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Oleg Sarafanov

Asymptotic Theory of Resonant Tunneling in Quantum Waveguides of Variable Cross-Section

JYVÄSKYLÄN YLIOPISTO

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ABSTRACT

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The narrows of a quantum waveguide with variable cross-section play the role of effective potential barriers for the electron longitudinal motion. Two narrows form a quantum resonator where a resonant tunneling can occur. It means that electrons with energy close to a resonant value pass through the resonator with probability near to 1. We give an asymptotic description of electron wave propagation in a quantum waveguide with two narrows. The wave number k is assumed to be between the first and the second thresholds, so only one incoming and one outgoing wave may propagate in every outlet of the waveguide to infinity. We present the asymptotic expansions of wave functions, the reflection and transition coefficients as the diameters of narrows tend to zero. Moreover, the asymptotic formulas for the resonant frequencies are obtained and the behavior of the coefficients is analyzed near a resonance.

Keywords: resonant tunneling, Helmholtz equation, scattering matrix, transition coefficient, reflection coefficient, radiation conditions, cylindrical outlets, compound asymptotics

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1 INTRODUCTION

Resonant tunneling can occur as electrons propagate through a potential barrier of complex shape (say, composed of two identical barriers placed one after another). In the process, electrons of energies close to some (resonant) values pass trough the barrier with probability near to one. In electronics, resonant devices (transistors, key devices, electron energy monochromators) based on one-dimensional hetero-structures consisting of layers of distinct chemical compositions, have gained widespread usage. To provide operating these devices in an optimal regime, one should know main characteristics of the process (the resonant energies, the shape of the transition coefficient near a resonance). These characteristics can be calculated with the help of one-dimensional models by the WKB method.

Such one-dimensional structures possess several disadvantages. That is why the creation of homogeneous structures with resonant tunneling conditions is a topical problem. Alternatively, one can consider two- and three-dimensional waveguides of variable cross-section; their narrows play the role of effective barriers. The behavior of the resonant tunneling characteristics in such waveguides has not been studied theoretically. One-dimensional models are ineffectual. Numerical modelling meets difficulties, when the narrows of the waveguide become "too narrow" and the resonant peak too sharp. In this work, we present an asymptotic description of the resonant tunneling in quantum waveguides with narrows as the diameters of the narrows tend to zero, which gives qualitative picture of the phenomenon.

In the introductory chapter we first consider the one-dimensional electron motion and demonstrate the occurrence of the resonant tunneling in the case (Sect. 1.1); then some applications of the one-dimensional resonant tunneling are listed, in particular, the operation of some mentioned resonant devices is explained (Sect. 1.2); motivations for studying two- and three-dimensional waveguides are given (Sect. 1.3); finally, the used mathematical methods are briefly described (Sect. 1.4).

1.1 One-dimensional resonant tunneling

The one-dimensional resonant tunneling (described by Schrödinger equation (1.1)) is a well studied phenomenon [1] – [3]. We remind some known results.

Consider an electron propagating through a potential barrier (Fig. 1) from $-\infty$ to $+\infty$. Its wave function Ψ_1 satisfies

$$-\frac{\hbar^2}{2m}\Psi''(x) + U(x)\Psi(x) = E\Psi(x). \tag{1.1}$$

Since $U(x) = 0$ as $x < x_1$ and $x > x_2$, we have

$$\Psi_1(x) = \begin{cases} e^{ikx} + re^{-ikx}, & \text{as } x < x_1; \\ te^{ikx}, & \text{as } x > x_2, \end{cases} \tag{1.2}$$

where $k^2 = 2mE/\hbar^2$, the summand re^{-ikx} is the reflected wave, and te^{ikx} is the transited one. The value $R = |r|^2$ is called the reflection coefficient and $T = |t|^2$ the transition coefficient, $R + T = 1$.

Let Ψ be a wave function such that

$$\Psi(x) = \begin{cases} A_1e^{ikx} + B_1e^{-ikx}, & \text{as } x < x_1; \\ A_2e^{ikx} + B_2e^{-ikx}, & \text{as } x > x_2. \end{cases}$$

Due to the linearity of the equation (1.1), the coefficients A_1 and B_1 depend on A_2 and B_2 linearly. More exactly, one can prove that

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = D \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}, \qquad \text{where} \quad D = \begin{pmatrix} 1/t & r/t \\ \bar{r}/\bar{t} & 1/\bar{t} \end{pmatrix},$$

r, t being the coefficients in (1.2).

Consider now a complex barrier composed of two simple barriers placed one after another (Fig. 2). Given the reflection and transition coefficients for each of the simple barriers, we can find such coefficients for the complex one. Assume that

$$\Psi(x) = te^{ikx} \qquad \text{as} \quad x > x_4.$$

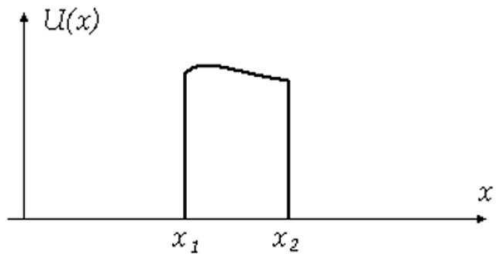

FIGURE 1 A potential barrier.

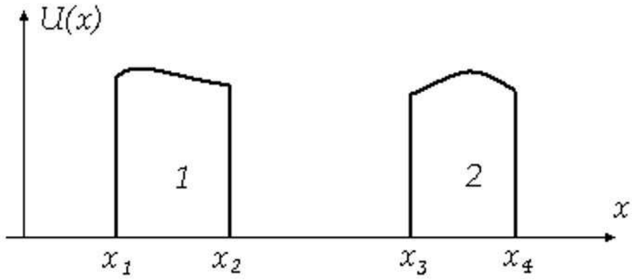

FIGURE 2 Two potential barriers.

Then

$$\Psi(x) = Ae^{ikx} + Be^{-ikx} \qquad \text{as} \quad x_2 < x < x_3,$$

where

$$\begin{pmatrix} A \\ B \end{pmatrix} = D_2 \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} 1/t_2 & r_2/t_2 \\ \bar{r}_2/\bar{t}_2 & 1/\bar{t}_2 \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} = t \begin{pmatrix} 1/t_2 \\ \bar{r}_2/\bar{t}_2 \end{pmatrix}.$$

Finally,

$$\Psi(x) = e^{ikx} + re^{-ikx} \qquad \text{as} \quad x < x_1,$$

where

$$\begin{pmatrix} 1 \\ r \end{pmatrix} = D_1 \begin{pmatrix} A \\ B \end{pmatrix} = t \begin{pmatrix} 1/t_1 & r_1/t_1 \\ \bar{r}_1/\bar{t}_1 & 1/\bar{t}_1 \end{pmatrix} \begin{pmatrix} 1/t_2 \\ \bar{r}_2/\bar{t}_2 \end{pmatrix} = t \begin{pmatrix} \dfrac{1}{t_1 t_2}\left(1 + \dfrac{t_2}{\bar{t}_2} r_1 \bar{r}_2\right) \\ \dfrac{1}{\bar{t}_1 t_2}\left(\bar{r}_1 + \dfrac{t_2}{\bar{t}_2} \bar{r}_2\right) \end{pmatrix}.$$

This equality implies

$$t = \frac{t_1 t_2}{1 + \dfrac{t_2}{\bar{t}_2} r_1 \bar{r}_2}, \qquad r = \frac{t_1}{\bar{t}_1} \frac{\bar{r}_1 + \dfrac{t_2}{\bar{t}_2}\bar{r}_2}{1 + \dfrac{t_2}{\bar{t}_2} r_1 \bar{r}_2}.$$

We set $\varphi = \arg(t_2 r_1 \bar{r}_2/\bar{t}_2)$, $T_j = |t_j|^2$, and $R_j = |r_j|^2$, $j = 1, 2$. Then

$$T = |t|^2 = \frac{T_1 T_2}{1 + R_1 R_2 + 2\sqrt{R_1 R_2}\cos\varphi}, \quad R = |r|^2 = \frac{R_1 + R_2 + 2\sqrt{R_1 R_2}\cos\varphi}{1 + R_1 R_2 + 2\sqrt{R_1 R_2}\cos\varphi}.$$

Since $R_j + T_j = 1$, we obtain $1 + R_1 R_2 = R_1 + R_2 + T_1 T_2$, which leads to

$$T = \frac{T_1 T_2}{T_1 T_2 + \left(\sqrt{R_1} - \sqrt{R_2}\right)^2 + 2\sqrt{R_1 R_2}(\cos\varphi + 1)},$$

$$R = \frac{\left(\sqrt{R_1} - \sqrt{R_2}\right)^2 + 2\sqrt{R_1 R_2}(\cos\varphi + 1)}{T_1 T_2 + \left(\sqrt{R_1} - \sqrt{R_2}\right)^2 + 2\sqrt{R_1 R_2}(\cos\varphi + 1)}.$$

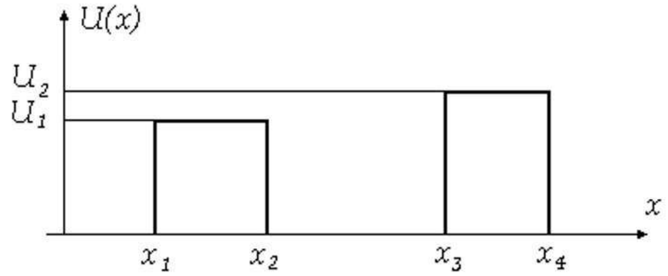

FIGURE 3 Rectangular potential barriers.

Thus, the T is maximal if $\cos\varphi = -1$. When $R_1 = R_2$, the maximum of T is equal to one. Such a phenomenon is called the resonant tunneling.

The coefficients $R(k)$ and $T(k)$ for a single barrier can be obtained approximately by the WKB-method (see, for instance, [4]). Let us consider a situation, where the coefficients can be explicitly calculated.

Consider two rectangular barriers of heights U_1, U_2 and widths $a_1 = x_2 - x_1$, $a_2 = x_4 - x_3$, respectively (Fig. 3). A direct calculation shows that, for energy E in the interval $0 < E < \min\{U_1, U_2\}$,

$$t_j = \frac{2ik\varkappa_j e^{-ik(x_{2j}-x_{2j-1})}}{(k^2 - \varkappa_j^2)\sinh(\varkappa_j a_j) + 2ik\varkappa_j \cosh(\varkappa_j a_j)},$$
$$r_j = \frac{(k^2 + \varkappa_j^2)\sinh(\varkappa_j a_j) e^{2ikx_{2j-1}}}{(k^2 - \varkappa_j^2)\sinh(\varkappa_j a_j) + 2ik\varkappa_j \cosh(\varkappa_j a_j)},$$

where $j = 1, 2$, $\varkappa_j^2 = 2m(U_j - E)/\hbar^2$, and $k^2 = 2mE/\hbar^2$. Therefore

$$T_j = \frac{4k^2\varkappa_j^2}{(k^2 + \varkappa_j^2)^2 \sinh^2(\varkappa_j a_j) + 4k^2\varkappa_j^2}, \qquad R_j = \frac{(k^2 + \varkappa_j^2)^2 \sinh^2(\varkappa_j a_j)}{(k^2 + \varkappa_j^2)^2 \sinh^2(\varkappa_j a_j) + 4k^2\varkappa_j^2}.$$

The equality $R_1 = R_2$ is obviously valid for barriers of equal heights and widths. As follows from the last formula, R_j monotonically increases when U_j or a_j is increasing. This means that $R_1 = R_2$ can be fulfilled for barriers of various shape. The condition $\cos\varphi = -1$ is independent of the condition $R_1 = R_2$. For rectangular barriers,

$$\varphi = 2k(x_4 - x_1) + \arctan\left(\frac{2k\varkappa_1}{k^2 - \varkappa_1^2}\coth(\varkappa_1 a_1)\right) + \arctan\left(\frac{2k\varkappa_2}{k^2 - \varkappa_2^2}\coth(\varkappa_2 a_2)\right).$$

The location of resonances (in the interval $0 < E < \min\{U_1, U_2\}$) is determined by the equality $\varphi = \pi(1 + 2n)$, $n \in \mathbb{Z}$; theirs height is determined by the difference $\sqrt{R_1} - \sqrt{R_2}$ (Fig. 4).

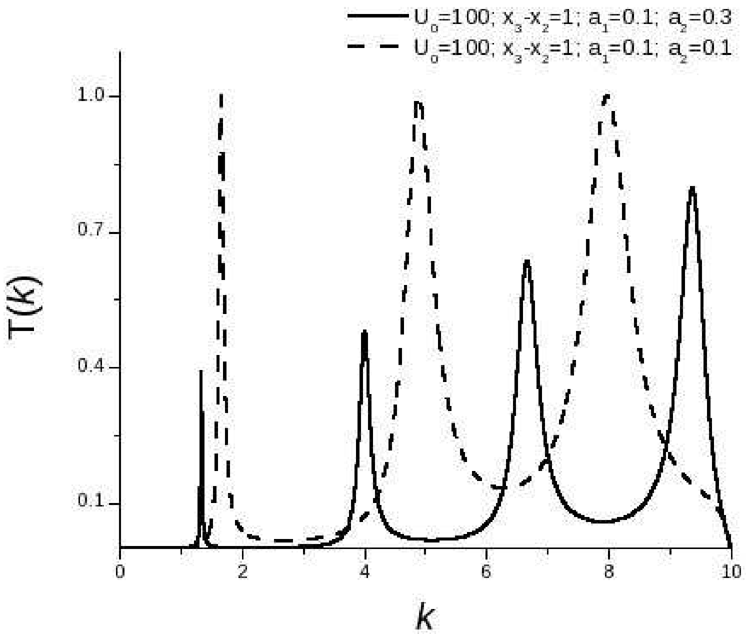

FIGURE 4 T(k) in the case of rectangular potential barriers of the same height U; here $U_0 = 2mU/\hbar^2$.

1.2 Applications of resonant tunneling

1.2.1 Field emission from adsorbate-covered surfaces

The phenomenon of resonant tunneling is used to interpret a wide range of experiments.

As known, electron removal from the surface of metal placed into vacuum won't occur, if the energy of the electron is under some level (work function of the metal). When an external electric field presents, the potential energy near the metal-vacuum interface has the shape of a barrier (Fig. 5). Due to the tunneling through the barrier, even "cold" electrons can escape from the metal surface. This effect is called the field emission.

Near an atom adsorbed on the surface of the metal, the potential energy has shape of two potential barriers (Fig. 6). If an electron in the metal has energy close to one of the allowed electron energy levels in the adsorbed atom, the escape probability of the electron from a neighborhood of an adsorbed atom (as a result of the resonant tunneling) will be much more than the escape probability from the pure surface. When the allowed energy levels of the adsorbate lie close to the Fermi level of the metal, the resonant tunneling can result in a valuable increasing

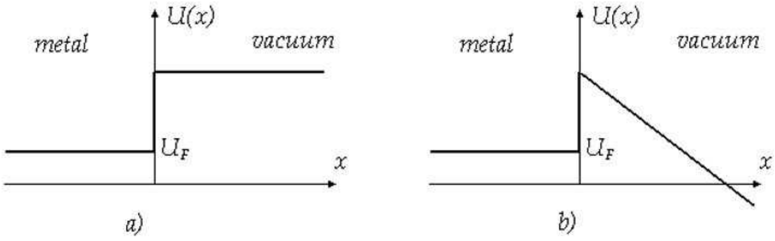

FIGURE 5 The potential energy near metal-vacuum interface a) without external field, b) with an external electric field.

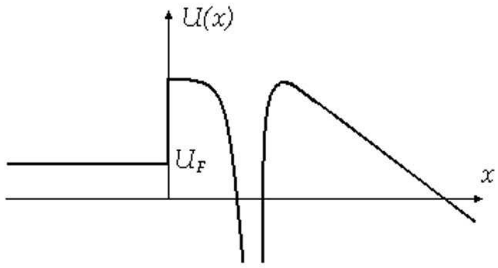

FIGURE 6 The potential energy near an adsorbed atom.

of the density of the emission current [5, Sect. 6.2.2, Fig. 6.7,a)]. The allowed levels of the adsorbate located much lower the Fermi energy practically have no influence on the emission current but dramatically change the electron energy distribution [5, Sect. 6.2.1, Fig. 6.6].

The resonant tunneling can play an important role for initiating the explosion electronic emission [6] – [8]. The explosion emission occurs whereas micro-pimples on the metal surface are being heated and exploding. Presence of atoms adsorbed on the metal surface results in increasing the local density of the initiating current and decreasing the explosion emission threshold.

1.2.2 Applications of the resonant tunneling in micro- and nano-electronics

The resonant tunneling can occur in hetero-structures consisting of layers of distinct chemical compositions (Fig. 7, a)). Such structures can be used as key-devices and amplifiers.

To explain the operation of such devices, consider Figure 7, b), where the

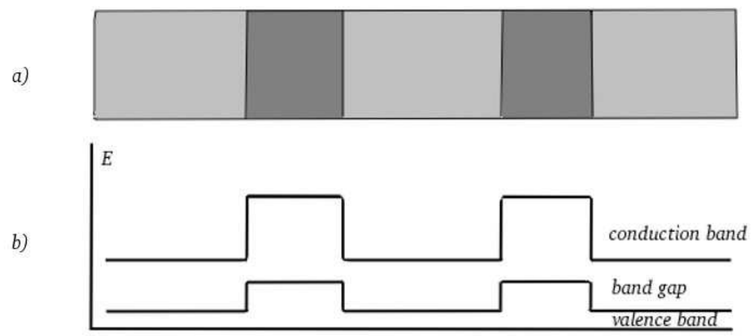

FIGURE 7 a) One-dimensional heterostructure. b) Band gap in this structure.

energy band gap for the structure is shown. The domain between the barriers we call the resonator. When a small potential difference between the emitter and the collector exists, in this structure, a very weak electric current flows due to the tunneling of electrons through the barriers. If, in the emitter, one of the levels occupied with a high probability coincides with one of the resonator allowed levels E_1, the conditions for the resonant tunneling will be provided and the current will become notable. Electric fields in a neighborhood of the resonator (created, say, by an external control electrode) can change the resonant levels. For a certain control potential U_c, the resonant level moves and the current in the system almost vanishes. Thus the structure works as a key device. For some densities of charge carriers in the emitter and the collector, one can provide a smooth variation of the current whereas U_c varies. Then the device can be used as a transistor.

