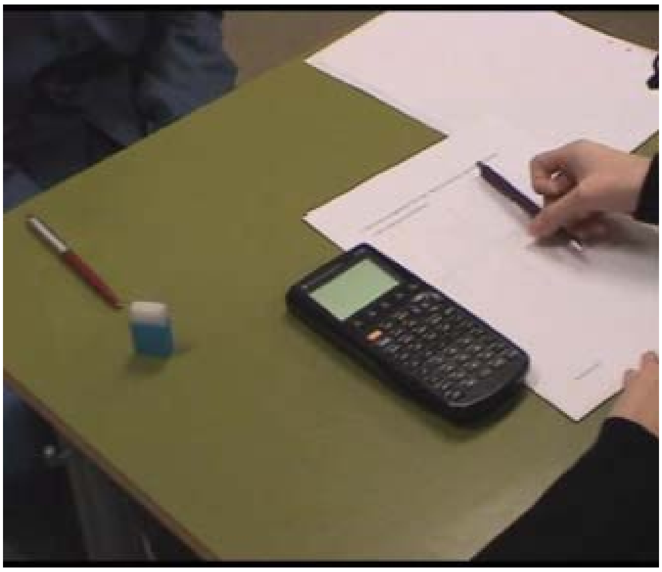


Figure 10. *Susanna places the pencil as a tangent to the graph at the point x = 1.9 in Task 2.*

All the students used also the horizontalness of a graph for thinking about the derivative. This means that a student refers to a graph being horizontal, uniform or something similar. Horizontalness seemed to be closely connected to the extreme values of a function and to the zero points of the derivative. As increase and steepness, also horizontalness was related to a change of the values of a function. It seemed to be an especially good tool for perceiving when the values are not changing. Moreover, the students seemed to regard it as a pointwise property. All the students, except Tommi, made also a gesture of drawing a horizontal line in the air when referring to the horizontalness. For example, Niina imagined magnifying the graph in Task 2 to explain why the derivative is zero at the point 0.9 but not at the point 2:

> If you would zoom in on here *(points to the graph at x = 0.9),* for example, it would be straight for a while *(draws a line with a finger)*, but not there *(points to the graph at x = 2).*

The students used the tangent in two ways. They calculated the slope of a drawn tangent and they perceived the tangent as such. Samuel is a good example, how these are really two different ways to use the tangent. In Task 2 Samuel seemed to see the magnitude of the derivative directly from the position of the tangent:

Samuel: Because the tangent is. Here it goes like this *(moves a ruler as a tangent along the graph from -3 to -1.5),* here it is still positive *(holds the ruler as a tangent at -2 and pretends to draw a line)*, but when it is here, it is horizontal *(holds the ruler as a tangent at -1.5 and pretends to draw a line)*. [...] And then it is zero. [...] From two to four the derivative is constant.

Interviewer: On what grounds would it be constant? Or why?

Samuel: If this would be. If you took a tangent, for instance, there *(points to the graph at a point 2.4 with a pencil and puts the ruler as a tangent there)* [...] and there *(holds the ruler in the same position, but moves the pencil to a point 3.6)*, so it would be still the same.

Also in Task 3 and Task 4 Samuel used the tangent to represent the "correct derivative" as secants were approaching it. Contrary to this efficient tool to think with, Samuel did not use the algorithm $\Delta y/\Delta x$ to calculate the slope of the tangent at the interview nor at the pre-test. For the other students the slope of a tangent offered an easy way to estimate the value of the derivative. Tommi even noticed in Task 5d that the value he calculated for the slope did not correspond to the drawn tangent. The tangent as such could be used in the same manner as steepness to perceive the magnitude of the derivative without calculation. Another similarity is the use of gestures. Both of these representations were accompanied by a similar gesture: placing a pencil or a ruler tangent-like to a graph (Fig. 10).

8.3 Hypothetical learning path to the derivative

On the basis of the analysis of the students' use of different kinds of representations, it was considered how these representations could be used in the learning of the derivative. These considerations and the resulting hypothetical learning path to the derivative are presented in the article [5]. In this section the hypothetical learning path is reviewed shortly. Figure 11 represents the overview of the hypothetical learning path.

