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Elena Kudryashova

# Cycles in Continuous and Discrete Dynamical Systems

## Computations, Computer-assisted Proofs, and Computer Experiments

JYVÄSKYLÄN YLIOPISTO

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## ABSTRACT

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The present work is devoted to calculation of periodic solutions and bifurcation research in quadratic systems, Lienard system, and nonunimodal one-dimensional discrete maps using modern computational capabilities and symbolic computing packages.

In the first Chapter the problem of Academician A.N. Kolmogorov on localization and modeling of cycles of quadratic systems is considered. For the investigation of small limit cycles (so-called local 16th Hilbert's problem) the method of calculation of Lyapunov quantities (or Poincaré-Lyapunov constants) is used. To calculate symbolic expressions for the Lyapunov quantities the Lyapunov method to the case of nonanalytical systems was generalized. Following the works of L.A. Cherkas and G.A. Leonov, the symbolic algorithms for transformation of quadratic systems to special Lienard systems were developed. For the first time general symbolic expressions of first four Lyapunov quantities for Lienard systems are obtained. The large limit cycles (or "normal" limit cycles) for quadratic and Lienard systems with parameters corresponding to the domain of existence of a large cycle, obtained by G.A. Leonov, are presented. Visualization on the plane of parameters of a Lienard system of the domain of parameters of quadratic systems with four limit cycles, obtained by S.L. Shi, is realized.

The second Chapter of the thesis is devoted to nonunimodal one-dimensional discrete maps describing operation of digital phase-locked loop (PLL).Qualitative analysis of PLL equations helps one to determine necessary system operating conditions (which, for example, include phase synchronization and clock skew elimination). In this work, application of the qualitative theory of dynamical systems, special analytical methods, and modern mathematical packages has helped us to advance considerably in calculation of bifurcation values and to define numerically fourteen bifurcation values of the DPLL's parameter. It is shown that for the obtained bifurcation values of a nonunimodal map, the effect of convergence similar to Feigenbaum's effect is observed.

Keywords: limit cycles, Kolmogorov's problem, Lyapunov quantities, four limit cycles of two-dimensional dynamical systems, Lienard system, phase locked loops, period-doubling bifurcations, bifurcation parameters

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## LIST OF FIGURES

# CONTENTS

## LIST OF INCLUDED ARTICLES

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# INTRODUCTION AND THE STRUCTURE OF THE WORK

This work is devoted to calculation of periodic solutions and bifurcation research in quadratic systems, Lienard system, and nonunimodal one-dimensional discrete maps using modern computational capabilities and symbolic computing packages.

The study of such systems is an actual problem and has great importance for various areas of applied research: biology, mechanics, electronics. For example, consideration of various population models in biology leads to the study of quadratic systems [Murray, 2003; Mark Kot, 2001; Rockwood, 2006], and the Lienard system describes dynamics of various mechanical and electronic systems [Andronov, Vitt & Khaiken, 1966; Leonov, 2006]. In such models, limiting periodic solutions [Leonov, 2006; Leonov, 2009; Kuznetsov, 2008] are of great importance. The one-dimensional discrete maps considered arise in analysis of operation of digital phase locked loops [Gardner, 1966; Lindsey, 1972; Lindsey & Chie, 1981; Leonov, Reitmann & Smirnova, 1992; Leonov, Ponomarenko & Smirnova, 1996; Leonov, 2002; Leonov, 2008[1]; Kroupa, 2003; Best, 2003; Abramovitch, 2002], where there arises the problem of determination of a sequence of bifurcation parameters of the system resulting in chaotic behavior [Osborn, 1980; Lapsley *et al.*, 1997; Leonov & Seledzhi, 2002; Leonov & Seledzhi, 2005; Banerjee & Sarkar, 2006; Banerjee & Sarkar, 2008].

The first chapter is devoted to the problem of Academician A.N. Kolmogorov [Arnold, 2005] on localization and modeling of cycles of quadratic systems. The first part of the chapter deals with small limit cycles (so-called local 16th Hilbert's problem [Yu, 2005; Yu & Han, 2005; Li, 2003; Chavarriga & Grau, 2003; Gine, 2007; Leonov, 2008; Christopher & Lloyd, 1996]). For this purpose, the method of calculation of Lyapunov quantities (or Poincaré-Lyapunov constants) is used [Bautin, 1952; Serebryakova, 1959; Lynch, 2005; Kuznetsov, 2008]. These constants determine stability and instability in a small neighborhood of weak focus, and the above-mentioned method had been suggested in the classical works of H. Poincaré [Poincaré, 1885] and A.M. Lyapunov [Lyapunov, 1892]. To calculate symbolic expressions for the Lyapunov quantities, in the work we generalize Lyapunov method to the case of nonanalytical systems and realize the algorithm in Matlab symbolic calculation package, in order to provide effective analysis of the Lyapunov quantities and for convenience of further computational study of limit cycles. We follow the works of L.A. Cherkas [Cherkas, 1976] and G.A. Leonov [Leonov, 1998; Leonov, 2006; Leonov, 2007; Leonov, 2008], and reduce quadratic systems to special Lienard system. The corresponding symbolic algorithms were developed. With the help of the developed algorithm, for the first time general symbolic expressions of first four Lyapunov quantities for Lienard system have been obtained. Following the classical Bautin's method, we show that small perturbations allow one to get either a pair of small limit cycles such that each of these cycles surrounds one of two equilibria or three small limit cycles surrounding one equilibrium of a quadratic system and the corresponding Lienard system.

It should be noted that calculation of the Lyapunov quantities is also related to an important problem of engineering mechanics on the behavior of a dynamical system when its parameters are close to the boundary of the stability domain. According to Bautin's work [Bautin, 1952], "dangerous" and "safe" boundaries are to be distinguished; small violation of such boundaries results in small (reversible) or irreversible changes of state of the system. Such changes correspond, for example, to the scenarios of "soft" and "stiff" oscillation excitation, considered by A.A. Andronov [Andronov & Leontovich, 1956; Andronov, Vitt & Khaiken, 1966]. In this case, a planar autonomous system is considered in a neighborhood of an equilibrium whose linearization has a pair of complex conjugate eigenvalues (critical case). If the stability boundary is crossed in the direction from negative real parts of the eigenvalues to positive ones and the first nonzero Lyapunov quantity is negative, then there appears a unique stable limit cycle that is retracted to a point at the reverse change of parameters, which corresponds to "safe" boundary. On the contrary, if the first nonzero Lyapunov quantity is positive, then, under small perturbations, a trajectory may go far away from the equilibrium, which corresponds to "dangerous" boundary.

The second part of the first chapter is devoted to study of large limit cycles (or "normal" limit cycles; this term has been introduced by L.M. Perko [Perko, 1990] for cycles that can be seen by means of numerical procedures). In this case, conversion to a Lienard system allows us to use Leonov's method [Leonov, 2009] of asymptotic integration and construction of a family of transverse curves for analytical determination of the domain of existence of a large cycle. It should be noted that such a reduction helps to reduce the problem of search for large limit cycles for a quadratic system with three small limit cycles to modeling of Lienard system that is determined by two coefficients. Such a consideration in this paper has resulted in formation of domains on the plane of two coefficients which correspond to the existence of four limit cycles in a quadratic system and in visualization of known Shi results [Shi, 1980]. In this work, the above-mentioned technique allowed us to extend results from [Leonov, Kuznetsov & Kudryashova, 2008] on the existence of large limit cycles.

The results of our study allowed us to discover the effect of "trajectory rigidity" of a system when strong flattening complicates numerical localization of a limit cycle. We also consider scenarios of "destruction" of large cycles as parameters approach the boundaries of parameter domains that correspond to the existence of a cycle. Results of the first chapter were presented in joint reports at international conferences [Kudryashova *et al.*, 2008; Leonov *et al.*, 2008], included into a plenary report [Leonov, Kuznetsov & Kudryashova, 2008[1]], and partially published in the article [Leonov, Kuznetsov & Kudryashova, 2008], included into an appendix and containing other approaches to calculation of Lyapunov quantities. The results of this chapter were also published in part in the article [Leonov, Kuznetsov & Kudryashova, 2009].

The second chapter of the thesis is devoted to nonunimodal one-dimensional discrete maps describing operation of digital phase-locked loop (PLL). Digital PLLs are widely used in computer architecture and telecommunications [Banerjee & Sarkar, 2006; Zoltowski, 2001; Mannino *et al.,* 2006; Leonov & Seledzhi, 2005; Dalt, 2005; Hussain & Boashash, 2002; Kudrewicz & Wasowicz, 2007; Kennedy, Rovatti & Sett, 2000]. Qualitative analysis of PLL equations helps one to determine necessary system operating conditions (which, for example, include phase synchronization and clock skew elimination) [Gardner, 1966; Lindsey, 1972; Lindsey & Chie, 1981; Nash, 1994; Leonov & Seledzhi, 2005; Lapsley *et al.,* 1997; Kroupa, 2003; Best, 2003; Abramovitch, 2002; Solonina, Ulahovich & Jakovlev, 2000; Shahgildyan *et al.,* 1989; Shakhtarin, 1977; Shakhtarin & Arkhangelśkiy, 1977]. In one of the first works dealing with analysis of digital PLLs [Osborne, 1980], an algorithm for study of periodic solutions was considered; it was shown that even a simple discrete model of PLL includes bifurcation phenomena resulting in occurrence of new stable periodic solutions and change of their period. Further, the works [Belykh & Lebedeva, 1983; Belykh & Maksakov, 1979] devoted to such systems included a model of transition to chaos through period-doubling bifurcations. Integration and development of these ideas in the works [Leonov & Seledzhi, 2002; Leonov & Seledzhi, 2005] has resulted in construction of a bifurcation tree of transition to chaos through period-doubling bifurcations. For this purpose, the first few bifurcation parameter values were obtained analytically, while calculation of subsequent bifurcation values and chaos analysis were based on computer modeling. The calculations have revealed an effect similar to Feigenbaum's effect for unimodal maps [Feigenbaum, 1978; Feigenbaum, 1980; Vul, Sinai & Khanin, 1984; Campanin & Epstain, 1981; Hu & Rudnick, 1982; Lanford, 1982; Kuznetsov, 2001].

In this work, application of the qualitative theory of dynamical systems, special analytical methods [Leonov & Seledzhi, 2002; Leonov & Seledzhi, 2005], and modern mathematical packages has helped us to advance considerably in calculation of bifurcation values and to define numerically fourteen bifurcation values of the system parameter. It is shown that for the obtained bifurcation values of a nonunimodal map, the effect of convergence similar to Feigenbaum's effect is observed. The work also demonstrates opportunities of Microsoft Excel modeling of digital PLLs. The article "Discrete Phase-Locked Loop Systems and Spreadsheets" by Kudryashova et al. in the journal "Spreadsheets in Education (eJSiE)" is presented in the appendix and covers this issue. The results of this chapter were also partially published in the article [Kudryashova, 2009] and discussed in joint reports at various international conferences [Kudryashova & Leonov, 2005; Kudryashova & Seledzhi, 2007; Kudryashova & Kuznetsov, 2009; Leonov, Kuznetsov & Kudryashova, 2008[1]; Leonov *et al.,* 2009].

The present work is based on more than 10 published papers and reports at international conferences which were mentioned above. In these papers, the statements of problems are due to the supervisor, while computational procedures, algorithms, and computer modeling are due to the author.

# 1 COMPUTATION OF LIMIT CYCLES FOR POLYNOMIAL SYSTEMS AND KOLMOGOROV'S PROBLEM

## 1.1 Introduction

Many applied problems such as oscillations of electronic generators and electrical machines, dynamics of populations, critical and admissible boundaries of stability, and also purely mathematical problems such as Hilbert's sixteenth problem and the center-and-focus problem stimulated the study of cycles for two-dimensional dynamical systems [Shilnikov, Turaev & Chua, 2001; Bautin & Leontovich, 1976; Andronov, Vitt & Khaiken, 1966; Andronov & Leontovich, 1956; Blows & Perko, 1990; Perko, 1990; Anosov et al., 1997, and others]. At present, there are many well-known results on existence and computation of limit cycles, but the problem of localization of limit periodic solutions for two-dimensional autonomous systems is still far from being solved in the general case even for Lienard system and quadratic systems.

V.I. Arnold writes (translated from [Arnold, 2005] into English): "To estimate the number of limit cycles of square vector fields on plane, A.N. Kolmogorov had distributed several hundreds of such fields (with randomly chosen coefficients of quadratic expressions) among a few hundreds of students of Mechanics and Mathematics Faculty of MGU as a mathematical practice. Each student had to find the number of limit cycles of his/her field. The result of this experiment was absolutely unexpected: not a single field had a limit cycle! It is known that a limit cycle persists under a small change of field coefficients. Therefore, the systems with one, two, three (and even, as has become known later, four) limit cycles form an open set in the space of coefficients, and so for a random choice of polynomial coefficients, the probability of hitting in it is positive. The fact that this did not occur suggests that the above-mentioned probabilities are, apparently, small."

For study of small and large limit cycles, various analytical and numerical methods are used. But while large limit cycles can be obtained with the help of computer modeling, small limit cycles can be effectively studied only by analytical methods. Thereby, Kolmogorov's problem on the study of limit cycles for differential systems can be divided into two parts.

While computation of small limit cycles is a very difficult task, it is possible to analytically construct small cycles in the neighborhood of an equilibrium for the critical case of linearization of the system when there is a nonzero Lyapunov quantity [Bautin, 1949]. Therefore, computation of the Lyapunov quantities is one of the central problems in the study of small cycles [Poincaré, 1885; Lyapunov, 1892; Cherkas, 1976; Marsden & McCracken, 1976; Lloyd, 1988; Yu, 1998; Yu & Han, 2005; Lynch, 2005; Roussarie, 1998; Reyn, 1994; Li, 2003; Chavarriga & Grau, 2003; Gine, 2007; Gasull, Guillamon & Manosa, 1997; Christopher & Li, 2007; Yu & Chen, 2008; Kuznetsov & Leonov, 2007].

While general form of the first and second Lyapunov quantities had been computed for two-dimensional real autonomous systems (in terms of coefficients of the right-hand side of the system) in the 40-50s of the last century [Bautin, 1949; Bautin, 1952; Serebryakova, 1959], the general form of the third Lyapunov quantity has been computed significantly later with the help of a special new algorithm and symbolic computation [Leonov, Kuznetsov & Kudryashova, 2008].

There are no general schemes for study of large (normal-amplitude or large-amplitude) limit cycles for polynomial systems. There are different schemes depending on the type of an investigated system. Examples of numerical computation of large limit cycles for specific systems can be found in many papers (see, e.g., [Cherkas, 1976; Perko, 1990; Shi, 1980] and others). There are also some analytical conditions of the existence of large limit cycles (see, e.g., [Shi, 1980; Leonov, 2009]). Following the work [Leonov, 2009], with the help of a method of asymptotic integration of trajectories for a Lienard system, the region of parameters of a Lienard system having large limit cycles can be determined. In [Shi, 1980], conditions of existence of four limit cycles for a quadratic system were obtained. With the help of reduction of a quadratic system into a Lienard system, the domain of parameters of a Lienard system corresponding to Shi's conditions has been determined.

In the present chapter, the method of Lyapunov quantities is applied to the study of small and large limit cycles. We extended the classical Lyapunov method for calculating Lyapunov quantities to the case of nonanalytic systems. Application of the classical analytic method and modern software tools for symbolic computing allowed us to obtain formulas for Lyapunov quantities in a general form. The program code for computation of Lyapunov quantities in mathematical package MatLab is presented. This result can be used for constructing small limit cycles.

It was discovered that, for study of large limit cycles of quadratic systems, it is useful to reduce such systems to Lienard system of special type. Reduction of a quadratic system into a Lienard system is described. The computation of the first four Lyapunov quantities for a Lienard system is presented. We present results of computer experiments for the domain of parameters where large limit cycles exist obtained by Leonov [Leonov, 2009]. Using small perturbations and an algorithm for constructing small limit cycles for a Lienard system with parameters from this domain, it is possible to construct systems with four cycles: three small limit cycles surround the zero equilibrium, and one large limit cycle surrounds another equilibrium. Large limit cycles for quadratic and Lienard systems with parameters corresponding to the described domain are presented. The peculiarities of behavior of trajectories and difficulties of modeling which were observed in computer experiments are formulated. A visualization on the plane of parameters of a Lienard system of the conditions of the existence of four limit cycles for a quadratic system obtained by S.L. Shi [Shi, 1980] is presented. Thus, Kolmogorov's problem on the study of limit cycles in quadratic systems can be solved by means of computer modeling.

## 1.2 Calculation of small limit cycles

Calculation of small limit cycles for a dynamical system in the neighborhood of an equilibrium is a very difficult problem. However, in the case where there is a nonzero Lyapunov quantity, small cycles can be obtained with the help of the algorithm for constructing of small limit cycles [Bautin, 1949; Lynch, 2005; Lloyd & Pearson, 1997]. Thus, Kolmogorov's problem for small limit cycles reduces to the problem of computation of the Lyapunov quantities.

At present, there exist various methods for determining Lyapunov quantities. Computer realizations of these methods allow us to find Lyapunov quantities in the form of symbolic expressions which depend on expansion coefficients of the right-hand sides of the equations of the system [Bautin, 1952; Serebryakova, 1959; Li, 2003; Lynch, 2005]. These methods differ in complexity of algorithms and compactness of the obtained symbolic expressions.

The first method for finding Lyapunov quantities was suggested by Poincaré [Poincaré, 1885]. This method consists of sequential constructing time-independent holomorphic integrals for approximations of the system. Further, different methods for computation that use the reduction of a system to normal forms, were developed in [Yu & Chen, 2008; Li, 2003].

Another approach to computation of Lyapunov quantities is connected with construction of approximations of a solution (as a finite sum of powers of the initial data) in the original Euclidean system of coordinates and in a time domain [Leonov, 2008; Kuznetsov & Leonov, 2008]. In [Leonov, Kuznetsov & Kudryashova, 2008], the formula of the third Lyapunov quantity has been obtained by means of this approach.

In the present work, in order to compute Lyapunov quantities, we will use another approach, which is related to finding approximations of a solution of the system. We write the system in polar coordinates and apply procedures for recurrent construction of approximations of solutions. This approach to computation of Lyapunov quantities is known as the classical Lyapunov method [Lyapunov, 1892]. Here this approach is extended to the case of nonanalitical systems.

### 1.2.1 Classical Lyapunov method

Following the work [Lyapunov, 1892], consider a system of two autonomous differential equations

$$\begin{aligned} \frac{dx}{dt} &= -y + f(x,y), \\ \frac{dy}{dt} &= x + g(x,y), \end{aligned} \tag{1}$$

where $x, y \in \mathbb{R}$, and the functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ have continuous partial derivatives of $(n+1)$st order in a small neighborhood of the point $(x,y) = (0,0)$,

$$f(\cdot,\cdot), g(\cdot,\cdot)\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \in \mathbb{C}^{(n+1)}, \tag{2}$$

in which we consider solutions.

Assume that the expansions of the functions $f, g$ have the following form:

$$\begin{aligned} f(x,y) &= \sum_{k+j=2}^{n} f_{kj} x^k y^j + o\big((|x| + |y|)^n\big), \\ g(x,y) &= \sum_{k+j=2}^{n} g_{kj} x^k y^j + o\big((|x| + |y|)^n\big). \end{aligned} \tag{3}$$

In polar coordinates,

$$x = r\cos(\theta), \quad y = r\sin(\theta),$$

the system takes the form

$$\dot{r}\cos\theta - \dot{\theta}r\sin\theta = -r\sin\theta + f(r\cos\theta, r\sin\theta),$$

$$\dot{r}\sin\theta + \dot{\theta}r\cos\theta = r\cos\theta + g(r\cos\theta, r\sin\theta).$$

We multiply the obtained equations by $\cos(\theta)$ and $\sin(\theta)$, add, and subtract to get the relations

$$\begin{aligned} \dot{r} &= f(r\cos\theta, r\sin\theta)\cos\theta + g(r\cos\theta, r\sin\theta)\sin\theta, \\ \dot{\theta} &= 1 - \frac{f(r\cos\theta, r\sin\theta)\sin\theta}{r} + \frac{g(r\cos\theta, r\sin\theta)\cos\theta}{r}. \end{aligned} \tag{4}$$

We note that the functions $f(r\cos\theta, r\sin\theta)$ and $g(r\cos\theta, r\sin\theta)$ contain only terms whose order in $r$ is not less than two. Thus, we can divide the first equation by $r$ for $r \neq 0$, and then extend the result by continuity to $r = 0$. Following [Bautin & Leontovich, 1976], we see that, for all sufficiently small $r$, i.e., in some neighborhood of the equilibrium, we have $\dot{\theta} \neq 0$, and $\theta$ increases (rotation is counterclockwise) as time increases ($\dot{\theta} > 0$), and we can replace the system of equations (4) by one equation

$$\frac{dr}{d\theta} = \frac{f(r\cos\theta, r\sin\theta)\cos\theta + g(r\cos\theta, r\sin\theta) sin\theta}{1 - \frac{f(r\cos\theta, r\sin\theta)\sin\theta}{r} + \frac{g(r\cos\theta, r\sin\theta)\cos\theta}{r}}. \tag{5}$$

Write equation (5) in the form

$$\frac{dr}{d\theta} = R(r, \theta).$$

By (2), (3), the function $R(r, \theta)$ is a sufficiently smooth periodic function of $\theta$ with period $2\pi$ for sufficiently small $r$. Moreover, $R(0, \theta) = 0$, i.e., $r = 0$ is a solution of equation (5). Therefore, the function $R(r, \theta)$ can be expressed as a finite sum of degrees of $r$ plus a reminder. Let us introduce the notation

$$R_{r^k}(\theta) = \frac{1}{k!}\frac{\partial^k R(\eta, \theta)}{\partial^k \eta}|_{\eta=0}.$$

Then it follows from (2) that

$$R(r, \theta) = \sum_{k=1}^{n} R_{r^k}(\theta) r^k + \frac{r^{n+1}}{(n+1)!}\frac{\partial^{n+1} R(\theta, \eta)}{\partial \eta^{n+1}}|_{\eta=r\theta_R(\theta,r)}, \quad 0 \leq \theta_R(\theta, r) \leq 1.$$

By (2), the function

$$\frac{r^{n+1}}{(n+1)!}\frac{\partial^{n+1} R(\theta, \eta)}{\partial \eta^{n+1}}|_{\eta=r\theta_R(\theta,r)}, \quad 0 \leq \theta_R(\theta, r) \leq 1,$$

is smooth in $\theta$ for sufficiently small $r$; moreover, this function is equal to $o(r^n)$ uniformly in $\theta$ on the finite interval $\theta \in [0, 2\pi]$. Thus, we can represent the function $R(r, \theta)$ in the following form:

$$\frac{dr}{d\theta} = R(r, \theta) = rR_1(\theta) + r^2 R_2(\theta) + ... + r^n R_n(\theta) + o(r^n), \tag{6}$$

where $R_i$ is a periodic function of $\theta$ of period $2\pi$. We will consider the solution of equation (6) which is equal to $r_0$ if $\theta = \theta_0$ :

$$r = f(\theta; \theta_0, r_0).$$

This solution is the polar equation of the trajectory of system (1) that passes through a point with polar coordinates $(\theta_0, r_0)$. The right-hand sides of system (1) are sufficiently smooth. Then the function $f(\theta; \theta_0, r_0)$ is a sufficiently smooth function of $\theta_0$ and $r_0$ [Hartman, 1984]. Since $r = 0$ is a solution of (5), we have

$$f(\theta; \theta_0, 0) \equiv 0. \tag{7}$$

From the theorem on continuous dependence on initial conditions and from (7) it is possible to make the following conclusion [Bautin & Leontovich, 1976]: all trajectories of system (1) that pass through a sufficiently small neighborhood of the origin of coordinates cross each of half-lines $\theta = const$, $0 \leq \theta \leq 2\pi$.