1.3 Resonant tunneling in deformed waveguides

The fabrication of the heterogeneous structures with given resonant properties is complicated from the technological point of view (it is hard to produce layers of given widths, to avoid defects arising at the interfaces between layers, etc.). As far as we can see, the homogeneous structures would be free from such disadvantages. That is why the creation of homogeneous structures with resonant tunneling conditions is a topical problem.

To this purpose, one can consider two- and three-dimensional waveguides of variable cross-section; their narrows play the role of effective barriers (Figures 8 and 13 below). This can be explained by the following reasons. If a waveguide is a cylinder (a strip), i.e. it has a constant cross-section, the full energy E of an electron is represented as the sum $E = E_\perp + E_\parallel$, $E_\perp$ being the (quantized) energy of the transverse motion and $E_\parallel$ the energy of the longitudinal motion; the energy $E_\perp$ is inversely proportional to the cross-section square. When a waveguide cross-section varies along the axis, one can consider $E \approx E_\perp + E_\parallel$ as approximate relation. In a narrow, $E_\perp$ is increasing and E remains constant, so $E_\parallel$ is decreasing.

That the resonant tunneling can occur in deformed waveguides was confirmed by numerical experiments [9], where the dependence of the transition coefficient T on the energy E of an electron was calculated. For some E, resonant peaks are present, where T is close to one. To analyze the operation of devices based on such waveguides it is useful to study the behavior of the coefficient T for energies close to a resonance. In particular, it is important to know the location of the resonance, the height of the resonant peak, and its width at the half-height, i.e., the resonant quality factor (Q-factor).

Approximate numerical calculations are effective only if the narrows of a waveguide are "not too narrow" (so that the resonant peak is sufficiently wide). When the peak is very sharp, known numerical procedures converge slowly and become unstable. That is why to obtain a detailed picture of the phenomenon it is important to combine both numerical and asymptotic methods supplementing each other.

1.4 Method of compound asymptotics

We construct an asymptotics of the wave function using the method of compound asymptotic expansions [10], [11].

Let the diameters ε_1 and ε_2 of the narrows of the waveguide $G(\varepsilon_1, \varepsilon_2)$ play the role of small parameters. The domain $G(0,0)$ obtained as the limit of $G(\varepsilon_1, \varepsilon_2)$ when $\varepsilon_1, \varepsilon_2 \to 0$ consists of three parts. The boundary value problem in any part of $G(0,0)$ is called the limit problem of the first kind. Solutions of the first kind problems serve as the principal term of approximation of the wave function in the corresponding part of $G(0,0)$ within a certain distance of the narrows. Intuitively, this means that we look at $G(\varepsilon_1, \varepsilon_2)$ with the naked eye observing no small details. Replacing the wave function by these solutions (in the corresponding part of $G(\varepsilon_1, \varepsilon_2)$) leads to an error which is supported in the very neighborhood of the narrows. With the help of the transformation of coordinates $x \to \varepsilon^{-1}x$ (with origin at a narrow) we enlarge the neighborhood of the narrow whereas the the remainder of the waveguide tends to infinity. As a result, we obtain the limit problems of the second kind in unbounded domains Ω_1 and Ω_2. Solutions of these problems together with solutions of the first kind problems give the first order approximation to the wave function.

More precisely, we first solve the limit problem of the first kind in one of the unbounded parts of $G(0,0)$. The leading part of the obtained discrepancy is compensated by a solution of the second kind limit problem. The discrepancy, given in turn by this solution, is concentrated, generally speaking, on the both sides of of the narrow. To continue the procedure, we need that the mentioned discrepancy would be concentrated outside of the part of $G(0,0)$, which was considered on the previous step. To this end, we apply the method of "redistribution of discrepancies" analogous to that used in [11]. Then the problem in the resonator must be solved. A new discrepancy is compensated by a solution of the second kind limit problem supported near the second narrow and so on. A more detailed but still short description of this procedure is given in Chapter 2.

An additional feature of our constructions in comparison with [10], [11] is using radiation conditions at infinity to fix necessary solutions of problems in unbounded domains. Moreover, that the limit problem in the resonator loses its unique solvability for some energies, causes difficulties in estimating remainders in the asymptotic expansions. Situations, when limit problems are not uniquely solvable, were considered in [10], [11], too. But our case is more complicated, because we derive an estimate uniform with respect both ε_1, ε_2 and the wave number k.

To make the material more understandable, in Chapter 2, we have presented the main asymptotic formulas in a more simple situation, when the waveguide is a strip or a cylinder with narrows. The particular case of symmetric waveguide is considered there as well. In Chapter 3, the problem in a "general" three-dimensional waveguide is stated. The limit problems are defined and some properties of these problems are listed. Then the tunneling in a waveguide with one narrow is studied. Finally, the resonant tunneling is investigated in a waveguide with two narrows.

2 FORMULATION OF MAIN RESULTS

2.1 Statement of the problem

2.1.1 Geometry of waveguide

We consider a waveguide with two narrows of small diameters ε_1 and ε_2. To describe the waveguide we first introduce three domains G, Ω_1, Ω_2 in $\mathbb{R}^n$ ($n = 2$ or 3) that are independent of the parameters ε_1 and ε_2.

Let G be a cylinder $\mathbb{R} \times D$, where $\mathbb{R}$ is a straight line and D is a cross-section that, for $n = 2$, is a segment and, for $n = 3$, is a domain bounded by a smooth closed path.

Pass on to Ω_1 (Fig. 8). For $n = 2$, we denote by K_1 a couple of vertical angles. For $n = 3$, by K_1 is meant a double cone which is symmetric about the coordinate origin and cuts out on the unit sphere centered at the origin two (symmetric) domains; each of them is bounded by a smooth contour. Assume that Ω_1 contains the cone K_1 and a neighborhood of its vertex, moreover, outside a large ball (with

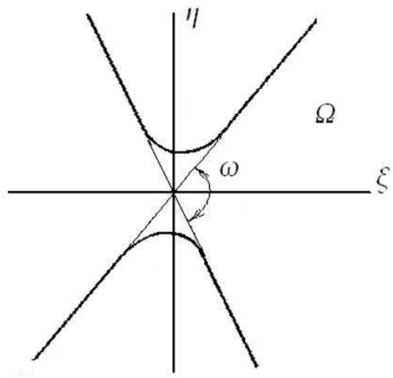

FIGURE 8 Geometry of a narrow.

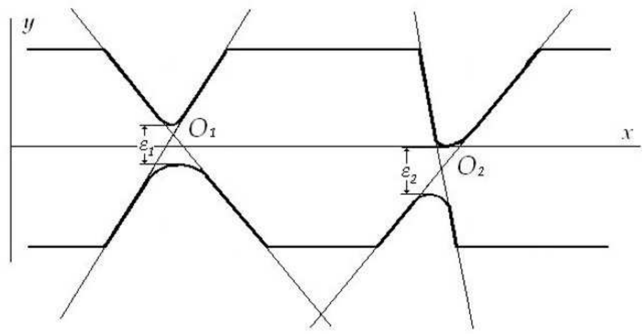

FIGURE 9 Geometry of the waveguide.

center at the vertex), Ω_1 coincides with K_1; the boundary of Ω_1 is assumed to be smooth. The domain Ω_2 is described analogously (with a cone K_2).

Now, we turn to the waveguide $G(\varepsilon_1, \varepsilon_2)$ (Fig. 9). For the time being, we let O_1 and O_2 be arbitrary (interior) points of the domain G. Introduce orthogonal coordinates (x_j, y_j, z_j) with origin O_j and axis x_j parallel to the generatrices of the cylinder G, $j = 1, 2$. Suppose the domain Ω_j to be located so that the vertex of K_j coincides with O_j and the whole axis x_j (except the origin) lies inside K_j. From now on we assume that the point O_1 is disposed far enough from the point O_2 so that the distance between the sets $\partial K_1 \cap \partial K_2$ and G is positive (as usual, ∂K_j stands for the boundary of K_j). Denote by $\Omega_j(\varepsilon_j)$ the domain obtained from Ω_j by the contraction with center at O_j and coefficient ε_j. In other words, $(x_j, y_j, z_j) \in \Omega_j(\varepsilon_j)$ if and only if $(x_j/\varepsilon_j, y_j/\varepsilon_j, z_j/\varepsilon_j) \in \Omega_j$. We put

$$G(\varepsilon_1, \varepsilon_2) = G \cap \Omega_1(\varepsilon_1) \cap \Omega_2(\varepsilon_2).$$

2.1.2 Boundary value problem

A wave function of a free electron of energy k^2 satisfies the boundary value problem

$$\begin{aligned} \Delta u + k^2 u &= 0 \quad \text{in} \quad G(\varepsilon_1, \varepsilon_2), \\ u &= 0 \quad \text{on} \quad \partial G(\varepsilon_1, \varepsilon_2). \end{aligned} \tag{2.1}$$

Moreover, u is subject to radiation conditions at infinity. To formulate the conditions we need the problem on the cross-section D of the waveguide:

$$\begin{aligned} \Delta v(y,z) + \lambda^2 v(y,z) &= 0, \quad (y,z) \in D, \\ v(y,z) &= 0, \quad (y,z) \in \partial D. \end{aligned} \tag{2.2}$$

The eigenvalues λ_q^2 of this problem, where $q = 1, 2, \ldots$, are called the thresholds. In the case of a two-dimensional waveguide $G(\varepsilon_1, \varepsilon_2)$, the domain D is the segment $(-l/2, l/2)$ and the thresholds form the sequence $\lambda_q^2 = (\pi q/l)^2$,

$q = 1, 2, \ldots$. In a three-dimensional waveguide, the thresholds form an increasing sequence of positive numbers tending to $+\infty$. We suppose that k^2 in (2.1) does not coincide with any of thresholds. For fixed (real) k, there exist finitely many linearly independent bounded wave functions. In the linear space of such wave functions corresponding to the given k, the basis is formed by wave functions that are subject to the radiation conditions

$$u_m(x,y,z) = \begin{cases} e^{i\nu_m x}\Psi_m(y,z) + \sum_{j=1}^{M} s_{mj}\, e^{-i\nu_j x}\Psi_j(y,z) + O(e^{\delta x}), & x \to -\infty, \\ \sum_{j=1}^{M} s_{m,M+j}\, e^{i\nu_j x}\Psi_j(y,z) + O(e^{-\delta x}), & x \to +\infty; \end{cases}$$

$$u_{M+m}(x,y,z) = \begin{cases} \sum_{j=1}^{M} s_{M+m,j}\, e^{-i\nu_j x}\Psi_j(y,z) + O(e^{\delta x}), & x \to -\infty, \\ e^{-i\nu_m x}\Psi_m(y,z) + \\ \qquad + \sum_{j=1}^{M} s_{M+m,M+j}\, e^{i\nu_j x}\Psi_j(y,z) + O(e^{-\delta x}), & x \to +\infty. \end{cases}$$

Here M is the number of thresholds satisfying the inequality $\lambda^2 < k^2$ (for a fixed k); $m = 1, 2, \ldots, M$; $\nu_m = \sqrt{k^2 - \lambda_m^2}$; Ψ_m is an eigenfunction of the problem (2.2) that corresponds to the eigenvalue λ_m^2 and is normalized by the condition

$$\nu_m \int_D |\Psi_m(y,z)|^2 dy\, dz = 1.$$

When the waveguide is two-dimensional,

$$\Psi_m(y) = \begin{cases} (2/l\nu_m)\sin\lambda_m y, & m \text{ even}, \\ (2/l\nu_m)\cos\lambda_m y, & m \text{ odd}. \end{cases} \tag{2.3}$$

The function U_j defined in the cylinder G by the equation

$$U_j(x,y,z) = e^{i\nu_j x}\Psi_j(y,z), \qquad j = 1, \ldots, M,$$

is a wave coming from $-\infty$ and going to $+\infty$. Analogously, the function

$$U_{M+j}(x,y,z) = e^{-i\nu_j x}\Psi_j(y,z), \qquad j = 1, \ldots, M,$$

is a wave going from $+\infty$ to $-\infty$. The scattering matrix

$$S = \|s_{mj}\|_{m,j=1,\ldots,2M}$$

is unitary. The value

$$R_m = \sum_{j=1}^{M} |s_{mj}|^2, \qquad m = 1, \ldots, M, \tag{2.4}$$

is called the reflection coefficient for the wave U_m, which comes to $G(\varepsilon_1, \varepsilon_2)$ from $-\infty$, the transition coefficient for this wave is defined by the equality

$$T_m = \sum_{j=1}^{M} |s_{m,\,M+j}|^2. \tag{2.5}$$

Similar definitions can be given for the wave U_{M+m} coming from $+\infty$.

In this work, we discuss only the situation, when the parameter k^2 is between the first and the second thresholds. Then the scattering matrix is of size 2×2 and (2.4) and (2.5) take the form

$$R = |s_{11}|^2, \qquad T = |s_{12}|^2.$$

We consider only the scattering of the wave coming from $-\infty$, that is why we omit the indices in the notation of the coefficients $R = R(k, \varepsilon_1, \varepsilon_2)$ and $T = T(k, \varepsilon_1, \varepsilon_2)$. The purpose is to find the "resonant" values $k_r = k_r(\varepsilon_1, \varepsilon_2)$ of the parameter k, at which the transition coefficient takes the maximal value. Moreover, we are interested in the behavior of $k_r(\varepsilon_1, \varepsilon_2)$ as $\varepsilon_1, \varepsilon_2 \to 0$.

2.2 Asymptotics of the wave function in a two-dimensional waveguide

To derive an asymptotics of a wave function (i.e. solution of the problem (2.1)) as ε_1, $\varepsilon_2 \to 0$ we use the method of compound asymptotic expansions. To this end we introduce "limit" boundary value problems independent of the parameters ε_1 and ε_2. Remind that G is the strip $\{(x, y) : -\infty < x < +\infty, -l/2 < y < l/2\}$, K_1 is a couple of vertical angles with vertex at the point $O_1 \in G$, and K_2 is a couple of vertical angles with vertex at the point $O_2 \in G$. We put $G(0, 0) = G \cap K_1 \cap K_2$ (Fig. 10). Thus, the set $G(0, 0)$ is divided into three parts $G^{(1)}$, $G^{(2)}$, and $G^{(3)}$, where $G^{(1)}$ and $G^{(3)}$ are infinite domains and $G^{(2)}$ is a bounded resonator.

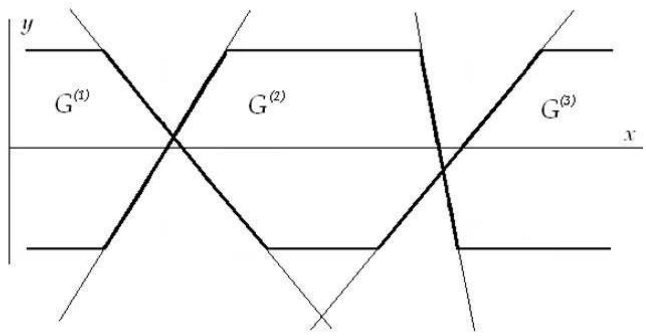

FIGURE 10 The limiting domain G(0,0).

The problems

$$\begin{aligned} \Delta v(x,y) + k^2 v(x,y) &= 0, \qquad (x,y) \in G^{(q)}, \\ v(x,y) &= 0, \qquad (x,y) \in \partial G^{(q)}, \end{aligned} \tag{2.6}$$

where $q = 1,2,3$ and $\partial G^{(q)}$ is the boundary of $G^{(q)}$, are called the first kind boundary value problems. Solutions $v^{(1)}$ and $v^{(3)}$ are subject to some radiation conditions at infinity and all three functions $v^{(1)}$, $v^{(2)}$, $v^{(3)}$ are subject to some conditions at the corner points. All of the conditions will be formulated as required.

Now we return to the domains Ω_1 and Ω_2 (see Fig. 8). Problems of the form

$$\begin{aligned} \Delta w(\xi_j,\eta_j) &= F(\xi_j,\eta_j) \qquad \text{in } \Omega_j, \\ w(\xi_j,\eta_j) &= 0 \qquad \text{on } \partial\Omega_j, \end{aligned} \tag{2.7}$$

are called the second kind boundary value problems. We seek solutions of these problems in the class of functions satisfying the condition

$$w(\xi_j,\eta_j) = O\left(\rho_j^{-3\pi/\omega_j}\right) \qquad \text{as } \rho_j \to \infty;$$

here, (ξ_j,η_j) are Cartesian coordinates in Ω_j with origin at O_j, ρ_j is the distance from (ξ_j,η_j) to O_j, and ω_j is the opening of K_j, $j = 1,2$.

In a two-dimensional waveguide $G(\varepsilon_1,\varepsilon_2)$, we consider the scattering of the wave $U(x,y) = e^{i\nu_1 x}\Psi_1(y)$ coming from $-\infty$ (see (2.3)). The main technical result is provided by the asymptotic formula (2.8) given below for the wave function. Although rather cumbersome, it results in much more explicit corollaries for basic physical characteristics of the process. The wave function admits the representation

$$\begin{aligned} &u(x,y;\varepsilon_1,\varepsilon_2) = \chi^{(1)}_{\varepsilon_1}(x,y)v^{(1)}(x,y;\varepsilon_1,\varepsilon_2) + \\ &+\Theta(r_1)w_1(\varepsilon_1^{-1}x_1,\varepsilon_1^{-1}y_1;\varepsilon_1,\varepsilon_2) + \chi^{(2)}_{\varepsilon_1,\varepsilon_2}(x,y)v^{(2)}(x,y;\varepsilon_1,\varepsilon_2) + \\ &+\Theta(r_2)w_2(\varepsilon_2^{-1}x_2,\varepsilon_2^{-1}y_2;\varepsilon_1,\varepsilon_2) + \chi^{(3)}_{\varepsilon_2}(x,y)v^{(3)}(x,y;\varepsilon_1,\varepsilon_2) + R(x,y;\varepsilon_1,\varepsilon_2). \end{aligned} \tag{2.8}$$

Let us explain the notation and the structure of this formula. When composing the formula, we first describe the behavior of the wave function to the right of the narrows, where the wave function can be approximated by a solution of problem (2.6) in the domain $G^{(3)}$. The solution of (2.6) is subject to the radiation condition

$$v^{(3)}(x,y;\varepsilon_1,\varepsilon_2) \sim s_{12}(\varepsilon_1,\varepsilon_2)e^{i\nu_1 x}\Psi_1(y) \qquad \text{as } x \to +\infty, \tag{2.9}$$

the element $s_{12}(\varepsilon_1,\varepsilon_2)$ of scattering matrix being yet unknown. Problem (2.6) does not contain ε_1, ε_2, nevertheless the function $v^{(3)}$ depends on the parameters because of $s_{12}(\varepsilon_1,\varepsilon_2)$.