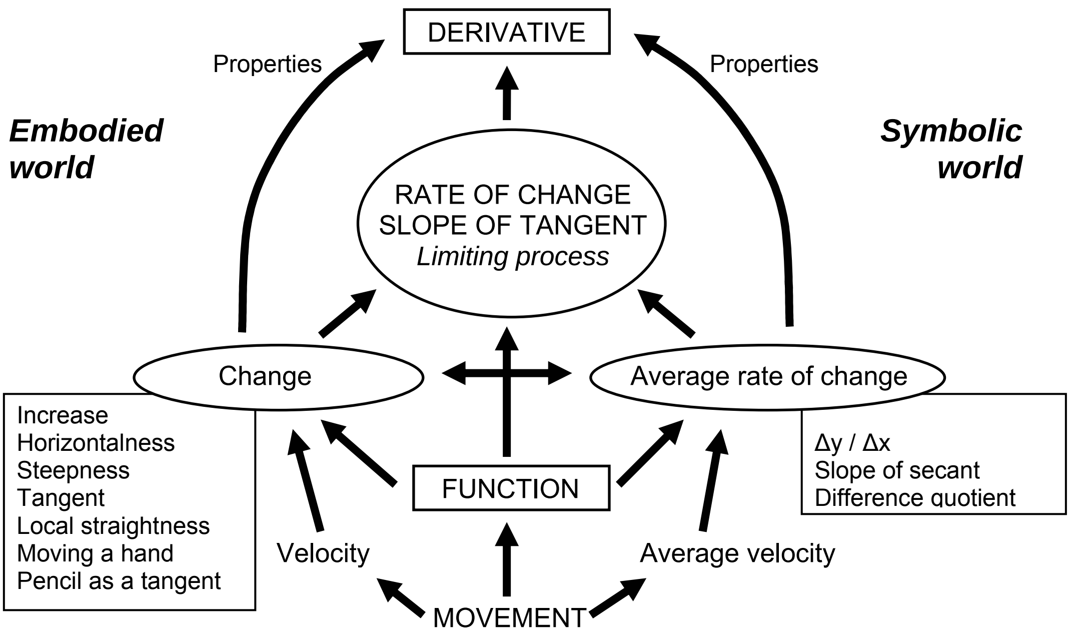


Figure 11. *The learning path to the derivative (the article [5]).*

The learning of the derivative could start from examining the motion, especially the average and the instant velocity. We can assume that students are usually able to consider some aspects of the instant velocity qualitatively. For example, they can notice when the velocity of a moving object is great and when small. At first, the students could work in the embodied world to learn to consider qualitatively the rate of change of a function from the graph of the function. Distance-time functions may help them to activate their past experiences. From the velocity as a special case of the rate of change, they could proceed to examine how the values of a function are changing. The representations discussed in the preceding section may help them to perceive the velocity or the rate of change qualitatively. The students could also calculate the average rate of change by different representations, for example, as the slope of the secant, by a change in values divided by a change in parameter, or by the difference quotient. At this point the students have worked in the embodied and in the symbolic worlds and become familiar with the instant rate of change. They have hopefully constructed many representations. Combining these worlds, students may begin to examine how to estimate the value of the instant rate of change. Solving this problem would give reason to define a new concept, the derivative.

In this way, the students have already learnt important properties of the derivative at the very beginning of the learning process of the derivative. The most important applications of the derivative in high school are examining the increase and the extreme values of a function and determining the slope of the tangent and the rate of change. The interviewed students had already made a connection between these properties of a function and its derivative. So the students following this path would have a good readiness for investigating related topics and for deepening their understanding.

9 CONCLUSIONS AND DISCUSSION

In this section the main conclusions of the study are presented. Conclusions are accompanied by discussions of them and related results.

Conclusion 1: Embodied world offered powerful thinking tools for the students

The above-discussed increase, steepness, horizontalness and tangent representations seemed to be those tools that the students used for thinking about the derivative qualitatively without calculating anything. Students used these tools literally for thinking and not just for carrying out algorithms. For example, the article [3] presents how Susanna used the differentiation rule incorrectly to solve Task 3 but noticed the mistake by considering the horizontalness and increase of the graph. Similarly, in Task 5b she perceived how the distance is changing but in Task 5c abandoned these perceptions and produced a contradictory graph using a symbolic formula. In the embodied world, Susanna, as well as the other students, had good readiness for reasoning. This is clearly illustrated by their observations of the derivative in Task 2 and their reasoning about the distance and acceleration in Task 5. Daniel made even observations of the change of the derivative in Task 2 (see the article [4]). Therefore, this study supports Tall's (2003, 2004a, 2004b, 2005) claim that working in the embodied world should not be underestimated even in the case of the concepts of advanced mathematics. Also many other studies support this claim in the case of the derivative concept (Berry & Nyman, 2003; Speiser & al., 2003; Repo, 1996; Heid, 1988). Some studies have shown how students may construct ideas related to calculus even before high school (Schorr, 2003; Wright, 2001; Radford & al., 2003).

The representation concept had to be re-characterized to better fit the ideas of constructivism and distributed cognition. The small adjustments of the concept seem to be simple and even trivial after the re-characterization was done. Still, that was an important step (and in no way simple) for the study and helped to characterize students' activity better. Particularly, the re-characterized representation concept fits better to the consideration of the embodied world. Especially, in the embodied world, it became clear that a representation has usually something visible and something invisible. Gestures were noticed to be an important visible part of the students' thinking. They were not just external representations of the students' ideas but they were an integral part of the idea. An extensive discussion on the role of gestures in reasoning is presented in the articles [3] and [4]. This study supports arguments that gestures are important for thinking and part of expressing, communicating and reorganizing one's thinking (McNeill, 1992; Radford & al., 2003; Rasmussen & al., 2004; Roth & Welzel, 2001; Moschkovich, 1996). For an advanced learner in mathematics, these gestures may seem meaningless or useless. I emphasize that they may *seem* useless, but in grasping a novel concept they may have a great role also to a mathematician. For example, Rasmussen et al. (2004) reported on the role of gestures in learning differential equations. For the students of this study the gestures were of great help and helped to focus attention on particular aspects, such as increase and steepness. Actually, as said before, the gestures did not just help but they were an integral part of the students' thinking. As in the study of Roth and Welzel (2001), the gestures seemed to help to make an abstract concept visible and concrete.