Therefore, we will consider the solution

$$r = f(\theta; 0, r_0).$$

Since $f(\theta; 0, r_0)$ is smooth, the above solution is a series of degrees of $r_0$:

$$r = f(\theta; 0, r_0) = u_1(\theta) r_0 + u_2(\theta) {r_0}^2 + u_3(\theta) {r_0}^3 + ... + u_n(\theta) {r_0}^n + o\big({r_0}^n\big).$$

Here $o\big({r_0}^n\big)$ is a continuously differentiable function of $\theta$ in the considered interval. Inserting the obtained representation into (6) and equating the expressions at the same powers of $r_0$, we obtain equations for determining the $u_i(\theta)$ one by one:

$$\begin{aligned}
&\dot{u}_1(\theta) = R_1(\theta) u_1(\theta),\\
&\dot{u}_2(\theta) = R_1(\theta) u_2(\theta) + R_2(\theta) {u_1}^2(\theta),\\
&\dot{u}_3(\theta) = R_1(\theta) u_3(\theta) + R_2(\theta) u_1(\theta) u_2(\theta) + R_3(\theta) {u_1}^3(\theta),\\
&...\\
&\dot{u}_n(\theta) = R_1(\theta) u_n(\theta) + \ldots + R_n(\theta) {u_1}^n(\theta).
\end{aligned} \tag{8}$$

From the condition

$$f(\theta; 0, r_0) = r_0$$

it obviously follows that

$$u_1(0) = 1, \quad u_{i>1}(0) = 0.$$

Hence, we can successively define $u_i(\theta)$ from the recurrent differential equations (8).

If we take $\theta = 2\pi$ in the solution $r = f(\theta; 0, r_0)$, then we get the values corresponding to the first (after the initial point) intersection of a trajectory with the positive semiaxis:

$$r = r(2\pi, 0, r_0) = \alpha_1 r_0 + \alpha_2 {r_0}^2 + \alpha_3 {r_0}^3 + ... + \alpha_n {r_0}^n + o\big({r_0}^n\big), \quad \alpha_j = u_j(2\pi). \tag{9}$$

It is easy to see that $\alpha_1 = 1$ and $\alpha_2 = 0$.

Let $n = 2m + 1$. If $\alpha_2 = ... = \alpha_{2m} = 0$, then $\alpha_{2m+1}$ is called the $m$th Lyapunov quantity.

Note that, according to the Lyapunov theorem, the first nonzero coefficient of the expansion $r(2\pi, 0, r_0)$ is always an odd number, and for sufficiently small initial data $r_0$, the sign of $\alpha_{2m+1}$ (of the Lyapunov quantity) determines the qualitative behavior (winding or unwinding) of the trajectory $\big(x(t,h), y(t,h)\big)$ on the plane [Lyapunov, 1892].

For the further research, we will need expressions of the first four Lyapunov values. The described classical Lyapunov method for computation of Lyapunov quantities has been programmed in mathematical package MatLab.

For calculation of the $m$th Lyapunov quantity we need the expansion of the right-hand side of the system up to $(2m + 1)$th order [Leonov, Kuznetsov & Kudryashova, 2008]. For example, for calculation of the third Lyapunov quantity, we have to consider a system with $n = 7$.

With the help of the following computer code, expressions of a necessary number of Lyapunov quantities for a quadratic system in the general form can be obtained.

```
% Computation of N Lyapunov quantities in general form
% for the quadratic system
% \dot x = -y + f(x,y)
% \dot y =  x + g(x,y)
% with the expansion of the right-hand side of the system
% up to (2N+1)th order

  clear all
  syms x y h v r

  N = 3;
  NS =2*N+1;

% Define the functions f(x,y),g(x,y) for the system of (NS)th order
% \dot x = f(x,y),
% \dot y = g(x,y).

  fxy=0; gxy=0;
  fxy_n(2:NS,1) = 0*x*y;
  gxy_n(2:NS,1) = 0*x*y;

  for n=2:NS
    for iy=0:n
        ix = n - iy;
        fxy_n(n,1)=fxy_n(n,1)+
              +sym(['f',int2str(ix),int2str(iy)],'real')*x^ix*y^iy;
        gxy_n(n,1)=gxy_n(n,1)+
              +sym(['g',int2str(ix),int2str(iy)],'real')*x^ix*y^iy;
    end;
    fxy = fxy + fxy_n(n);
    gxy = gxy + gxy_n(n);
  end;

% turn to the polar coordinate system
  frv = subs(fxy, [x y], [r*cos(v) r*sin(v)]);
  grv = subs(gxy, [x y], [r*cos(v) r*sin(v)]);
  dr = frv*cos(v)+grv*sin(v);
  dv = simplify(1 - frv*sin(v)/r+grv*cos(v)/r);
  Rrv = dr/dv;

% Computation of coefficients R_k(0)
  Rv(1:NS) = 0*h;
  for i=1:NS
      Rv(i) = subs((diff(Rrv,r,i)/factorial(i)),r,'0');
  end;

% Generation of symbolic representation of series of solution rvh_s
  rv_s(1:NS) = 0*h; rvh_s = 0*h;
  for n=1:NS
      rv_s(n) = sym(['rv_',int2str(n)],'real');
      rvh_s = rvh_s + rv_s(n)*h^n;
  end

% Generation of symbolic representation of series of the right part
% Rvr_s = Rv_1*r + Rv_2*r^2 + ... + Rv_n*r^n
  Rv_s(1:NS) = 0*h; Rvr_s = 0*r;
  for n=1:NS
      Rv_s(n) = sym(['Rv_',int2str(n)],'real');
      Rvr_s = Rvr_s + Rv_s(n)*r^n;
  end

% Define the right part in series of h from coefficients
  uv_Rrs(1:NS) = 0*h;
  uvh_Rrs  = subs(Rvr_s, r, rvh_s);

  for n=1:NS
  uv_Rrs(n)=simplify(subs(diff(uvh_Rrs,'h',n)/factorial(n),h,'0'));
  end
```

```

uvh_Rrs = simplify(subs(uvh_Rrs,rv_s(1),'1'));
uv_Rrs = simplify(subs(uv_Rrs,rv_s(1),'1'));
uvh_Rrs = simplify(subs(uvh_Rrs,Rv_s(1),'0'));
uv_Rrs = simplify(subs(uv_Rrs,Rv_s(1),'0'));

uv(1:NS) = 0*v;
rv(1:NS)=0*v; rv(1) = 1;
rv_cur = rv(1)*h;
Lv(1:NS) = 0*v; Lv(1) = 1;

for i=2:NS
    uv_rs = subs(uv_Rrs(i),Rv_s,Rv);
    uv(i)= subs(uv_rs,rv_s, rv);
    Iv = int(uv(i),v);
    I_v0 = Iv - subs(Iv,v,'0');
    rv(i) = simplify(I_v0);
    rv_cur = rv_cur + rv(i)*h^i;
    Lv(i) = simplify(subs(rv(i),'v','2*pi'));
end;

L(1:N)=0*x;    %Column of Lyapunov quantities for the computation
G(1:N)=0*x;    %Column of coefficients which will be nulled

for i=1:N
  L(i) = Lv(2*i +1);
  for n=1:i -1
      L(i) = subs(L(i),['g',num2str(2*n),'1'],G(n),0);
  end ;
  L(i) = simplify(L(i));
  G(i) = solve(L(i),['g',num2str(2*i),'1']);
end ;
```

For the first time, the expressions for the first and second Lyapunov quantities had been obtained by N. Bautin [Bautin, 1952] and N. Serebryakova [Serebryakova, 1959], respectively. In [Leonov, Kuznetsov & Kudryashova, 2008], which can be found in the included articles, formulas for the first three Lyapunov quantities are presented.

#### 1.2.1.1 Perturbation of the system and small cycles

According to representation (9), we have

$$r = r_0 + \alpha_3 {r_0}^3 + \alpha_5 {r_0}^5 + o\left({r_0}^5\right).$$

Suppose that $\alpha_3 = 0$, and the first nonzero Lyapunov quantity is $\alpha_5 > 0$.

Following Bautin's method [Bautin, 1952] described in the included article [Leonov, Kuznetsov & Kudryashova, 2008], we can perturb the coefficient $\alpha_3 > 0$ so that for the perturbed system, the following conditions will be valid:

$$\widetilde{\alpha}_3 < 0, \quad \widetilde{\alpha}_5 > 0.$$

Then, for sufficiently small initial data ${r_0}^I$, trajectories of the perturbed system will spiral in, and for some initial data ${r_0}^{II}$ (${r_0}^{II} >> {r_0}^I$), trajectories will spiral out. Thus, we can obtain an unstable small limit cycle (Fig. 1) near the zero equilibrium.

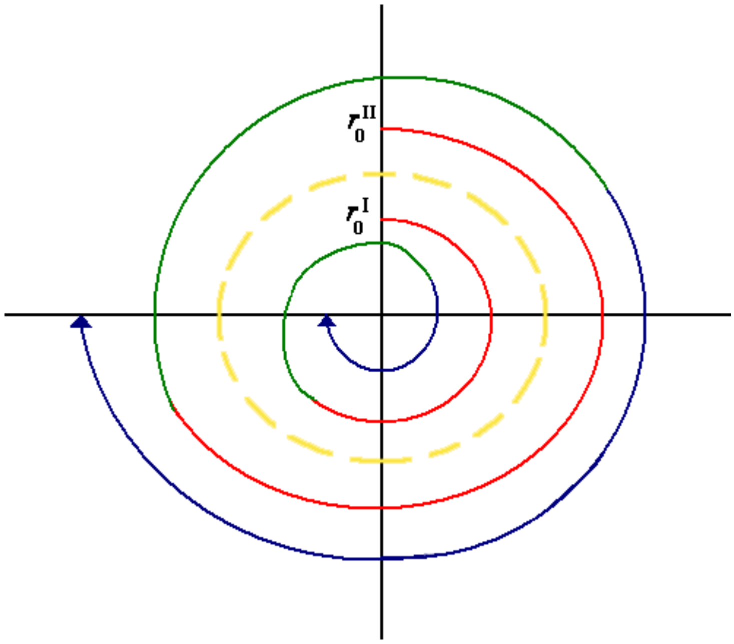


FIGURE 1 Unstable limit cycle.

Similarly, one can perturb the coefficient $\alpha_1$ (Lyapunov 0th quantity) and obtain a second small limit cycle.

This procedure for construction of small cycles can be extended to the case of a larger number of zero Lyapunov quantities [Lynch, 2005; Lloyd & Pearson, 1997].

#### 1.2.1.2 Computation of Lyapunov quantities for the Lienard system

For the study of limit cycles of quadratic systems, it was found useful to reduce quadratic systems into Lienard system. In computer experiments, results of which are presented later, formulas for the first four Lyapunov quantities for a Lienard system are used. Thereby, the expression of the forth Lyapunov quantity is presented only for a Lienard system.

Consider a Lienard system,

$$\dot{x} = -y,$$
$$\dot{y} = x + g_{x1}(x)y + g_{x0}(x),$$

or the equivalent Lienard equation,

$$\ddot{x} + x + \dot{x}g_{x1}(x) + g_{x0}(x) = 0.$$

Let $g_{x1}(x) = g_{11}x + g_{21}x^2 + \dots$ and $g_{x0}(x) = g_{20}x^2 + g_{30}x^3 + \dots$ .
Then

$$L_1 = -\frac{\pi}{4}(g_{20}g_{11} - g_{21}).$$

If $g_{21} = g_{20}g_{11}$, then $L_1 = 0$ and

$$L_2 = \frac{\pi}{24}(3g_{41} - 5g_{20}g_{31} - 3g_{40}g_{11} + 5g_{20}g_{30}g_{11}).$$

If $g_{41} = \frac{5}{3} g_{20} g_{31} + g_{40} g_{11} - \frac{5}{3} g_{20} g_{30} g_{11}$, then $L_2 = 0$ and

$$L_3 = -\frac{\pi}{576}(70 g_{20}^3 g_{30} g_{11} + 105 g_{20} g_{51} + 105 g_{30}^2 g_{11} g_{20} + 63 g_{40} g_{31} \\ -63 g_{11} g_{40} g_{30} - 105 g_{30} g_{31} g_{20} - 70 g_{20}^3 g_{31} - 45 g_{61} - 105 g_{50} g_{11} g_{20} + 45 g_{60} g_{11}).$$

If $g_{61}$ is determined from the equation $L_3 = 0$, then

$$L_4 = \frac{\pi}{17280}(945 g_{81} + 4158 g_{20}^2 g_{40} g_{31} + 2835 g_{20} g_{30} g_{51} - 5670 g_{20} g_{30} g_{11} g_{50} \\ -4158 g_{20}^2 g_{30} g_{11} g_{40} + 2835 g_{20} g_{11} g_{70} + 1215 g_{30} g_{11} g_{60} + 1701 g_{40} g_{11} g_{50} \\ -4620 g_{20}^3 g_{11} g_{50} - 8820 g_{20}^3 g_{30} g_{31} + 1701 g_{30} g_{40} g31 + 2835 g_{20} g_{50} g_{31} \\ -2835 g_{20} g_{30}^2 g_{31} - 1701 g_{30}^2 g_{11} g_{40} + 8820 g_{20}^3 g_{30}^2 g_{11} + 3080 g_{20}^5 g_{11} g_{30} \\ +2835 g_{20} g_{30}^3 g_{11} + 4620 g_{20}^3 g_{51} - 1701 g_{40} g_{51} - 945 g_{11} g_{80} \\ -3080 g_{20}^5 g_{31} - 1215 g_{60} g_{31} - 2835 g_{20} g_{71}).$$

To obtain the described formulas, it is enough to omit the corresponding coefficients in the program code presented earlier.

In the further symbolic calculations, the following functions for computation of Lyapunov quantities are used.

```

% Function is useful for computation  of a coefficient in the series
  function coef = fCoefInSeries(fx,n,x_val)
    coef = (subs(diff(fx,'x',n),'x',x_val))/factorial(n);
```

Function for computation of the first Lyapunov quantity for a Lienard system.

```

% Computation of the first Lyapunov quantity for the Lienard  system,
% where G(x) = x + (the terms of high orders)
  function L1 = fLs_L1_FxGx(Fx,Gx)
    g20 = fCoefInSeries(Gx,2,0);
    g11 = fCoefInSeries(-Fx,1,0);
    g21 = fCoefInSeries(-Fx,2,0);
% First Lyapunov quantity
  L1 = -1/4*pi*(g20*g11-g21)
```

Function for computation of the second Lyapunov quantity for a Lienard system.

```

% Computation of the second Lyapunov quantity for the Lienard system,
% where G(x) = x + (the terms of high orders)
  function L2 = fLs_L2_FxGx(Fx,Gx)
    g20 = fCoefInSeries(Gx,2,0);
    g30 = fCoefInSeries(Gx,3,0);
    g40 = fCoefInSeries(Gx,4,0);
    g11 = fCoefInSeries(-Fx,1,0);
    g31 = fCoefInSeries(-Fx,3,0);
    g41 = fCoefInSeries(-Fx,4,0);
% Second Lyapunov quantity
  L2 = 1/24*pi*(3*g41-5*g20*g31-3*g40*g11+5*g20*g30*g11);
```

Function for computation of the third Lyapunov quantity for a Lienard system.

```
% Computation of the third Lyapunov quantity for the Lienard system,
% where G(x) = x + (the terms of high orders)
  function L3 = fLs_L3_FxGx(Fx,Gx)
    g20 = fCoefInSeries(Gx,2,0);
    g30 = fCoefInSeries(Gx,3,0);
    g40 = fCoefInSeries(Gx,4,0);
    g50 = fCoefInSeries(Gx,5,0);
    g60 = fCoefInSeries(Gx,6,0);

    g11 = fCoefInSeries(-Fx,1,0);
    g31 = fCoefInSeries(-Fx,3,0);
    g51 = fCoefInSeries(-Fx,5,0);
    g61 = fCoefInSeries(-Fx,6,0);

% Third Lyapunov quantity
  L3 = -1/576*pi*(70*g20^3*g30*g11+105*g20*g51+105*g30^2*g11*g20
         +63*g40*g31-63*g11*g40*g30-105*g30*g31*g20-70*g20^3*g31
         -45*g61-105*g50*g11*g20+45*g60*g11);
```

### 1.2.2 Transformation between quadratic and Lienard systems

For the study of limit cycles for quadratic systems it was found useful to reduce them to Lienard system of special type. For example, to solve the local 16th Hilbert's problem, it is necessary to have the possibility of finding symbolic zeros of Lyapunov quantities. For the Lienard system, this procedure is not difficult. It is impossible to find symbolic independent zeros of Lyapunov quantities (at which only one Lyapunov quantity is equal to zero) for cubic systems, and there are no methods to reduce them into Lienard system. For this reason, the local 16th Hilbert's problem in the general case has not been solved for cubic systems.

To solve Kolmogorov's problem, we consider conversion between quadratic and Lienard systems [Cherkas, 1973; Leonov, 1997; Leonov, 1998; Leonov, 2006; Leonov, 2007; Leonov, 2008].

#### 1.2.2.1 Reduction of a quadratic system to a Lienard system

Consider a two-dimensional polynomial quadratic system in the general form:

$$\begin{aligned} \dot{x} &= a_1x^2 + b_1xy + c_1y^2 + \alpha_1 x + \beta_1 y + d_1, \\ \dot{y} &= a_2x^2 + b_2xy + c_2y^2 + \alpha_2 x + \beta_2 y + d_2. \end{aligned} \tag{10}$$

Following the paper [Petrovskii, 2009], we state a definition of a limit cycle for system (10).

**Definition** *A closed integral line is called a limit cycle if all its points are regular, and some other integral line approaches it asymptotically.*

For limit cycles of quadratic systems, the following result is well known:

**Lemma 1 (Ye Yian-Qian, 1986)** *Let C be a limit cycle for a quadratic system. Then in the domain limited by the limit cycle, there is a unique singular point (focus).*

From this it follows that in the study of limit cycles of quadratic system, we may shift the origin of coordinates $(x, y)$ to a singular point and assume, without loss of generality, that $d_1 = d_2 = 0$.

We also assume that system (10) is *nontrivial*, i.e., that no linear change of variables $\mu x + \nu y$ can reduce one of the equations in new variables to an equation whose right-hand side is a function of one variable (in this case, this equation is integrable, and the system obviously has no limit cycles).

Now, following [Cherkas, 1973; Leonov, 1997], we prove a lemma.

**Lemma 2** *A nontrivial quadratic system*

$$\begin{aligned} \dot{x} &= \alpha_1 x + \beta_1 y + a_1 x^2 + b_1 xy + c_1 y^2, \\ \dot{y} &= \alpha_2 x + \beta_2 y + a_2 x^2 + b_2 xy + c_2 y^2 \end{aligned} \tag{11}$$

*can be reduced to a Lienard system*

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= -F(x)y + G(x) \end{aligned} \tag{12}$$

with the help of a nonsingular transformation.

A detailed description of this statement can be found in [Leonov, Kuznetsov & Kudryashova, 2008] in the included articles. Let us consider some results necessary for Kolmogorov's problem.

**Proposition 1** *Without loss of generality, we may assume that $c_1 = 0$, in system (11).*

For the proof of this result, consider a nonsingular change of variables

$$x = \tilde{x} + \nu y,$$

where $\nu \in \mathbb{R}$, $\nu \neq 0$ such that $\tilde{c}_1 = 0$ [Leonov, 1997].

```
% Reduction of quadratic system
% dx = al1*x+bt1*y+a1*x^2+b1*x*y+c1*y^2
% dy = al2*x+bt2*y+a2*x^2+b2*x*y+c2*y^2
% into quadratic system in simple form, where c1=0

function [a1,b1,c1,al1,bt1,a2,b2,c2,al2,bt2]
                  = fDefSimpleQs(a1,b1,c1,al1,bt1,a2,b2,c2,al2,bt2)
 syms v
 if (c1 ~= 0)
    if (a2 ~= 0) %else x->y y->x
       fv = ((solve('-a2*v^3+(a1-b2)*v^2+(b1-c2)*v+c1')));
        for i=1:3   % looking for real solution
            v_s = eval(fv(i));
            % check on real number
            if ( (v_s*conj(v_s)) == real(v_s)*real(v_s))
                v_s = real(v_s);
                a1=a1-v_s*a2;
                b1=-2*v_s^2*a2+(2*a1-b2)*v_s+b1;
                c1=0;
                al1=al1-v_s*al2;
                bt1=v_s*al1+bt1-v_s*(v_s*al2+bt2);
                break;
            end;
        end;
```

```
  else % x->y y->x
    warning('c1~=0, a2=0 => x->y y->x')
    a1_save=a1; b1_save=b1; c1_save=c1; al1_save=al1; bt1_save=bt1;
    a1=c2; b1=b2; c1=a2; al1=bt2; bt1=al2;
    a2=c1_save; b2=b1_save; c2=a1_save; al2=bt1_save; bt2=al1_save;
  end;
end;
```

Further we assume that $c_1 = 0$.
Introducing a proper notation, we write the system in the form

$$\begin{aligned} \dot{x} &= \alpha_1 x + \beta_1 y + a_1 x^2 + b_1 xy = p_0(x) + p_1(x)y, \\ \dot{y} &= \alpha_2 x + \beta_2 y + a_2 x^2 + b_2 xy + c_2 y^2 = q_0(x) + q_1(x)y + q_2(x)y^2. \end{aligned} \tag{13}$$