Let $\chi^{(3)}_{\varepsilon_2}(x,y)$ be the cut-off function defined by

$$\chi^{(3)}_{\varepsilon_2}(x,y) = (1 - \Theta(r_2/\varepsilon_2))\,\mathbf{1}_{G^{(3)}}(x,y),$$

where $r_2 = \sqrt{x_2^2 + y_2^2}$ and (x_2, y_2) are the coordinates of a point (x, y) in the system obtained by shifting the origin to the point O_2; $\mathbf{1}_{G^{(3)}}$ is the characteristic function of $G^{(3)}$ (equal to one in $G^{(3)}$ and to zero outside $G^{(3)}$), and $\Theta(\rho)$ is a smooth positive function on the half-axis $0 \leqslant \rho < +\infty$ and is equal to one as $0 \leqslant \rho \leqslant \delta$ and to zero as $\rho \geqslant 2\delta$ (δ being a fixed small positive number). Note that the function $\chi^{(3)}_{\varepsilon_2}$ turns out to be defined on the whole waveguide $G(\varepsilon_1, \varepsilon_2)$ as well as the function $\chi^{(3)}_{\varepsilon_2} v^{(3)}$ in (2.8).

Being substituted to the problem (2.1), the function $\chi^{(3)}_{\varepsilon_2} v^{(3)}$ gives a discrepancy in the right-hand side of the Helmholtz equation; the discrepancy is supported near the second narrow (to the right of it). We compensate the principal part of the discrepancy with the help of the second kind limit problem in the domain Ω_2. Namely, the discrepancy is rewritten into coordinates (ξ_2, η_2) in Ω_2 and is taken as a right-hand side for the Laplace equation. The solution w_2 of the corresponding problem (2.7) has to be rewritten into coordinates (x_2, y_2) and multiplied by a cut-off function. As a result, there arises the term $\Theta(r_2) w_2(\varepsilon_2^{-1} x_2, \varepsilon_2^{-1} y_2; \varepsilon_1, \varepsilon_2)$ in (2.8).

Now we substitute the sum of two obtained terms into the problem (2.1). The principal part of the corresponding discrepancy is supported in $G^{(2)}$ near the second narrow. We compensate it by solving the problem (2.6) in $G^{(2)}$ and obtain the term $\chi^{(2)}_{\varepsilon_1,\varepsilon_2}(x, y) v^{(2)}(x, y; \varepsilon_1, \varepsilon_2)$ with

$$\chi^{(2)}_{\varepsilon_1,\varepsilon_2}(x, y) = \Big(1 - \Theta(\varepsilon_1^{-1} r_1) - \Theta(\varepsilon_2^{-1} r_2)\Big) \mathbf{1}_{G^{(2)}}(x, y).$$

After that the summands

$$\Theta(r_1) w_1(\varepsilon_1^{-1} x_1, \varepsilon_1^{-1} y_1; \varepsilon_1, \varepsilon_2) \text{ and } \chi^{(1)}_{\varepsilon_1}(x, y) v^{(1)}(x, y; \varepsilon_1, \varepsilon_2)$$

arise in a similar way.

At the last step, we find the function $v^{(1)}$ that satisfies both the limit problem (2.6) in $G^{(1)}$ and the radiation condition

$$v^{(1)}(x, y; \varepsilon_1, \varepsilon_2) \sim s_{12}(\varepsilon_1, \varepsilon_2)\alpha(\varepsilon_1, \varepsilon_2) e^{i\nu_1 x} \Psi_1(y) + s_{12}(\varepsilon_1, \varepsilon_2)\beta(\varepsilon_1, \varepsilon_2) e^{-i\nu_1 x} \Psi_1(y)$$

as $x \to -\infty$. The coefficients α, β and the elements s_{11}, s_{12} of the scattering matrix turn out to be uniquely determined by a relation between α and β that assures compensation of the principal part of the discrepancy arising when the problem is being solved in $G^{(1)}$, and by the following requirements:

$$s_{12}(\varepsilon_1, \varepsilon_2)\alpha(\varepsilon_1, \varepsilon_2) = 1, \quad s_{12}(\varepsilon_1, \varepsilon_2)\beta(\varepsilon_1, \varepsilon_2) = s_{11}(\varepsilon_1, \varepsilon_2), \quad |\alpha|^2 = |\beta|^2 + 1;$$

the last equality means the scattering matrix is unitary.

The remainder $R(x, y; \varepsilon_1, \varepsilon_2)$ in (2.8) is small in comparison with the principal part of (2.8) as $\varepsilon_1, \varepsilon_2 \to 0$.

We specify (2.8) for a "symmetric" waveguide, where $\varepsilon_1 = \varepsilon_2 = \varepsilon$, both narrows have the same opening ω, and the resonator $G^{(2)}$ is invariant with respect to the transformation $(x, y) \mapsto (d - x, -y)$, while d is the distance between the points O_1 and O_2. A specification can be obtained without any assumptions concerning symmetry, however the formulas would be much more cumbersome. Nevertheless, we will present asymptotic expansions for the most important characteristics (say, the transition coefficient) in the general case as well.

1. The function $v^{(3)}(x, y; \varepsilon)$ is defined by the equality

$$v^{(3)}(x, y; \varepsilon) = I(\varepsilon)\mathbf{v}^{(3)}(x, y), \tag{2.10}$$

where $I(\varepsilon)$ is a constant depending on ε and given by (2.15) below; $\mathbf{v}^{(3)}(x, y)$ is a solution of the first kind limit problem in $G^{(3)}$ and satisfies

$$\mathbf{v}^{(3)}(x, y) \sim r_2^{-\pi/\omega}\Phi(\varphi_2) \quad \text{as } r_2 \to 0$$

in a neighborhood of the point O_2; here (r_2, φ_2) are polar coordinates with center O_2, $\Phi(\varphi) = \cos(\pi\varphi/\omega)$, and

$$\mathbf{v}^{(3)}(x, y) \sim Ae^{i\alpha^{(3)}}e^{i\nu_1 x}\Psi_1(y) \quad \text{as } x \to +\infty, \tag{2.11}$$

where $A > 0$. The problem for $\mathbf{v}^{(3)}$ is uniquely solvable.

2. $w_2(\xi_2, \eta_2; \varepsilon) = I(\varepsilon)\varepsilon^{-\pi/\omega}\mathbf{w}_2(\xi_2, \eta_2)$,
where $\mathbf{w}_2$ is the solution of the second kind problem in Ω_2 with right-hand side

$$\begin{aligned} F_2(\rho_2, \varphi_2) &= -[\Delta, \zeta_+]\left(\rho_2^{-\pi/\omega}\Phi(\varphi_2)\right) - \\ &\quad -[\Delta, \zeta_-]\left(H_{21}\rho_2^{-\pi/\omega} + H_{22}\rho_2^{\pi/\omega}\right)\Phi(\pi - \varphi_2). \end{aligned}$$

Here, ζ_+ is a cut-off function equal to one in the right half-plane as $\rho_2 > d$ and to zero anywhere in the left half-plane and, in the right half-plane, for $\rho_2 < d/2$ (d being a fixed sufficiently large number); $\zeta_-(\rho_2, \varphi_2) = \zeta_+(\rho_2, \pi - \varphi_2)$; the constants H_{21} and H_{22} are uniquely determined by the requirement $\mathbf{w}_2 = O\left(\rho_2^{-3\pi/\omega}\right)$ as $\rho_2 \to \infty$.

3. $v^{(2)}(x, y; \varepsilon) = I(\varepsilon)\left(\varepsilon^{-2\pi/\omega}\dfrac{H_{22}}{c_1(k)}\mathbf{v}_-^{(2)}(x, y) + q_0 H_{21}\mathbf{v}^{(2)}(x, y)\right)$,
where $\mathbf{v}_-^{(2)}$ and $\mathbf{v}^{(2)}$ are solutions of the problem (2.6) in $G^{(2)}$, while

$$\mathbf{v}^{(2)} \sim \begin{cases} r_2^{-\pi/\omega}\Phi(\pi - \varphi_2) & \text{near } O_2, \\ q_0 r_1^{-\pi/\omega}\Phi(\varphi_1) & \text{near } O_1; \end{cases}$$

$$\mathbf{v}_-^{(2)} \sim \begin{cases} c_1(k)r_2^{\pi/\omega}\Phi(\pi - \varphi_2) & \text{near } O_2, \\ (k^2 - k_0^2)r_1^{-\pi/\omega}\Phi(\varphi_1) + b_1(k)r_1^{\pi/\omega}\Phi(\varphi_1) & \text{near } O_1; \end{cases}$$

where k_0^2 is any eigenvalue of the resonator that is between the first and the second thresholds. Given the increasing terms in the asymptotics near any of the corner points, the solutions $\mathbf{v}_-^{(2)}$ and $\mathbf{v}^{(2)}$ are defined uniquely; at the same time the constants q_0, $c_1(k)$, and $b_1(k)$ are defined, too. The constant q_0 is independent of k; for a symmetric waveguide, $q_0 = \pm 1$.

4. $w_1(\xi_1,\eta_1;\varepsilon) =$

$$= \varepsilon^{-\pi/\omega}\left[\left(\frac{H_{22}}{c_1(k)}\frac{k^2-k_0^2}{\varepsilon^{2\pi/\omega}} + q_0 H_{21}\right)\mathbf{w}_1^-(\xi_1,\eta_1) + H_{22}\frac{b_1(k)}{c_1(k)}\mathbf{w}_1^+(\xi_1,\eta_1)\right],$$

where $\mathbf{w}_1^\pm$ is the solution of the problem (2.7) in the domain Ω_1 for

$$\begin{aligned} F_1^\pm(\rho_1,\varphi_1) &= -[\Delta,\zeta_+]\left(\rho_1^{\pm\pi/\omega}\Phi(\varphi_1)\right) - \\ &\quad -[\Delta,\zeta_-]\left(H_{11}^\pm\rho_1^{-\pi/\omega} + H_{12}^\pm\rho_1^{\pi/\omega}\right)\Phi(\pi-\varphi_1), \end{aligned}$$

the constants $H_{11}^\pm$ and $H_{12}^\pm$ are uniquely defined by the condition $\mathbf{w}_1^\pm = O\left(\rho_1^{-3\pi/\omega}\right)$ as $\rho_1 \to \infty$.

5. $v^{(1)}(x,y;\varepsilon) = \overline{\mathbf{v}^{(1)}(x,y)} + I(\varepsilon)\left(\frac{1}{2}\left(a_1(\varepsilon) + \frac{1}{a_1(\varepsilon)}\right) - \frac{a_0(\varepsilon)}{2iA}\right)\mathbf{v}^{(1)}(x,y),$
where $\mathbf{v}^{(1)}$ is the solution of the problem (2.6) in $G^{(1)}$ subject to the conditions

$$\mathbf{v}^{(1)}(x,y) \sim r_1^{-\pi/\omega}\Phi(\varphi_1) \quad \text{as } r_1 \to 0,$$

and

$$\mathbf{v}^{(1)}(x,y) \sim Ae^{i\alpha^{(1)}}e^{-i\nu_1 x}\Psi_1(y) \quad \text{as } x \to -\infty, \tag{2.12}$$

$A > 0$ is the same constant as in the radiation conditions for $\mathbf{v}^{(3)}$, $\overline{\mathbf{v}^{(1)}}$ stands for the function complex-conjugated with $\mathbf{v}^{(1)}$. Moreover,

$$a_1(\varepsilon) = H_{22}H_{11}^-\frac{k^2-k_0^2}{\varepsilon^{2\pi/\omega}c_1(k)} + H_{22}H_{11}^+\frac{b_1(k)}{c_1(k)} + H_{21}H_{11}^- q_0, \tag{2.13}$$

$$a_0(\varepsilon) = \varepsilon^{-2\pi/\omega}\left(H_{22}H_{12}^-\frac{k^2-k_0^2}{\varepsilon^{2\pi/\omega}c_1(k)} + H_{22}H_{12}^+\frac{b_1(k)}{c_1(k)} + H_{21}H_{12}^- q_0\right), \tag{2.14}$$

$$I(\varepsilon) = \left(\frac{1}{2}\left(a_1(\varepsilon) + \frac{1}{a_1(\varepsilon)}\right) - \frac{a_0(\varepsilon)}{2iA}\right)^{-1}(Ae^{-i\alpha^{(1)}})^{-1}. \tag{2.15}$$

The $a_1(\varepsilon)$, $a_0(\varepsilon)$, A, $\alpha^{(1)}$, $\alpha^{(3)}$, and $I(\varepsilon)$ depend on k.

6. The remainder $R(x,y;\varepsilon)$ can be represented in the form $\varepsilon^{2-\delta}\widetilde{R}(x,y;\varepsilon)$, where δ is an arbitrary small positive number, while $\widetilde{R}(x,y;\varepsilon)$ is small by comparison with the principal part of (2.8) as $\varepsilon \to 0$.

2.3 Resonant tunneling in a two-dimensional waveguide

First, we discuss a symmetric waveguide using the formula for the wave function in Section 2.2. When considering asymmetric waveguides in the last part of this section, we do not write out the unwieldy expression for the wave function and restrict ourselves to more comprehensible formulas for the resonant frequency, the reflection and transition coefficients, the height of the resonant peak, and its width at half-height.

2.3.1 Resonant frequency

A resonant frequency is a value $k = k_r(\varepsilon)$ at which the transition coefficient $T = T(k, \varepsilon)$ has a (local) maximum, i. e. $T(k, \varepsilon) \leqslant T(k_r(\varepsilon), \varepsilon)$ for any k in a small neighborhood of $k_r(\varepsilon)$.

The formulas (2.10), (2.11), and (2.15) result in the expression for the transition coefficient:

$$\begin{aligned} T(k,\varepsilon) &= A^2(k)|I(k,\varepsilon)|^2 = \\ &= \left(\frac{1}{4}\left(a_1(k,\varepsilon) + \frac{1}{a_1(k,\varepsilon)}\right)^2 + \frac{a_0(k,\varepsilon)^2}{4A^2(k)}\right)^{-1} \left(1 + o(\varepsilon^{2-\delta})\right). \quad (2.16) \end{aligned}$$

From the representations (2.13) and (2.14) for a_1 and a_0 it follows that, as ε is small, the coefficient $T(k, \varepsilon)$ reaches its maximum, if $a_0(k, \varepsilon) = 0$.

We rewrite this condition (using (2.14)) in the $k^2 - k_0^2$-resolved form. Note that from the equation obtained, k^2 can be found by the step-by-step method (since ε is small). Taking only the leading summand in the series for $k^2 - k_0^2$ by powers of ε, we obtain the leading term in the asymptotics of the resonant frequency:

$$k_r^2(\varepsilon) = k_0^2 - \left(q_0 c_1(k_0)\frac{H_{21}}{H_{22}} + b_1(k_0)\frac{H_{12}^+}{H_{11}^-}\right)\varepsilon^{2\pi/\omega} + o(\varepsilon^{2\pi/\omega+2-\delta}). \quad (2.17)$$

A more detailed analysis of the solutions of the limit problems involved in the asymptotics of the wave function shows that the coefficient of $\varepsilon^{2\pi/\omega}$ in (2.17) is negative.

2.3.2 The asymptotics of the wave function near a resonance (symmetric waveguide)

For k close to a resonant frequency, the expression for the wave function obtained in Section 2.2 can be somewhat simplified.

We first consider (2.13) and (2.14). Let us expand all the functions in k involved in these formulas in power series of $k^2 - k_r^2$ and, under the assumption $k^2 - k_r^2 = o(\varepsilon^{2\pi/\omega})$, take only the principal terms of the expansions:

$$\begin{aligned} a_1(k,\varepsilon) &= \frac{H_{22}}{H_{12}^-}\left(H_{11}^+H_{12}^- - H_{11}^-H_{12}^+\right)\frac{b_1(k_r)}{c_1(k_r)} + O\left(\frac{k^2-k_r^2}{\varepsilon^{2\pi/\omega}}\right), \\ a_0(k,\varepsilon) &= H_{22}H_{12}^-\frac{k^2-k_r^2}{\varepsilon^{4\pi/\omega}c_1(k_r)} + O\left(\frac{k^2-k_r^2}{\varepsilon^{2\pi/\omega}}\right). \end{aligned}$$

The $b_1(k_r)$ and $c_1(k_r)$ depend on ε due to k_r. To avoid such a dependence we use the following obvious relations

$$\begin{aligned} b_1(k_r) &= b_1(k_0) + O(k_r^2 - k_0^2) = b_1(k_0) + O(\varepsilon^{2\pi/\omega}), \\ c_1(k_r) &= c_1(k_0) + O(k_r^2 - k_0^2) = c_1(k_0) + O(\varepsilon^{2\pi/\omega}). \end{aligned}$$

Then

$$\begin{aligned} a_1(k,\varepsilon) &= \frac{H_{22}}{H_{12}^-}\left(H_{11}^+ H_{12}^- - H_{11}^- H_{12}^+\right)\frac{b_1(k_0)}{c_1(k_0)} + \\ &+ O\left(\max\left\{\varepsilon^{2\pi/\omega}, \frac{|k^2 - k_r^2|}{\varepsilon^{2\pi/\omega}}\right\}\right), \qquad (2.18) \\ a_0(k,\varepsilon) &= H_{22}H_{12}^- \frac{k^2 - k_r^2}{\varepsilon^{4\pi/\omega} c_1(k_0)} + O\left(\frac{k^2 - k_r^2}{\varepsilon^{2\pi/\omega}}\right). \qquad (2.19) \end{aligned}$$

One can show that, in a symmetric waveguide, the leading term in the right-hand side of (2.18) is equal to $q_0 = \pm 1$. Taking that into account, we analogously rewrite (2.15) :

$$I(k,\varepsilon) = \frac{2ie^{i\alpha^{(1)}(k_0)}}{2iA(k_0)q_0 + \dfrac{H_{22}H_{12}^-}{c_1(k_0)}\left(\dfrac{k^2 - k_r^2}{\varepsilon^{4\pi/\omega}}\right)} + O\left(\max\left\{\varepsilon^{2\pi/\omega}, \frac{|k^2 - k_r^2|}{\varepsilon^{2\pi/\omega}}\right\}\right). \qquad (2.20)$$

As $k^2 = k_r^2 + o(\varepsilon^{2\pi/\omega})$, we obtain the following expressions for the elements of the asymptotics of the wave function:

$$\begin{aligned} v^{(1)}(x,y;k,\varepsilon) &= \frac{e^{i\alpha^{(1)}(k_0)}}{A(k_0)}\overline{\mathbf{v}^{(1)}(x,y;k_0)} - \frac{I(k,\varepsilon)a_0(k,\varepsilon)}{2iA(k_0)}\mathbf{v}^{(1)}(x,y;k_0) + O(\varepsilon^{2\pi/\omega}), \\ w_1(\xi,\eta;k,\varepsilon) &= \varepsilon^{-\pi/\omega} I(k,\varepsilon)\left[H_{22}\frac{b_1(k_0)}{c_1(k_0)}\left(\frac{H_{12}^+}{H_{12}^-}\mathbf{w}_1^+(\xi,\eta) - \mathbf{w}_1^-(\xi,\eta)\right) + \right. \\ &+ \left. O\left(\max\left\{\varepsilon^{2\pi/\omega}, \frac{|k^2 - k_r^2|}{\varepsilon^{2\pi/\omega}}\right\}\right)\right], \\ v^{(2)}(x,y;k,\varepsilon) &= \varepsilon^{-2\pi/\omega} I(k,\varepsilon) H_{22}\frac{1}{c_1(k_0)}\mathbf{v}_-^{(2)}(x,y;k_0) + O(1), \\ w_2(\xi,\eta;k,\varepsilon) &= \varepsilon^{-\pi/\omega} I(k,\varepsilon)\mathbf{w}_2(\xi,\eta), \\ v^{(3)}(x,y;k,\varepsilon) &= I(k,\varepsilon)\mathbf{v}^{(3)}(x,y;k_0) + O(\varepsilon^{2\pi/\omega}), \end{aligned}$$

where $a_0(k,\varepsilon)$ and $I(k,\varepsilon)$ are given by (2.19) and (2.20) respectively. In the formulas for $v^{(1)}$, $v^{(2)}$, and $v^{(3)}$, the neglected terms are infinitely large near the points O_1 and O_2, so their estimates $O(\varepsilon^{2\pi/\omega})$ and $O(1)$ (as $\varepsilon \to 0$) are uniform with respect to (x,y) only on sets that are separated from O_1 and O_2 (and independent of ε). Analogously, the remainder estimate in the formula for w_1 is uniform with respect to (ξ,η) on bounded sets independent of ε.