Of course, there were also other external sides than the gestures. For example, the students' utterances and the inscriptions were external sides of the representations. It is noteworthy that all the inferences of the students' representations were based on the visible external sides. For the purpose of any study, representations which do not have any external side are useless because we cannot get any information about them. When an indication of some representation is detected, there has to be also something which is not visible. These internal sides of representations are important for the use of external sides. As Meira (1998) has pointed out, the expert-designed powerful external representations are not necessarily powerful for a student. For example, the article [3] reports how Susanna had difficulties in perceiving the minimum point of the derivative although she used a "good" external side of a representation. To use a representation effectively, the external side of the representation has to be coordinated with an appropriate internal side. The use of a representation is neither internal nor external but more like an interplay between these two sides.

Conclusion 2: The students considered the derivative as an object at the early stage of the learning process

It seems that the students considered the derivative as an object which has some properties, such as sign and magnitude. This happened at the very early stage of their learning of the derivative. Moreover, this was the case with all the students, with the successful and the less successful ones. This result suggests that theories like the APOS theory (Asiala & al., 1997) and the reification theory (Sfard, 1991), which propose that learning proceeds from process conception to object conception, do not take into account this very meaningful alternative. Because the students seemed to consider the derivative as an object especially in the embodied world, this result fits to the claim of Gray and Tall (2001) and to Tall's (2003, 2004a, 2004b, 2005) evolving theory of three worlds of mathematics. Gray and Tall (2001) have claimed that students do not necessarily learn mathematics by performing actions on some object and then constructing a new object from these actions. According to Gray and Tall, students may also learn by acting with the object and learn its properties. This seemed to happen to the students of this study. They acted with the derivative in the embodied world and learnt it properties. Therefore, this object is different from the object which is encapsulated or reified.

It should be noted that in this study, the derivative as an object does not refer to the derivative as a function as it is sometimes assumed. At this stage the students considered the derivative at a point. However, they considered the derivative at many points, and Daniel even observed how the derivative varied in coordination with the variable. Therefore, they also started to move towards conceiving the derivative as a function.

According to Sfard (1991, 1992) and Sfard and Linchevsky (1994), one exception to the reification theory is pseudo-structural conception. This means that a concept is treated like an object, but this object does not have an internal structure, that is, it is not reified from a process. It may be true that the students of this study did not learn the derivative in the embodied world through reification. However, it is not true that the derivative did not have an internal structure. The structure is just not being built in the form of processes but as acting with the derivative. Moreover, if the derivative were acquired only through reification, there would be a danger that students would not succeed in reification and would learn the derivative only as a pseudo-structural object which is calculated according to the differentiation rules. If students have an opportunity to learn the derivative also in the embodied world, the derivative will have some internal structure even if reification fails. Even if the students accept the differentiation rules as such, they can use these rules and their knowledge in the embodied world, for example, to find out the extreme values of a function (without being given an algorithm for this).

Conclusion 3: Learning in the embodied world may be described by the increasing transparency of representations

In the article [3] it was found that Susanna perceived some aspects of the derivative but seemed to focus on the graph as a physical object. For example, she noticed such things as the graph going upward and the steepness of the graph. She also used physical objects (e.g., a pencil) to see these aspects and had problems with the minimum point of the derivative in Task 2. Susanna did not seem to use her representations of increase, steepness and horizontalness very transparently, because she focused more on these tools than on the derivative which can be seen through them. These representations seemed to be aspects that Susanna could recognize in the graph. According to Noble et al. (2004), the experiences of recognizing something familiar in the picture may cause a person to see the picture in a new way. Susanna may still be on her way toward seeing the graph of a function as a representation of the derivative.

In the article [4] it was found that Daniel seemed to see the derivative transparently through the representations. He even perceived how the rate of change of the derivative was changing. This required perceiving aspects that expect a more disciplined way of seeing because in the teaching-learning sequence, no methods (e.g., curvature) were discussed to see when the derivative is increasing and when decreasing. Daniel also perceived estimation for the value of the derivative in Task 3 without using any physical materials.

Therefore, the students seemed to be at different levels concerning the transparency of the representations of the embodied world. The transparency concept seems to offer meaningful conceptualization for learning in the embodied world. This was one of the theoretical problems faced in the study. Learning in the symbolic world is described clearly and straightforwardly in Tall's (2003, 2004a, 2004b, 2005) theory in the same way as in the APOS theory. However, the same kind of development is not possible in the embodied world, and it was not very clear how to describe the differences among the students. Tall (2005), proposes that students may learn in the embodied world by shifting their focus from actions to the effects of those actions. This bears some resemblance to the transparency concept as discussed in section 5.5. One of the achievements of this study was finding a way to describe learning in the embodied world using the transparency concept.