Let us note that, due to the nontriviality, $|b_1| + |\beta_1| \neq 0$; otherwise, the first equation is integrable, and the system obviously has no cycle (thus, $p_1(x)$ is not equally identical to zero).
**Proposition 2** [Leonov, 1997] *For $b_1 \neq 0$, the straight line*

$$p_1(x) = \beta_1 + b_1 x = 0 \tag{14}$$

*is either invariant or transversal for system (13).*
**Proof** The proof follows from the equality

$$(\beta_1 + b_1 x)^\bullet = b_1[(\beta_1 + b_1 x)y + a_1 x^2 + \alpha_1 x],$$

where $x = -\dfrac{\beta_1}{b_1}$ (here the symbol $^\bullet$ means the derivative with respect to the system).
**Proposition 3** *For $x$ such that*

$$p_1(x) = b_1 x + \beta_1 \neq 0,$$

*system (13) can be reduced to a Lienard system (12), where*

$$F(x) = -(p_0' - p_0 p_1' p_1^{-1} + q_1 - 2p_0 q_2 p_1^{-1}) p_1(x)^{-1} e^{\phi(x)},$$

$$G(x) = (p_0 q_1 - q_0 p_1 - p_0^2 q_2 p_1^{-1}) p_1(x)^{-2} e^{2\phi(x)},$$

*and $\phi(x)$ is a primitive of the function $\big(-q_2(x)p_1(x)^{-1}\big)$.*

The given transformation is obviously valid also for more general form of the polynomials $p_i, q_i$ [Christopher, Lloyd & Pearson, 1995].

```

% Function for reduction of quadratic system with c1=0
% dx = al1*x+bt1*y+a1*x^2+b1*x*y
% dy = al2*x+bt2*y+a2*x^2+b2*x*y+c2*y^2
% into Lienard system
% dx = -y
% dy = -fx*y+gx
% with the functions
% fx =  (A*x^2+B*x+C)/p1^(-1)*exp(phi)
% gx = (C1*x^4+C2*x^3+C3*x^2+x)/p1^(-2)*exp(2*phi)

 function [fx,gx,fx_n,gx_n,p1,phi]
                         =fLsFromQs(a1,b1,al1,bt1,a2,b2,c2,al2,bt2)
 syms x y 'real'

 dx = al1*x+bt1*y+a1*x^2+b1*x*y;
 dy = al2*x+bt2*y+a2*x^2+b2*x*y+c2*y^2;

 p0 = subs(dx,y,0);
 p1 = diff(dx,y);

 q0 = subs(dy,y,0);
 q1 = subs(diff(dy,y),y,0);
 q2 = diff(dy,y,2)/2;

 phi = simplify(p1^(-1)*abs(exp(int(q2/(-p1),x))));
% If (a,b) is not contain 1, then exp(int(1/(x-1),a,b))=abs(b-a)

 fx_n = -(diff(p0,x)- p0*diff(p1,x)*p1^(-1)+q1-2*p0*q2*p1^(-1))*p1;
 fx_n = collect(simplify(fx_n),x);
 fx = fx_n*p1^(-1)*phi;

 gx_n = (p0*q1-q0*p1-p0^2*q2*p1^(-1))*p1;
 gx_n = collect(simplify(gx_n),x);
 gx =  gx_n*p1^(-1)*phi^2;
```

Let us note that a limit cycle cannot have common points with an invariant curve (by the definition of a limit cycle) or with the transverse curve $p_1(x) = 0$. It follows that the specified reduction to a Lienard system is nonsingular in terms of consideration of limit cycles. The reasoning and results described above prove Lemma 2. To apply the method of Lyapunov quantities and obtain analytical conditions of existence of small limit cycles, let us consider the case of a complex focus or a center for a quadratic system in which the matrix of first approximation of system (13) at the point $(0,0)$,

$$A_{(0,0)} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

has two purely imaginary eigenvalues, i.e., the following conditions hold:

$$\alpha_1 + \beta_2 = 0, \quad \Delta = \alpha_1\beta_2 - \beta_1\alpha_2 > 0. \tag{15}$$

**Proposition 4** [Leonov, Kuznetsov & Kudryashova, 2008] *If condition (15) is fulfilled, then system (13) can be reduced to the form*

$$\begin{aligned} \dot{x} &= -y + a_1x^2 + b_1xy, \quad b_1 \in \{0,1\}, \\ \dot{y} &= \ \ x + a_2x^2 + b_2xy + c_2y^2. \end{aligned} \tag{16}$$

Let us find symbolic expressions for parameters of the Lienard system (12) obtained from the simplified quadratic system (16).

```

% Function for transformation of a quadratic system
% with c1=0, al1=0, bt1=-1, al2=1, bt0=0
% dx = -y+a1*x^2+b1*x*y
% dy = +x+a2*x^2+b2*x*y+c2*y^2
% into a Lienard system
% dx = -y
% dy = -fx*y+gx
% with the functions
% fx =  (A*x+B)*x/p1^(-1)*exp(phi)
% gx = (C1*x^3+C2*x^2+C3*x+1)*x/p1^(-2)*exp(2*phi)

 function[A,B,q,C1,C2,C3,fx,gx,p1,phi]
                                         =fLsFromSimpleQs(a1,b1,a2,b2,c2)
 syms x y 'real'

 dx = -y + a1*x^2 + b1*x*y;
 dy =  x + a2*x^2 + b2*x*y + c2*y^2;

 p0 = subs(dx,y,0);
 p1 = diff(dx,y);

 q0 = subs(dy,y,0);
 q1 = subs(diff(dy,y),y,0);
 q2 = diff(dy,y,2)/2;

 phi = simplify(p1^(-1)*abs(exp(int(q2/(-p1),x))));
 %If (a,b) is not contain 1, then exp(int(1/(x-1),a,b))=abs(b-a)

 fx_n = -(diff(p0,x)-p0*diff(p1,x)*p1^(-1)+ q1-2*p0*q2*p1^(-1))*p1;
 fx_n = collect(simplify(fx_n),x);
 fx = fx_n*p1^(-1)*phi;
 gx_n =(p0*q1- q0*p1 - p0^2*q2*p1^(-1))*(-p1);% denominator (-p1)^3
 gx_n = collect(simplify(gx_n),x);
 gx =  gx_n*(simple(-p1^(-1))*phi^2);

 q = -c2;
 A= subs(diff(fx_n,x,2)/factorial(2),'x',0);
 B= subs(diff(fx_n,x,1)/factorial(1),'x',0);

 C1= simplify(subs(diff(gx_n,x,4)/factorial(4),'x',0));
 C2= simplify(subs(diff(gx_n,x,3)/factorial(3),'x',0));
 C3= simplify(subs(diff(gx_n,x,2)/factorial(2),'x',0));
```

Often, a Lienard system is considered in the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -F(x)y - G(x), \end{aligned} \tag{17}$$

with the following functions $F(x), G(x)$:

$$\begin{aligned} F(x) &= (Ax + B)x|x+1|^{q-2}, \\ G(x) &= (C_1x^3 + C_2x^2 + C_3x + 1)x\frac{|x+1|^{2q}}{(x+1)^3}. \end{aligned} \tag{18}$$

If $b_1 = 1$ for the simplified quadratic system (16), formulas of Proposition 3 give us the following formulas for parameters of the Lienard system (18):

$$
\begin{aligned}
A &= 2c_2a_1 - b_2 - a_1, \\
B &= 1 - b_2 + 2c_2 - 2a_1, \\
q &= -\frac{c_2}{b_1}, \\
C_1 &= b_2a_1 - c_2a_1^2 - a_2, \\
C_2 &= 2 + b_2a_1 - a_1 + b_2 - 2a_2 - 2c_2a_1, \\
C_3 &= b_2 - a_2 - c_2 - a_1 + 3.
\end{aligned}
$$

#### 1.2.2.2 The reverse conversion of a Lienard system to a quadratic system

Consider the Lienard system (17) with the functions (18). Note that the matrix of first approximation $A_{(x_{sp},y_{sp})}$ of system (17) at the stationary point $(x_{sp} = 0, y_{sp} = 0)$ has two purely imaginary eigenvalues.

The Lienard system (17) can be reduced to a quadratic system with coefficients

$$
b_1 = 1, \quad c_1 = 0, \quad \alpha_1 = 1, \quad \beta_1 = 1, \quad c_2 = -q, \quad \alpha_2 = -2, \quad \beta_2 = -1,
$$

$$
\begin{aligned}
a_1 &= 1 + \frac{B - A}{2q - 1}, \\
a_2 &= -(q+1)a_1^2 - Aa_1 - C_1, \\
b_2 &= -A - a_1(2q+1),
\end{aligned}
$$

if for the coefficients $A, B, q, C_1, C_2, C_3$, the relations

$$
C_1 = \frac{2(A-B)(B + A(2q-2))}{(2q-1)^2} + \frac{(A-B)(B(q-1) - A(3q-2))}{(2q-1)^2} + C_3 - 2 \quad (19)
$$

$$
C_2 = \frac{3(A-B)(B + A(2q-2))}{(2q-1)^2} + \frac{2(A-B)(B(q-1) - A(3q-2))}{(2q-1)^2} + 2C_3 - 3 \quad (20)
$$

are satisfied.

A proof of this statement is presented in the included article [Leonov, Kuznetsov & Kudryashova, 2008].

```
% Function for conversion of parameters
% A,B,q,C1,C2,C3 of Lienard system
% into parameters a1,b1,c1,al1,bt1,a2,b2,c2,al2,bt2
% of quadratic system
% dx = a1*x^2 + b1*x*y +c1*y^2+ al1*x + bt1*y;
% dy = a2*x^2 + b2*x*y +c2*y^2+ al2*x + bt2*y;

 function [a1,b1,c1,al1,bt1,a2,b2,c2,al2,bt2]=
                                     = fDefQsFromLs(A,B,q,C1,C2,C3)

 a1=0; b1=0; c1=0; al1=0; bt1=0;
 a2=0; b2=0; c2=0; al2=0; bt2=0;

 eq1 = simplify(eval(((B-A)/(2*q-1)^2)*((1-q)*B+(3*q-2)*A)-
                                               -(2*C2-3*C1-C3)));
 eq2 = simplify(eval(((B-A)/(2*q-1)^2)*(B+2*(q-1)*A)-(C2-2*C1-1)));

 if or(eq1 == 0, eq2 == 0)
    return;
 end;

 b1 = 1; c1 = 0;  al1 = 1;  bt1=1;
 c2 = -q;  al2 = -2;  bt2 = -1;
 a1 = 1 + (B-A)/(2*q-1);
 a2 = -(q+1)*a1^2-A*a1-C1;
 b2 = -A-a1*(2*q+1);
```

The described conversions between quadratic and Lienard systems allow us to obtain expressions of Lyapunov quantities and their zeros that are useful for study.

#### 1.2.2.3 Study of a Lienard system

For experiments, the nontrivial Lienard system (17) with

$$A \neq B, \quad AB \neq 0, \quad q \neq \frac{1}{2}$$

has been considered.
Consider the zero equilibrium $x_{sp} = 0$. Let $L_1 = 0$ and assume that (19), (20) are satisfied. Then

$$\mathrm{L}_2(x_{sp}) = \frac{\pi(A-B)(5A-4B-2Bq)(2BA-B^2-4q^3-1-B^2q+3q)}{24B(2q-1)^2}. \tag{21}$$

This result can be obtained with the help of the following code. The functions $fLs\_L1\_FxGx$, $fLs\_L1\_FxGx$, $fLs\_L1\_FxGx$ which were described above for computation of Lyapunov quantities are used.

```
clear all
syms A B q C1 C2 C3 x 'real'

% we can consider absolute value abs(1+x)=1+x in the zero equilibrium
  Fx = (A*x+B)*x*(1+x)^(q-2);
  Gx = (C1*x^3+C2*x^2+C3*x^1+1)*x*(1+x)^(2*q)/(1+x)^3;

  L1 = fLs_L1_FxGx(Fx,Gx);
  Eq1 = ((B-A)/(2*q-1)^2)*((1-q)*B+(3*q-2)*A) -(2*C2-3*C1-C3);
  Eq2 = ((B-A)/(2*q-1)^2)*(B+2*(q-1)*A) - (C2-2*C1-1);

  [C1_s C2_s C3_s] = solve(Eq1 ,Eq2, L1, C1, C2, C3);
  L2 = subs(fLs_L2_FxGx(Fx,Gx),[C1 C2 C3],[C1_s C2_s C3_s], 0)
```

From (21) it follows that the condition $L_2 = 0$ is satisfied if

$$\text{either} \quad (5A - 4B - 2Bq) = 0 \quad \text{or} \quad (2BA - B^2 - 4q^3 - 1 - B^2q + 3q) = 0.$$

But, if

$$(2BA - B^2 - 4q^3 - 1 - B^2q + 3q) = 0,$$

then $L_3 = 0$. This can be proved if, after the previous calculations, we compute

```
  A_s = -(-B^2-4*q^3-1-B^2*q+3*q)/(2*b)
  L3 = subs(fLs_L3_FxGx(Fx,Gx),[C1 C2 C3 A],[C1_s C2_s C3_s A_s], 0)
```

Therefore, let for equation (17), the conditions $L_1 = 0$, (19), and (20) be satisfied. In addition, let the following condition be valid:

$$5A - 2Bq - 4B = 0. \tag{22}$$

Then $\mathrm{L}_2(x_{sp}) = 0$, and for the values $C_1, C_2, C_3$, which are determined by equations (19), (20) and condition (22), we obtain the following formulas:

$$\begin{aligned} C_1 &= (q+3)\frac{B^2}{25} - \frac{(1+3q)}{5}, \\ C_2 &= (B^2 - 10q + 5)\frac{3}{25}, \\ C_3 &= \frac{3(3-q)}{5}. \end{aligned} \tag{23}$$

For the value $\mathrm{L}_3(x_{sp})$ determined by the obtained values $C_1, C_2, C_3$, we have the formula

$$\mathrm{L}_3(x_{sp}) = -\frac{\pi B(q+2)(3q+1)[5(q+1)(2q-1)^2 + B^2(q-3)]}{20000}. \tag{24}$$

```
clear all
syms A B q C1 C2 C3 x 'real'

% in the zero equilibrium x = 0
% we can consider abs(1+x) = (1+x)
  Fx = (A*x+B)*x*(1+x)^(q-2);
  Gx = (C1*x^3+C2*x^2+C3*x^1+1)*x*(1+x)^(2*q)/(1+x)^3;
  A_s = 2/5*B*(q + 2);
  L1 = subs(fLs_L1_FxGx(Fx,Gx), A, A_s);
  Eq1 = subs(((B-A)/(2*q-1)^2)*((1-q)*B+(3*q-2)*A)-
                                        -(2*C2-3*C1-C3),A,A_s);
  Eq2 = subs(((B-A)/(2*q-1)^2)*(B+2*(q-1)*A) - (C2-2*C1-1),A,A_s);

  [C1_s C2_s C3_s] = solve(Eq1 ,Eq2, L1, C1, C2, C3)
  L3=simplify(subs(fLs_L3_FxGx(Fx,Gx),[A C1 C2 C3],
                                          [A_s C1_s C2_s C3_s],0))
```

For the Lienard system (17), for which conditions (19) and (20) are valid, $L_1 = L_2 = 0$, and $L_3 \neq 0$, parameters $A, C_1, C_2, C_3$ can be expressed via parameters $B$ and $q$.

Thus, we have obtained symbolic expressions of Lyapunov quantities for a Lienard system. The algorithm for conversion of a Lienard system into a quadratic system allows us to obtain symbolic expressions for Lyapunov quantities for a quadratic system.

Thus, in the context of Kolmogorov's problem, application of the described algorithms for calculation of Lyapunov quantities and the method of construction of small limit cycles allow us to determine a class of quadratic systems with three small limit cycles and to begin search for large limit cycles in a domain of parameters $B$ and $q$.

## 1.3 Computation of large limit cycles

The term "normal" limit cycle was introduced by L.M. Perko [Perko, 1990] for cycles that can be seen by means of numerical procedures. Now the term "large" cycle is more often used for such cycles.

Large limit cycles for some quadratic systems can be obtained with the help of computer modeling. Consider the quadratic system

$$\begin{aligned} \dot{x} &= y - x + xy, \\ \dot{y} &= \frac{1}{4}y^2 - xy - x^2 + \alpha_2 x + 2y, \end{aligned} \tag{25}$$

where $\alpha_2$ is a parameter.

A plot of a large cycle for system (25) can be obtained, for example, in mathematical package MatLab. For computation, it is necessary to determine an additional function.

```
function dz = qsys(t,z)
global a1 b1 c1 al1 bt1 a2 b2 c2 al2 bt2
dz = zeros(2,1);
dz(1) = a1*z(1)^2 + b1*z(1)*z(2) + c1*z(2)^2 + al1*z(1) + bt1*z(2);
dz(2) = a2*z(1)^2 + b2*z(1)*z(2) + c2*z(2)^2 + al2*z(1) + bt2*z(2);
```

In computer experiments it is useful to determine following function for construction of trajectory painted in three different colors.

```
% Function for construction of trajectory, painted in three different
% colors.
  function f = fPlotTrajectory(X,Y,Color1,Color2,Color3)
  lenTr = length(X); lenTr3 = round(lenTr/3);
  lenColor1 = round(lenTr3); lenColor2 = round(2*lenTr3);
  plot(X(1:lenColor1),Y(1:lenColor1),Color1);
  hold on;
  plot(X(lenColor1:lenColor2),Y(lenColor1:lenColor2),Color2);
  hold on;
  plot(X(lenColor2:length(X)),Y(lenColor2:length(Y)),Color3);
```

With the help of the following code, a plot of a large limit cycle for system (25) with $\alpha_2 = -1000$ can be obtained.

```
% Code for construction  of a plot  of a large limit cycle for system
% dx = a1*x^2+b1*x*y+c1*y^2+al1*x+bt1*y,
% dy = a2*x^2+b2*x*y+c2*y^2+al2*x+bt2*y.
% Functions qsys and fPlotTrajectory are used.

  clear all;
  global a1 b1 c1 al1 bt1 a2 b2 c2 al2 bt2

% Parameters of the system
  a1 = 0;  b1 = 1;  c1 = 0;   al1 = -1;    bt1 = 1;
  a2 = -1; b2 = -1; c2 = 1/4; al2 = -1000; bt2 = 2;

% Options for computation
  RelTol_value = 2.22045e-014;
  AbsTol_value = 1e-16;
  options = odeset('RelTol',RelTol_value,'AbsTol',AbsTol_value);
  figure('Name','Large limit cycle for a quadratic system');

% Initial data for the "unwinding" trajectory
  x0_out =-0.1; y0_out = 0; T_out = 12;
% Computation of the "unwinding" trajectory
  [T1,Z1] = ode45(@qsys,[0 T_out],[x0_out y0_out],options);
% Plotting of the "unwinding" trajectory
  fPlotTrajectory(Z1(:,1),Z1(:,2),'red','green','blue');
  hold on;

% Initial data for the "winding" trajectory
  x0_in =6; y0_in = 0; T_in = 5;
% Computation of the "unwinding" trajectory
  [T2,Z2] = ode45(@qsys,[0 T_in],[x0_in y0_in],options);
% Plotting of the "winding" trajectory
  fPlotTrajectory(Z2(:,1),Z2(:,2),'red','green','blue');
  hold on; grid on;
  hold off;
```

The result of code execution is shown in Fig. 2.

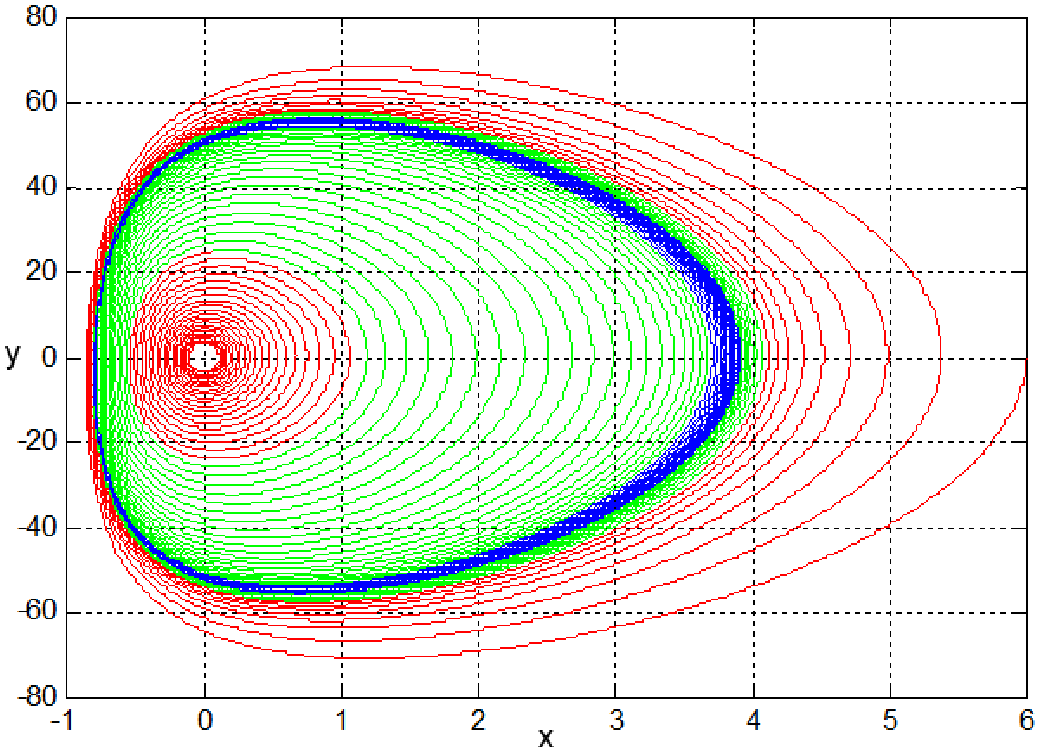


FIGURE 2 Large limit cycle for a quadratic system. Example 1.

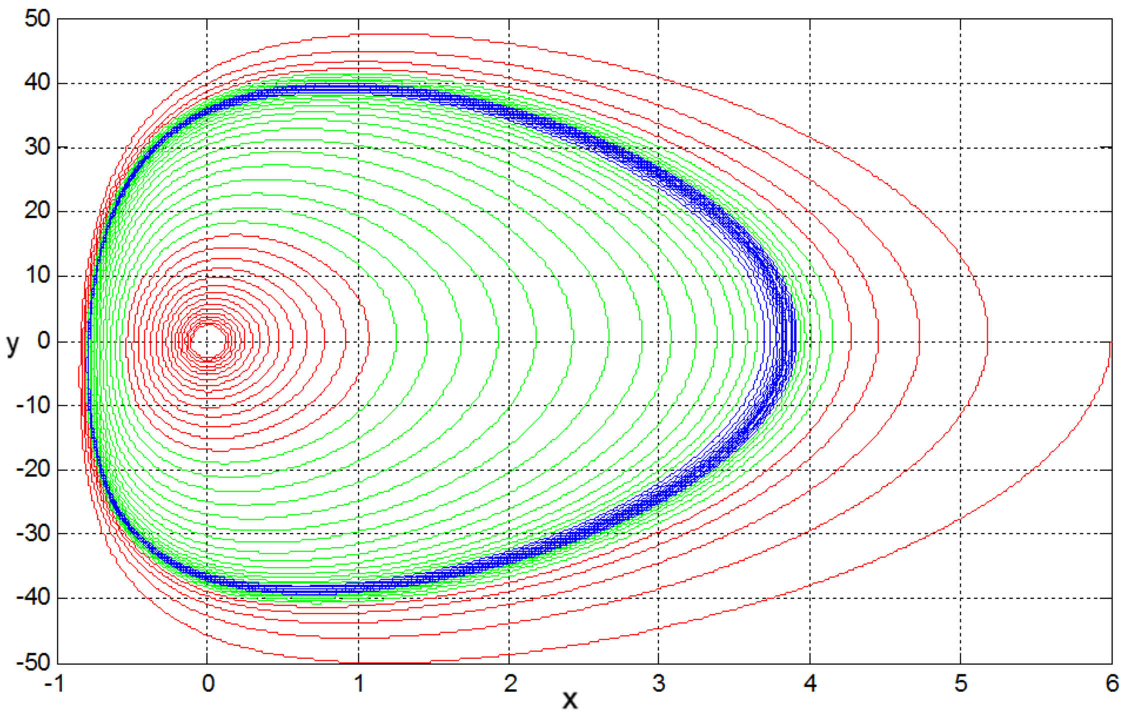


FIGURE 3 Large limit cycle for a quadratic system. Example 2.