2.3.3 Reflection and transition coefficients

Using (2.11), (2.20), and the expression for $v^{(3)}$, we find the amplitude $t(k,\varepsilon)$ of the transited wave,

$$t(k,\varepsilon) = \frac{2iA(k_0)e^{i(\alpha^{(1)}(k_0)+\alpha^{(3)}(k_0))}}{2iA(k_0)q_0 + \dfrac{H_{22}H_{12}^-}{c_1(k_0)}\left(\dfrac{k^2-k_r^2}{\varepsilon^{4\pi/\omega}}\right)} + o\left(\varepsilon^{2-\delta}\right).$$

Suppose that $k^2 - k_r^2 = O(\varepsilon^{2\pi/\omega+2})$. Then the remainder

$$O\left(\max\{\varepsilon^{2\pi/\omega}, |k^2-k_r^2|/\varepsilon^{2\pi/\omega}\}\right)$$

arisen after substituting the formulas (2.18)-(2.20) can be united with the summand $o\left(\varepsilon^{2-\delta}\right)$, which arises from the remainder in the formula (2.8). As shown below, the width of the resonant peak is $O(\varepsilon^{4\pi/\omega})$. Hence the condition put on k^2 is not very burdensome; it allows us to use the stated asymptotics for $t(k,\varepsilon)$ in the most interesting region, i. e. in a neighborhood of the resonant peak. However the condition can be weakened; to this end, in the expansions (2.18)-(2.20) one should take two or more terms.

Analogously, from (2.12), (2.20), and expression for $v^{(1)}$ we find the amplitude of the reflected wave

$$r(k,\varepsilon) = \frac{e^{2i\alpha^{(1)}(k_0)}\dfrac{H_{22}H_{12}^-}{c_1(k_0)}\left(\dfrac{k^2-k_r^2}{\varepsilon^{4\pi/\omega}}\right)}{2iA(k_0)q_0 + \dfrac{H_{22}H_{12}^-}{c_1(k_0)}\left(\dfrac{k^2-k_r^2}{\varepsilon^{4\pi/\omega}}\right)}\left(1+o\left(\varepsilon^{2-\delta}\right)\right).$$

This leads to the asymptotics for the transition and reflection coefficients,

$$\begin{aligned} T(k,\varepsilon) &= |t(k,\varepsilon)|^2 = \frac{1}{1+Q\left(\dfrac{k^2-k_r^2}{\varepsilon^{4\pi/\omega}}\right)^2} + o\left(\varepsilon^{2-\delta}\right), \qquad (2.21)\\ R(k,\varepsilon) &= |r(k,\varepsilon)|^2 = \frac{Q\left(\dfrac{k^2-k_r^2}{\varepsilon^{4\pi/\omega}}\right)^2}{1+Q\left(\dfrac{k^2-k_r^2}{\varepsilon^{4\pi/\omega}}\right)^2}\left(1+o\left(\varepsilon^{2-\delta}\right)\right), \end{aligned}$$

where $Q = \left(H_{22}H_{12}^-/2c_1(k_0)A(k_0)\right)^2$. It can be seen that the principal term of the asymptotics of $T(k,\varepsilon)$ has at $k=k_r$ a peak of height 1 and of width (at the half-height)

$$\Delta(\varepsilon) = \frac{2}{\sqrt{Q}}\varepsilon^{4\pi/\omega}.$$

2.3.4 Asymmetric waveguide

Let a waveguide have two narrows of distinct diameters ε_1 and ε_2. Moreover let the narrows have the distinct openings ω_1 and ω_2. In such a situation, one can repeat all the above reasoning. The values H_{22}/H_{12}^- and $(b_1(k_0)/c_1(k_0))^2$ ($= q_0^2$) must no longer be equal to one. At the same time, the amplitudes of the outgoing waves in the asymptotics (2.11) and (2.12) can be distinct; we denote these amplitudes by $A^{(3)}$ and $A^{(1)}$. As a result, we obtain

$$\begin{aligned} k_r^2(\varepsilon_1,\varepsilon_2) &= k_0^2 - b_1(k_0)\frac{H_{12}^+}{H_{11}^-}\varepsilon_1^{2\pi/\omega_1} - q_0 c_1(k_0)\frac{H_{21}}{H_{22}}\varepsilon_2^{2\pi/\omega_2} + \\ &\qquad + o\left(\varepsilon_1^{2\pi/\omega_1+2-\delta} + \varepsilon_2^{2\pi/\omega_2+2-\delta}\right), \quad (2.22) \\ T(k,\varepsilon_1,\varepsilon_2) &= \frac{1}{\frac{1}{4}\left(z+\frac{1}{z}\right)^2 + P\left(\frac{k^2-k_r^2}{\varepsilon_1^{2\pi/\omega_1}\varepsilon_2^{2\pi/\omega_2}}\right)^2} + o\left(\varepsilon_1^{2-\delta}+\varepsilon_2^{2-\delta}\right), \quad (2.23) \end{aligned}$$

where $k^2 = k_r^2 + O\left(\min\{\varepsilon_1^{2\pi/\omega_1+2}, \varepsilon_2^{2\pi/\omega_2+2}\}\right)$,

$$z = \frac{A^{(1)}(k_0)H_{22}b_1(k_0)\varepsilon_1^{2\pi/\omega_1}}{A^{(3)}(k_0)H_{12}^-c_1(k_0)\varepsilon_2^{2\pi/\omega_2}}, \qquad P = \left(\frac{H_{22}H_{12}^-}{2c_1(k_0)A^{(1)}(k_0)A^{(3)}(k_0)}\right)^2.$$

Thus, in an asymmetric waveguide, the principal term of the asymptotics of T is less than one. The width of the peak at half-height (calculated by means of the principal term of the asymptotics of the transition coefficient) equals

$$\Delta(\varepsilon_1,\varepsilon_2) = \frac{|z+z^{-1}|}{\sqrt{P}}\,\varepsilon_1^{2\pi/\omega_1}\varepsilon_2^{2\pi/\omega_2}. \quad (2.24)$$

2.4 Resonant tunneling in a three-dimensional waveguide

In this section, we present asymptotics for resonant frequencies and the transition coefficients in a three-dimensional waveguide. To derive these formulas we have first constructed (as well as in a two-dimensional waveguide) an asymptotics of the wave function. The formula is rather unwieldy, and we do not present it here (see the final formulas of Subsection 3.4.2).

2.4.1 Limit problems

Recall that G is a cylinder $\mathbb{R}\times D$, K_j a cone with vertex at O_j cutting out a domain S_j on the unit sphere centered at the vertex of the cone, $j = 1,2$. The set $G(0,0) = G\cap K_1\cap K_2$ is divided into three parts $G^{(1)}$, $G^{(2)}$, and $G^{(3)}$, where $G^{(1)}$ and $G^{(3)}$ are infinite domains and $G^{(2)}$ is a bounded resonator.

We consider the first kind boundary value problems

$$\begin{aligned} \Delta v(x,y,z) + k^2 v(x,y,z) &= 0, \qquad (x,y,z) \in G^{(q)}, \\ v(x,y,z) &= 0, \qquad (x,y,z) \in \partial G^{(q)}, \end{aligned}$$

where $q = 1,2,3$. Let $\mathbf{v}^{(1)}$ be a solution of the problem in $G^{(1)}$ satisfying

$$\mathbf{v}^{(1)}(x,y,z) \sim \begin{cases} A^{(1)} e^{i\alpha^{(1)}} e^{-i\nu_1 x} \Psi_1(y,z) & \text{as} \quad x \to -\infty, \\ r_1^{-\mu_{11}-1} \Phi_{11}(\varphi_1) & \text{as} \quad r_1 \to 0, \end{cases}$$

where $A^{(1)} > 0$, $\alpha^{(1)} \in \mathbb{R}$, $\nu_1 = \sqrt{k^2 - \lambda_1^2}$, λ_1^2 is the first eigenvalue of the operator $-\Delta$ in the domain D, Ψ_1 is an eigenfunction corresponding to the eigenvalue λ_1^2, and (r_1, φ_1) are polar coordinates with center at O_1. The $\mu_{11}(\mu_{11}+1)$ stands for the first eigenvalue of the Laplace–Beltrami operator on the base S_1 of the cone K_1 and Φ_{11} is an eigenfunction corresponding to the eigenvalue $\mu_{11}(\mu_{11}+1)$. Moreover let $\mathbf{v}^{(2)}$ and $\mathbf{v}_-^{(2)}$ be solutions of the problem in $G^{(2)}$ such that

$$\begin{aligned} \mathbf{v}^{(2)}(x,y,z) &\sim \begin{cases} r_2^{-\mu_{21}-1} \Phi_{21}(\varphi_2) & \text{near } O_2, \\ q_0 r_1^{-\mu_{11}-1} \Phi_{11}(\varphi_1) & \text{near } O_1; \end{cases} \\ \mathbf{v}_-^{(2)}(x,y,z) &\sim \begin{cases} c_1(k) r_2^{\mu_{21}} \Phi_{21}(\varphi_2) & \text{near } O_2, \\ (k^2 - k_0^2) r_1^{-\mu_{11}-1} \Phi_{11}(\varphi_1) + b_1(k) r_1^{\mu_{11}} \Phi_{11}(\varphi_1) & \text{near } O_1, \end{cases} \end{aligned}$$

where (r_2, φ_2) are polar coordinates with center at the point O_2, μ_{21} is such that $\mu_{21}(\mu_{21}+1)$ is the first eigenfunction of the Laplace-Beltrami operator on S_2, Φ_{21} is a corresponding eigenfunction, and k_0^2 an eigenvalue of the operator $-\Delta$ in $G^{(2)}$. At last, let $\mathbf{v}^{(3)}$ be a solution of the problem in $G^{(3)}$,

$$\mathbf{v}^{(3)}(x,y,z) \sim \begin{cases} A^{(3)} e^{i\alpha^{(3)}} e^{i\nu_1 x} \Psi_1(y,z) & \text{as} \quad x \to +\infty, \\ r_2^{-\mu_{21}-1} \Phi_{21}(\varphi_2) & \text{as} \quad r_2 \to 0, \end{cases}$$

where $A^{(3)} > 0$, $\alpha^{(3)} \in \mathbb{R}$. We did not indicate the dependence of $\mathbf{v}^{(j)}$, $A^{(j)}$, and $\alpha^{(j)}$ on the variable k for simplicity of notation.

Now, we consider the second kind boundary value problems

$$\begin{aligned} &\Delta w(\xi,\eta) = F(\xi,\eta) - [\Delta, \zeta_-] \left(H_{j1}^{\pm} \rho^{\mu_{j1}} + H_{j2}^{\pm} \rho^{-\mu_{j1}-1} \right) \Phi(\varphi) && \text{in } \Omega_j, \\ &w(\xi,\eta) = 0 && \text{on } \partial\Omega_j, \end{aligned}$$

where $j = 1,2$, the domains Ω_j are defined in Subsection 2.1.1, (ρ, φ) are polar coordinates in Ω_j centered at O_j, and the cut-off functions ζ_- are similar to those described in §3. The constants $H_{j1}^{\pm}$, $H_{j2}^{\pm}$ are chosen so that $w = o(\rho^{-\mu_{j1}-1})$ as $\rho \to \infty$. By $\{\mathbf{w}_j^+, H_{j1}^+, H_{j2}^+\}$ $(j = 1,2)$, we denote the solution of the problem in Ω_j with right-hand side

$$F_j(\rho,\varphi) = -[\Delta, \zeta_+] \rho^{\mu_{j1}} \Phi_{j1}(\varphi),$$

and by $\{\mathbf{w}_j^-, H_{j1}^-, H_{j2}^-\}$ the solution of the problem with right-hand side

$$F_j(\rho,\varphi) = -[\Delta, \zeta_+] \rho^{-\mu_{j1}-1} \Phi_{j1}(\varphi).$$

2.4.2 Asymptotic formulas

Let k_0^2 be an eigenvalue of the operator $-\Delta$ in the resonator $G^{(2)}$ and let k_0^2 lie between the first and the second thresholds. Near such an eigenvalue there is a frequency that resonant tunneling occurs at. It is expressed by the formula

$$\begin{aligned} k_r^2(\varepsilon_1,\varepsilon_2) = k_0^2 - b_1(k_0)\frac{H_{12}^+}{H_{11}^-}\varepsilon_1^{2\mu_{11}+1} - q_0 c_1(k_0)\frac{H_{21}^-}{H_{22}^-}\varepsilon_2^{2\mu_{21}+1} + \\ + o\left(\varepsilon_1^{2\mu_{11}+3-\delta} + \varepsilon_2^{2\mu_{21}+3-\delta}\right). \end{aligned} \tag{2.25}$$

Near a resonance (as $k^2 - k_r^2 = O(\min\{\varepsilon_1^{2\mu_{11}+3}, \varepsilon_2^{2\mu_{21}+3}\})$), the transition coefficient satisfies

$$T(k,\varepsilon_1,\varepsilon_2) = \frac{1}{\frac{1}{4}\left(z+\frac{1}{z}\right)^2 + P\left(\frac{k^2-k_r^2}{\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}}\right)^2}\left(1 + o\left(\varepsilon_1^{2-\delta} + \varepsilon_2^{2-\delta}\right)\right), \tag{2.26}$$

where

$$z = \frac{A^{(1)}(k_0)H_{22}^- b_1(k_0)\varepsilon_1^{2\mu_{11}+1}}{A^{(3)}(k_0)H_{12}^- c_1(k_0)\varepsilon_2^{2\mu_{21}+1}}, \qquad P = \left(\frac{H_{22}^- H_{12}^-}{2c_1(k_0)A^{(1)}(k_0)A^{(3)}(k_0)}\right)^2.$$

The width of the resonant peak at its half-height calculated by means of the principal term of the asymptotics for $T(k,\varepsilon_1,\varepsilon_2)$ is

$$\Delta(\varepsilon_1,\varepsilon_2) = \frac{|z+z^{-1}|}{\sqrt{P}}\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}. \tag{2.27}$$

The formulas of this Chapter were presented in [14] and the detailed proofs exposed in the following Chapters were given in [15].

3 ASYMPTOTIC THEORY OF ELECTRON TUNNELING IN THREE-DIMENSIONAL WAVEGUIDES

3.1 Statement of the problem in a three-dimensional waveguide

3.1.1 Geometry of waveguide

We consider a waveguide with two narrows of small diameters ε_1 and ε_2. To describe the waveguide, we first introduce three domains G, Ω_1, Ω_2 in $\mathbb{R}^3$, which are independent of the parameters ε_1 and ε_2.

Let G (Fig. 11) be a domain in $\mathbb{R}^3$ that coincides outside of a large ball with the union of two non-overlapping half-cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$; their cross-sections are denoted by D_1 and D_2. Each of D_1 and D_2 is a domain bounded by a (simple) smooth closed path. The boundary of G is assumed to be smooth.

Pass on to Ω_1 (Fig. 12). We denote by K_1 and L_1 open cones in $\mathbb{R}^3$ whose closures $\overline{K}_1$ and $\overline{L}_1$ have no common points except vertex. Suppose that there ex-

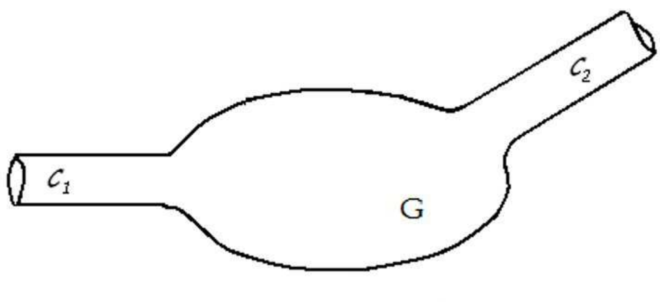

FIGURE 11 The domain G.