Conclusion 4: The students used various representations of the limiting process and made associative and reflective connections from these to the limit of the difference quotient

The students used various different limiting processes when considering the derivative and connected them in different ways to the limit of the difference quotient. In the article [1] the following four options for the students' conceptual knowledge of the limit of the difference quotient were presented:

1. A student does not use any idea where limiting is explicitly involved when considering the derivative.
2. A student uses an idea of limiting when considering the derivative.
3. A student uses an idea of limiting and associates it with the limit of the difference quotient.
4. A student uses an idea of limiting together with the limit of the difference quotient and uses one to explain the other.

The options 3 and 4 correspond to associative and reflective connections. Susanna and Tommi demonstrated option 3. It seems that such a connection means that students know that two representations represent or can be used for thinking about the same thing. This kind of connection does not mean that they necessarily understand why it is so. Instead, the reflective connection demonstrated by Daniel and Samuel means that the students also, at some level, understand why the representations can be used for thinking about the same thing.

These connections were noticed during the data analysis. They enabled analyzing students' dynamic solution drives from their point of view instead of testing whether they were able to make some specific connection. This focus indicates a shift from viewing connections as static structures toward viewing them as dynamic constructs in students' drive along knowledge networks. This corresponds to the dynamic feature of conceptual knowledge that Haapasalo and Kadijevich (2000) highlight in their definition of conceptual knowledge. Afterwards, the characterizations of associative and reflective connections were compared to those found in the literature (see section 5.3).

It is well documented that students have difficulties to understand the limiting process inherent in the derivative (Orton, 1983; Heid, 1988; Tall, 1992, 1991; Tall & Vinner, 1981; Repo, 1996; Zandieh, 2000). In this study, it was found that the students used good ideas of limiting processes. The difficulty seemed to be in the structure of the representations of limiting and in their connections to formal mathematics. There should also be some other limiting process than the notation $\lim_{x \to a}$. Along the lines of Cottrill et al. (1996), the limiting process should be a coordination of two processes. For example, in Task 3 Samuel coordinated "x approaching 1" and "difference quotients over corresponding intervals approaching the derivative". Using the limit of the difference quotient does not necessarily include limiting, however skilfully it is done. If other limiting processes are not discussed in the instruction, the representations of the embodied world may stay detached from the symbolic world. A good example of how even at the university level a student may use the detached embodied world misleadingly is presented in Viholainen (2006). According to his analysis an almost qualified mathematics teacher used the tangent inappropriately to conclude that the discontinuous function is differentiable. Another university student came to the same conclusion by using the limit of the difference quotient inadequately (ibid.). Therefore, it is important to emphasize the various limiting processes in teaching from the beginning, as limiting processes may serve as a bridge between the embodied and the symbolic world.

Conclusion 5: It is possible to use conceptual knowledge of the limit of the difference quotient without being able to carry out the procedure

One interesting result earns its place for a separate discussion. This issue concerns the students' procedural and conceptual knowledge of the limit of the difference quotient. The limit of the difference quotient is the most important symbolic representation at this stage. Therefore, the students' procedural and conceptual knowledge of this notion was analyzed thoroughly. It was found that Susanna, Niina and Daniel had difficulties in using procedural knowledge, while Tommi and Samuel used it fluently. On the other hand, Daniel and Samuel used conceptual knowledge efficiently. The other students did not demonstrate so efficient conceptual knowledge as they did.

The interesting result is that Daniel used conceptual knowledge efficiently but had difficulties with the procedural knowledge. Tommi had it the other way around. Also the studies of Zandieh (2000), Repo (1996) and Orton (1983) have shown that students may have procedural fluency of using the limit of the difference quotient but still have difficulties with the conceptual knowledge of the limiting process. This is similar to Tommi's case. However, Daniel's case shows that even at the beginning of the learning of the derivative, it is possible to use the conceptual knowledge of this notion without being able to use the algorithm. Thus, if a student is not able to use some notion, it does not mean that he would not have knowledge of the underlying structure of the notion.

Another similar case is that of Samuel. He used both the procedural and conceptual knowledge of the limit of the difference quotient efficiently. However, he was not able to use the procedural knowledge of the slope of the tangent despite the fact that he used the tangent representation as part of his conceptual knowledge. This is another example of how a student uses conceptual knowledge competently but something prevents him from using procedural knowledge of the same notion.

Haapasalo and Kadijevich (2000) and Haapasalo (2003) have discussed whether it is necessary to have conceptual knowledge in order to have procedural knowledge or vice versa. The results of this study suggest that neither is necessary. However, for learning to be meaningful, students should at some point attain both kind of knowledge and build connections between them, as Haapasalo (2003) has stated. Both Daniel and Samuel had difficulties in their problem solving as regards procedural knowledge. Still both of them had something that they could build their knowledge on.