There are two trajectories in Fig. 2. Both trajectories start in red color, end in blue color, and form a stable limit cycle.

Large limit cycles for system (25) with the parameter $\alpha_2 = -500$, $\alpha_2 = -100$ and $\alpha_2 = -30$ are presented in Fig. 3, Fig. 4, Fig. 5, correspondingly.

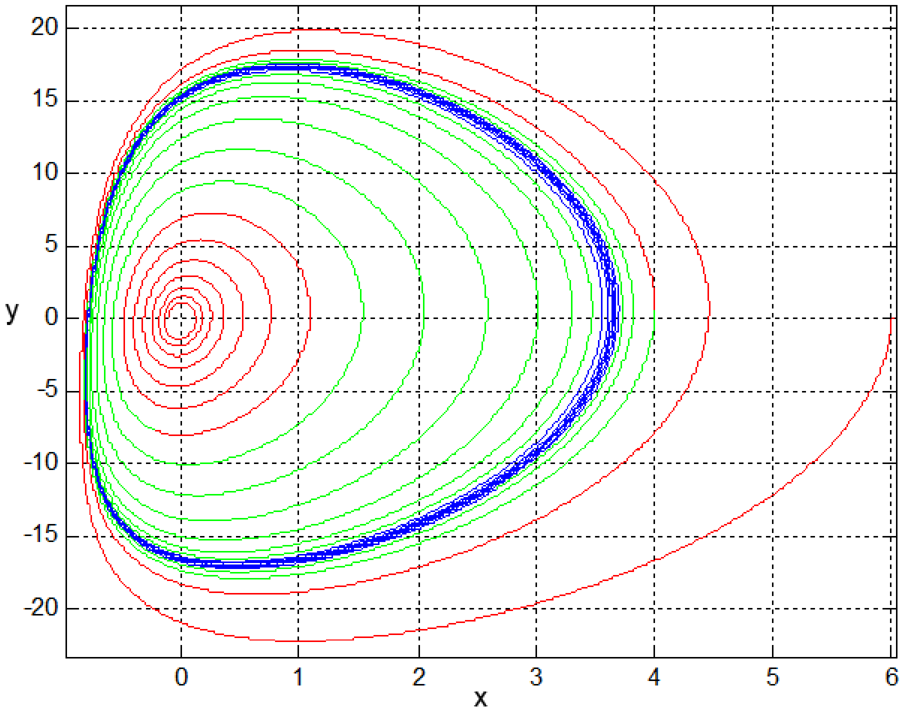


FIGURE 4    Large limit cycle for a quadratic system. Example 3.

It is not always easy to obtain a plot of a large limit cycle for a system for which such cycle exists. In each case, it is necessary to find the corresponding initial data for trajectories, time and options are needed for calculating. For example, Lienard systems require higher accuracy of calculations, and, consequently, more time.

However, at first there arises the problem of determining classes of systems for which it is possible to obtain a large limit cycle. Some theoretical and numerical approaches were considered in [Shi, 1980; Blows & Perko, 1990; Blows & Lloyd, 1984; Cherkas, 1976; Christopher & Li, 2007; Lynch, 2005; Leonov, 2006; Leonov, 2007; Leonov, 2008; Leonov, 2009].

Following the paper [Leonov, 2009], let us describe how to apply the method of asymptotic integration of trajectories for a Lienard system in the search of the region of parameters of a Lienard system where large limit cycles exist.

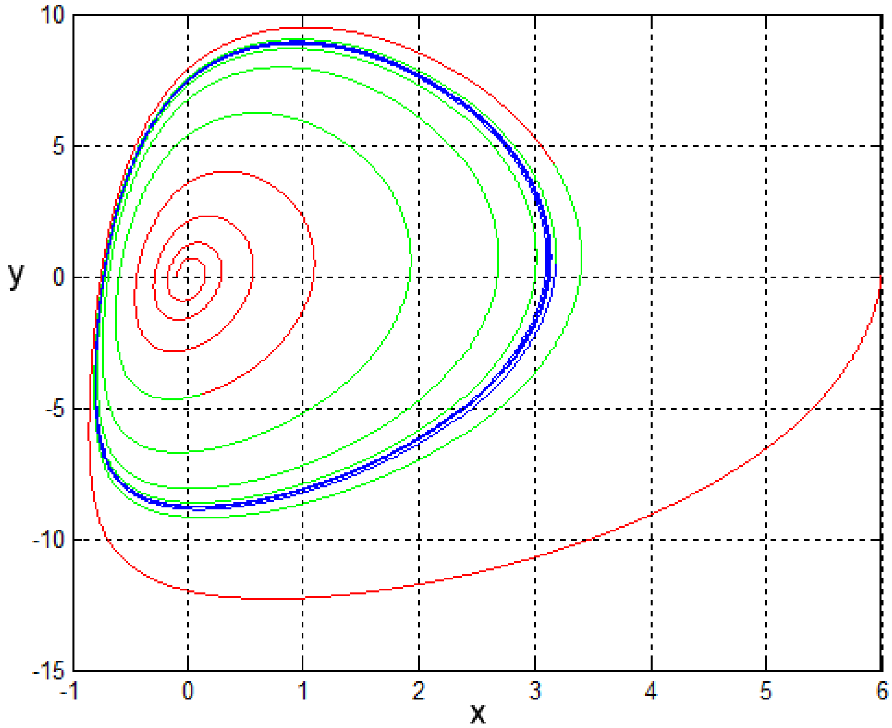


FIGURE 5 Large limit cycle for a quadratic system. Example 4.

### 1.3.1 Method of asymptotic integration

Consider the method of asymptotic integration of trajectories for the Lienard equation

$$\ddot{x} + F(x)\dot{x} + G(x) = 0. \tag{26}$$

Denote $y = \dot{x}$, then system (26) can be rewritten as following first-order equation

$$\frac{dy}{dx}y + F(x)y + G(x) = 0. \tag{27}$$

Suppose that in the interval $(a, b)$ representation

$$\begin{aligned} F(x) &= F_0 + F_1(x), \quad F_0 \neq 0, \ \lim_{x \to a} F_1(x) = 0, \\ G(x) &= G_1 x + G_2(x), \quad \lim_{x \to a} G_2(x) = 0 \end{aligned} \tag{28}$$

is valid. Then consider approach of the system (27) in the interval $(a, b)$

$$\frac{\mathrm{d}y_{app}}{\mathrm{d}x} y_{app} + F_0 y_{app} + G_1 x = 0. \tag{29}$$

The solution $y_{app}(x)$ of the equation (29), by virtue of (28), is an approach of the solution $y(x)$ of initial nonlinear system (27) with initial data $(x_0, y_{app}(x_0))$ for $x_0$ close to $a$.

Passing from (29) back to the equation of the second order we obtain

$$\ddot{x} + F_0\dot{x} + G_1 x = 0. \tag{30}$$

Introduce a new variables

$$\lambda = \frac{-F_0}{2},$$

$$\Delta = -(\frac{F_0^2}{4} - G_1).$$

If condition

$$\Delta = -(\frac{F_0^2}{4} - G_1) > 0 \tag{31}$$

is valid, then solution (30) can be rewritten in the following form:

$$x = e^{\lambda}(C_1 \sin(\sqrt{\Delta}\, t) + C_2 \cos(\sqrt{\Delta}\, t)) \tag{32}$$

else

$$x = C_1 e^{-\lambda+\sqrt{-\Delta}} + C_2 e^{-\lambda-\sqrt{-\Delta}}. \tag{33}$$

For reduction of functions in the equation (27) to the form (28) we will use the change of variables $x = D(z)$, where $D(z) : (a_z, b_z) \to (a, b)$ is a monotonous smooth function.

Denote

$$\widetilde{F}(z) = \frac{\mathrm{d}D(z)}{\mathrm{d}z} F(D(z)),$$

$$\widetilde{G}(z) = \frac{\mathrm{d}D(z)}{\mathrm{d}z} G(D(z)).$$

Then we obtain following equation

$$\frac{\mathrm{d}y}{\mathrm{d}z} + \widetilde{F}(z) + \frac{\widetilde{G}(z)}{y} = 0, \quad z \in (a_z, b_z). \tag{34}$$

Here, for the functions $\widetilde{F}(z)$ and $\widetilde{G}(z)$ representation (28) is valid for monotonous increasing function $D(z)$ at $z \to a_z$, and for monotonous decreasing function $D(z)$ at $z \to b_z$.

### 1.3.2 Investigation of large cycles for quadratic and Lienard systems

Consider described above special class of Lienard system, which can be obtained from nontrivial quadratic system (11) with the zero equilibrium, such that $c_1 = 0$ (it is always possible to obtain by replacement $x \to x + \nu y$ at $a_1 \neq 0$ or by reassignment $x \leftrightarrow y$) and $b_1\beta_1 \neq 0$.

In this case, system (11) can be transformed to Lienard system (17) with the functions (18). Taking into account (14), functions $F(x)$, $G(x)$ in (18) can be rewritten as:

$$\begin{aligned} F(x) &= (Ax^2 + Bx + C)|x+1|^{q-2}, \\ G(x) &= (C_1x^4 + C_2x^3 + C_3x^2 + C_4x + C_5)\frac{|x+1|^{2q}}{(x+1)^3}. \end{aligned} \tag{35}$$

From [Li, 2003; Lynch, 2005; Yu, 2005; Yu & Chen, 2008], it is known that if $L_1(0) = L_2(0) = 0$ and $L_3(0) \neq 0$, then using small perturbations of parameters of a quadratic system and, consequently, of system (17), it is possible to determine classes of systems having tree small limit cycles in a neighborhood of the point $x = y = 0$.

In this case we have conditions (23) for $C_1$, $C_2$, $C_3$, condition (24) for $L_3$, and

$$A = \frac{2}{5}B(q+2), \quad C_4 = 1, \quad C_5 = 0. \tag{36}$$

For such Lienard system (and corresponding quadratic system) can be analytically obtained estimations of domain of parameters $(B, q)$, corresponding to the existence of a large cycle on the left side from discontinuity line $x = -1$. It is important to notice that obtained domain expands the class of quadratic systems, considered by Shi [Shi, 1980].

For the further studying of this domain we will use numerical procedures and the method of asymptotic integration. Let's use in the Lienard equation the change of variables $x = -w - 2$ (symmetric map with respect to discontinuity line $x = -1$). Then, grouping coefficients at the degrees of $(1 + w)$, we will obtain

$$\begin{aligned}
&\ddot{w} + F(w)\dot{w} + G(w) = 0,\\
&F(w) = (\widetilde{A}_w(1+w)^2 + \widetilde{B}_w(w+1) + \widetilde{C}_w)|w+1|^{q-2},\\
&G(w) = (\widetilde{C}_{w1}(1+w)^4 + \widetilde{C}_{w2}(1+w)^3 + \widetilde{C}_{w3}(1+w)^2 + \widetilde{C}_{w4}(1+w) + \widetilde{C}_{w5})\frac{|w+1|^{2q}}{(w+1)^3}.
\end{aligned} \tag{37}$$

Here, taking into account conditions (23), (24), (36) we have

$$\begin{aligned}
\widetilde{A}_w &= \frac{2}{5}B(2+q),\\
\widetilde{B}_w &= \frac{1}{5}B(3+4q),\\
\widetilde{C}_w &= \frac{1}{5}B(-1+2q),\\
\widetilde{C}_{w1} &= \frac{1}{25}((3+q)B^2 - 15q - 5),\\
\widetilde{C}_{w2} &= \frac{1}{25}((4q+9)B^2 - 35 - 30q),\\
\widetilde{C}_{w3} &= \frac{1}{25}((6q+9)B^2 - 30 - 15q),\\
\widetilde{C}_{w4} &= \frac{1}{25}B^2(3+4q),\\
\widetilde{C}_{w5} &= \frac{1}{25}B^2q.
\end{aligned}$$

These formulas can be obtained with the help of following code.

```
clear all
syms A B q C1 C2 C3 C4 C5 x w
syms wC1 wC2 wC3 wC4 wC5 wD1 wD2 wD3 wD4 wD5 wA wB wC wDA wDB wDC

A = solve(5*A-4*B-2*B*q,A);
C1 =(B^2*q - 15*q + 3*B^2 - 5)/25;
C2 =(3*(B^2 - 10*q + 5))/25 ;
C3 =9/5 - (3*q)/5;

Fx = (A*x+B)*x*abs(x+1)^(q-2);
Gx = (C1*x^3+C2*x^2+C3*x^1+1)*x*abs(x+1)^(2*q)/(x+1)^3;

Fw = factor(subs(Fx,x,-w-2));
numFw = simplify(Fw/abs(w+1)^(q-2));
vA = subs(diff(numFw,w,2)/factorial(2),w,0);
vB = subs(diff(numFw,w,1)/factorial(1),w,0);
vC = subs(diff(numFw,w,0)/factorial(0),w,0);

Gw = -factor(subs(Gx,x,-w-2));
numGw = simplify(Gw/(abs(w+1)^(2*q)/(w+1)^3));
vC1 = subs(diff(numGw,w,4)/factorial(4),w,0);
vC2 = subs(diff(numGw,w,3)/factorial(3),w,0);
vC3 = subs(diff(numGw,w,2)/factorial(2),w,0);
vC4 = subs(diff(numGw,w,1)/factorial(1),w,0);
vC5 = subs(diff(numGw,w,0)/factorial(0),w,0);

Fw_Z =  (vA*w^2+vB*w+vC);
Fw_Zn =  collect((wDA*(1+w)^2+wDB*(1+w)+wDC),w);
w2 = subs(diff(Fw_Zn-Fw_Z,w,2)/factorial(2),’w’,0,0);
w1 = subs(diff(Fw_Zn-Fw_Z,w,1)/factorial(1),’w’,0,0);
w0 = subs(diff(Fw_Zn-Fw_Z,w,0)/factorial(0),’w’,0,0);
fC = solve(w2,w1,w0,wDA,wDB,wDC);
wA = factor(fC.wDA)
wB = factor(fC.wDB)
wC = factor(fC.wDC)

Gw_Z =  (vC1*w^4+vC2*w^3+vC3*w^2+vC4*w+vC5);
Gw_Zn=collect(wD1*(1+w)^4+wD2*(1+w)^3+wD3*(1+w)^2+wD4*(1+w)+wD5,w);
w4 = subs(diff(Gw_Zn-Gw_Z,w,4)/factorial(4),’w’,0,0);
w3 = subs(diff(Gw_Zn-Gw_Z,w,3)/factorial(3),’w’,0,0);
w2 = subs(diff(Gw_Zn-Gw_Z,w,2)/factorial(2),’w’,0,0);
w1 = subs(diff(Gw_Zn-Gw_Z,w,1)/factorial(1),’w’,0,0);
w0 = subs(diff(Gw_Zn-Gw_Z,w,0)/factorial(0),’w’,0,0);

fC = solve(w4,w3,w2,w1,w0,wD1,wD2,wD3,wD4,wD5);
wC1 = factor(fC.wD1)
wC2 = factor(fC.wD2)
wC3 = factor(fC.wD3)
wC4 = factor(fC.wD4)
wC5 = factor(fC.wD5)
```

Note that, in according to formulas (23), (24), (36), for studying of existence of cycles it is enough to consider $B$ with one sign only. Let’s consider further

$$B < 0,$$

$B/A > 1$ and $q \in (-1, 0)$.

Note that in these conditions function $G(w)$ has one zero (i.e. one equilibrium of system) on the right side from discontinuity line $w = -1$.

Let's rewrite the equation (37) in the form

$$\begin{aligned} \dot{w} &= y, \\ \dot{y} &= -F(w)y - G(w). \end{aligned} \tag{38}$$

Applying various replacements $w = D(z)$ we will use described above method of asymptotic reduction of the system (38) to the system

$$\begin{aligned} \dot{z} &= y, \\ \dot{y} &= -\widetilde{F}_0 y - \widetilde{G}_1 z. \end{aligned} \tag{39}$$

As before we will denote

$$\Delta = -(\frac{\widetilde{F}_0^2}{4} - \widetilde{G}_1),$$

$$\lambda = \frac{-\widetilde{F}_0}{2}.$$

Fix some numbers $a > -1$ and $\delta > 0$, and consider sufficiently large numbers $R$ and $\widetilde{R}$. Then the following lemmas take place.

1) For $w \in (-1, +\infty)$ in the equation (37) change the variables

$$w = D(z) = z^{\frac{1}{q+1}} - 1 : \ z \in (0, +\infty) \nearrow w \in (-1, +\infty),$$

and obtain

$$\begin{aligned} \widetilde{F}(z) &= \frac{1}{(q+1)}(\widetilde{A}_w + \widetilde{B}_w z^{-\frac{1}{q+1}} + \widetilde{C}_w z^{-\frac{2}{q+1}}), \\ \widetilde{G}(z) &= \frac{1}{(q+1)} z(\widetilde{C}_{w1} + z^{-\frac{1}{q+1}}\widetilde{C}_{w2} + z^{-\frac{2}{q+1}}\widetilde{C}_{w3} + z^{-\frac{3}{q+1}}\widetilde{C}_{w4} + z^{-\frac{4}{q+1}}\widetilde{C}_{w5}). \end{aligned} \tag{40}$$

Here

$$\widetilde{F}_0 = \frac{\widetilde{A}_w}{(q+1)},$$

$$\widetilde{G}_1 = \frac{\widetilde{C}_{w1}}{(q+1)}.$$

**Lemma 3** *Let $\Delta > 0$. Then for the solution of system (38) with initial data*

$$w(0) = a, \ y(0) = R$$

*there exists a number $T > 0$ such that*

$$w(T) = a, \ y(T) < 0,$$

$$w(t) > a, \ \forall\, t \in (0, T),$$

$$R \exp\left(\frac{\lambda\pi}{\sqrt{\Delta}} - \delta\right) < |y(T)| < R \exp\left(\frac{\lambda\pi}{\sqrt{\Delta}} + \delta\right).$$

2) For $w \in (-1, 0)$ in the equation (37) change the variables

$$w = D(z) = z^{\frac{1}{q-1}} - 1 : \; z \in (1, +\infty) \searrow w \in (-1, 0),$$

and obtain

$$\begin{aligned}\widetilde{F}(z) &= \frac{1}{(q-1)}(\widetilde{A}_w z^{\frac{2}{q-1}} + \widetilde{B}_w z^{\frac{1}{q-1}} + \widetilde{C}_w), \\ \widetilde{G}(z) &= \frac{1}{(q-1)} z(z^{\frac{4}{q-1}}\widetilde{C}_{w1} + z^{\frac{3}{q-1}}\widetilde{C}_{w2} + z^{\frac{3}{q-1}}\widetilde{C}_{w3} + z^{\frac{4}{q-1}}\widetilde{C}_{w4} + \widetilde{C}_{w5}).\end{aligned} \tag{41}$$

Here

$$\widetilde{F}_0 = \frac{\widetilde{C}_w}{(q-1)},$$

$$\widetilde{G}_1 = \frac{\widetilde{C}_{w5}}{(q-1)}.$$

**Lemma 4** *Let $\Delta < 0$. Then for the solution of system (38) with initial data*

$$w(0) = a, \quad y(0) = -\widetilde{R}$$

*there exists a number $T > 0$ such that*

$$w(T) = a, \; 0 < y(T) < \delta\widetilde{R},$$

$$w(t) \in (-1, a), \; \forall\, t \in (0, T).$$

Here the forms of functions $\widetilde{F}(z)$ and $\widetilde{G}(z)$ are well adapted to asymptotic analysis of trajectories with large initial conditions since the terms

$$z^{-\frac{k}{q+1}} \quad \text{and} \quad z^{\frac{k}{q-1}}, \; k = 1, \ldots, 4,$$

are infinitesimal at infinity, and omitting these terms, it is possible to pass to analysis of second-order equations with constant coefficients.

These lemmas imply the following theorem.

**Theorem 1** *Let*

$$B^2 < -5(q+1)(3q+1) \tag{42}$$

*is valid. Then system (17) has a limit cycle in the half-plane $\{x < -1, \quad y \in R^1\}$.*

Thus, small perturbations and the conditions described in Theorem 1 determine classes of systems (17) and (11) with four limit cycles.

The region of parameters $B$, $q$ for which condition (42) is satisfied is presented in Fig. 6. The region bounded by lines in Fig. 6 corresponds to the lines of reversed sign of the third Lyapunov quantity.

For constructing of this region the following procedure can be determined.

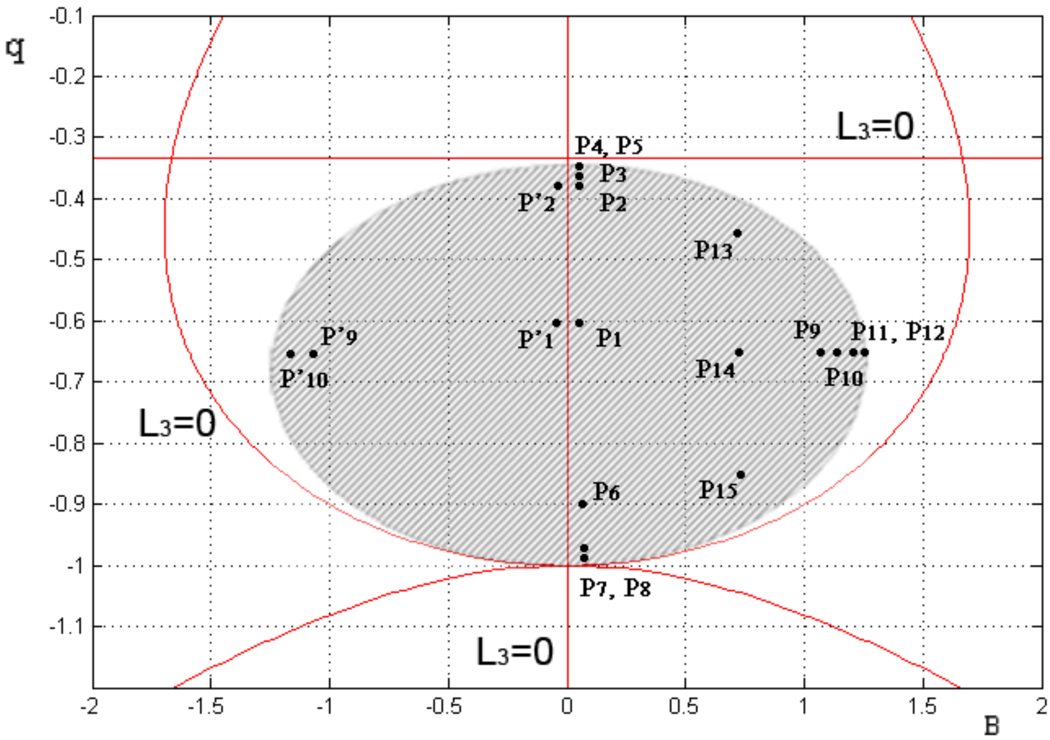


FIGURE 6 Region of parameters for computer modeling of large limit cycles for quadratic and Lienard systems.

```
function d=plotd(fxy,x,y,xd,yd,color)
for xi=1:length(xd)
    for yi=1:length(yd)
      if (subs( subs(fxy,[x y],[xd(xi) yd(yi)]))> 0)
         plot( xd(xi), yd(yi),'--rs','LineWidth',1,'Color',color);
         hold on;
      end;
    end;
end;
end;
```

Then the region in Fig. 6 can be constructed with the help of following code.

```
clear all
syms B q
q_min = -2; q_max = 2; q_step  = 0.03;
B_min = -2; B_max = 2; B_step  = 0.03;
Bq = -B^2-5*(q+1)*(3*q+1);
plotd(Bq, B, q, [B_min:B_step:B_max], [q_min:q_step:q_max],'red');
hold on; grid on;
hold off;
```

Thus, using small perturbations and the algorithm for constructing small cycles for a Lienard system with parameters from this region, it is possible to construct systems with four cycles: three small cycles around the zero equilibrium and one large cycle around another equilibrium.