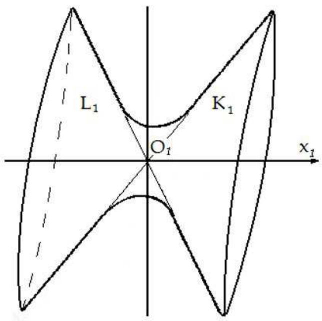

FIGURE 12 Geometry of a narrow.

ists a straight line s_1 passing through the vertex of K_1 and L_1 and lying (except the vertex) in $K_1 \cup L_1$. (The last condition is assumed only to simplify the description in what follows.) The cone K_1 (L_1) cuts out on the unit sphere centered at the vertex a domain $S(K_1)$ ($S(L_1)$) bounded by a smooth closed path. Suppose that Ω_1 contains both cones K_1 and L_1 as well as a neighborhood of their vertex, moreover, outside a large ball (with center at the vertex) Ω_1 coincides with $K_1 \cup L_1$; the boundary of Ω_1 assumed to be smooth. The domain Ω_2 is described analogously with cones K_2, L_2 and a straight line s_2.

Now, we turn to the waveguide $G(\varepsilon_1, \varepsilon_2)$ (Fig. 13). For the time being, we let O_1 and O_2 be arbitrary (interior) points of the domain G placed (for the sake of simplicity) in the half-cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Introduce orthogonal coordinates $x^j = (x_1^j, x_2^j, x_3^j)$ with origin O_j and axis x_1^j parallel to the generatrices of the half-cylinder $\mathcal{C}_j$, $j = 1, 2$; the positive half-axis x_1^j lies inside $\mathcal{C}_j$. Suppose the domain Ω_j to be located so that the vertex of K_j and L_j coincides with O_j, the straight line s_j coincides with the axis x_1^j, and the positive half-axis x_1^j lies inside K_j. From now on we assume that the points O_1 and O_2 are disposed far enough from the "non-cylindric" part of G so that the nearest to O_j connected component

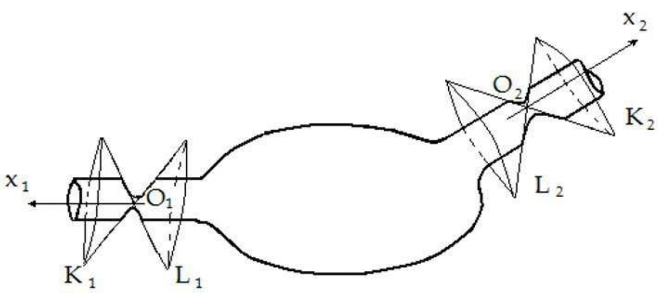

FIGURE 13 Geometry of the waveguide.

of the set $\partial G \cap \partial L_j$ coincides with $\partial \mathcal{C}_j \cap \partial L_j$. Denote by $\Omega_j(\varepsilon_j)$ the domain obtained from Ω_j by the contraction with center at O_j and coefficient $\varepsilon_j > 0$. In other words, $x^j \in \Omega_j(\varepsilon_j)$ if and only if $(x^j/\varepsilon_j) \in \Omega_j$. Let $G(\varepsilon_1, \varepsilon_2)$ be the domain obtained from G by changing $\mathcal{C}_1$ and $\mathcal{C}_2$ for $\mathcal{C}_1 \cap \Omega_1(\varepsilon_1)$ and $\mathcal{C}_2 \cap \Omega_2(\varepsilon_2)$, respectively.

3.1.2 Boundary value problem

A wave function of a free electron of energy $E = \hbar^2 k^2/2m$ satisfies the boundary value problem

$$\begin{aligned} \Delta u + k^2 u &= 0 \quad \text{in} \quad G(\varepsilon_1, \varepsilon_2), \\ u &= 0 \quad \text{on} \quad \partial G(\varepsilon_1, \varepsilon_2), \end{aligned} \tag{3.1}$$

where $\partial G(\varepsilon_1, \varepsilon_2)$ is the boundary of $G(\varepsilon_1, \varepsilon_2)$. Moreover, u is subject to radiation conditions at infinity. To formulate the conditions, we need the problem on the cross-section D_j of the half-cylinder $\mathcal{C}_j$, $j = 1, 2$:

$$\begin{aligned} \Delta v + \lambda^2 v &= 0 \quad \text{in} \quad D_j, \\ v &= 0 \quad \text{on} \quad \partial D_j. \end{aligned} \tag{3.2}$$

The eigenvalues λ_{jm}^2 of this problem, where $m = 1, 2, \ldots$, are called the thresholds; they form an increasing sequence of positive numbers tending to $+\infty$. Denote by $\Psi_{j\,m}$ an eigenfunction of the problem (3.2) that corresponds to the eigenvalue $\lambda_{j\,m}^2$ and is normalized by

$$\nu_{j\,m} \int_{D_j} |\Psi_{j\,m}(x_2, x_3)|^2 dx_2\, dx_3 = 1, \tag{3.3}$$

where $\nu_{j\,m} = \sqrt{k^2 - \lambda_{j\,m}^2}$. Let M_j be the number of thresholds of the problem on D_j, $j = 1, 2$, satisfying the inequality $\lambda^2 < k^2$ (for a fixed k). The function U_m^+ defined in the half-cylinder $\mathcal{C}_1$ by

$$U_m^+(x^1) = \exp\left(-i\nu_{1\,m} x_1^1\right) \Psi_{1\,m}(x_2^1, x_3^1), \qquad m = 1, \ldots, M_1,$$

is a wave coming in $\mathcal{C}_1$ from infinity (remind that the positive half-axis x_1^1 lies in $\mathcal{C}_1$). Analogously, the function

$$U_{M_1+m}^+(x^2) = \exp\left(-i\nu_{2\,m} x_1^2\right) \Psi_{2\,m}(x_2^2, x_3^2), \qquad m = 1, \ldots, M_2,$$

is a wave coming from infinity in $\mathcal{C}_2$. The outgoing waves U_m^-, $m = 1, \ldots, M_1 + M_2$, are obtained from the incoming ones by complex conjugation: $U_m^- = \overline{U_m^+}$.

It is well known (see, e.g., [12, Chapter 5]) that if k^2 is not a threshold, then there exist (smooth) solutions u_m, $m = 1, \ldots, M_1 + M_2$, to problem (3.1) satisfying the radiation conditions

$$u_m(x) = \begin{cases} U_m^+(x^1) + \sum_{p=1}^{M_1} s_{m\,p}\, U_p^-(x^1) + O(\exp(-\delta x_1^1)), & x_1^1 \to +\infty, \\ \sum_{p=1}^{M_2} s_{m,p+M_1}\, U_{p+M_1}^-(x^2) + O(\exp(-\delta x_1^2)), & x_1^2 \to +\infty, \end{cases}$$
$$m = 1, \dots, M_1,$$
$$u_m(x) = \begin{cases} \sum_{p=1}^{M_1} s_{m\,p}\, U_p^-(x^1) + O(\exp(-\delta x_1^1)), & x_1^1 \to +\infty, \\ U_m^+(x^2) + \sum_{p=1}^{M_2} s_{m,\,p+M_1}\, U_{p+M_1}^-(x^2) + O(\exp(-\delta x_1^2)), & x_1^2 \to +\infty, \end{cases}$$
$$m = M_1 + 1, \dots, M_1 + M_2, \quad (3.4)$$

where δ is a sufficiently small positive number. The functions u_m form a basis modulo $O(\exp -\delta|x|)$ in the space of bounded solutions of the problem (3.1) that is any bounded solution to (3.1) is a linear combination of the functions u_m up to a term $O(\exp -\delta|x|)$; if for a given k there is no nonzero solutions to (3.1) exponentially decaying at infinity, then the functions u_m form a basis in the usual sense. The scattering matrix $S = \|s_{p\,q}\|_{p,q=1,\dots,M_1+M_2}$ is unitary.

The value

$$R_m = \sum_{q=1}^{M_1} |s_{m\,q}|^2, \qquad m = 1, \dots, M_1, \tag{3.5}$$

is called the reflection coefficient for the wave U_m^+, which comes in $G(\varepsilon_1, \varepsilon_2)$ from $\mathcal{C}_1$; the transition coefficient for this wave is defined by

$$T_m = \sum_{q=1}^{M_2} |s_{m,\,q+M_1}|^2. \tag{3.6}$$

Similar definitions can be given for the wave $U_{M_1+m}^+$, which comes from $\mathcal{C}_2$.

In this work, we discuss only the situation where the parameter k^2 is "between the first and the second thresholds" or, more precisely, in the interval $(\lambda_{11}^2, \lambda_{12}^2) \cap (\lambda_{21}^2, \lambda_{22}^2)$ (supposed to be nonempty). Then the scattering matrix is of size 2×2 and (3.5) and (3.6) take the form

$$R = |s_{11}|^2, \qquad T = |s_{1\,2}|^2.$$

We consider only the scattering of the wave coming from $\mathcal{C}_1$ and omit the indices in the notation of the coefficients $R = R(k, \varepsilon_1, \varepsilon_2)$ and $T = T(k, \varepsilon_1, \varepsilon_2)$. The purpose is to find the "resonant" values $k_r = k_r(\varepsilon_1, \varepsilon_2)$ of the parameter k at which the transition coefficient takes the maximal value. Moreover, we are interested in the behavior of $k_r(\varepsilon_1, \varepsilon_2)$, $T(k, \varepsilon_1, \varepsilon_2)$, and $R(k, \varepsilon_1, \varepsilon_2)$ as $\varepsilon_1, \varepsilon_2 \to 0$.

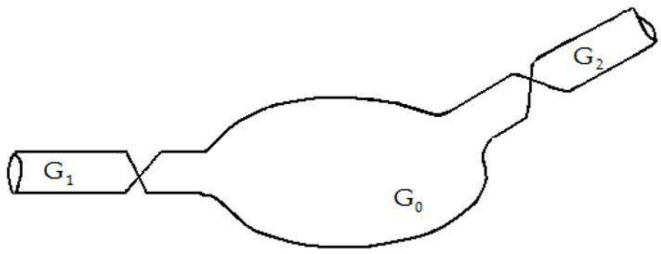

FIGURE 14 The domain G(0,0).

3.2 Limit problems

3.2.1 First kind limit problems

Recall that the limit domain $G(0,0)$ consists of the unbounded parts G_1, G_2 and the bounded resonator G_0 (Fig. 14). The boundary value problems

$$\begin{aligned} \Delta v(x) + k^2 v(x) &= f, & x \in G_j, \\ v(x) &= 0, & x \in \partial G_j, \end{aligned} \tag{3.7}$$

are called the first kind limit problems; here $j = 0, 1, 2$, and ∂G_j is the boundary of G_j.

We introduce function spaces for the problem (3.7) in G_0. Let ϕ_1, and ϕ_2 be smooth real functions in the closure $\overline{G}_0$ of G_0 such that $\phi_j = 1$ in a neighborhood of O_j, $j = 1, 2$, and $\phi_1^2 + \phi_2^2 = 1$. For $l = 0, 1, \dots$ and $\gamma_j \in \mathbb{R}$, the space $V^l_{\gamma_1,\gamma_2}(G_0)$ is the completion in the norm

$$\|v; V^l_{\gamma_1,\gamma_2}(G_0)\| = \left(\int_{G_0} \sum_{|\alpha|=0}^{l} \sum_{j=1}^{2} \phi_j^2(x) r_j(x)^{2(\gamma_j - l + |\alpha|)} |\partial^\alpha v(x)|^2 \, dx \right)^{1/2} \tag{3.8}$$

of the set of smooth functions in $\overline{G}_0$ vanishing near O_1 and O_2; here $r_j(x) = \operatorname{dist}(x, O_j)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, and $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}$.

Let $S(L_j)$ be the domain that the cone L_j cuts out on the unit sphere centered at O_j and let $0 < \mu_{j1} < \mu_{j2} < \dots$ stand for the numbers such that $\mu_{jm}(\mu_{jm} + 1)$, $m = 1, 2, \dots$, are the eigenvalues of the Dirichlet problem for the Beltrami operator in $S(L_j)$. Proposition 3.1 follows from the well known general results [13].

Proposition 3.1. *Assume that $|\gamma_j - 1| < \mu_{j1} + 1/2$. Then for every $f \in V^0_{\gamma_1,\gamma_2}(G_0)$ and any k^2 except the positive increasing sequence $\{k_p^2\}_{p=1}^\infty$ of eigenvalues, $k_p^2 \to \infty$,* *there exists a unique solution* $v \in V^2_{\gamma_1,\gamma_2}(G_0)$ *to the problem (3.7) in* G_0*. The estimate*

$$\|v; V^2_{\gamma_1,\gamma_2}(G_0)\| \leq c\|f; V^0_{\gamma_1,\gamma_2}(G_0)\| \tag{3.9}$$

holds with a constant c *independent of* f*. If* f *is a smooth function in* $\overline{G}_0$ *vanishing near* O_1 *and* O_2 *and* v *is any solution in* $V^2_{\gamma_1,\gamma_2}(G_0)$ *of the problem (2.6), then* v *is smooth in* $\overline{G}_0$ *except at* O_1 *and* O_2 *and admits the asymptotic representations*

$$v(x) = b_j \frac{1}{\sqrt{r_j}} \widetilde{J}_{\mu_{j1}+1/2}(kr_j)\Phi^L_{j1}(\varphi_j) + O(r_j^{\mu_{j2}}), \quad r_j \to 0,\ j = 1, 2,$$

near the points O_1 *and* O_2*, where* (ρ_j, φ_j) *are polar coordinates with center at* O_j*,* $\widetilde{J}_\mu$ *stands for the Bessel function multiplied by a constant such that*

$$\frac{1}{\sqrt{r}} \widetilde{J}_{\mu_{j1}+1/2}(kr) = r^{\mu_{j1}} + o(r^{\mu_{j1}}),$$

Φ^L_{j1} *is an eigenfunction of the Beltrami operator corresponding to the eigenvalue* $\mu_{j1}(\mu_{j1} + 1)$ *and normalized by the condition*

$$(2\mu_{j1} + 1) \int_{S(L_j)} |\Phi^L_{j1}(\varphi)|^2 d\varphi = 1,$$

and b_j *are some constant coefficients.*

If $k^2 = k_0^2$ *is an eigenvalue of problem (3.7) then the problem (3.7) in* G_0 *will be solvable only if* $(f, v_0)_{G_0} = 0$ *for any eigenfunction* v_0 *corresponding to* k_0^2*. Under such conditions there exists a unique solution* v *to the problem (3.7) that is orthogonal to the eigenfunctions and satisfies (3.9).*

We turn to the problems (2.6) for $j = 1, 2$. Let $\chi_{0,j}$ and $\chi_{\infty,j}$ be smooth real functions in the closure $\overline{G}_j$ of G_j such that $\chi_{0,j} = 1$ in a neighborhood of O_j, $\chi_{0,j}$ vanishes outside a compact set, and $\chi^2_{0,j} + \chi^2_{\infty,j} = 1$. We also assume that the support $\operatorname{supp}\chi_{\infty,j}$ is located in the cylindrical part $\mathcal{C}_j$ of G_j. For $\gamma \in \mathbb{R}$, $\delta > 0$, and $l = 0, 1, \dots$, the space $V^l_{\gamma,\delta}(G_j)$ is the completion in the norm

$$\|v; V^l_{\gamma,\delta}(G_j)\| = \left(\int_{G_j} \sum_{|\alpha|=0}^{l} (\chi^2_{0,j} r_j^{2(\gamma - l + |\alpha|)} + \chi^2_{\infty,j} \exp(2\delta x_1^j)) |\partial^\alpha v|^2 \, dx^j \right)^{1/2} \tag{3.10}$$

of the set of smooth functions in $\overline{G}_j$ vanishing near O_j and having compact supports.

Let $S(K_j)$ be the domain that the cone K_j cuts out on the unit sphere centered at O_j and let $0 < \varkappa_{j1} < \varkappa_{j2} < \dots$ stand for the numbers such that $\varkappa_{jm}(\varkappa_{jm} + 1)$, $m = 1, 2, \dots$, are the eigenvalues of the Dirichlet problem for the Beltrami operator in $S(K_j)$. As was mentioned, in what follows we assume that k^2 lies between the first and the second thresholds, so in every G_j there is the only outgoing wave U^- (we drop the subscript in the notation because confusions will be excluded by the context). The next proposition follows, e.g., from Theorem 5.3.5 in [12].

Proposition 3.2. *Let* $|\gamma - 1| < \varkappa_{j1} + 1/2$ *and suppose that there is no nontrivial solution to the the homogeneous problem* (2.6) *(where* $f = 0$*) in* $V^2_{\gamma,\delta}(G_j)$ *with arbitrary small positive* δ*. Then for any* $f \in V^0_{\gamma,\delta}(G_j)$ *there exists a unique solution* v *to the problem* (2.6) *that admits the representation*

$$v = u + A_j \chi_{\infty,j} U^-$$

where $A_j = const$ *and* $u \in V^2_{\gamma,\delta}(G_j)$*, the* δ *being sufficiently small, while the estimate*

$$\|u; V^2_{\gamma,\delta}(G_j)\| + |A_j| \leq c\|f; V^0_{\gamma,\delta}(G_j)\| \tag{3.11}$$

holds with a constant c *independent of* f*. If, in addition, the* f *is smooth and vanishes near* O_j*, then the solution* v *satisfies*

$$v(x^j) = a_j \frac{1}{\sqrt{r_j}} \widetilde{J}_{\varkappa_{j1}+1/2}(kr_j)\Phi^K_{j1}(\varphi_j) + O(r_j^{\varkappa_{j2}}), \quad r_j \to 0,$$

Φ^K_{j1} *denotes an eigenfunction to the Beltrami operator corresponding to* $\varkappa_{j1}(\varkappa_{j1}+1)$ *and normalized by*

$$(2\varkappa_{j1}+1)\int_{S(K_j)} |\Phi^K_{j1}(\varphi)|^2 d\varphi = 1,$$

a_j *is a constant.*

3.2.2 Second kind limit problems

In the domains Ω_j, $j = 1, 2$, introduced in Subsection 3.1.1, we consider the boundary value problems

$$\begin{aligned} \triangle w(\xi^j) &= F(\xi^j) \quad \text{in } \Omega_j, \\ w(\xi^j) &= 0 \qquad \text{on } \partial\Omega_j, \end{aligned} \tag{3.12}$$

which are called the second kind limit problems; by $\xi^j = (\xi^j_1, \xi^j_2, \xi^j_3)$ we mean Cartesian coordinates with origin at O_j.