Conclusion 6: The hypothetical learning path to the derivative

As one result of this study a hypothetical learning path to the derivative was constructed. The validity of the learning path is based on the detailed analysis of how the five students used representations at the interviews. These representations could be used similarly by other students in other classrooms. Tests were not implemented to claim that teaching had been efficient. The studies of Repo (1996), Asiala et al. (1997) and Heid (1988) have already shown that it is possible to achieve good learning outcomes. However, from these studies it is difficult to say which factor of experimental teaching was beneficial for students and why it was advantageous. This study answers these questions by focusing on a small number of students. It was found that the students' uses of representations had certain characteristics and they used these representations for specific purposes. Being aware of these issues is important, as according to Davis and Maher (1997), a teacher has to recognize the representations that students are using and to design them opportunities to further develop these representations. In constructing the hypothetical learning path I have considered what opportunities are opened by certain elements of the students' conceptions which is one principle in educational reconstruction (Kattmann & al., 1998). I believe that some other teacher could teach his/her class efficiently, taking into account the representations that are proposed by the hypothetical learning path. If this happens, then the practical aims of the study are fulfilled.

The hypothetical learning path is in line with studies which have argued that the APOS theory suits well for designing the teaching of the derivative (Asiala & al., 1997; Repo, 1996). However, this study also suggests that acting with an object and making perceptions are a powerful stage in the learning of the derivative. Thus, the study supports Tall's (2003, 2004a, 2004b, 2005) evolving theory of the three worlds of mathematics. Particularly, it localizes Tall's theory to the case of the derivative concept. In the symbolic world students learn the derivative as they interiorize actions of calculating average rates of change to a limiting process where the base interval of the average rate of change decreases. This process may be encapsulated to a procept which means that a student is able to think about the limiting process also as a resulting rate of change. At the same time students may perceive the derivative as an object in the embodied world. In this world students learn as the representations become transparent allowing seeing the derivative through the representations.

10 REFLECTIONS ON THE QUALITY OF THE STUDY

In this section I reflect on the quality of the study. I will use the criteria provided by Sierpinska (1993): relevance, validity, objectivity, originality, rigor and precision, predictability, reproducibility, and relatedness. Also Kilpatrick's (1993) discussion on these and Lincoln and Guba's (1985) discussion on similar criteria is utilized. I supplement these criteria also with the criteria provided by Clement (2000).

Relevance

Sierpinska (1993) differentiates between the theoretical and pragmatic relevance of a research. According to her, an educational research is theoretically relevant if it "broadens and deepens our understanding of the teaching and learning phenomena" (p. 38). A study is pragmatically relevant if it "has some positive impact on the practice of teaching" (p. 38). In the following, I cannot reflect on whether this study has actually caused such things to happen but I will reflect on whether it has potential to do so.

This study is theoretically relevant as it contributes to the theoretical framework of how learners build mathematical concepts. In particular, it deepens our understanding of how students may use different representations for thinking about the derivative. Particular indications of the theoretical relevance of the study are the re-characterization of the representation concept, the characterizations of the associative and reflective connections, support to the evolving theory of the three worlds of mathematics, and suggesting that learning in the embodied world may be described using the transparency concept.

The pragmatic relevance of the study resides on the hypothetical learning path to the derivative. I hope that some teachers could in their teaching take into account these ways how students may use different representations. Particularly, I hope that teachers will notice that students may build knowledge of the derivative before the definition of the derivative is introduced. The study aims at encouraging teachers to emphasize perceptual activity in their teaching.

Validity

According to Kilpatrick (1993), validity refers to how we justify the interpretations we make from the research. The criterion has a different meaning in the qualitative research paradigm (Ernest, 1997) from studies grounded in the positivistic tradition. Some authors have suggested replacing the term with viability (Clement, 2000) or with credibility and transferability (Lincoln & Guba, 1985). Synthesizing these views, considering validity means examining how plausible the claims are, what empirical support there exists for the claims, and how these claims are extended beyond the situation that was studied.

Before beginning to consider these aspects of validity, it has to be noted that there are conclusions (see section 9). As Sierpinska (1993) indicated, we have to take a critical standpoint towards validity of (case) studies that describe in detail students' mathematical behaviour but do not actually make any conclusions. This is not the case in this study, as I have made interpretations which are stronger than the observations of students' behaviour.

The claims made in this study are plausible because they have explanatory adequacy and internal coherence (Clement, 2000). This means that the interpretations that I have made give a description of students' thought processes that can be imagined to take place. The interpretations are also internally coherent in that, for example, there are no contradictory claims.

In addition to the plausibility of claims, there also has to be empirical support for them. In this study, the interpretations of students' use of representations are based on several observations of their behaviour at the interviews. It was also looked after that there were no deviations of the observed behaviour. According to Clement (2000), these are factors that have to be taken into account when considering the empirical support for the claims. The third factor, according to Clement, is evaluating the strength of the connection from interpretation to each observation. The reader can evaluate this on the basis of the excerpts provided from the interviews. In addition to presenting excerpts, I have also explained how the data support the interpretation.