### 1.3.3 Computer modeling of large limit cycles

The computer modeling and computational experiments were performed with the help of mathematical package MatLab R2008b. For construction of trajectories of quadratic systems and Lienard system of the form (17), built-in functions of approximate solution of differential equations of ode packet MatLab were used. However, in each case it is necessary to match individual parameters which are necessary for computing the trajectories of motion, such as absolute and relative accuracy, original and maximal steps, and the computing interval. For construction of phase portraits of some unstable cycles, it was necessary previously to study the system at inverse time. It is also essential that even on a powerful computer, computation of one trajectory for certain values of parameters requires a considerable time. For example, the study of the region of parameters of the Lienard system that correspond to the existence of four cycles requires a few months of computer time (approximately a month of pure machine time on dual cores Pentium 4).

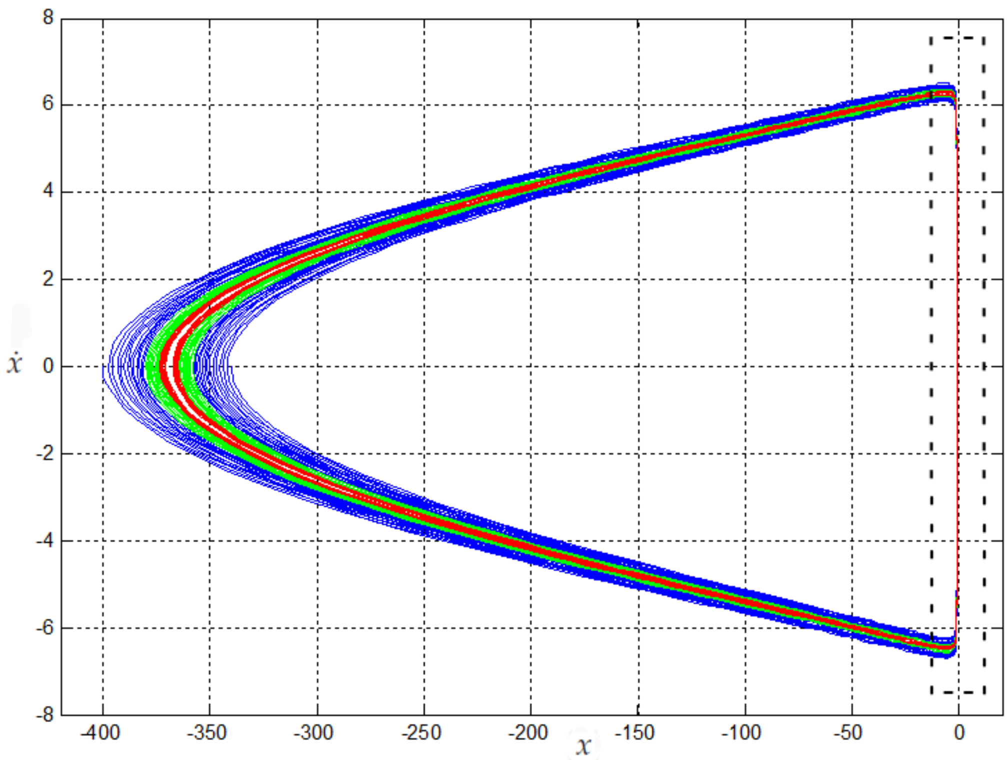


FIGURE 7 Limit cycle of a Lienard system.

Thus, even by using modern powerful hardware and special-purpose mathematical packages, study of systems of differential equations with the help of computer modeling takes a lot of time.

The most illustrative example of a large unstable limit cycle (Fig. 7) in a neighborhood of the nonzero stationary point for the Lienard system was ob-

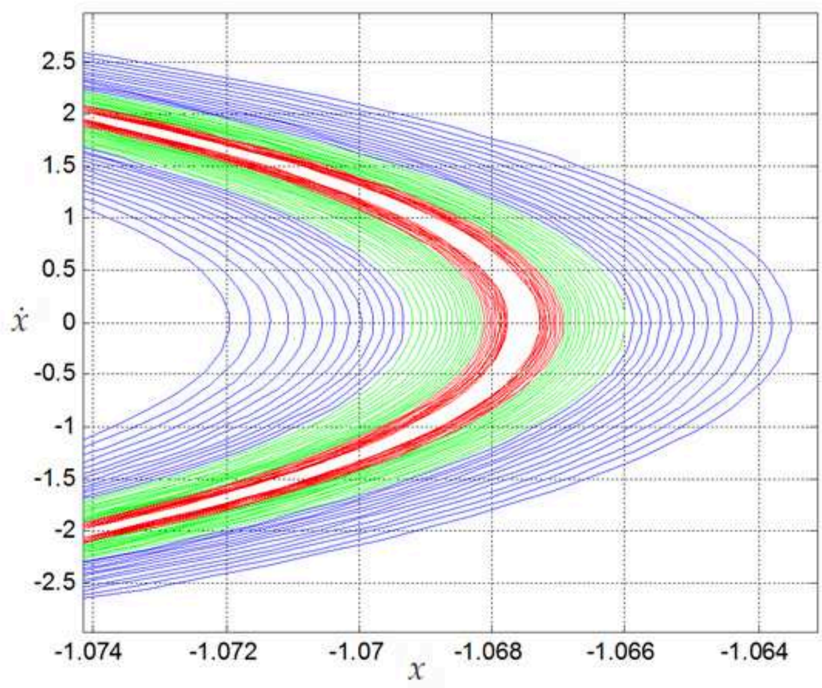


FIGURE 8 Limit cycle of a Lienard system. Flattening domain.

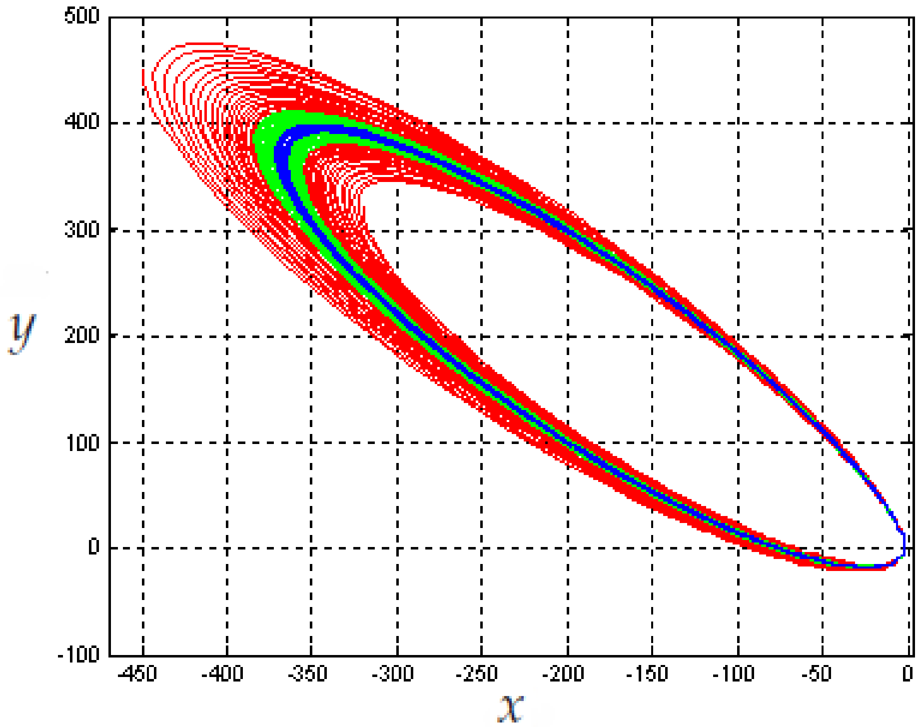


FIGURE 9 Stable limit cycle of a quadratic system (point P1).

tained for the system with parameters $B = 0.01$ and $q = -0.6$ (corresponding to the point $P1$ in Fig. 6).

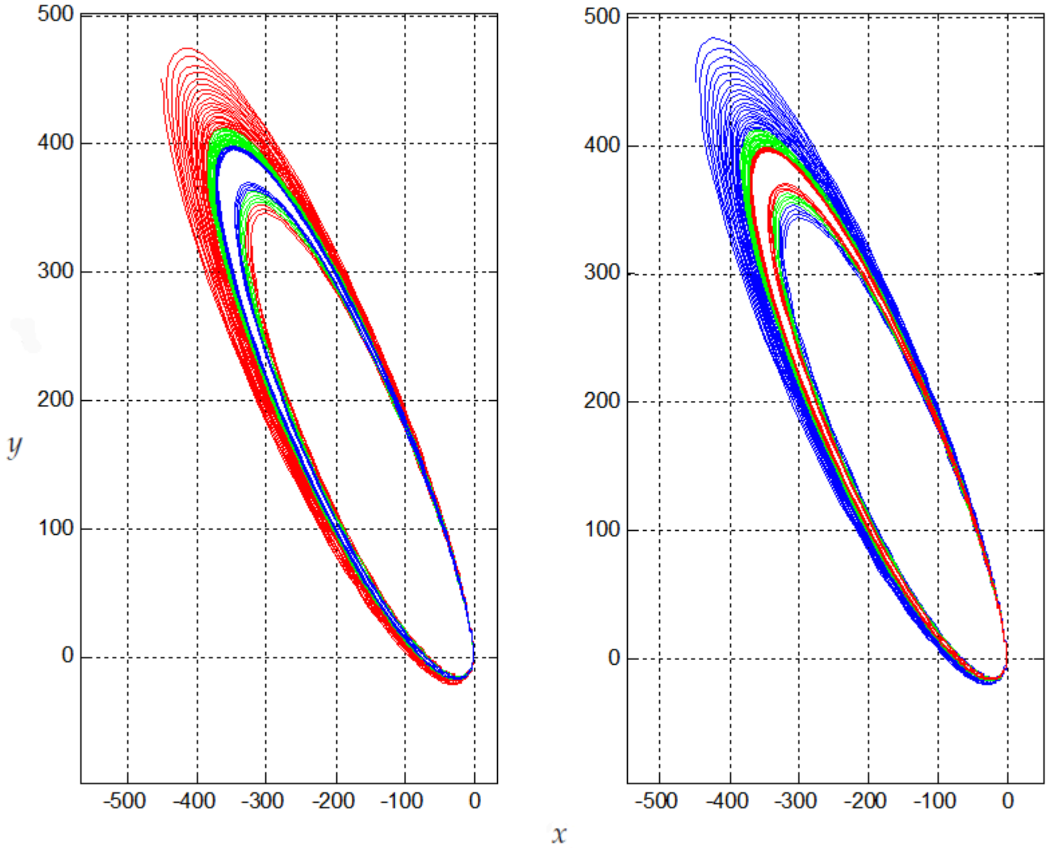


FIGURE 10 Stable and unstable limit cycles of quadratic systems (points P1 and P'1).

We indicate two trajectories starting near the cycle situated in the center of the red domain. The first trajectory winds around a stationary point, and the second trajectory unwinds and goes away from the cycle.

In the right-hand side of the graph, intensive "flattening" of trajectories in a neighborhood of the straight line $x = -1$ is observed.

This effect of "trajectory rigidity" complicates calculations considerably, especially for Lienard system in a neighborhood of straight line $x = -1$.

A scaled-up picture of the flattening domain (indicated in Fig. 7 by dotted line) is represented in Fig. 8.

For the above Lienard system, we obtained the corresponding quadratic system using the conversion of the Lienard system into a quadratic system. The large limit cycle for the indicated quadratic system is shown in Fig. 9.

Note that the two halves of the region in Fig. 6 are symmetric [Leonov, Kuznetsov & Kudryashova, 2008]. They differ by the sign of the third Lyapunov quantity. If $B > 0$, we have $L_3 < 0$, and if $B < 0$, we have $L_3 > 0$. Thus, for a quadratic system with $B > 0$, we can obtain a stable large limit cycle, and with $B < 0$, we can obtain an unstable large limit cycle. For a Lienard system with $B > 0$, we can obtain an unstable large limit cycle and with $B < 0$, we can obtain a stable large limit cycle.

Symmetry can be seen in Fig. 10, where the left picture shows a stable large

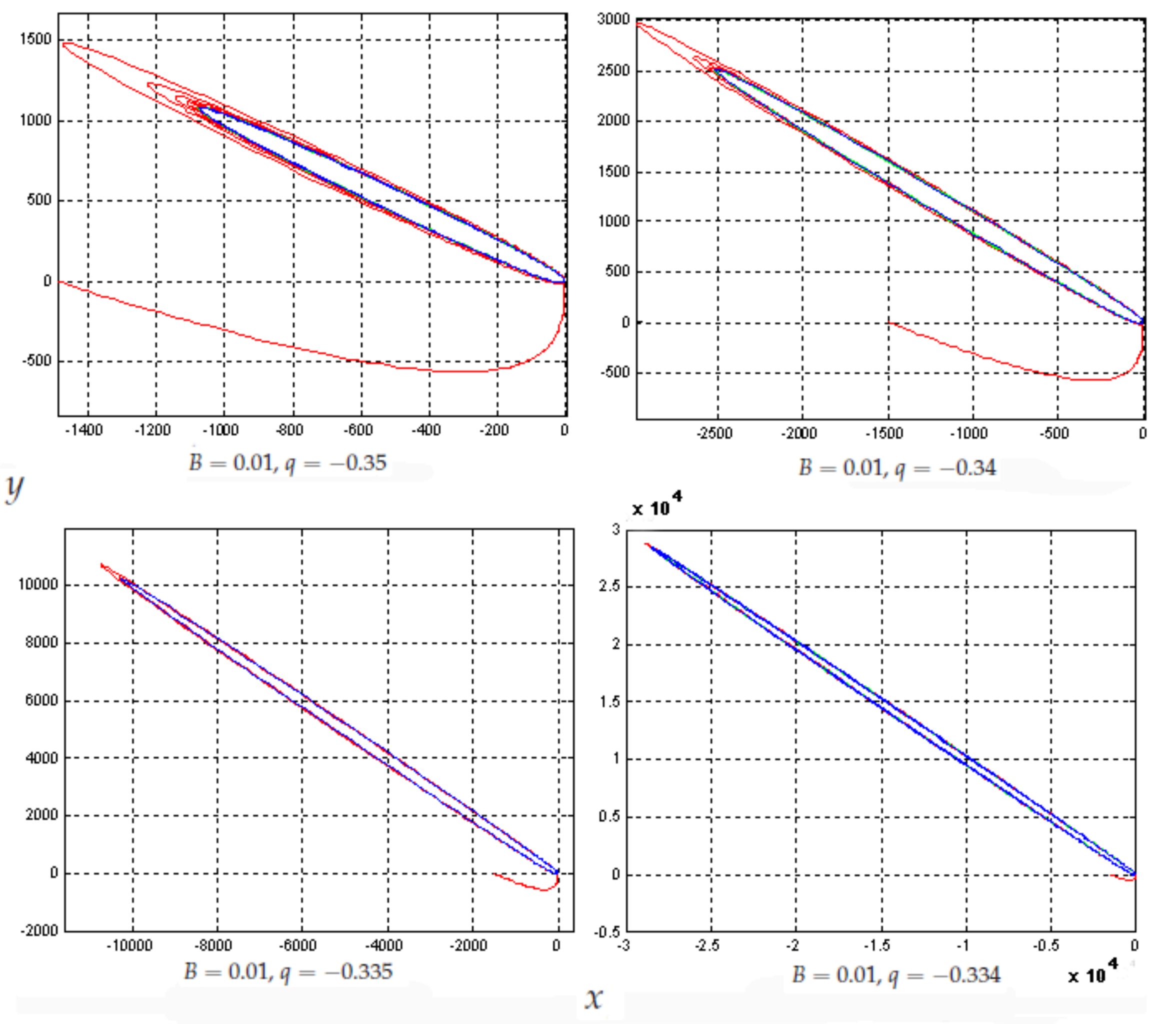


FIGURE 11 Increase of the size of a limit cycle of a quadratic system (points P2, P3, P4, P5).

limit cycle for the quadratic system which corresponds to the point $P1$ from Fig. 6. In the right picture of Fig. 10, we show an unstable large limit cycle for the point $P'1$ from Fig. 6.

It is clear that stable cycles are easier to simulate than unstable ones. Because of symmetry, we can model limit cycles for quadratic systems with $B > 0$ and for Lienard system with $B < 0$.

In computer experiments, some dependence of change of the cycle size on change of parameters of system has been found. Let us describe what happens to a large limit cycle if we take systems with parameters $B, q$ near the region border.

Increase of the size of a limit cycle of quadratic systems which correspond to points $P2$, $P3$, $P4$, $P5$ is shown in Fig. 11. A small change of value of parameter $q$ leads to increase of the size of a cycle.

The same effect is shown in Fig. 12, Fig. 13, Fig. 14, where limit cycles for points $P6$, $P7$, $P8$ are presented. For all three points, parameter $B$ is equal to 0.01.

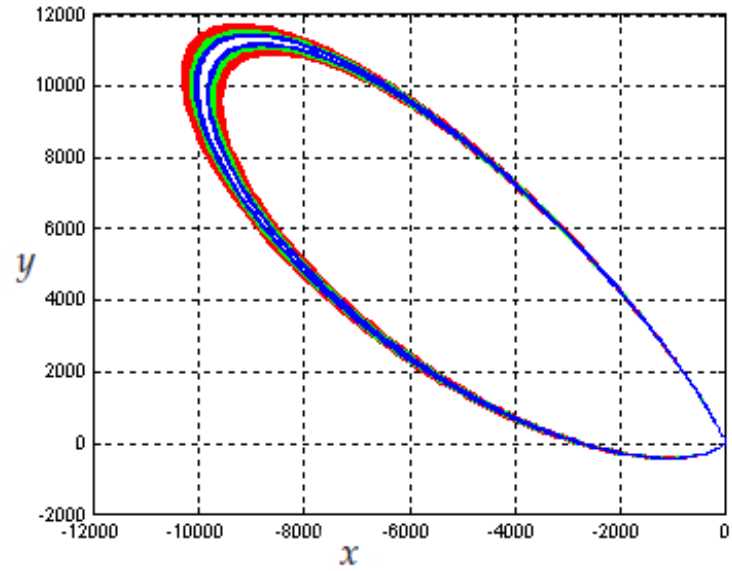


FIGURE 12 Limit cycle of a quadratic system (point P6).

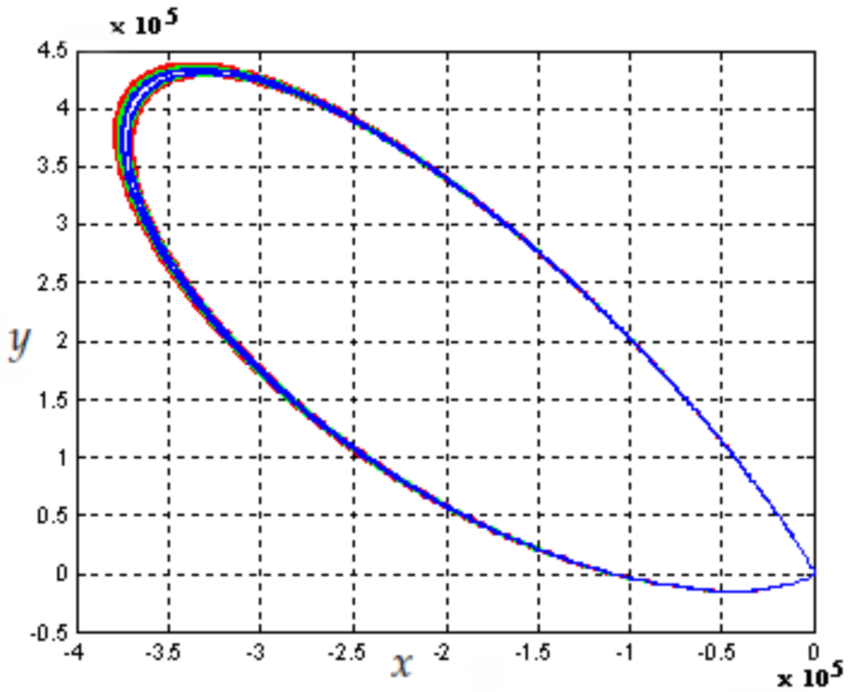


FIGURE 13 Limit cycle of a quadratic system (point P7).

In Fig. 12, the cycle corresponds to a system with $q = -0.9$ and has size smaller than $10^4$. In Fig. 13 and Fig. 14, where $q = -0.99$ and $q = -0.999$, correspondingly, one can observe cycles with size larger than $10^5$.

Note that the further approach to the top and bottom borders of the domain in Fig. 6 is accompanied by increase of complexity of calculations. It is neces-

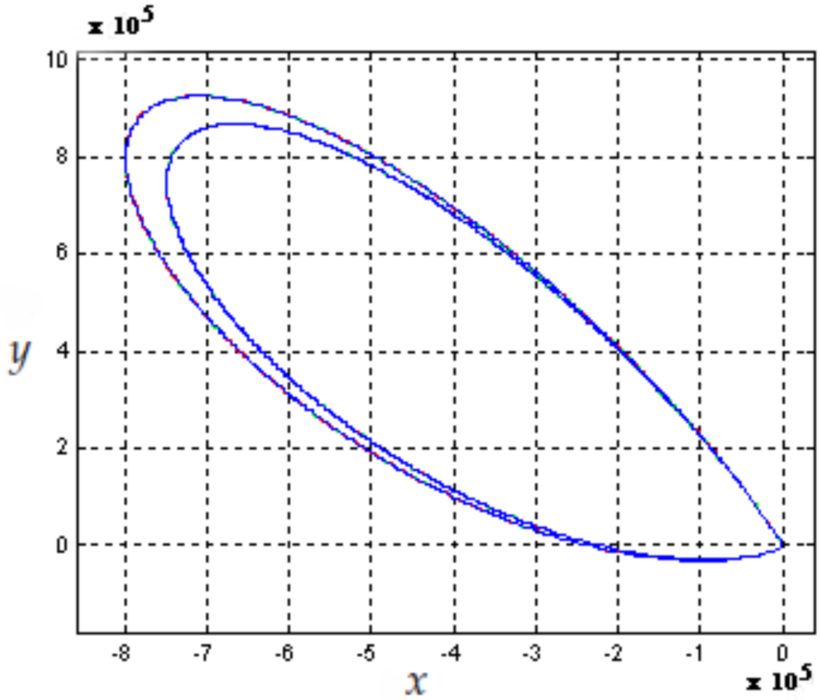


FIGURE 14 Limit cycle of a quadratic system (point P8).