Let $\rho_j(\xi^j) = \operatorname{dist}(\xi^j, O_j)$ and let $\psi_{0,j}$, $\psi_{\infty,j}$ be smooth real functions in $\overline{\Omega}_j$ such that $\psi_{0,j}$ equals 1 for $\rho_j < N/2$, vanishes for $\rho_j > N$, and $\psi^2_{0,j} + \psi^2_{\infty,j} = 1$, the N being a sufficiently large positive number. For $\gamma \in \mathbb{R}$ and $l = 0, 1, \ldots,$ the space $V^l_\gamma(\Omega_j)$ is the completion of the set $C^\infty_c(\overline{\Omega}_j)$ of smooth functions in $\overline{\Omega}_j$ with compact supports in the norm

$$\|v; V^l_\gamma(\Omega_j)\| = \left(\int_{\Omega_j} \sum_{|\alpha|=0}^{l} (\psi_{0,j}(\xi^j)^2 + \psi_{\infty,j}(\xi^j)^2 \rho_j(\xi^j)^{2(\gamma-l+|\alpha|)}) |\partial^\alpha v(\xi^j)|^2 \, d\xi^j \right)^{1/2}. \tag{3.13}$$

The next proposition is a corollary of Theorem 4.3.6 [12].

Proposition 3.3. *Let $|\gamma - 1| < \min\{\mu_{j1}, \varkappa_{j1}\} + 1/2$. Then for every $F \in V^0_\gamma(\Omega_j)$ there exists a unique solution $w \in V^2_\gamma(\Omega_j)$ to problem (2.7) and*

$$\|w; V^2_\gamma(\Omega_j)\| \leq c\|F; V^0_\gamma(\Omega_j)\| \tag{3.14}$$

holds with a constant c independent of F. If $F \in C^\infty_c(\overline{\Omega}_j)$, then the w is infinitely differentiable in $\overline{\Omega}_j$ and admits the representation

$$w(\xi^j) = \alpha_j \rho_j^{-\varkappa_{j1}-1} \Phi^K_{j1}(\varphi_j) + O(\rho_j^{-\varkappa_{j2}-1}), \quad \rho_j \to \infty, \tag{3.15}$$

in the cone K_j; here (ρ_j, φ_j) are polar coordinates in Ω_j with center at O_j, the $\varkappa_{jp}$, Φ^K_{j1} are the same as in Proposition 3.2, α_j is a constant coefficient. In the cone L_j, a similar expansion holds with β_j, μ_{jp}, and Φ^L_{j1} instead of α_j, $\varkappa_{jp}$, and Φ^K_{j1}. The α_j and β_j are defined by

$$\alpha_j = -(F, w^K_j)_\Omega, \qquad \beta_j = -(F, w^L_j)_\Omega,$$

where w^K_j and w^L_j are unique solutions to the homogeneous problem (2.7) such that, as $\rho_j \to \infty$,

$$w^K_j = \begin{cases} \left(\rho_j^{\varkappa_{j1}} + \alpha^K_j \rho_j^{-\varkappa_{j1}-1}\right) \Phi^K_{j1}(\varphi_j) + O(\rho_j^{-\varkappa_{j2}-1}) & \text{in } K_j; \\ \beta^K_j \rho_j^{-\mu_{j1}-1} \Phi^L_{j1}(\varphi_j) + O(\rho_j^{-\mu_{j2}-1}) & \text{in } L_j; \end{cases} \tag{3.16}$$

$$w^L_j = \begin{cases} \delta^L_j \rho_j^{-\varkappa_{j1}-1} \Phi^K_{j1}(\varphi_j) + O(\rho_j^{-\varkappa_{j2}-1}) & \text{in } K_j; \\ \left(\rho_j^{\mu_{j1}} + \gamma^L_j \rho_j^{-\mu_{j1}-1}\right) \Phi^L_{j1}(\varphi_j) + O(\rho_j^{-\mu_{j2}-1}) & \text{in } L_j, \end{cases} \tag{3.17}$$

the coefficients α^K_j, β^K_j, γ^L_j, δ^L_j being constant.

Proposition 3.4. *For any $F \in C^\infty_c(\overline{\Omega}_j)$ there exists a unique solution $w_j \in C^\infty(\overline{\Omega}_j)$ to (2.7) satisfying, as $\rho_j \to \infty$, the asymptotic formulas*

$$w_j(\xi^j) = \begin{cases} H_{j1}\rho_j^{-\varkappa_{j1}-1} + H_{j2}\rho_j^{\varkappa_{j1}} + O(\rho_j^{-\varkappa_{j2}-1}) & \text{in } K_j, \\ O(\rho^{-\mu_{j2}-1}) & \text{in } L_j, \end{cases} \tag{3.18}$$

the H_{j1} and H_{j2} being constant.

Proof. First, we prove that the constant β^K_j in (3.16) is nonzero. As is known, for the first eigenvalue of the Beltrami operator, one can choose a positive eigenfunction. However, every eigenfunction Φ_m, $m \geqslant 2$, corresponding to any other eigenvalue, is not of the fixed sign. When Φ^K_{j1} is positive in $S(K_j)$, there exists no any subdomain $\widetilde{\Omega}_j$ of Ω_j where $w^K_j < 0$. Indeed, if such a subdomain $\widetilde{\Omega}_j$ would exist, then the restriction of w^K_j to $\widetilde{\Omega}_j$ is a solution of the Dirichlet problem in $\widetilde{\Omega}_j$ vanishing at infinity (if $\widetilde{\Omega}_j$ is unbounded). We arrive at a contradiction, because $w^K_j = 0$ in $\widetilde{\Omega}_j$ and consequently in Ω_j.

In L, the expansion $w_j^K(\xi^j) \sim \sum_m \beta_{jm}\rho_j^{-\varkappa_{jm}-1}\Phi_{jm}^K(\varphi_j)$ holds. Let $\beta_{j1}^K = 0$, then all the coefficients β_{jm} must vanish. Otherwise there would exist a subdomain with $w_j^K < 0$, since the eigenfunctions Φ_{jm}^K, $m \geqslant 2$, take both positive and negative values. This leads to a contradiction, because, as is known, a harmonic function decreasing at infinity faster any power of ρ_j vanishes anywhere in Ω_j.

Now, we prove the existence. Let $\widehat{w}_j$ be a bounded solution to the problem (3.12). By Proposition 3.3

$$\widehat{w}_j = \begin{cases} \alpha_j\rho_j^{-\varkappa_{j1}-1}\Phi_{j1}^K(\varphi_j) + O(\rho_j^{-\varkappa_{j2}-1}) & \text{in } K_j; \\ \beta_j\rho_j^{-\mu_{j1}-1}\Phi_{j1}^L(\varphi_j) + O(\rho_j^{-\mu_{j2}-1}) & \text{in } L_j. \end{cases} \tag{3.19}$$

The function $w = \widehat{w}_j - \frac{\beta_j}{\beta_j^K}w_j^K$ is the desired solution w, here β_j and β_j^K are the coefficients in the expansions (3.19) and (3.16). The mentioned expansions result in

$$\widehat{w} - \frac{\beta_j}{\beta_j^K}w_j^K = \begin{cases} O(\rho_j^{-\varkappa_{j2}-1}) & \text{in } K_j, \\ \left(H_{j1}\rho_j^{-\mu_{j1}-1} + H_{j2}\rho_j^{\mu_{j1}}\right)\Phi_{j1}(\varphi_j) + O(\rho_j^{-\mu_{j2}-1}) & \text{in } L_j, \end{cases}$$

where

$$H_{j1} = \alpha_j - \beta_j\alpha_j^K/\beta_j^K, \qquad H_{j2} = -\beta_j/\beta_j^K. \tag{3.20}$$

To prove the uniqueness, it is enough to verify that $F_j = 0$ leads to $w_j = 0$. When $F_j = 0$, the difference $w_j - H_{j2}w_j^K$ solves the homogeneous Dirichlet problem in Ω_j, vanishes at infinity, and, hence, must be zero. Comparing asymptotic expansions of $H_{j2}w_j^K$ and w_j as $\rho_j \to \infty$ in L, we obtain $H_{j2}\beta_j^K = 0$. Since $\beta_j^K \neq 0$, we have $H_{j2} = 0$ and $w_j = H_{j2}w_j^K = 0$. □

In Ω_j, consider the problem to find a function w_j and numbers H_{j1}, H_{j2} such that

$$\triangle w_j = F - [\triangle, \zeta_j^K]\left(H_{j1}\rho_j^{-\varkappa_{j1}-1} + H_{j2}\rho_j^{\varkappa_{j1}}\right)\Phi_{j1}^K(\varphi_j) \quad \text{in } \Omega_j, \tag{3.21}$$

$$w_j = 0 \quad \text{on } \partial\Omega_j, \tag{3.22}$$

$$w_j = \begin{cases} O(\rho_j^{-\varkappa_{j2}-1}) & \text{as } \rho_j \to \infty \text{ in the cone } K_j, \\ O(\rho_j^{-\mu_{j2}-1}) & \text{as } \rho_j \to \infty \text{ in the cone } L_j, \end{cases} \tag{3.23}$$

where $j = 1, 2$; the polar coordinates (ρ_j, φ_j) are the same as in Proposition 3.3; the cut-off function ζ_j^K is nonzero only in K, equals 1 as $\rho_j > \delta$ and 0 as $\rho_j < \delta/2$, δ being a positive number. The next proposition follows from Proposition 3.4.

Proposition 3.5. *The problem (3.21)–(3.23) has a unique solution for any right-hand side $F \in C_c^\infty(\overline{\Omega}_j)$.*

3.3 Tunneling in a waveguide with one narrow

The purpose of this section is to carry out preliminary constructions which are necessary in further steps but not related with the phenomenon of resonance. We thereby lighten the exposition of the next section and by the way demonstrate the compound asymptotic method in a more simple situation.

We consider the electron motion in a waveguide $G(\varepsilon)$ with one narrow. The role of such a waveguide is played by $G(\varepsilon,\varepsilon_0)$ (see Subsection 3.1.1), where ε_0 is a fixed number, ε remains an infinitesimal parameter. Since only the first narrow is considered, we omit the index "1" in the notation of its attributes; for instance, we write "point O" instead of "point O_1", etc.

3.3.1 Special solutions to the first kind homogeneous problems

The limit waveguide $G(0)$ consists of two parts G_1 and G_2; each of them has one conic point and one cylindric end at infinity. Let us consider G_1. Suppose that the homogeneous problem (3.7) in G_1 has no nontrivial bounded solutions. In what follows, to construct an asymptotics of a wave function, we will use special solutions to the homogeneous problem (3.7) unbounded near the point O.

In the cone K, consider the problem

$$\begin{aligned} \Delta u + k^2 u &= 0 \quad \text{in} \quad K, \\ u &= 0 \quad \text{on} \quad \partial K, \end{aligned} \tag{3.24}$$

The function

$$v_1^K(r,\varphi) = \frac{1}{\sqrt{r}}\widetilde{N}_{\varkappa_1+1/2}(kr)\Phi_1^K(\varphi), \tag{3.25}$$

satisfies (3.24); here $\widetilde{N}_\varkappa$ stands for the Neumann function multiplied by a constant such that

$$\frac{1}{\sqrt{r}}\widetilde{N}_{\varkappa_1+1/2}(kr) = r^{-\varkappa_1-1} + o(r^{-\varkappa_1-1});$$

$\varkappa_1, \Phi_1^K$ are the same as in Proposition 3.1. Let $t \mapsto \Theta(t)$ be a cut-off function on $\mathbb{R}$ equal to one for $t < \delta/2$ and zero for $t > \delta$, δ being a positive number. Introduce a solution of the problem (3.7) in G_1 by the formula

$$\mathbf{v}_1(x) = \Theta(r)v_1^K(x) + \widetilde{v}_1(x), \tag{3.26}$$

where $\widetilde{v}_1$ is the bounded solution of the problem (3.7) with right-hand side $f = -[\triangle,\Theta]v_1^K$. By Proposition 3.2, the $\mathbf{v}_1$ exists, is uniquely determined, and admits the asymptotic expansions

$$\mathbf{v}_1 = \begin{cases} \frac{1}{\sqrt{r}}\left(\widetilde{N}_{\varkappa_1+1/2}(kr) + a_1\widetilde{J}_{\varkappa_1+1/2}(kr)\right)\Phi_1^K(\varphi) + O(r^{\varkappa_2}), & r \to 0, \\ A_1U^-(x) + O(e^{-\delta x_1}), & x_1 \to +\infty, \end{cases} \tag{3.27}$$

where $\widetilde{J}_\varkappa$ is the same as in Propositions 3.1 and 3.2.

Lemma 3.1. *There holds the equality* $|A_1|^2 = \operatorname{Im} a_1$.

Proof. Denote by $G_{N,\delta}$ the truncated domain $G_1 \cap \{x_1 < N\} \cap \{r > \delta\}$. By the Green formula

$$0 = (\triangle \mathbf{v} + k^2\mathbf{v}, \mathbf{v})_{G_{N,\delta}} - (\mathbf{v}, \triangle \mathbf{v} + k^2\mathbf{v})_{G_{N,\delta}} = (\partial \mathbf{v}/\partial n, \mathbf{v})_{\partial G_{N,\delta}} - (\mathbf{v}, \partial \mathbf{v}/\partial n)_{\partial G_{N,\delta}} = 2i \operatorname{Im}(\partial \mathbf{v}/\partial n, \mathbf{v})_{\partial G_{N,\delta}}.$$

The integral in the right-hand side is supported by the vertical part $\{x_1 = N\}$ of the boundary and by the sphere $\{r = \delta\}$. Taking account of (3.27) as $x_1 \to +\infty$ and (3.3), we have

$$\begin{aligned}(\partial \mathbf{v}/\partial n, \mathbf{v})_{x_1=N} &= \int_{D_1} A_1 \frac{\partial U^-}{\partial x_1}(x)\overline{A_1 U^-(x)}\Big|_{x_1=N} dx_2\, dx_3 + o(1) = \\ &= |A_1|^2 i\nu_1 \int_{G_{N,\delta}} |\Psi_1(x_2,x_3)|^2 dx_2\, dx_3 + o(1) = i|A_1|^2 + o(1),\end{aligned}$$

Using (3.27) as $r \to 0$ and the normalization of Φ_1 (see proposition (3.2)), we obtain

$$\begin{aligned}(\partial \mathbf{v}/\partial n, \mathbf{v})_{r=\delta} &= \int_{S(K)} \left[-\frac{\partial}{\partial r}\frac{1}{\sqrt{r}} \left(\widetilde{N}_{\varkappa_1+1/2}(kr) + a_1 \widetilde{J}_{\varkappa_1+1/2}(kr) \right) \right] \Phi_1^K(\varphi) \times \\ &\times \frac{1}{\sqrt{r}} \left(\widetilde{N}_{\varkappa_1+1/2}(kr) + \overline{a}_1 \widetilde{J}_{\varkappa_1+1/2}(kr) \right) \Phi_1^K(\varphi) r^2 \Big|_{r=\delta} d\varphi + o(1) = \\ &= -a_1(2\varkappa_1+1) \int_{G_{N,\delta}} |\Phi_1(\varphi)|^2 d\varphi + o(1) = -a_1 + o(1).\end{aligned}$$

Thus $|A_1|^2 - \operatorname{Im} a_1 + o(1) = 0$ as $N \to \infty$ and $\delta \to 0$, which completes the proof. □

Assume that v satisfies the homogeneous problem (2.6) in G_1, and

$$v = \begin{cases} \dfrac{1}{\sqrt{r}} \left(a^- \widetilde{N}_{\varkappa_1+1/2}(kr) + a^+ \widetilde{J}_{\varkappa_1+1/2}(kr) \right) \Phi_1^K(\varphi) + O(r^{\varkappa_2}), & r \to 0; \\ A^- U^-(x) + A^+ U^+(x) + O(e^{-\delta x_1}), & x_1 \to +\infty. \end{cases} \tag{3.28}$$

We find a relation between the coefficients $a^\pm$ and $A^\pm$.

By (3.27) and (3.28), the function $v - (a^- - A^+/\overline{A}_1)\mathbf{v}_1 - (A^+/\overline{A}_1)\overline{\mathbf{v}}_1$ is a bounded solution to the homogeneous problem (2.6) satisfying the natural radiation conditions. Applying the Green formula as in the proof of Lemma 3.1, we obtain that the amplitude $A^- - A_1(a^- - A^+/\overline{A}_1)$ of the outgoing wave in the asymptotics of the function at infinity must be 0. Under assumptions of Proposition 3.2 any such a solution is trivial. Thus, $a^- - A^+/\overline{A}_1 = A^-/A_1$ and

$$v = \frac{A^-}{A_1}\mathbf{v}_1 + \frac{A^+}{\overline{A}_1}\overline{\mathbf{v}}_1. \tag{3.29}$$

Equating the coefficients in the asymptotics of the both sides as $r \to 0$, we obtain the relations

$$a^- = \frac{A^-}{A_1} + \frac{A^+}{\overline{A}_1}, \quad a^+ = \frac{A^-}{A_1}a_1 + \frac{A^+}{\overline{A}_1}\overline{a}_1.$$

Rewrite them in the matrix form:

$$\begin{pmatrix} a^- \\ a^+ \end{pmatrix} = \frac{1}{|A_1|^2} \begin{pmatrix} \overline{A}_1 & A_1 \\ a_1\overline{A}_1 & \overline{a}_1 A_1 \end{pmatrix} \begin{pmatrix} A^- \\ A^+ \end{pmatrix}. \tag{3.30}$$

Because of the assumption $A_1 \neq 0$ and Lemma 3.1, we have

$$\begin{pmatrix} A^- \\ A^+ \end{pmatrix} = -\frac{1}{2i\,\mathrm{Im}\, a_1} \begin{pmatrix} \overline{a}_1 A_1 & -A_1 \\ -a_1\overline{A}_1 & \overline{A}_1 \end{pmatrix} \begin{pmatrix} a^- \\ a^+ \end{pmatrix}. \tag{3.31}$$

In a similar way, one can treat a solution of the problem in G_2.

3.3.2 Passing through a narrow

Assume that a wave function in $G(\varepsilon)$ is approximated, to the left of the narrow, by a solution v_1 to the first kind limit problems in G_1 and, to the right of the narrow, by a solution v_2 to the first kind limit problem in G_2; moreover,

$$\begin{aligned} v_1 &= \frac{1}{\sqrt{r}}\left(a_1^-\widetilde{N}_{\varkappa_1+1/2}(kr) + a_1^+\widetilde{J}_{\varkappa_1+1/2}(kr)\right)\Phi_1^K(\varphi) + O(r^{\varkappa_2}), \quad r \to 0, \\ v_2 &= \frac{1}{\sqrt{r}}\left(a_2^-\widetilde{N}_{\mu_1+1/2}(kr) + a_2^+\widetilde{J}_{\mu_1+1/2}(kr)\right)\Phi_1^L(\varphi) + O(r^{\mu_2}), \quad r \to 0. \end{aligned} \tag{3.32}$$

We seek a relation between the coefficients $a_1^\pm$ and $a_2^\pm$. To this end, a formal asymptotics of the wave function (more precisely, the principal term of the asymptotics) is constructed by the method of compound expansions.