Evaluating (external) validity includes also considering the generalizability of the findings. Also this concept depends on the paradigm. In this study, generalizability means that the students' ways of using the representations serve as an illustrative example of how representations could be used for thinking about the derivative. Depending on the context, also other students could use the representations similarly. As the analysis of the five students' uses of representations is described in very detail, the readers may adapt these results into their context.

Objectivity

The paradigm of this study rejects the possibility of objective knowledge. However, this does not mean that every interpretation is acceptable. As Sierpinska (1993) states, a researcher has to make effort towards "objectivity". This does not mean total objectivity but the aim is to rule out obvious biases. According to Lincoln and Guba (1985), the objectivity of knowledge means its conformability. In this study an effort towards objectivity is made by presenting a detailed account of the theoretical framework, aims of the study and methodology. Thus, I have made explicit (for myself and for the reader) the background from which I made interpretations. Without this, implicit assumptions (of which neither the researcher nor the reader is aware) would had guided interpretations. Being aware of one's theories is a necessary condition for being independent enough of these theories. Clement (2000) emphasizes this kind of "objectivity". This is also related to the openness of the research. One factor of openness that increases the objectivity of this research is that the study was presented in different phases instead of presenting only the final product. For example, the first research plans (which were afterwards modified) were publicly presented for criticism and suggestions at national seminars. Also the (preliminary) results of the study were reported already during the first years of the study (e.g., Hähkiöniemi, 2004).

Originality

The originality of the study comes from a fresh question formulation in a well studied topic of learning and understanding the derivative. I did not try to confirm the results provided by the previous studies, for example, of Heid (1988), Repo (1996) and Asiala et al. (1997). Instead, I utilized the method of the task-based interview to gain qualitative information on a few students' reasoning. Although the method of a task-based interview itself is not very original, I designed the interviews as well as the data analysis procedure specifically for this study. Instead of strictly following some well known analysis procedures, I created my own procedure by taking some aspects from the procedures presented in the literature. Also the theoretical framework is original as many theoretical perspectives are synthesized and reflected, and Tall's (2004a) evolving theory of the three worlds of mathematics is a new theory which is constantly developing.

Rigor and precision

According to Kilpatrick (1993), this criterion concerns the care taken in observations, the attention to detail and the willingness to test alternatives. According to him, the researcher has to show sensitivity to the meanings of the phenomenon, and he/she has to design and conduct the study carefully by anticipating possible misinterpretations. To fulfil this criterion, the data collection and analysis were carefully designed as described in section 7. In the presentation of the results, the interpretations are tried to express clearly and differentiate them from observations. Likewise, the theoretical terms used in the study are defined. The terms are not used to make simple things complex but to construct explanations of complex phenomena.

Predictability

According to the paradigm of this study, we cannot predict anything to happen under given conditions. Instead, this criterion is replaced by examining the generalizability of the findings of this study (see validity). The study aims to offer some possibilities that could happen but not to predict anything about what will happen under certain conditions.

Reproducibility

In this study the research setting and procedures are described to allow other researchers to conduct a similar study. According to the paradigm of this study, it does not make sense to try to achieve the same results by following the presented procedures. Instead, reproducibility means that the results of the study can be compared, clarified, extended and challenged by other studies that are conducted in a similar setting according to similar procedures.

Relatedness

This criterion is especially important as this thesis is submitted to the Department of Mathematics and Statistics. According to Kilpatrick (1993) and Sierpinska (1993), a research in mathematics education should be related both to mathematics and to education. For example, proving a theorem is not research in mathematics education. In my opinion, neither are designs for mathematics courses which are based solely on the designer's rational thinking of the mathematical content. Instead, a research in mathematics education should study the phenomenon of learning and/or teaching. However, studying only learning and/or teaching is not enough. Learning and/or teaching has to be especially related to mathematics. This means that the teaching/learning phenomenon could not have been researched, for example, in the context of history education. Mathematics should not act only as a placeholder (Kilpatrick, 1993).

Obviously, this study is related to education, as students' use of different representations is studied using methods of educational research. From the students' use of representations, I do not only analyze, for example, whether the representations are pictorial, verbal, or symbolic. Instead, the analysis is closely related to mathematics. For example, I analyzed for which mathematical purpose they used the representations. Also the conclusions concerning, for example, process-object dualism, are related to a special kind of nature of mathematical thinking. Conducting such a study as this would not be possible without having considerable understanding of mathematics and practice in doing mathematics. Good practical criteria, which are filled by this study, for relatedness are that the study should not be conductible by an educational researcher (without gaining considerable expertise in mathematics) nor by a mathematician (without gaining considerable expertise in education).