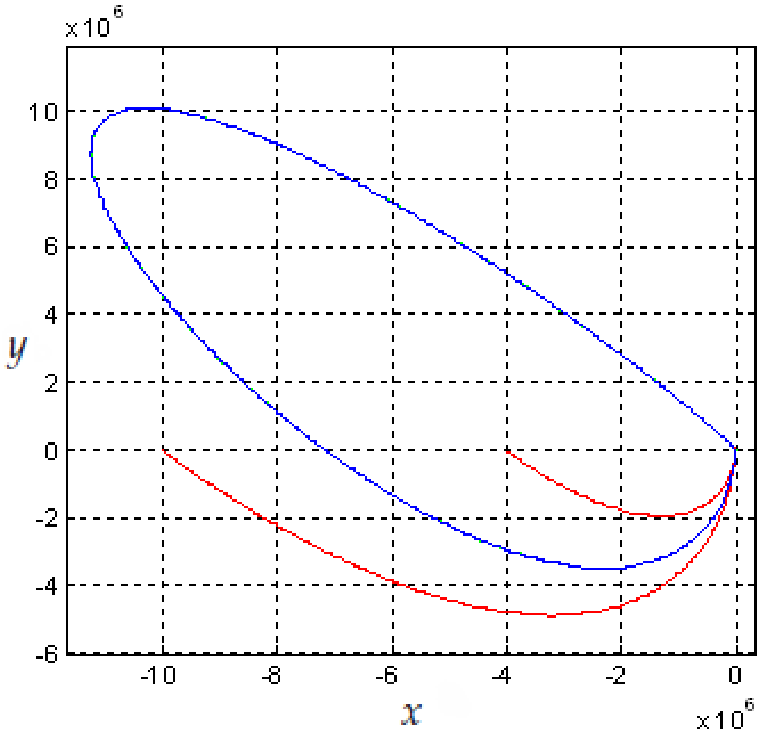


FIGURE 15 Limit cycle of a quadratic system (point P9).

sary to increase accuracy of calculations, and the time of constructing of cycles accordingly increases.

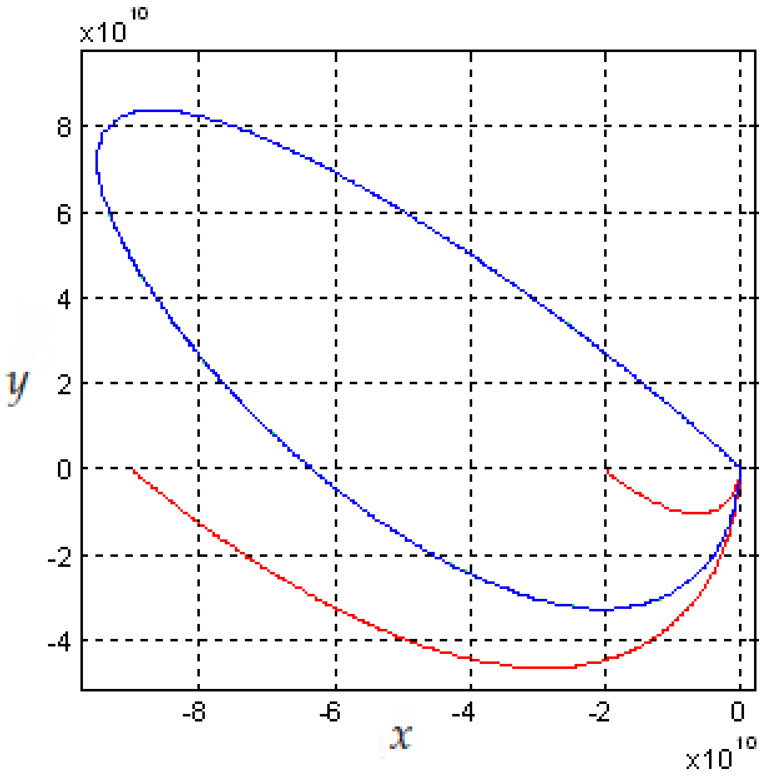


FIGURE 16 Limit cycle of a quadratic system (point P10).

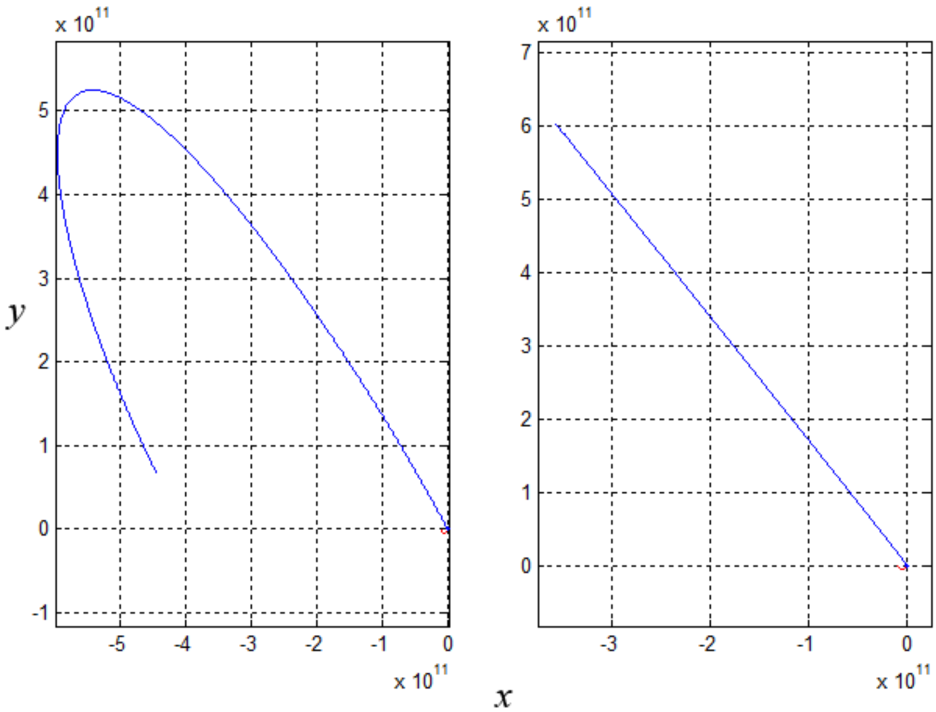


FIGURE 17 Disappearing of cycles of a quadratic system (points P11 and P12).

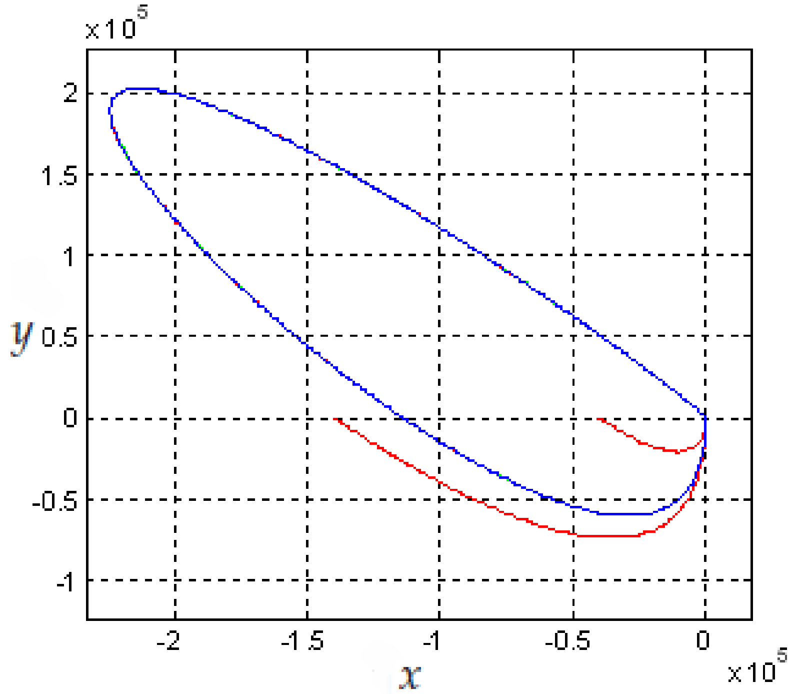


FIGURE 18  Limit cycle of a quadratic system (point P13).

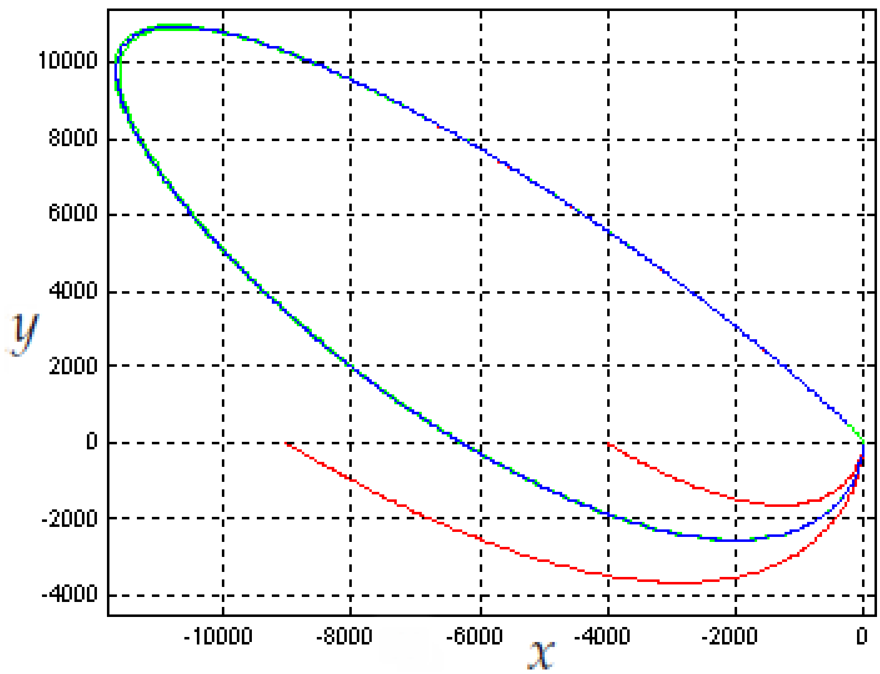


FIGURE 19  Limit cycle of a quadratic system (point P14).

Now we describe the deformation of a limit cycle of a system with parameters $B$, $q$ close to the right border of region from Fig. 6. Due to the symmetry of the region, the same happens near the left border.

Fig. 15 presents a stable large cycle of a quadratic system with parameters

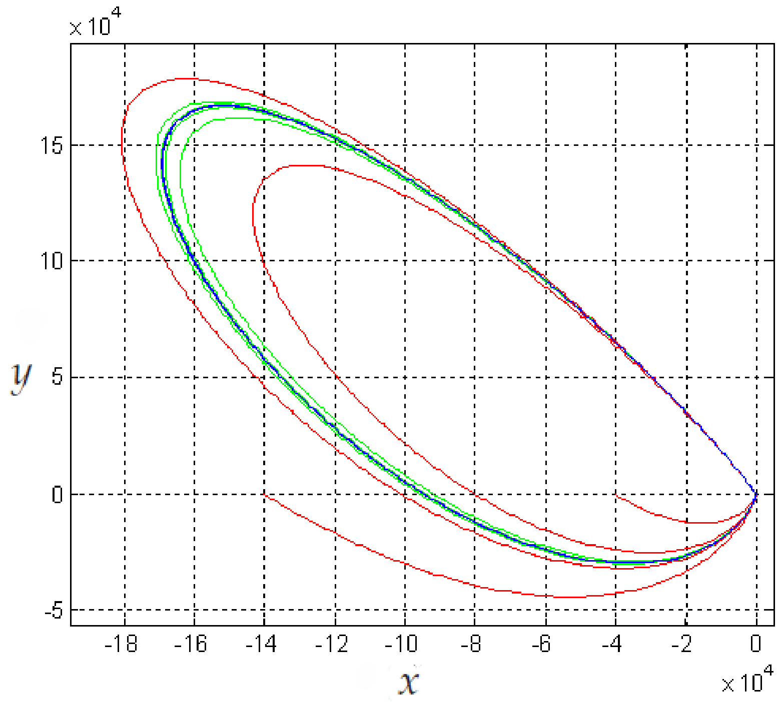


FIGURE 20 Limit cycle of a quadratic system (point P15).

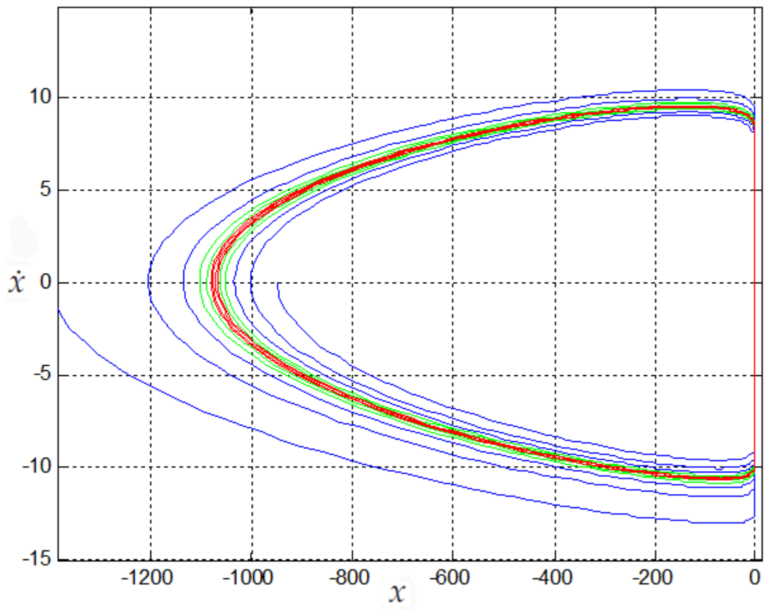


FIGURE 21 Limit cycle of a Lienard system (point P'2).

corresponding to $B = 1.1$, $q = -0.65$. This system corresponds to the point $P9$, and order of the cycle is $10^4$.

In Fig. 16, we show a cycle with order $10^{10}$. This cycle was constructed for a quadratic system with parameters $B = 1.2$, $q = -0.65$ (point $P10$).

From the figures one can conclude that the size of a cycle has increased in

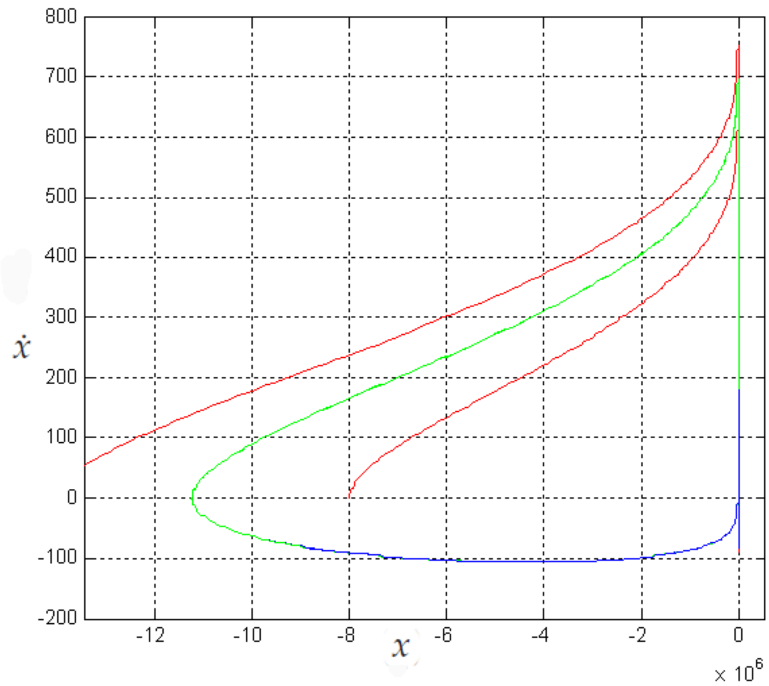


FIGURE 22 Limit cycle of a Lienard system (point P’9).

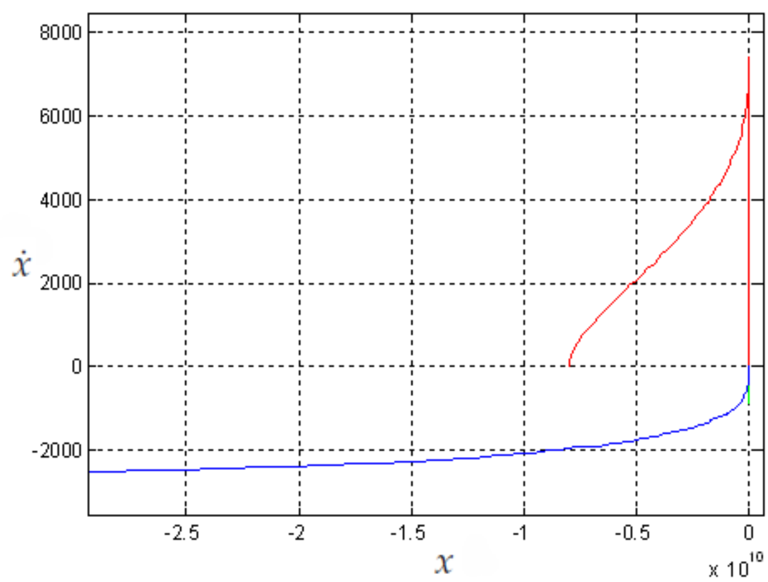


FIGURE 23 Unwinding trajectory of a Lienard system (point P’10).

comparison with the size of a cycle in Fig. 15.

In Fig. 17, we can see how a limit cycle disappears. In the left picture, the unwinding trajectory with order $10^{11}$ corresponds to a quadratic system with parameters $B = 1.21$, $q = -0.65$ (point $P11$). In the right picture unwinding trajectory with parameters $B = 1.25$, $q = -0.65$ (point $P12$) is shown.

The further approach to the left and right borders of the domain from Fig. 6

is accompanied by quick increase of the cycle, and it becomes impossible to construct a cycle by means of built-in possibilities of MatLab.

We can note one more feature of trajectories: the speed of winding of trajectories decreases with increase of $|q|$. Consider the points $P13$ (Fig. 18), $P14$ (Fig. 19), and $P15$ (Fig. 20). In Fig. 18, trajectories reach the area of the cycle faster, while in Fig. 20 trajectories at first have some turns, and only then reach the area of the cycle. It follows from this that modeling of cycles from the lower part of the area requires more time.

As was stated earlier, it is more difficult to construct a limit cycle for the Lienard system. But all the described effects are observed for Lienard system too. In Fig. 21, we show a stable cycle for a Lienard system with parameters $B = -0.01$, $q = -0.35$ (point $P'2$). The size of the cycle has also increased compared with the size of the cycle in Fig. 7.

In Fig. 22, we show a stable limit cycle for a Lienard system with parameters $B = -1.1, q = -0.65$ (point $P'9$).

In Fig. 23, we can see only one unwinding trajectory for a Lienard system with parameters $B = -1.2$, $q = -0.65$ (point $P'10$). The order of the cycle is more than $10^{10}$. Thus, by means of computer modeling, we have studied the region of parameters of a Lienard system for which the existence of large limit cycles was proved by G.A. Leonov. Large cycles for Lienard and quadratic systems with parameters from the specified domain are obtained. As a result of the performed research, we have established the presence of a "trajectory rigidity" effect, when major flattening complicates numerical localization of a limit cycle. We have also shown scenarios of "destruction" of large cycles related to approaching the boundaries of the coefficients domain. These results were partly presented at international conferences [Kudryashova *et al.,* 2008; Leonov *et al.,* 2008], were included in the plenary report [Leonov, Kuznetsov & Kudryashova, 2008[1]], and were partly published in the article [Leonov, Kuznetsov & Kudryashova, 2008].

### 1.3.4 Visualization of results of S.L. Shi

For a long time, it was believed that the maximal number of limit cycles of a quadratic system is three. This belief was based on a paper by Petrovskii and Landis [Petrovskii & Landis, 1957], which later turned out to be erroneous [Petrovskii & Landis, 1959]. Probably, S.L. Shi [Shi, 1980] was the first one to produce an example of a quadratic system with four limit cycles.

In [Shi, 1980] the following quadratic system was considered:

$$\begin{aligned} \dot{x} &= \lambda x - y + lx^2 + (5a+\delta)xy + ny^2, \\ \dot{y} &= x + ax^2 + (3l + 5n + (\delta(l+n) + 8\epsilon)/a)xy, \end{aligned} \tag{43}$$

and the following theorem was stated.

**Theorem 2** *If*

$$a(2a^2 + 2n^2 + ln)(a(5l+6n) - 3(l+2n)(l+n^2)) \neq 0,$$
$$\delta a(2a^2 + 2n^2 + ln) > 0, \quad \delta\epsilon < 0, \quad \epsilon\lambda < 0,$$

$$
\begin{gathered}
0 < |\lambda| << |\epsilon| << |\delta| << 1, \\
3a^2 - l(l+2n) < 0, \\
25a^2 + 12n(l+2n) < 0, \\
l(3l+5n)^2 - 5a^2(3l+5n) + na^2 < 0, \\
a^2(5l+8n) - ((2l+5n)^2 + 15a^2)(25a^2 + 3n(2l+5n)) > 0,
\end{gathered}
$$

*then there are three limit cycles around the origin (0,0), and there is a limit cycle around the equilibrium point* $(0, 1/n)$.

The system with parameters

$$l = -10, \quad a = 1, \quad \delta = -10^{-13}, \quad \epsilon = 10^{-52}, \quad \lambda = -10^{-240},$$

was offered as an example of system with four limit cycles. No representation of the region of all parameters corresponding to conditions of Theorem 2 has been obtained.

A descriptive representation of the region of parameters satisfying Theorem 2 was constructed in the present work with the help of conversion of system (43) into a Lienard system. At first, system (43) was converted into a quadratic system of form (16), and then into a Lienard system.