Introduce a cut-off function $\chi_{\varepsilon,2}$ on G_2 by

$$\chi_{\varepsilon,2}(x) = \left(1 - \Theta(\varepsilon^{-1}r)\right)\mathbf{1}_{G_2}(x),$$

where the cut-off function Θ was defined before the relation (3.26) and $\mathbf{1}_{G_2}$ is the characteristic function of the domain G_2 (equal to one in G_2 and to zero outside G_2). The product $\chi_{\varepsilon,2}v_2$ turns out to be defined on the whole waveguide $G(\varepsilon)$. Substitute it to the problem (3.1). The boundary condition is fulfilled and we get the following discrepancy in the equation:

$$(\triangle + k^2)\chi_{\varepsilon,2}v_2 = [\triangle, \chi_{\varepsilon,2}]v_2 + \chi_{\varepsilon,2}(\triangle + k^2)v_2 = [\triangle, 1 - \Theta(\varepsilon^{-1}r)]v_2.$$

Clearly, the discrepancy is non-zero only in a small neighborhood of the narrow, in which v_2 can be replaced by its asymptotics. Write out the principal part of the discrepancy and transform it passing to the variables (ρ, φ), where $\rho = \varepsilon^{-1}r$:

$$\begin{aligned} (\triangle + k^2)\chi_{\varepsilon,2}v_2 &\sim [\triangle, 1 - \Theta(\varepsilon^{-1}r)]\left(a_2^- r^{-\mu_1-1} + a_2^+ r^{\mu_1}\right)\Phi_1^L(\varphi) = \\ &= \varepsilon^{-2}[\triangle_{(\rho,\varphi)}, 1 - \Theta(\rho)]\left(a_2^-\varepsilon^{-\mu_1-1}\rho^{-\mu_1-1} + a_2^+\varepsilon^{\mu_1}\rho^{\mu_1}\right)\Phi_1^L(\varphi). \end{aligned}$$

Now, introduce the solutions $\{\mathbf{w}^\pm, H_1^\pm, H_2^\pm\}$ of the problem (3.21)—(3.23) where

$$F^\pm(\rho, \varphi) = -[\triangle, \zeta^L]\rho^{\pm(\mu_1+1/2)-1/2}\Phi_1^L(\varphi),$$

the ζ^L denotes the function $1-\Theta$ restricted to the cone L and then extended by zero to the whole Ω. We add to $\chi_{\varepsilon,2}v_2$ the function $\Theta(r)w(\varepsilon^{-1}x)$, where

$$w = a_2^- \varepsilon^{-\mu_1-1}\mathbf{w}^- + a_2^+ \varepsilon^{\mu_1}\mathbf{w}^+, \tag{3.33}$$

and substitute the sum to (3.1):

$$\begin{aligned} &(\triangle + k^2)\left(\chi_{\varepsilon,2}(x)v_2(x) + \Theta(r)w(\varepsilon^{-1}x)\right) = \\ = \;& [\triangle, \chi_{\varepsilon,2}(x)]\left(v_2(x) - \left(a_2^- r^{-\mu_1-1} + a_2^+ r^{\mu_1}\right)\Phi_1^L(\varphi)\right) + k^2\Theta(r)w(\varepsilon^{-1}x) - \\ - \;& [\triangle, \zeta^K(\varepsilon^{-1}x)]\left\{a_2^- \varepsilon^{-\mu_1-1}\left(H_1^-(\varepsilon^{-1}r)^{-\varkappa_1-1} + H_2^-(\varepsilon^{-1}r)^{\varkappa_1}\right) + \right. \\ + \;& \left. a_2^+\varepsilon^{\mu_1}\left(H_1^+(\varepsilon^{-1}r)^{-\varkappa_1-1} + H_2^+(\varepsilon^{-1}r)^{\varkappa_1}\right)\right\}\Phi_1^K(\varphi). \end{aligned}$$

Thus, we have compensated the leading terms of the discrepancy with support to the right of the narrow. As is shown below in the proof of Theorem 3.1, the summand $k^2\Theta w$ is small. The remaining summands are supported to the left of the narrow and cancel after adding $\chi_{\varepsilon,1}v_1$ to $\chi_{\varepsilon,2}v_2 + \Theta w$; the cut-off function $\chi_{\varepsilon,1}$ in G_1 is defined similar to $\chi_{\varepsilon,2}$, and v_1 satisfies the homogeneous first kind limit problem in G_1 and admits the expansion (3.32) near O with the coefficients

$$\begin{aligned} a_1^- &= a_2^- H_1^- \varepsilon^{\varkappa_1-\mu_1} + a_2^+ H_1^+ \varepsilon^{\varkappa_1+\mu_1+1}, \\ a_1^+ &= a_2^- H_2^- \varepsilon^{-\mu_1-\varkappa_1-1} + a_2^+ H_2^+ \varepsilon^{\mu_1-\varkappa_1}. \end{aligned}$$

These equations imply

$$\begin{pmatrix} a_1^- \\ a_1^+ \end{pmatrix} = \begin{pmatrix} H_1^-\varepsilon^{\varkappa_1-\mu_1} & H_1^+\varepsilon^{\varkappa_1+\mu_1+1} \\ H_2^-\varepsilon^{-\mu_1-\varkappa_1-1} & H_2^+\varepsilon^{\mu_1-\varkappa_1} \end{pmatrix}\begin{pmatrix} a_2^- \\ a_2^+ \end{pmatrix}, \tag{3.34}$$

$$\begin{pmatrix} a_2^- \\ a_2^+ \end{pmatrix} = \frac{1}{H_1^-H_2^+ - H_2^-H_1^+}\begin{pmatrix} H_2^+\varepsilon^{\mu_1-\varkappa_1} & -H_1^+\varepsilon^{\varkappa_1+\mu_1+1} \\ -H_2^-\varepsilon^{-\mu_1-\varkappa_1-1} & H_1^-\varepsilon^{\varkappa_1-\mu_1} \end{pmatrix}\begin{pmatrix} a_1^- \\ a_1^+ \end{pmatrix}. \tag{3.35}$$

Lemma 3.2. *There holds the equality* $H_1^-H_2^+ - H_2^-H_1^+ = -1$.

Proof. First of all, we express $H_{1,2}^{\pm}$ in terms of the coefficients in (3.16) and (3.17). Remind that $\{\mathbf{w}^-, H_1^-, H_2^-\}$ is the solution of the problem (3.21)—(3.23) with

$$F(\rho,\varphi) = -[\triangle, \zeta^L]\rho^{-\mu_1-1}\Phi_1^L(\varphi).$$

The solution of (3.12) with that right-hand side F is

$$\widehat{w}(\rho,\varphi) = -\zeta^L(\rho,\varphi)\,\rho^{-\mu_1-1}\Phi_1^L(\varphi).$$

The coefficients in (3.19) are $\alpha = 0$ and $\beta = -1$. From (3.20) we obtain that $H_1^- = \alpha^K/\beta^K$ and $H_2^- = 1/\beta^K$.

Turn to the solution $\{\mathbf{w}^+, H_1^+, H_2^+\}$ of the problem (3.21)—(3.23) with

$$F(\rho,\varphi) = -[\triangle, \zeta^L]\rho^{\mu_1}\Phi_1^L(\varphi).$$

The solution of the problem (3.12) corresponding to that right-hand side is

$$\widehat{w}(\rho,\varphi) = w_1^L(\rho,\varphi) - \zeta^L(\rho,\varphi)\,\rho^{\mu_1}\Phi_1^L(\varphi).$$

The coefficients in (3.19) are $\alpha = \gamma^L$ and $\beta = \delta^L$. From (3.20) it follows that $H_1^+ = \gamma^L - \delta^L\alpha^K\beta^K$, $H_2^+ = -\delta^L/\beta^K$.

The obtained expressions lead to $H_1^- H_2^+ - H_2^- H_1^+ = -\gamma^L/\beta^K$. It remains to prove that $\gamma^L/\beta^K = 1$. Denote by Ω_R the truncated domain $\Omega \cap \{\rho < R\}$. By the Green formula

$$0 = (\triangle w^K, w^L)_{\Omega_R} - (w^K, \triangle w^L)_{\Omega_R} = (\partial w^K/\partial n, w^L)_{\partial\Omega_R} - (w^K, \partial w^L/\partial n)_{\partial\Omega_R}.$$

The right-hand side is supported by the sphere $\rho = R$. To calculate the integrals, replace w^K and w^L by theirs asymptotics (3.16) and (3.17). As a result (compare with the proof of Lemma 3.1) we get $0 = \gamma^L - \beta^K + o(1)$ as $R \to \infty$, which completes the proof. □

3.3.3 Formal asymptotic expressions

Here, we obtain asymptotic formulas for the amplitudes of the reflected and transited waves. We do not need the asymptotic formula for the wave function to this end. In fact, in the preceding subsection, we employed the formula to find the relation between coefficients in asymptotics of solutions to the first kind limit problems on the opposite sides of the narrow. The asymptotics of the wave function will be explicitly exposed at the end of the subsection. We will use the formula, when estimating remainders in asymptotic formulas.

Suppose that, in the domain G_2, the wave function is approximated by a solution of the first kind limit problem that admits an asymptotic expansion at infinity of the form (3.28) with coefficients $A_2^- = \widetilde{s}_{12}$ and $A_2^+ = 0$, where $\widetilde{s}_{12}$ is the yet unknown amplitude of the transited wave. According to (3.30), this solution has the asymptotics near the point O with coefficients

$$\begin{pmatrix} a_2^- \\ a_2^+ \end{pmatrix} = \frac{1}{|A_2|^2}\begin{pmatrix} \overline{A}_2 & A_2 \\ a_2\overline{A}_2 & \overline{a}_2 A_2 \end{pmatrix}\begin{pmatrix} A_2^- \\ A_2^+ \end{pmatrix} = \frac{\widetilde{s}_{12}}{A_2}\begin{pmatrix} 1 \\ a_2 \end{pmatrix},$$

where a_2, A_2 are similar to a_1, A_1 and are defined by the asymptotics of the form (3.27) of the special solution $\mathbf{v}_2$ to the homogeneous limit problem in G_2, which is defined by an equality similar to (3.26).

As was shown in the previous subsection, in the domain G_1, the wave function is approximated by the solution v_1 of the homogeneous limit problem, which, near the point O, admits the asymptotics of the form (3.32) with coefficients (cf. (3.34))

$$\begin{aligned}\begin{pmatrix} a_1^- \\ a_1^+ \end{pmatrix} &= \frac{\widetilde{s}_{12}}{A_2}\begin{pmatrix} H_1^-\varepsilon^{\varkappa_1-\mu_1} & H_1^+\varepsilon^{\varkappa_1+\mu_1+1} \\ H_2^-\varepsilon^{-\varkappa_1-\mu_1-1} & H_2^+\varepsilon^{-\varkappa_1+\mu_1} \end{pmatrix}\begin{pmatrix} 1 \\ a_2 \end{pmatrix} = \\ &= \frac{\widetilde{s}_{12}}{A_2}\begin{pmatrix} H_1^-\varepsilon^{\varkappa_1-\mu_1} + a_2 H_1^+\varepsilon^{\varkappa_1+\mu_1+1} \\ H_2^-\varepsilon^{-\varkappa_1-\mu_1-1} + a_2 H_2^+\varepsilon^{-\varkappa_1+\mu_1} \end{pmatrix}.\end{aligned}$$

According to (3.31) and Lemma 3.1, the coefficients in the asymptotics of v_1 at infinity (see (3.27)) are given by

$$\begin{pmatrix} A_1^- \\ A_1^+ \end{pmatrix} = -\frac{1}{2i\,\mathrm{Im}\,a_1}\begin{pmatrix} \overline{a}_1 A_1 & -A_1 \\ -a_1\overline{A}_1 & \overline{A}_1 \end{pmatrix}\begin{pmatrix} a_1^- \\ a_1^+ \end{pmatrix} = -\frac{\widetilde{s}_{12}}{2iA_2}\begin{pmatrix} I^- \\ I^+ \end{pmatrix},$$

where

$$\begin{aligned} I^- &= \frac{1}{\overline{A}_1}(\overline{a}_1 H_1^- \varepsilon^{\varkappa_1-\mu_1} + \overline{a}_1 a_2 H_1^+ \varepsilon^{\varkappa_1+\mu_1+1} - H_2^- \varepsilon^{-\varkappa_1-\mu_1-1} - a_2 H_2^+ \varepsilon^{-\varkappa_1+\mu_1}), \\ I^+ &= -\frac{1}{A_1}(a_1 H_1^- \varepsilon^{\varkappa_1-\mu_1} - a_1 a_2 H_1^+ \varepsilon^{\varkappa_1+\mu_1+1} + H_2^- \varepsilon^{-\varkappa_1-\mu_1-1} + a_2 H_2^+ \varepsilon^{-\varkappa_1+\mu_1}). \end{aligned}$$

The value $A_1^+ = -\widetilde{s}_{12} I^+/2iA_2$ is the amplitude of the incoming wave and supposed to equal one. This gives the first order approximations $\widetilde{s}_{12}$ and $\widetilde{s}_{11}$ to the amplitudes of the transited and reflected waves:

$$\widetilde{s}_{12} = -\frac{2iA_2}{I^+}, \qquad \widetilde{s}_{11} = A_1^- = -\frac{\widetilde{s}_{12} I^-}{2iA_2} = \frac{I^-}{I^+}.$$

Substituting the expressions for I^+ and I^- and omitting terms of higher orders, we obtain

$$\begin{aligned} \widetilde{s}_{12} &= -\frac{2iA_1A_2}{H_2^-}\varepsilon^{\varkappa_1+\mu_1+1} + O\left(\varepsilon^{4\kappa_1+2}\right), \\ \widetilde{s}_{11} &= -\frac{A_1}{\overline{A}_1}\left(1 + 2i\,\mathrm{Im}\,a_1 \frac{H_1^-}{H_2^-}\varepsilon^{2\varkappa_1+1} + O\left(\varepsilon^{4\kappa_1+2}\right)\right), \end{aligned}$$

where $\kappa_1 = \min\{\varkappa_1, \mu_1\}$. Using $\widetilde{s}_{12}$, we obtain the approximation to the transmission coefficient:

$$\widetilde{R} = |\widetilde{s}_{12}|^2 = \left|\frac{2A_1A_2}{H_2^-}\right|^2 \varepsilon^{2\varkappa_1+2\mu_1+2} + O\left(\varepsilon^{6\kappa_1+3}\right)$$

A direct calculation shows that

$$\frac{1}{4|A_2|^2}(|I^+|^2 - |I^-|^2) = -\frac{\mathrm{Im}\,a_1}{|A_1|^2}\frac{\mathrm{Im}\,a_2}{|A_2|^2}(H_1^- H_2^+ - H_2^- H_1^+) = 1.$$

Hence, $\widetilde{R} + \widetilde{T} = 1$ with $\widetilde{T} = |\widetilde{s}_{11}|^2$ and

$$\widetilde{T} = 1 - \left|\frac{2A_1A_2}{H_2^-}\right|^2 \varepsilon^{2\varkappa_1+2\mu_1+2} + O\left(\varepsilon^{6\kappa_1+3}\right).$$

Emphasize that the remainders in the above formulas denote the summands, which were omitted in the explicit expressions for the first order approximations, and do not show the distinction between the kept terms and the real values of the coefficients we are interested in. We estimate this distinction in the next subsection.

The first order approximation to the wave function is of the form

$$\widetilde{u}_1(x;\varepsilon) = \chi_{\varepsilon,1}(x)v_1(x;\varepsilon) + \Theta(r)w(\varepsilon^{-1}x;\varepsilon) + \chi_{\varepsilon,2}(x)v_2(x;\varepsilon), \tag{3.36}$$

where, owing to (3.29) and (3.33),

$$v_2(x;\varepsilon) = \frac{\widetilde{s}_{12}(\varepsilon)}{A_2}\mathbf{v}_2(x), \tag{3.37}$$

$$w(\xi;\varepsilon) = a_2^-(\varepsilon)\varepsilon^{-\mu_1-1}\mathbf{w}^-(\xi) + a_2^+(\varepsilon)\varepsilon^{\mu_1}\mathbf{w}^+(\xi), \tag{3.38}$$

$$v_1(x;\varepsilon) = \frac{A_1^-(\varepsilon)}{A_1}\mathbf{v}_1(x) + \frac{A_1^+(\varepsilon)}{\overline{A}_1}\overline{\mathbf{v}}_1(x). \tag{3.39}$$

3.3.4 Estimates of remainders

Introduce function spaces for the problem

$$\triangle u + k^2 u = f \quad \text{in } G(\varepsilon), \qquad u = 0 \quad \text{on } \partial G(\varepsilon). \tag{3.40}$$

Let Θ be the same as was introduced before (3.26) and let η_j, $j = 1, 2$, be supported by G_j and satisfy $\eta_1(x) + \Theta(r) + \eta_2(x) = 1$ in $G(\varepsilon)$. For $\gamma \in \mathbb{R}$, $\delta > 0$, and $l = 0, 1, \dots$, the space $V^l_{\gamma,\delta}(G(\varepsilon))$ is the completion in the norm

$$\|v; V^l_{\gamma,\delta}(G(\varepsilon))\| = \left(\int_{G(\varepsilon)} \sum_{|\alpha|=0}^{l} \left(\Theta^2\, (r^2+\varepsilon^2)^{\gamma-l+|\alpha|} + \sum_{j=1}^{2} \eta_j^2 e^{2\delta x_1^j} \right) |\partial^\alpha v|^2 \, dx \right)^{1/2} \tag{3.41}$$

of the set of smooth functions in $\overline{G(\varepsilon)}$ having compact supports.