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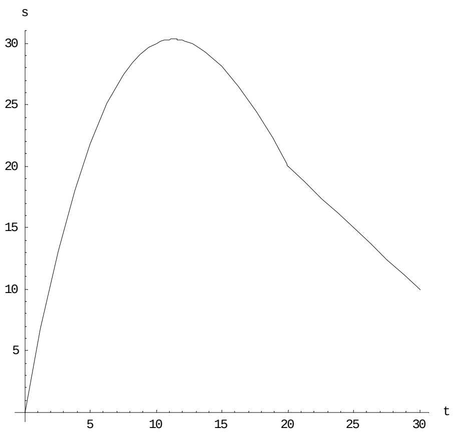
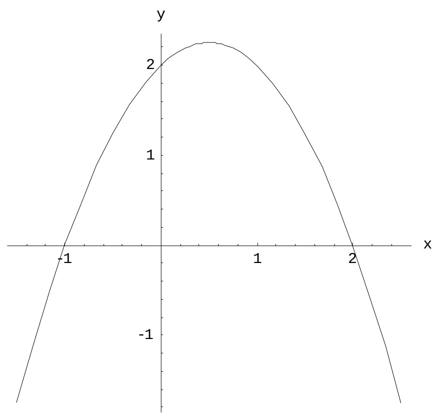
APPENDIX 1: Pre-test

Differential calculus 1: Pre-test Name:
The mark of the previous mathematics course:

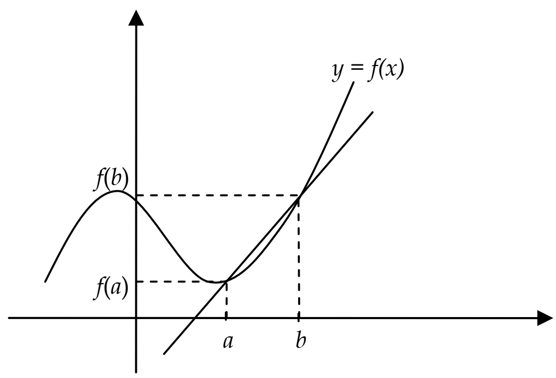
1. The figure represents the distance *s* (m) of a car from the starting point as a function of time *t* (s).
   a) What is the distance of the car from the starting point after 10 seconds from the start?
   b) When is the car furthest away from the starting point?
   c) What is the average velocity of the car at the time interval 5 s ... 10 s?
   d) At what time interval is the velocity of the car positive and at what interval negative? When is the velocity of the car zero?
   e) At what time interval is the velocity of the car constant?
   f) Determine the constant velocity in the subquestion e.
   g) What is the velocity of the car at the moment $t = 8$? Give a short argument.



2. The figure represents the graph of the function $f(x) = -x^2 + x + 2$.
   a) Determine $f(-0.5)$.
   b) What is the largest value that the function *f* takes?
   c) When are the values of the function *f* negative?
   d) Draw a tangent to the function *f* at the point $x = 1$. Determine the slope of the tangent.
   e) Draw a tangent to the function *f* that has the slope zero.
   f) How much do the values of the function *f* change from the point $x = -0.5$ to the point $x = 1$?



3. a) Interpret from the figure what the quotient $\frac{f(b) - f(a)}{b - a}$ means.
   b) Interpret from the figure what happens to the quotient $\frac{f(b) - f(a)}{b - a}$ when the point *b* is displaced closer to the point *a*.



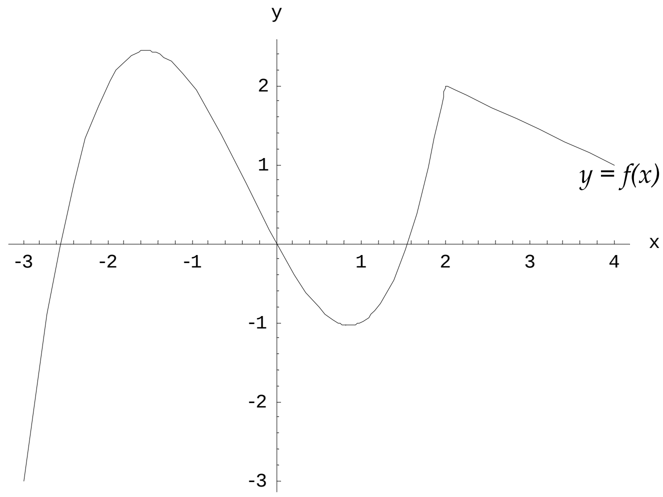
APPENDIX 2: Interview tasks

The most important possibilities for follow-up questions are listed after each question.

1. a) Explain in your own words what the derivative is.
    - What does the derivative mean in practice?
    - What is the derivative good for? For what can it be used?
    - How can the derivative be determined?

   b) The derivative of the function *f* at the point x = -5 is 3. What does this mean?

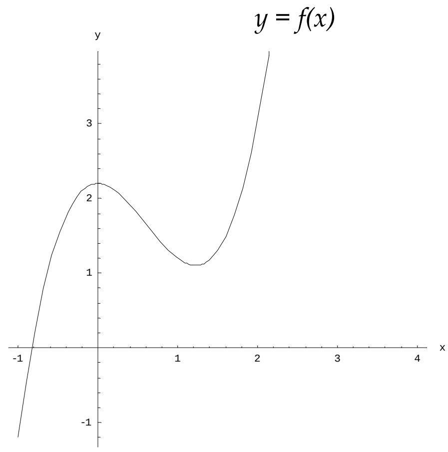
2. The graph of a function *f* is given in the figure. What observations can you make about the derivative of the function *f* at different points?
    - Where is the derivative positive?
    - Where is the derivative negative?
    - Where is the derivative zero?
    - Is the derivative defined everywhere?
    - Where is the derivative constant?
    - Where is the derivative at its greatest and where at its smallest?