The following function is used to determine the matrix of coefficients for the quadratic system.

```
% This function is used to determine the
% matrix of coefficients  [c11 c12 c13 c14 c15; c21 c22 c23 c24 c25]
% from the quadratic system
% dx = c11*x^2+ c12*x*y +c13*y^2 + c14*x+ c15*y;
% dy = c21*x^2+ c22*x*y +c23*y^2 + c24*x+ c25*y;
  function coeffsQsys = fDefCoefQs(dx,dy)
  syms x y
  c14 = simplify(subs(diff(subs(dx,y,0),x),x,0));
  c24 = simplify(subs(diff(subs(dy,y,0),x),x,0));
  c15 = simplify(subs(diff(subs(dx,x,0),y),y,0));
  c25 = simplify(subs(diff(subs(dy,x,0),y),y,0));
  c11 = simplify(diff(subs(dx,y,0)/x,x));
  c21 = simplify(diff(subs(dy,y,0)/x,x));
  c13 = simplify(diff(subs(dx,x,0)/y,y));
  c23 = simplify(diff(subs(dy,x,0)/y,y));
  c12 = simplify(diff(diff(dx,x),y));
  c22 = simplify(diff(diff(dy,x),y));
  coeffsQsys = [c11 c12 c13 c14 c15; c21 c22 c23 c24 c25];
```

The following function is used to replace variables in the quadratic system.

```
% In quadratic system \dot_x= dx, \dot_y = dy
% we do replacement
% x_new = x*h1+y*v1;
% y_new = x*h2+y*v2;
% and have got a new quadratic system
% dx_new = 1/h1 * (dx-v1*dy); dy_new  = 1/v2 * (dy-h2*dx)
  function [dx_new, dy_new] = fReplaceInQs(dx,dy,h1,v1,h2,v2)
  syms x y
  dx_new = simplify(subs(dx,[x y],[x*h1+y*v1 x*h2+y*v2])/h1 -
                          - v1*subs(dy,[x y],[x*h1+y*v1 x*h2+y*v2])/h1);
  dy_new = simplify(subs(dy,[x y],[x*h1+y*v1 x*h2+y*v2])/v2 -
                          - h2*subs(dx,[x y],[x*h1+y*v1 x*h2+y*v2])/v2);
```

The following code can be used to conversion of the quadratic system (43) into a Lienard system.

```
% Transformation of the quadratic system, considered in [Shi, 1980],
% dot_x = l*x^2+ (5*a+delta)*(x*y) +n*y^2 + lambda*x-y
% dot_y = a*x^2+ (3*l+5*n+(delta*(l+n)+8*eps)/a)*(x*y) + x
% into Lienard system
% dot_x = y
% dot_y = -F*y-G
  clear all
  syms a l n x y k a1 b1 c1 al1 bt1 a2 b2 c2 al2 bt2

% Consider the quadratic system, described in [Shi, 1980], where
% parameters delta =-10^(-13), eps =10^(-52) and lambda =-10^(-240)
% are small disturbances of the system.
% So, for our task we can suppose, that
  delta=0; eps = 0; lambda = 0;

  dx = l*x^2+ (5*a+delta)*(x*y) +n*y^2 + lambda*x-y;
  dy = a*x^2+ (3*l+5*n+(delta*(l+n)+8*eps)/a)*(x*y) + x;

% Following [Leonov, Kuznetsov & Kudryashova, 2008],
% consider transformation of investigated system into
% a quadratic system in the simple form
% dx = a1*x^2 + x*y + x + y;
% dy = a2*x^2 + b2*x*y +c2*y^2-2*x-y;

% 1.Transformation of system into a quadratic system, where c1 = 0.
% Following [Leonov, Kuznetsov & Kudryashova, 2008], Proposition 1,
% consider replacement x_new=x + k*y; y_new = y
  [dx,dy] = fReplaceInQs(dx,dy,1,k,0,1);
% Determine value of k for which c1=0:
  Qs = fDefCoefQs(dx,dy);
  k_tmp_array = solve(Qs(1,3),k);
% For symbolic computation in MatLab
% consider the first solution as Real value
  k_tmp = k_tmp_array(1);
% So, we have got quadratic system, where c1=0
  dx = subs(dx, k, k_tmp);
  dy = subs(dy, k, k_tmp);

% 2. Transformation into a quadratic system,
% where al2=-2al1^2/bt1.
% Following [Leonov, Kuznetsov & Kudryashova, 2008], Proposition 2,
% consider replacement x_new = x; y_new = k*x + y,
% where k = -al1/bt1+(-al1^2-bt1*al2)^(1/2)/bt1
  Qs = fDefCoefQs(dx,dy);
  [dx,dy] = fReplaceInQs(dx,dy,1,0,-Qs(1,4)/Qs(1,5)+ (-(Qs(1,4))^2 -
                    - Qs(1,5)*Qs(2,4))^(1/2)/Qs(1,5),1);

% 3. Transformation into a quadratic system,
% where bt1 = 1, al2 = -2 and b1 = 1,
% Following [Leonov, Kuznetsov & Kudryashova, 2008], Proposition 3,
% we have b1<>0, then consider replacement x_new = k*x; y_new = m*y,
% where k = bt1/b1, m = al1/b1
  Qs = fDefCoefQs(dx,dy);
  [dx,dy] =fReplaceInQs(dx,dy,Qs(1,5)/Qs(1,2),0,0,Qs(1,4)/Qs(1,2));
```

```
% So, we have got a quadratic system in the simple form
% dx = a1*x^2 + b1*x*y + x + y,  b1=1;
% dy = a2*x^2 + b2*x*y + c2*y^2 - 2*x - y;
% For transformation into Lienard system,
% we will return to determining of parameters of quadratic system
  Qs = fDefCoefQs(dx,dy);
[a1 b1 c1 al1 bt1 a2 b2 c2 al2 bt2]=[Qs(1,1) Qs(1,2) Qs(1,3) Qs(1,4)
                   Qs(1,5) Qs(2,1) Qs(2,2) Qs(2,3) Qs(2,4) Qs(2,5)];

% Transformation into a Lienard system
Fx = (-b2+2*c2*a1-a1)*x+(1-b2+2*c2-2*a1)*x*abs(x+1)^(-c2) /(x+1)^2;
Gx = -((-b2*a1+c2*a1^2+a2)*x^3 + (-2-b2*a1+a1-b2+2*a2+2*c2*a1)*x^2 +
              + (a2+c2+a1-3-b2)*x -1)*x *abs(x+1)^(2*(-c2))/(x+1)^3;
```

Thus, for parameters of system (43) satisfying Theorem 2, the corresponding parameters $B$ and $q$ of a Lienard system were found. The obtained region of parameters of a Lienard system is presented in Fig. 24.

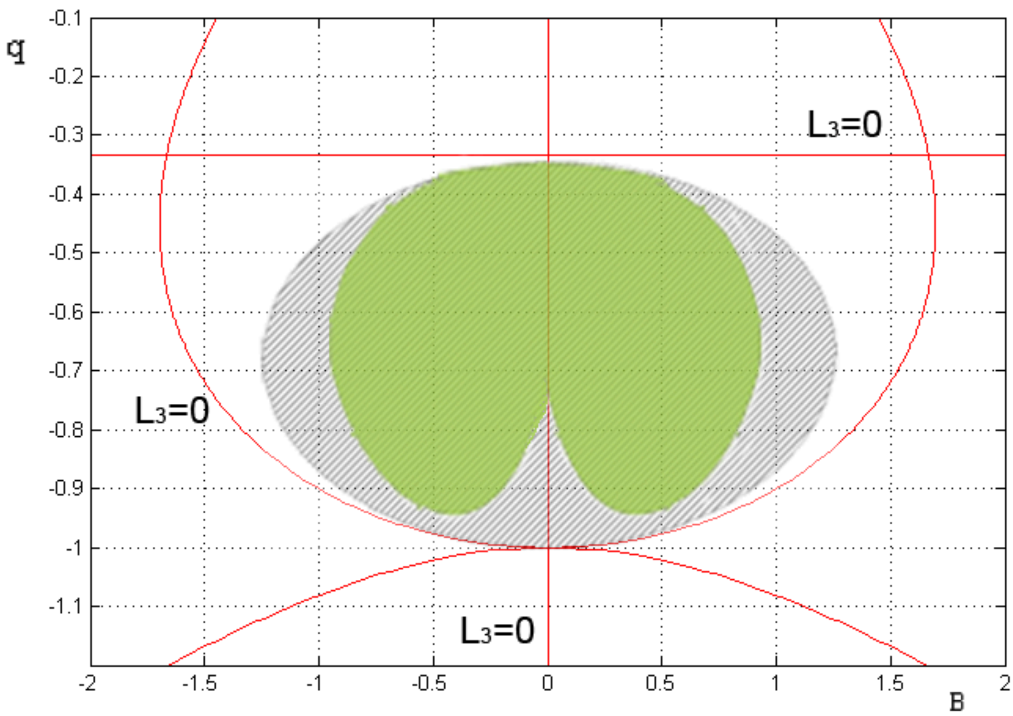


FIGURE 24 Region of parameters of a Lienard system for which there exist four limit cycles satisfying the results of S.L. Shi for quadratic systems.

The visualization constructed in the present paper allows us to see (Fig. 24) that the domain obtained by theoretical methods in [Shi, 1980] completely lies in the domain obtained by theoretical methods in [Leonov, 2009].

## 1.4 Conclusion of Chapter 1

The use of classical analytical methods and modern software tools for symbolic computing enables one to obtain formulas for calculation of Lyapunov quantities in the general form. Using the described algorithms for computation of Lyapunov quantities and a method of constructing small limit cycles allows one to solve Kolmogorov's problem about small limit cycles.

We present large limit cycles for quadratic and Lienard systems with parameters corresponding to the domain obtained by G.A. Leonov [Leonov, 2009], where large limit cycles exist. Using small perturbations and an algorithm for constructing small limit cycles for a Lienard system with parameters from this domain, it is possible to construct systems with four cycles: three small limit cycles around the zero equilibrium and one large limit cycle around another equilibrium. Visualization on the plane of parameters of a Lienard system corresponding to the conditions of S.L. Shi [Shi, 1980] is obtained. Using analytical methods and computer modeling allows us to solve Kolmogorov's problem about large limit cycles.

Thus, Kolmogorov's problem about study of limit cycles can be solved by means of computer modeling.

# 2 COMPUTER MODELING OF BIFURCATIONS OF A DISCRETE MODEL OF PHASE-LOCKED LOOPS

The study of limit cycles for discrete dynamical systems has a long history and important applications in various areas of research. The following nonlinear difference equation of the first order:

$$x(t+1) = rx(t)(1 - x(t)), \quad t \in N, \quad r > 0, \tag{44}$$

was introduced by Pierre Verhulst in 1845 as a mathematical model of population dynamics within a closed environment that takes into account internal competition [Schuster, 1984]. Logistic equation (44), which can be generalized to the form

$$x(t+1) = f(x(t)), \quad x \in R, \quad t \in N, \tag{45}$$

has an extremely complex limiting structure of solutions and was intensively studied in the second part of the $20^{th}$ century [Sharkovsky, 1995; Li & Yorke, 1975; Sharkovsky *et al.*, 1997; May, 1976; Metropolis, M. Stein & P. Stein, 1973; Feigenbaum, 1978; Weisstein, 1999]. In particular, in equation (44), period-doubling bifurcations were discovered.

Surprisingly, while solutions to a linear multidimensional discrete equation

$$x(t+1) = Ax(t), \quad x \in R^n, \quad t \in N,$$

and its continuous analogue

$$\dot{x} = Ax(t), \quad x \in R^n, \quad t \in R$$

(where $A$ is a constant $n \times n$ matrix), to a large extent, possess similar behavior, solutions to equation (45) and its continuous one-dimensional analogue

$$\dot{x} = f(x(t)), \quad x, t \in R,$$

bear qualitatively different structure (see for example [Leonov & Seledzhi, 2002; Neittaanmäki & Ruotsalainen, 1985; Keller, 1977; Marsden & McCracken, 1976]).

One of examples of a nonlinear difference equation which are important in applications is the following equation of the first order:

$$x(t+1) = x(t) - \alpha \sin x(t) + \gamma, \quad t \in N, \tag{46}$$

where $\alpha$ and $\gamma$ are nonnegative parameters. Over the last 40 years, many authors conducted rigorous studies of equation (46) both as a pure mathematical object [Arnold, 1983; Jakobson, 1971] and as a mathematical model of a Phase-locked loop [Osborne, 1980; Gupta, 1975; Lindsey, 1972; Lindsey & Chie, 1981].

Equation (46) with $\gamma = 0$ describes a wide class of Digital Phase-locked loops (DPLLs) with the sinusoidal characteristic of phase detector [Banerjee & Sarkar, 2005-2008; Leonov *et al.,* 1992; Leonov *et al.,* 1996; Leonov, 2001; Leonov, 2002; Leonov & Seledzhi, 2005]. Osborne [Osborne,1980] pioneered in the use of exact methods, such as the Contraction Mapping Theorem, applicable to the direct study of nonlinear effects in this system. However, even the exact methods used by Osborne, while revealing what then appeared as multiple cycle slipping, followed by a divergent behavior of iterations, did not allow for a precise interpretation of nonlinear effects discovered at transition from global asymptotic stability to chaos through period doubling bifurcations [Leonov & Seledzhi, 2005].

The research and development of mathematical theory of DPLLs for array processors are commonly used in radio engineering, communication, and computer architecture [Banerjee & Sarkar, 2005–2008; Zoltowski, 2001; Mannino *et al.,* 2006; Hussain & Boashash, 2002; Kudrewicz & Wasowicz, 2007; Gardner, 1966; Lindsey, 1972; Lindsey & Chie, 1981; Leonov, Reitmann & Smirnova, 1992; Kuznetsov, Leonov & Seledzi, 2006; Leonov, Ponomarenko & Smirnova, 1996; Lapsley *et al.,* 1997; Kroupa, 2003; Best, 2003; Abramovitch, 2002]. For example, such digital control systems exhibit high efficiency in eliminating clock skew - an undesirable phenomenon arising in parallel computing [Leonov & Seledzhi, 2002; Leonov & Seledzhi, 2005]. DPLLs have gained widespread recognition and preference over their analog counterparts because of their ability to deal with this phenomenon effectively. From a mathematical perspective, this gives rise to a problem associated with the analysis of global stability of nonlinear difference equations that serve as mathematical models of discrete phase-locked loops [Leonov, 2001]; that is, the analysis can be formulated in terms of parameters for such systems.

The present chapter is devoted to study of bifurcations of discrete system (46) with $\gamma = 0$ and to computation of bifurcation parameters. Using the qualitative theory of dynamical systems, special analytical methods, and advanced mathematical packages designed to work with long numbers allowed us to succeed in computation of bifurcation values of parameter of the investigated system. The first 14 bifurcation values are calculated with good accuracy. Also it is shown that for the obtained bifurcation values of investigated system, which is not a unimodal map, an effect of convergence similar to famous Feigenbaum's effect is observed.

## 2.1 Digital phase-locked loops

The nonlinear dynamics of different nonlinear electronic systems is studied by researchers for at least three decades. Occurrence of such complex behaviors as bifurcation, chaos, intermittency etc. in electronic systems has been revealed from these studies [Banerjee & Sarkar, 2008; Kilias *et al.*, 1995; Chen, Chau & Chan, 1999; Giannakopoulos & Deliyannis, 2005]. In addition, control chaos and bifurcation in electronic circuits and systems is an active area of research [Chen, Hill & Yu, 2003; Collado & Suarez, 2005].

By controlling chaos and bifurcation, one can suppress chaotic behavior where it is unwanted (e.g. in power electronics and mechanical systems). On the other hand, in electronic systems one can harness the richness of chaotic behavior in chaos based electronic communication system. Possibility of exploiting the chaotic signal in chaos based secure communication system has boosted up the research on the chaotic dynamics of electronic circuits and systems [Kennedy, 2000].

Owing to potential application in synchronous communication system and rich nonlinear dynamical behavior, PLL is probably the most widely studied system among all electrical systems [Gardner, 1966; Kudrewicz & Wasowicz, 2007]. At the advent of digital communication systems, DPLLs have rapidly replaced the conventional analog PLLs because they overcome the problems of sensitivity to DC drift, periodic adjustment, and the building of higher order loops [Lindsey & Chie, 1981].

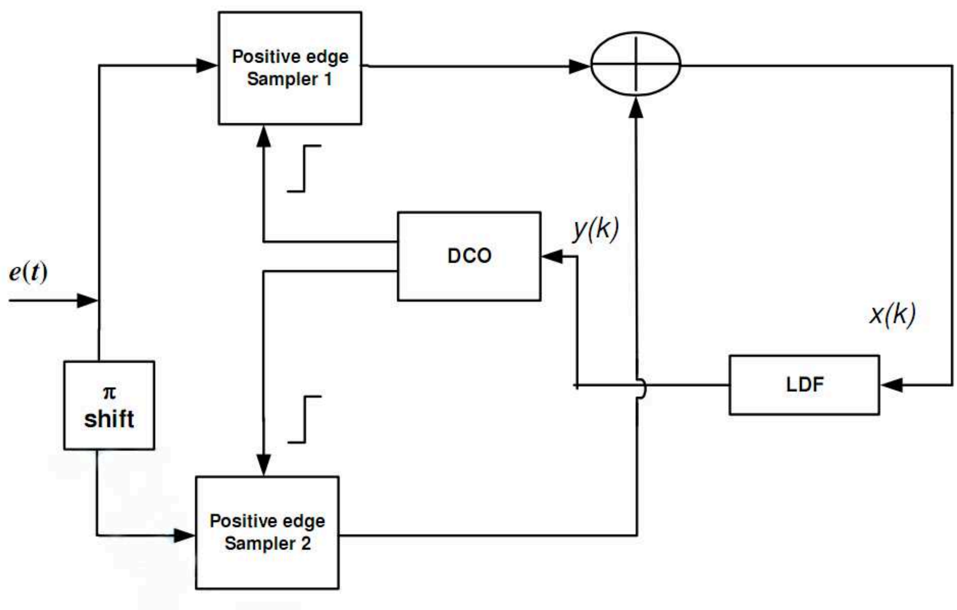


FIGURE 25 Functional block diagram of a $ZC_2 - DPLL$, [Banerjee & Sarkar, 2008].

DPLLs are widely used in frequency demodulators, frequency synthesizers, data and clock synchronizers, modems, digital signal processors, and hard disk drives to name a few [Banerjee & Sarkar, 2008; Zoltowski, 2001; Mannino *et al.*, 2006]. A DPLL is a discrete time nonlinear feedback controlled system whose nonlinear behavior is complicated, and it poses exact solutions only in particular cases. To understand the complete behavior of a DPLL, it is necessary to resort to modern nonlinear dynamical tools of bifurcation and chaos theories. Also in this regard bifurcation control of DPLL has not been explored yet. The study of nonlinear dynamics of DPLL has two fold applications. First, using the insight of nonlinear behaviors of DPLLs, an optimum DPLL system can be designed. Second, by characterizing the chaos from DPLLs, one can explore the possibility of using DPLLs in chaos based secure electronic communication systems. Thus, research of nonlinear dynamics of DPLLs is an important problem.

There are different types of DPLLs: positive zero crossing DPLLs (ZC1-DPLL) [Bernstein, Liberman & Lichtenberg, 1989; Banerjee & Sarkar, 2005; Banerjee & Sarkar, 2005[1]; Banerjee & Sarkar, 2008; Leonov & Seledzhi, 2005], uniform sampling DPLL [Zoltowski, 2001], bang-bang DPLLs [Dalt, 2005], and tanlock DPLLs [Hussain & Boashash, 2002].

In the present paper, we will consider a discrete dynamical system which describes the nonlinear dynamics of dual sampler based zero crossing DPLLs (ZC2-DPLL). Unlike ZC1-DPLL, in a ZC2-DPLL sampling is done at the positive and negative zero crossings of the input signal [Banerjee & Sarkar, 2008[1]]. For this particular sampling technique, it has a wide frequency acquisition range in comparison with a ZC1-DPLL, and that is why ZC2-DPLLs have drawn the attention of researchers for a long time [Majumdar, 1979; Frias & Rocha, 1980; Banerjee & Sarkar, 2006].

Following [Banerjee & Sarkar, 2008[1]; Majumdar, 1979; Leonov & Seledzhi, 2002], we formulate the equation of DPLL. Fig. 25 shows the block diagram of a ZC2-DPLL. It contains two positive edge triggered samplers. Input signal is fed directly into sampler-1, and a $\pi$ shifted version of input signal is fed into sampler-2. Let $e(t)$ be the noise-free analog input signal to the system with a phase angle $\theta_i(t)$ relative to the loop DCO phase. Then the system equation can be written as

$$e(t) = A_0 sin\left[\omega_0 t + \theta_i(t)\right],$$

where $\theta_i(t) = (\omega_i - \omega_0)t + \theta_0$. Here $A_0$ is the amplitude, and $\omega_i$ and $\theta_0$ are the angular frequency and phase of the input signal, respectively. $\omega_0$ is the nominal angular frequency of the DCO having time period $T$. Writing the sampled version of $e(t)$ at the $k$th sampling instant (SI) $t(k)$ as $x(k)$, one can write the output signals of sampler-1 and sampler-2, respectively, as follows [Majumdar, 1979]:

$$x_1(k') = A_0 sin[\omega_0 t(k') + \theta_i(k')], \quad k' = 2k,$$

$$x_1(k'') = A_0 sin[\omega_0 t(k'') + \theta_i(k'') - \pi], \quad k'' = (2k+1),$$

where $k = 0, 1, 2, 3 \ldots$. Here sampling instants (SIs) are occurring at the end of each half period of the DCO.

The sequence $x(k)$, $k = 0, 1, 2, \ldots$, is filtered digitally by a loop digital filter (LDF). The transfer function of LDF in a first order loop is written as a constant gain G1 (s/volt). The LDF output sequence is given by $y_k = G_1 x(k)$. The sequences $y(k)$ are used to control the next half period of the DCO. The $k$th half period $T'(k)$ of DCO can be written as

$$T'(k) = T(k)/2 = t(k+1) - t(k),$$

in terms of the $k$th and $(k+1)$st SIs, respectively. DCO period $T(k+1)$ at that instant is governed by the relation

$$T(k+1) = \frac{T}{2} - y(k).$$

In case $t(0) = 0$ , we get

$$t(k) = \frac{kT}{2} - \sum_{i=0}^{k-1} y(i).$$

Thus, the sampler output at the $k$th instant is

$$x(k) = A_0 sin[\phi(k)],$$

where

$$\phi(k) = \theta_i(k) - \omega_0 \sum_{i=0}^{k-1} y(i)$$

is the phase error between the input signal and the DCO output at $t(k)$.

Then, the equation for the phase of the ZC2-DPLL can be written as

$$\phi(k+1) = \phi(k) + \pi(z-1) - \frac{1}{2} z K_1 sin\phi(k), \qquad (47)$$

where $z$ has been substituted in place of $(\omega_i/\omega_0)$ and $K_1 = A_0\omega_0 G_1$ is the closed loop gain of ZC2-DPLL.

## 2.2 Analytical investigation

In engineering DPLL's practice, the case of initial frequency of master and local generators coincidence is very important [Banerjee & Sarkar, 2008[1]; Leonov & Seledzhi, 2002]. For this instance, in (47),

$$z = (\omega_i/\omega_0) = 1,$$

and the equation for DPLL can be given by

$$\sigma(t+1) = \sigma(t) - r \sin \sigma(t), \quad t \in N, \qquad (48)$$

where $r = \frac{1}{2} A_0 \omega_0 G_1$ is a positive number [Leonov & Seledzhi, 2002].

One of the first works dedicated to analysis of system (48) belongs to Osborne. In [Osborne, 1980], was considered the algorithm of investigation of periodic solutions, and it was shown that even in a simple discrete model of PLL, the bifurcation phenomenon involved to arising of new stable periodical solutions and to changing of their period are observed. Later, in the papers [Belykh & Maksakov, 1979; Belykh & Lebedeva, 1983] for such systems, a model of transition to chaos through a cascade of period-doubling bifurcations was considered. Association and development of these ideas in the works [Leonov & Seledzhi, 2002; Leonov & Seledzhi, 2005] has allowed to construct bifurcation tree of transition to chaos through a cascade of period doubling (Fig. 26).