3. Estimate as accurately as possible the value of the derivative of the function $f(x) = 2^x$ at the point $x = 1$.
    - How can the exact value of the derivative be determined?

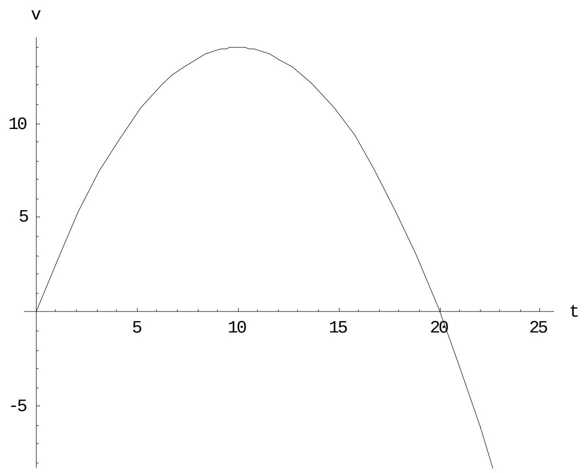
4. a) Interpret from the figure what the quotient $\frac{f(1+h) - f(1)}{h}$ means.
    - What does $f(1)$ mean?
    - What does $f(1 + h)$ mean? For example, if $h = 0.2$?
    - What does $1 + h$ mean?
    - Give a graph where $1 + h$, $f(1 + h)$ and $f(1)$ are marked.
    - How much does x change from the point 1 to $1 + h$?
    - How much do the values of the function change?

   b) Interpret from the figure what the limit $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ means.



c) Estimate the value of the limit $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$.

5. A car starts at the time t = 0 from the starting point. The figure represents the velocity v (m/s) of the car as a function of time t (s).
   a) What is the velocity of the car at the point t = 7?
      - Read the task carefully.
   b) When does the distance travelled by the car increase and when does it decrease?
      - What happens to the car when the distance decreases? What happens to the velocity?
   c) Sketch the graph of the distance travelled s (m) by the car as a function of time t (s) in the given (t, s)-coordinates.
      - When is the distance at its greatest?
   d) What is the average acceleration of the car at the interval 2s - 7s?
      - What does acceleration mean?
   e) What is the acceleration of the car at the point t = 7s?
   f) Sketch the graph of the acceleration a (m/s^2) of the car as a function of time t (s) in the given (t, a)-coordinates.
      - When is the acceleration zero?
      - How much is the acceleration at the beginning? How does it change?
      - When is the acceleration positive and when negative?
      - When does the acceleration increase and when decrease?



APPENDIX 3: The included articles

The articles are included only to the printed thesis.

[1] Hähkiöniemi, M. 2006. Associative and reflective connections between the limit of the difference quotient and limiting process. Journal of Mathematical Behavior, 25(2), 170-184. Reprinted from Journal of Mathematical Behavior, Vol. 25, Hähkiöniemi, M., Associative and reflective connections between the limit of the difference quotient and limiting process, Pages 170-184, Copyright (2006), with permission from Elsevier.

[2] Hähkiöniemi, M. 2006. Is there a limit in the derivative? – Exploring students' understanding of the limit of the difference quotient. In M. Bosch (Ed.) Proceedings of the fourth congress of the European society for research in mathematics education (CERME 4), Sant Feliu de Guíxols, Spain – 17 - 21 February 2005, 1758-1767. [http://ermeweb.free.fr/CERME4/].

[3] Hähkiöniemi, M. 2006. Perceiving the derivative: the case of Susanna. Nordic Studies in Mathematics Education, 11(1), 51-73. Reprinted with permission.

[4] Hähkiöniemi, M. Submitted. How the derivative becomes visible: the case of Daniel. Submitted to Teaching Mathematics and Computer Science.

An earlier version of this paper is published as

[4]* Hähkiöniemi, M. 2006. Ajattelun apuvälineet – tapaustutkimus derivaatan representaatioista. [Tools of thinking – A case study of representations of the derivative.] University of Jyväskylä. Department of Teacher Education. Research report 82. 79 p.

[5] Hähkiöniemi, M. Submitted. Hypothetical learning path to the derivative. Submitted to Mathematical Thinking and Learning.

An earlier version of this paper is published as

[5]* Hähkiöniemi, M. 2005. The role of different representations in teaching and learning of the derivative through open approach. In E. Pehkonen (Ed.) Problem solving in mathematics education. Proceedings of the ProMath meeting June 30 – July 2, 2004 in Lahti. University of Helsinki, Department of Applied Sciences of Education. Research Report 261, 71-82.