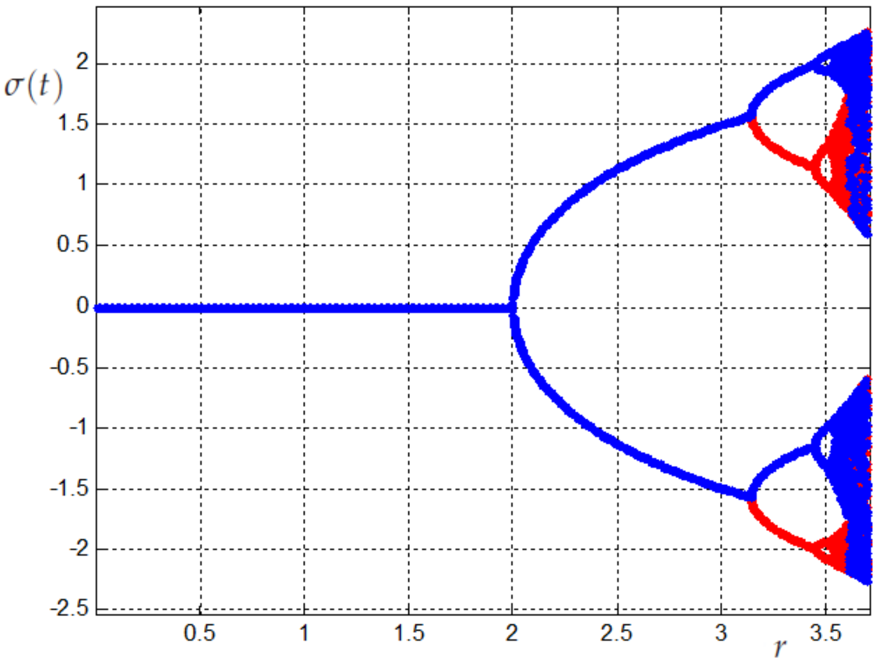


FIGURE 26 Seledzhi's bifurcation tree.

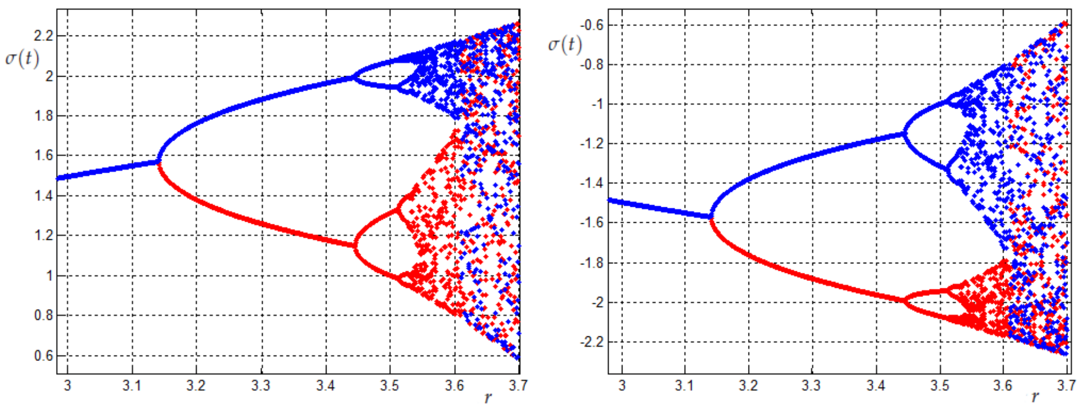


FIGURE 27 Seledzhi's bifurcation tree. Enlarged domains.

In computer modeling of Seledzhi's bifurcation tree the following function can be used.

```
% Function for construction of a plot of dependence limit values from
% parameter r for the system x(t+1)=x(t)-r*sin(x(t))

  function bf = fBifTree(beg_r, end_r, step_r, x0, Iter, color)
  j=1;
  for r= beg_r:step_r:end_r
    x=x0;
    for i=1:Iter
        x= x-r*sin(x);
    end
    res(j,:) = [r x];
    j=j+1;
  end

  plot(res(1:length(res),1),res(1:length(res),2),color);
```

With the help of following code, bifurcation tree for the discrete dynamical system (48) can be constructed.

```
% Constructing  of bifurcation tree for the discrete dynamical system
% x(t+1)= x(t) - r sin(x(t)) for initial data x0=x1 and x0=x2

  clear all
  j = 1;k = 1;
  beg_r = 0.01; end_r = 3.7; step_r = 0.001;
  Iter = 1000; n = 8;
  x1 = 1; color1 = '.red';
  x2 = -1; color2 = '.blue';

  for j=1:n
    fBifTree(beg_r, end_r, step_r, x1, Iter+j, color1);
    hold on;
  end

  for j=1:n
    fBifTree(beg_r, end_r, step_r, x2, Iter+j, color2);
    hold on;
  end

  grid on;
  hold off;
```

In [Leonov & Seledzhi, 2002], it was proved that system (48) is globally asymptotically stable for $r \in (0, 2)$.

Following the works [Osborne, 1980; Leonov & Seledzhi, 2002], let us consider behavior of periodic solutions of system (48) for $r \geq 2$.

Let $r \in (2, r_1)$, where $r_1$ is the root of the equation

$$\sqrt{r^2 - 1} = \pi + \arccos \frac{1}{r}.$$

Then the following theorem takes place.

**Theorem 3** *If $r \in (2, r_1)$ and $\sigma(0) \in [-\pi, \pi]$, then $\sigma(t) \in [-\pi, \pi]$ for all $t = 1, 2, \ldots$.*

This theorem determines system (48) as a map of the interval $[-\pi, \pi]$ into itself for $r \in (2, r_1)$.

For $r < 2$, equation (48) is global asymptotically stable: $\sigma(t)$ with any initial conditions $\sigma(0)$ aspire to the states of equilibrium $\sigma = 2\pi j$, $j \in Z$, $t \mapsto +\infty$.

The value $r = 2$ is the first point of bifurcation. For $r > 2$ all the stationary points become Lyapunov unstable. There is an asymptotically stable, symmetric with respect to $\sigma = 0$, solution with period 2 for $r \in (2, \pi)$.

Let $\sigma(t+1) = -\sigma(t)$, then $2\sigma(t) = r\sin(\sigma(t))$, and the symmetric solutions of equation (48) with period 2 have the properties:

$$\sigma(2j) = \sigma(0), \quad j \in Z,$$

$$\sigma(2j+1) = -\sigma(0), \quad j \in Z,$$

where the initial conditions of $\sigma(0)$ satisfy equality

$$2\sigma(0) = r\sin(\sigma(0)). \tag{49}$$

Equation (49) in the interval $[-\pi, \pi]$ has two roots for all $r > 2$: $\sigma(0)$ and $-\sigma(0)$.

For $r \in (\pi, \beta)$, where $\beta = \sqrt{\pi^2 + 2} \approx 3.445229$, there are two asymptotically stable solutions with period 2 which satisfy the relation

$$\sigma(t+1) = \sigma(t) \pm \pi.$$

From here it follows that

$$\pi = r\sin(\sigma(t)).$$

Then, the first periodic solution of period 2 for $r \in (\pi, \beta)$ has the properties

$$\sigma(2j) = \sigma(0), \quad \forall j \in Z,$$

$$\sigma(2j+1) = \sigma(0) - \pi, \quad \forall j \in Z,$$

where the initial conditions of $\sigma(0)$ satisfy the equality

$$\sin(\sigma(0)) = \frac{\pi}{r}. \tag{50}$$

The second periodic solution of period 2 for $r \in (\pi, \beta)$ has the properties

$$\sigma(2j) = \sigma(0), \quad \forall j \in Z,$$

$$\sigma(2j+1) = \sigma(0) + \pi, \quad \forall j \in Z,$$

where the initial conditions of $\sigma(0)$ satisfy the equality

$$\sin(\sigma(0)) = -\frac{\pi}{r}. \tag{51}$$

Equations (50), (51) have on $[-\pi, \pi]$ two roots for all $r > 2$.

According to analytical investigations described above, the following is known:

For $r = r_1 = 2$, the first bifurcation occurs. The global asymptotic stability of the stationary set vanishes, and a globally asymptotically stable cycle of period 2 appears.

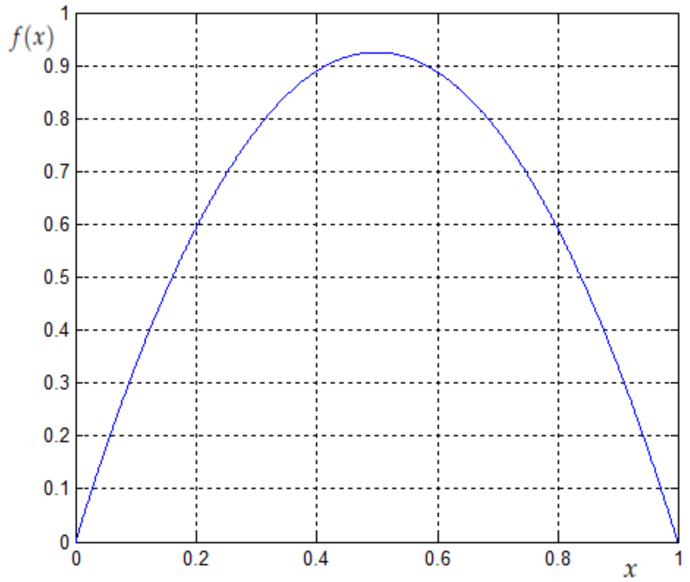


FIGURE 28 Plot of the function $f(x) = \lambda x(1-x), \quad \lambda = 3.7.$

The second bifurcation value $r = r_2 = \pi$ value corresponds to bifurcation of splitting: the globally stable cycle of period 2 loses its stability, and two locally stable cycles of period 2 appear. Note that for the first time, this phenomena was described in [Leonov & Seledzhi, 2002]: a cycle of some period $T$ loses its stability, and two cycles of the same period $T$ appear.

For $r = r_3 = \sqrt{\pi^2 + 2} \approx 3.4452$, the third bifurcation occurs: two cycles of period 2 lose stability, and two 4-periodical cycles appear.

In Fig. 29, we show an enlarged domain of the bifurcation tree where the second and third bifurcations are clearly seen.

Further transition to chaos through a cascade of period-doubling bifurcations takes place.

Note that the phenomenon of transition to chaos through a cascade of period-doubling bifurcations is well studied for the whole class of maps of an interval into itself. In 1975 M. Feigenbaum noticed that for the equation

$$x_{n+1} = \lambda x(1-x),$$

the following is observed: if

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = \delta_n,$$

then

$$\lim_{n\to\infty} \delta_n = 4.6692...,$$

where $\lambda_{n-1}, \lambda_n, \lambda_{n+1}$ are consecutive bifurcation values.

He performed similar calculations with another logistical map, and found a geometric progression with the same denominator. After that, the hypothesis that $\delta$ does not depend on the type of a specific map was born.

It was found that the convergence is universal for one-dimensional one-parameter families of maps of an interval into itself [Campanin & Epstain, 1981; Ostlund *et al.,* 1983; Lanford, 1982; Hu & Rudnick, 1982]. The value $\delta = 4.6692...$ is the famous Feigenbaum's constant. The Renorm-group Theory explains this phenomenon for the class of unimodal maps (continuous map an interval into itself which has a unique critical point in the interval and is strictly monotonous on either side of extremum (Fig. 28)) and for some special cases [Feigenbaum, 1980; Vul, Sinai & Khanin, 1984; Cvitanovich, 1989; Bensimon, Jensen & Kadanoff, 1986; Kuznetsov, 2001; Shirkov, Kazakov & Vladimirov, 1988].

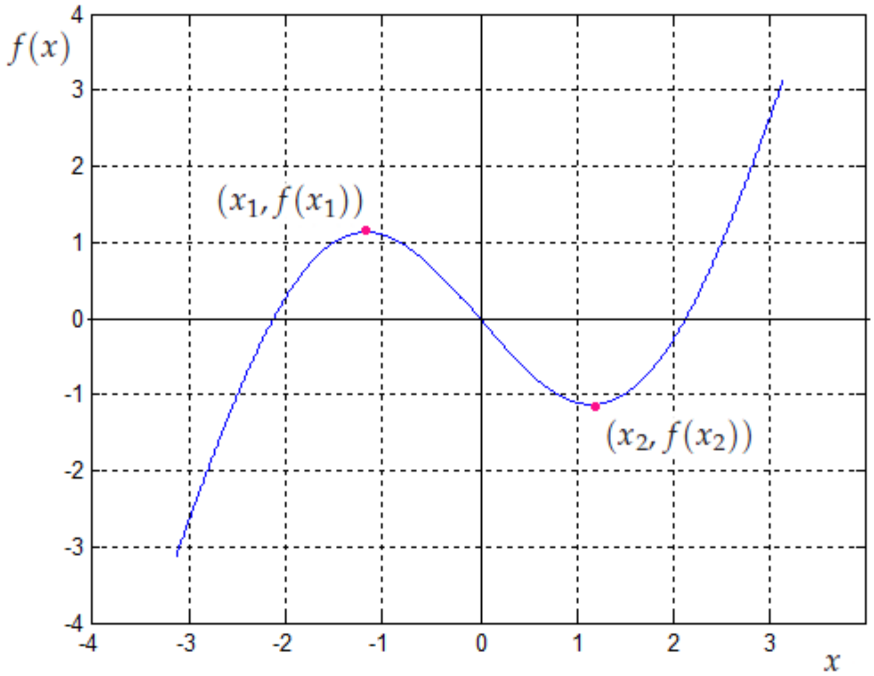


FIGURE 29 Plot of the function $f(x) = x - rsin(x), \quad r = 2.5$.

The calculations obtained in the present work allow us to show that, for system (48), the effect of convergence similar to Feigenbaum's effect is observed.

Note that for the function $f(x) = x - rsin(x)$, which has two critical points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ in the interval $[-\pi, \pi]$ (Fig. 29), and for the function $f(f(x)) = f(x) - rsin(f(x))$ (Fig. 30) we have for $r > 2$:

$$f(x_1) \neq x_1, \quad f(x_1) \neq x_2, \quad f(f(x_1)) \neq x_1, \quad f(f(x_1)) \neq x_2.$$

## 2.3 Computer modeling

First numerical calculations of bifurcation values of parameter $r$ for system (48) are presented in [Osborne, 1980; Banerjee & Sarkar, 2006; Leonov & Seledzhi, 2002]. An included article [Abramovich *et al.*, 2005] is devoted to the possibility of investigation of bifurcation behavior for (48) by using Microsoft Excel.

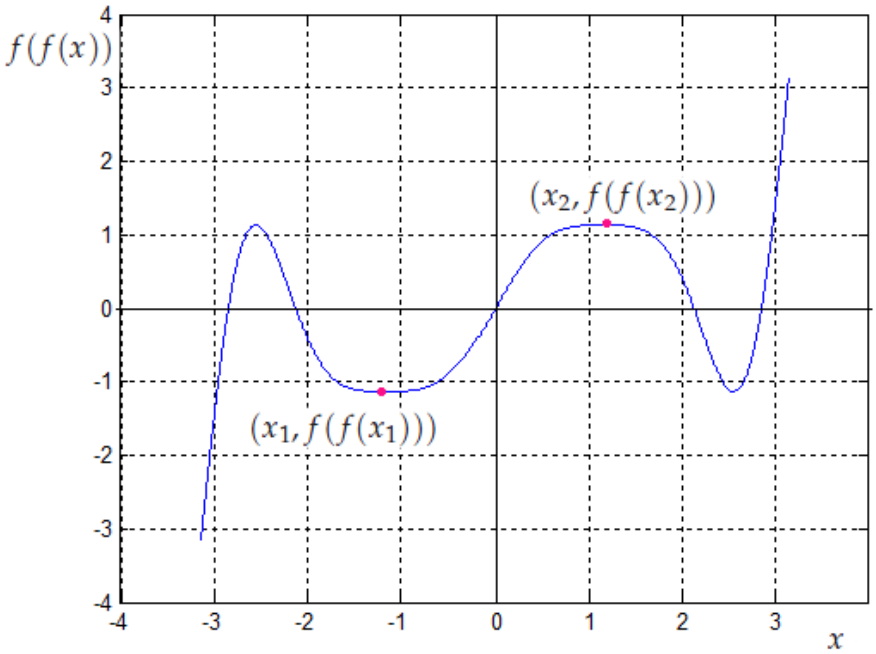


FIGURE 30 Plot of the function $f(f(x)) = f(x) - rsin(f(x)), \quad r = 2.5$.

In [Leonov & Seledzhi, 2002; Leonov & Seledzhi, 2005], methods for analysis of behavior of periodic trajectories of the system were developed. They allowed to justify the using of computational procedures for calculating the following bifurcation values. With the help of these analytical methods and specialized mathematical packages, the first 14 bifurcation values of parameter $r$ were obtained.

The algorithm for computation is based on application of the method of multipliers [Vul, Sinai & Khanin, 1984; Kuznetsov, 2001].

The multiplier of a periodic trajectory of period $T$ for a discrete dynamical system

$$x_{n+1} = f(x_n, r)$$

can be written as

$$M_T(r) = \prod_{i=1}^{T} f'(x_i(r), r),$$

where $x_i(r), i = 1, \ldots, T$, are the points (limit values) which form a stable periodic trajectory of period $T$. The multiplier is responsible for the stability of the cycle: for $r = r_T$ for which

$$M_T(r_T) = -1,$$

there occurs a period-doubling bifurcation at which the periodic trajectory of period $T$ loses stability and there appears a periodic trajectory of period $2T$.

With a good accuracy, exact values of the first 14 bifurcation values of parameter $r$ for initial data $\sigma(0) = 1$ for system (48) were obtained. For computation, multipliers of all periods 4, 16, 32, ..., 8192, for each value of parameter $r$ greater than the analytically obtained $r_3$, with small step were calculated. This allowed to avoid admission of bifurcation values.

TABLE 1 Values of bifurcation parameters and Feigenbaum's numbers for a discrete dynamical system.

| Number of bifurcation $j$ | Period of bifurcation (before / after) | Bifurcation parameter, $r_j$ | Feigenbaum's number, $\delta_j$ |
|---|---|---|---|
| 1 | 1/2 | 2 | |
| 2 | 1/2 | $\pi$ | 3.7597337326 |
| 3 | 2/4 | 3.445229223301312 | 4.4874675842 |
| 4 | 4/8 | 3.512892457411257 | 4.6240452067 |
| 5 | 8/16 | 3.527525366711579 | 4.6601478320 |
| 6 | 16/32 | 3.530665376391086 | 4.6671765089 |
| 7 | 32/64 | 3.531338162105000 | 4.6687679883 |
| 8 | 64/128 | 3.531482265584890 | 4.6690746582 |
| 9 | 128/256 | 3.531513128976555 | 4.6691116965 |
| 10 | 256/512 | 3.531519739097210 | 4.6690257365 |
| 11 | 512/1024 | 3.531521154835959 | 4.6686408913 |
| 12 | 1024/2048 | 3.531521458080261 | 4.6678177276 |
| 13 | 2048/4096 | 3.531521523045159 | 4.6657974003 |
| 14 | 4096/8192 | 3.531521536968802 | |

The values are obtained under the condition of convergence of limiting values up to 15 signs after comma:

$$|\sigma(t) - \sigma(t+T)| < 10^{-15}.$$

The specified condition demands $t = 2 \times 10^8$ iterations.

Table (1) shows the first 14 calculated bifurcation values of parameter $r$ for system (48).

As was said earlier, the bifurcation parameter $r = r_2 = \pi$ does not correspond to period-doubling bifurcation. There is a bifurcation of splitting of the cycle: the cycle of period 2 loses its stability, and two locally stable cycles of period 2 appear.

For the calculated bifurcation values of parameters $r_j$ (Table 1), with the help of the relation

$$\delta_j = \frac{r_j - r_{j-1}}{r_{j+1} - r_j},$$

the values of Feigenbaum's numbers $\delta_j$ are calculated. They are presented in the last column of Table 1.

Note that the obtained Feigenbaum's numbers $\delta_j$ have a good convergence to Feigenbaum's constant $\delta = 4.6692016....$ Thus, for system (48) an effect of convergence of bifurcation values of parameter $r$ similar to Feigenbaum's effect is observed.

## 2.4 Conclusion of Chapter 2

Application of qualitative theory of dynamical systems, special analytical methods [Leonov & Seledzhi, 2002; Leonov & Seledzhi, 2005], and modern mathematical packages has helped to promote considerably in calculation of bifurcation values of parameter for a one-dimensional discrete system describing operation of digital phase-locked loop. Numerically calculated fourteen bifurcation values of parameter of the investigated system are presented. It is shown that for the obtained bifurcation values of nonunimodal map, an effect of convergence similar to Feigenbaum's effect is observed.

The results described in this chapter are continuation of research presented in the included article [Abramovich *et al.*, 2005] and have been partially presented in [Kudryashova, 2009], and at the international conferences [Kudryashova & Leonov, 2005; Kudryashova & Seledzhi, 2007; Kudryashova *et al.*, 2008].

# YHTEENVETO (FINNISH SUMMARY)

Tässä väitöskirjassa, ”Rajajaksot jatkuvissa ja diskreeteissä järjestelmissä”, käsitellään jaksollisten ratkaisujen laskemista sekä bifurkaatioita neliöllisissä tehtävissä, Lienardin järjestelmissä ja ei-unimodaalisissa yksiulotteisissa diskreeteissä kuvauksissa käyttäen nykyaikaisia laskentamahdollisuuksia ja symbolisen laskennan ohjelmistoja.

Ensimmäisessä luvussa tarkastellaan akateemikko A.N. Kolmogorovin tehtävää neliöllisten järjestelmien rajajaksojen paikallistamiselle ja simuloinnille. Pienten rajajaksojen tutkimisessa (niin sanottu lokaali Hilbertin 16. ongelma) käytetään Lyapunovin suureiden (eli Poincaren ja Lyapunovin vakioiden) laskentamenetelmiä. Lyapunovin suureiden symbolisten lausekkeiden laskemiseksi työssä yleistettiin Lyapunovin ei-analyyttisille järjestelmille tarkoitettu menetelmä. L.A. Cherkasin ja G.A. Leonovin töitä seuraten kehitettiin symboliset algoritmit, joilla neliölliset järjestelmät muunnetaan erityisiksi Lienardin järjestelmiksi. Ensimmäistä kertaa saatiin muodostettua yleiset symboliset lausekkeet neljälle ensimmäiselle Lyapunovin suureelle Lienardin järjestelmissä. Työssä esitetään suurten rajajaksojen (eli ”normaalien” rajajaksojen) simulaatiotuloksia neliöllisille ja Lienardin järjestelmille, joiden parametrit kuuluvat G.A. Leonovin löytämälle suuren rajajakson olemassaolon alueelle. Työssä toteutetaan myös S.L. Shin löytämän neljän rajajakson neliöllisten järjestelmien parametrialueen muunnos Lienardin järjestelmän kaksiulotteiseksi parametrialueeksi ja visualisoidaan sitä tasossa.

Toisessa luvussa tarkastellaan ei-unimodaalisia yksiulotteisia diskreettejä kuvauksia, jotka kuvaavat tietokonearkkitehtuureissa ja tietoliikenteessä laajasti käytettyjen digitaalisten vaihelukitusjärjestelmien toimintaa. Vaihelukitusjärjestelmien yhtälöiden laadullinen analyysi auttaa määrittämään tarvittavat toimintaedellytykset (esimerkiksi, taajuuden synkronisointi ja kellovääristymien korjaaminen). Käyttämällä dynaamisten järjestelmien laadullista teoriaa, erityisiä analyyttisiä menetelmiä ja nykyaikaisia matemaattisia ohjelmistoja edistyttiin huomattavasti bifurkaatioarvojen laskemisessa ja onnistuttiin määräämään numeerisesti vaihelukitusjärjestelmän parametrien neljätoista bifurkaatioarvoa. Työssä myös osoitettiin, että saaduille ei-unimodaalisen kuvauksen bifurkaatioarvoille on ominaista suppeneminen, joka muistuttaa Feigenbaumin ilmiötä.

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