Proposition 3.6. *Let $|\gamma - 1| < \min\{\varkappa_1, \mu_1\} + 1/2$, $f \in V^0_{\gamma,\delta}(G(\varepsilon))$, and let u be a solution to the problem (3.40) that admits the representation*

$$u = \widetilde{u} + \eta_1 A_1^- U_1^- + \eta_2 A_2^- U_2^-, \tag{3.42}$$

where $A_j^- = \mathrm{const}$ and $\widetilde{u} \in V^2_{\gamma,\delta}(G(\varepsilon))$, δ being a small positive number. Then the estimate

$$\|\widetilde{u}; V^2_{\gamma,\delta}(G(\varepsilon))\| + |A_1^-| + |A_2^-| \leqslant c\|f; V^0_{\gamma,\delta}(G(\varepsilon))\| \tag{3.43}$$

holds with a constant c independent of f and ε.

Proof. Here, we adapt to our purpose the proof of Theorem 5.5.1 in [10]. For the sake of simplicity, denote the left-hand sides of (3.11) and (3.43) by

$$\|v; V^2_{\gamma,\delta,-}(G_j)\|, \qquad \|u; V^2_{\gamma,\delta,-}(G(\varepsilon))\|,$$

respectively. Let the cut-off functions $\chi_{\varepsilon,j}$ be the same as in Subsection 3.3.2. Rewrite the right-hand side f of the problem (3.40) in the form

$$f(x) = f_1(x;\varepsilon) + \varepsilon^{-\gamma-3/2}F(\varepsilon^{-1}x;\varepsilon) + f_2(x;\varepsilon), \tag{3.44}$$

where x are Cartesian coordinates with center at O,

$$f_j(x;\varepsilon) = \chi_{\sqrt{\varepsilon},j}(x)f(x), \quad F(\xi;\varepsilon) = \varepsilon^{\gamma+3/2}\Theta(\sqrt{\varepsilon}\rho)f(\varepsilon\xi).$$

From the definitions of the norms is follows that

$$\begin{aligned} &\|f_j; V^0_{\gamma,\delta}(G_j)\| \leqslant c_j\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|, \qquad j = 1, 2, \\ &\|F; V^0_\gamma(\Omega)\| \leqslant C\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|, \end{aligned} \tag{3.45}$$

where the constants c_j and C are independent of ε. Consider solutions v_j and w of the problems

$$\begin{aligned} \triangle v + k^2 v = f_j, \quad \text{in } G_j, \qquad v = 0 \quad \text{on } \partial G_j; \\ \triangle w = F, \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

respectively. Owing to Propositions 3.2 and 3.3, these problems are uniquely solvable and the estimates

$$\begin{aligned} &\|v_j; V^2_{\gamma,\delta,-}(G_j)\| \leqslant \widetilde{c}_j\|f_j; V^0_{\gamma,\delta}(G_j)\|, \\ &\|w; V^2_\gamma(\Omega)\| \leqslant \widetilde{C}\|F; V^0_\gamma(\Omega)\| \end{aligned} \tag{3.46}$$

hold with constants $\widetilde{c}_j$ and $\widetilde{C}$ independent of ε. Put

$$U(x;\varepsilon) = \chi_{\varepsilon,1}(x)v_1(x;\varepsilon) + \varepsilon^{-\gamma+1/2}\Theta(r)w(\varepsilon^{-1}x;\varepsilon) + \chi_{\varepsilon,2}(x)v_2(x;\varepsilon).$$

The mapping $R_\varepsilon : f \mapsto U(f)$ is bounded uniformly with respect to ε, which follows from the chain of inequalities

$$\begin{aligned} &\|U; V^2_{\gamma,\delta,-}(G(\varepsilon))\| \leqslant \widetilde{\widetilde{c}}_1 \Big(\|v_1; V^2_{\gamma,\delta,-}(G_1)\| + \|w; V^2_\gamma(\Omega)\| + \|v_2; V^2_{\gamma,\delta,-}(G_2)\| \Big) \leqslant \\ &\leqslant \widetilde{\widetilde{c}}_2 \Big(\|f_1; V^0_{\gamma,\delta}(G_1)\| + \|F; V^0_\gamma(\Omega)\| + \|f_2; V^0_{\gamma,\delta}(G_2)\| \Big) \leqslant \widetilde{\widetilde{c}}_3 \|f; V^0_{\gamma,\delta}(G(\varepsilon))\|, \end{aligned} \tag{3.47}$$

where we took into account the estimates

$$\|x \mapsto \Theta(r)w(\varepsilon^{-1}x;\varepsilon); V^2_\gamma(G(\varepsilon))\| \leqslant c\,\varepsilon^{\gamma+3/2}\|w; V^2_\gamma(\Omega)\|$$

(with c independent of ε), (3.46), and (3.45). Clearly, U vanishes on the boundary of $G(\varepsilon)$. The discrepancy given by U in the Helmholtz equation is of the form

$$\begin{aligned} &\triangle U(x;\varepsilon) + k^2 U(x;\varepsilon) = f_1(x;\varepsilon) + [\triangle, \chi^{(1)}_\varepsilon(x)]v^{(1)}(x;\varepsilon) + \\ &+ \varepsilon^{-\gamma-3/2}F(\varepsilon^{-1}x;\varepsilon) + \varepsilon^{-\gamma+1/2}[\triangle, \Theta(r)]w(\varepsilon^{-1}x;\varepsilon) + \\ &+ \varepsilon^{-\gamma+1/2}k^2\Theta(r)w(\varepsilon^{-1}x;\varepsilon) + f_2(x;\varepsilon) + \\ &+ [\triangle, \chi_{\varepsilon,2}(x)]v_2(x;\varepsilon) = f(x) + S_\varepsilon f(x;\varepsilon). \end{aligned} \tag{3.48}$$

Below, we prove that the operator S_ε has small norm in the space $V^0_{\gamma,\delta}(G(\varepsilon))$. Hence, the operator $I + S_\varepsilon$ is invertible, the same is true for the operator of the problem (3.40)

$$A_\varepsilon : u \mapsto \triangle u + k^2 u : \mathring{V}^2_{\gamma,\delta,-}(G(\varepsilon)) \mapsto V^0_{\gamma,\delta}(G(\varepsilon)),$$

where $\mathring{V}^2_{\gamma,\delta,-}(G(\varepsilon))$ stands for the subspace in $V^2_{\gamma,\delta,-}(G(\varepsilon))$ consisting of the functions equal to zero on the boundary. Moreover, the inverse operator $A_\varepsilon^{-1} = R_\varepsilon(I+S_\varepsilon)^{-1}$ is uniformly bounded with respect to ε. This gives the inequality (3.43).

In the last part of the proof, we establish

$$\|S_\varepsilon f; V^0_{\gamma,\delta}(G(\varepsilon))\| \leqslant c\,\varepsilon^{d/2}\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|,$$

where d is a small positive number, such that $|\gamma-1|+d < \min\{\varkappa_1,\mu_1\}+1/2$; here and further, c denotes, generally speaking, different constants independent of ε.

To begin with, we estimate the norm of the operator $f \mapsto [\triangle, \chi_{\varepsilon,1}]v_1$. Since the function $[\triangle,\chi_{\varepsilon,1}]v_1$ is supported in the region $c\varepsilon \leqslant r \leqslant C\varepsilon$,

$$\|[\triangle,\chi_{\varepsilon,1}]v_1; V^0_{\gamma,\delta}(G(\varepsilon))\| \leqslant c\varepsilon^d\|[\triangle,\chi_{\varepsilon,1}]v_1; V^0_{\gamma-d,\delta}(G(\varepsilon))\|,$$

d being a small positive number, such that $\gamma-d-1 > -\min\{\varkappa_1,\mu_1\}-1/2$. According to (3.46)

$$\|[\triangle,\chi_{\varepsilon,1}]v_1; V^0_{\gamma-d,\delta}(G(\varepsilon))\| \leqslant c\|v_1; V^2_{\gamma-d,\delta}(G_1)\| \leqslant c\|f_1; V^0_{\gamma-d,\delta}(G_1)\|.$$

Because $r > c\sqrt{\varepsilon}$ on the support of f_1,

$$\|f_1; V^0_{\gamma-d,\delta}(G_1)\| \leqslant c\varepsilon^{-d/2}\|f_1; V^0_{\gamma,\delta}(G_1)\|$$

From the last three inequalities and (3.45), the estimate

$$\|[\triangle,\chi_{\varepsilon,1}]v_1; V^0_{\gamma,\delta}(G(\varepsilon))\| \leqslant c\varepsilon^{d/2}\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|$$

follows. In a similar way, one can estimate $[\triangle,\chi_{\varepsilon,1}]v_1$.

Now, consider the summand $\varepsilon^{-\gamma+1/2}\Theta(r)w(\varepsilon^{-1}x;\varepsilon)$. Assume that d satisfies $\gamma+d-1 < \min\{\varkappa_1,\mu_1\}+1/2$. Then, taking into account (3.46), we have

$$\begin{aligned}
&\|x \mapsto [\triangle,\Theta(r)]w(\varepsilon^{-1}x;\varepsilon); V^0_{\gamma,\delta}(G(\varepsilon))\| \leqslant\\
&\leqslant c\|x \mapsto [\triangle,\Theta(r)]w(\varepsilon^{-1}x;\varepsilon); V^0_{\gamma+d,\delta}(G(\varepsilon))\| \leqslant\\
&\leqslant c\varepsilon^{\gamma+d-1/2}\|\xi \mapsto [\triangle_\xi,\Theta(\varepsilon\rho)]w(\xi;\varepsilon); V^0_\gamma(\Omega)\| \leqslant\\
&\leqslant c\varepsilon^{\gamma+d-1/2}\|w; V^2_{\gamma+d}(\Omega)\| \leqslant c\varepsilon^{\gamma+d-1/2}\|F; V^0_{\gamma+d}(\Omega)\|.
\end{aligned}$$

Since the function F is nonzero only as $\rho < c/\sqrt{\varepsilon}$,

$$\|F; V^0_{\gamma+d}(\Omega)\| \leqslant c\varepsilon^{-d/2}\|F; V^0_\gamma(\Omega)\|.$$

From here and (3.45), we obtain that

$$\varepsilon^{-\gamma+1/2}\|x \mapsto [\triangle,\Theta(r)]w(\varepsilon^{-1}x); V^0_{\gamma,\delta}(G(\varepsilon))\| \leqslant c\varepsilon^{d/2}\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|.$$

It remains only to estimate the summand $\varepsilon^{-\gamma+1/2}k^2\Theta(r)w(\varepsilon^{-1}x;\varepsilon)$:

$$\begin{aligned}
&\|x \mapsto \varepsilon^{-\gamma+1/2}\Theta(r)w(\varepsilon^{-1}x;\varepsilon); V^0_{\gamma,\delta}(G(\varepsilon))\| \leqslant c\varepsilon^2\|w; V^0_\gamma(\Omega)\| \leqslant\\
&\leqslant c\varepsilon^2\|w; V^2_{\gamma+2}(\Omega)\| \leqslant c\varepsilon^d\|w; V^2_{\gamma+d}(\Omega)\| \leqslant c\varepsilon^{d/2}\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|.
\end{aligned}$$

□

The following theorem contains the main result of this section. Let u_1 be a solution of the problem (3.1) in $G(\varepsilon)$ defined by (3.4) (it is supposed that $M_1 = M_2 = 1$); s_{11} and s_{12} are the entries of scattering matrix elements determined by this solution (the amplitudes of outgoing waves in its asymptotics at infinity). The function $\widetilde{u}_1$ and the numbers $\widetilde{s}_{11}$, $\widetilde{s}_{12}$ are constructed in Subsection 3.3.3.

Theorem 3.1. *Assume the hypotheses of Proposition 3.2 to be fulfilled and the constant A_1 in (3.27) to be nonzero. Then the following estimate is valid:*

$$\sup_{x\in G(\varepsilon)} |u(x;\varepsilon) - \widetilde{u}(x;\varepsilon)| + |s_{11}(\varepsilon) - \widetilde{s}_{11}(\varepsilon)| + |s_{12}(\varepsilon) - \widetilde{s}_{12}(\varepsilon)| \leqslant \leqslant c(\varepsilon^{\kappa_2+1} + \varepsilon^{\gamma+3/2})\varepsilon^{\varkappa_1}, \tag{3.49}$$

where $\gamma > 0$ satisfies $|\gamma - 1| < \kappa_1 + 1/2$; $\kappa_l = \min\{\varkappa_l, \mu_l\}$, $l = 1,2$; $\varkappa_l(\varkappa_l + 1)$ (resp., $\mu_l(\mu_l + 1)$) is the first eigenvalue of the Beltrami operator on the base of the cone K (resp., L); the constant c does not depend on ε.

Proof. The difference $u - \widetilde{u}$ satisfies the problem (3.40), where, according to (3.36),

$$\begin{aligned} f(x;\varepsilon) = &-[\triangle, \chi_{\varepsilon,1}] \left(v_1(x;\varepsilon) - (a_1^-(\varepsilon)r^{-\varkappa_1-1} + a_1^+(\varepsilon)r^{\varkappa_1})\Phi_1^K(\varphi)\right) - \\ &-[\triangle, \chi_{\varepsilon,2}] \left(v_2(x;\varepsilon) - (a_2^-(\varepsilon)r^{-\mu_1-1} + a_2^+(\varepsilon)r^{\mu_1})\Phi_1^L(\varphi)\right) - \\ &-[\triangle, \Theta]w(\varepsilon^{-1}x;\varepsilon) - k^2\Theta(r)w(\varepsilon^{-1}x;\varepsilon). \end{aligned} \tag{3.50}$$

Moreover, $u - \widetilde{u}$ is subject to the natural radiation conditions at infinity, i.e. its asymptotics contains only outgoing waves (indeed both u and $\widetilde{u}$ have in their asymptotics the incoming wave with amplitude 1. We are going to use Proposition 3.6. To this end, we estimate $\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|$.

Consider the first summand in the right hand side of (3.50). Since it is supported by the region $r = O(\varepsilon)$, one can replace v_1 by the leading term of its asymptotics as $r \to 0$. Then

$$\begin{aligned} &\|x \mapsto [\triangle, \chi_{\varepsilon,1}] \left(v_1(x;\varepsilon) - (a_1^-(\varepsilon)r^{-\varkappa_1-1} + a_1^+(\varepsilon)r^{\varkappa_1})\Phi_1^K(\varphi)\right); V^0_{\gamma,\delta}(G(\varepsilon))\|^2 \\ &\leqslant c\int_{G(\varepsilon)} (r^2+\varepsilon^2)^\gamma \left|[\triangle, \chi_{\varepsilon,1}](a_1^-(\varepsilon)r^{-\varkappa_1+1} + a_1^+(\varepsilon)r^{\varkappa_1+2})\Phi_1^K(\varphi)\right|^2 dx; \end{aligned}$$

the integration can be carried out only over the domain G_1 (even over the cone K), where $\chi_{\varepsilon,1}(x)$ is equal to $1 - \Theta(\varepsilon^{-1}r)$. Passing to the variables $\xi = \varepsilon^{-1}x$ and taking into account (3.34), we obtain

$$\begin{aligned} &\|x \mapsto [\triangle, \chi_{\varepsilon,1}] \left(v_1(x;\varepsilon) - (a_1^-(\varepsilon)r^{-\varkappa_1-1} + a_1^+(\varepsilon)r^{\varkappa_1})\Phi_1^K(\varphi)\right); V^0_{\gamma,\delta}(G(\varepsilon))\| \leqslant \\ &c\varepsilon^{\gamma+3/2}\left(|a_1^-(\varepsilon)|\varepsilon^{-\varkappa_1-1} + |a_1^+(\varepsilon)|\varepsilon^{\varkappa_1}\right) \leqslant c\varepsilon^{\gamma+3/2}\left(|a_2^-(\varepsilon)|\varepsilon^{-\mu_1-1} + |a_2^+(\varepsilon)|\varepsilon^{\mu_1}\right). \end{aligned}$$

Analogously,

$$\begin{aligned} &\|x \mapsto [\triangle, \chi_{\varepsilon,2}] \left(v_2(x;\varepsilon) - (a_2^-(\varepsilon)r^{-\mu_1-1} + a_2^+(\varepsilon)r^{\mu_1})\Phi_1^L(\varphi)\right); V^0_{\gamma,\delta}(G(\varepsilon))\| \\ &\leqslant c\varepsilon^{\gamma+3/2}\left(|a_2^-(\varepsilon)|\varepsilon^{-\mu_1-1} + |a_2^+(\varepsilon)|\varepsilon^{\mu_1}\right). \end{aligned}$$

Turn to the summands containing w. Taking into consideration (3.38) and the fact that, at infinity, $\mathbf{w}^{\pm}$ behaves as $O(\rho^{-\varkappa_2-1})$ in K and as $O(\rho^{-\mu_2-1})$ in L (since it solves the problem (3.21)–(3.23)), we get the estimate

$$\int_{G(\varepsilon)}(r^2+\varepsilon^2)^\gamma \left|[\triangle,\Theta]w(\varepsilon^{-1}x;\varepsilon)\right|^2 dx \leqslant c\left(|a_2^-(\varepsilon)|\varepsilon^{-\mu_1-1}+|a_2^+(\varepsilon)|\varepsilon^{\mu_1}\right)^2 \times$$
$$\times\left(\int_K (r^2+\varepsilon^2)^\gamma \left|[\triangle,\Theta](\varepsilon^{-1}r)^{-\varkappa_2-1}\Phi_2^K(\varphi)\right|^2 dx+\right.$$
$$\left.+\int_L (r^2+\varepsilon^2)^\gamma \left|[\triangle,\Theta](\varepsilon^{-1}r)^{-\mu_2-1}\Phi_2^L(\varphi)\right|^2 dx\right)\leqslant$$
$$\leqslant c\left(|a_2^-(\varepsilon)|\varepsilon^{-\mu_1-1}+|a_2^+(\varepsilon)|\varepsilon^{\mu_1}\right)^2 \varepsilon^{2\kappa_2+2},$$

where $\kappa_2=\min\{\varkappa_2,\mu_2\}$. Finally, again due to (3.38), we see that

$$\int_{G(\varepsilon)}(r^2+\varepsilon^2)^\gamma \left|\Theta(r)w(\varepsilon^{-1}x;\varepsilon)\right|^2 dx = \varepsilon^{2\gamma+3}\int_\Omega(\rho^2+1)^\gamma\left|\Theta(\varepsilon\rho)w(\xi;\varepsilon)\right|^2 d\xi \leqslant$$
$$\leqslant c\left(|a_2^-(\varepsilon)|\varepsilon^{-\mu_1-1}+|a_2^+(\varepsilon)|\varepsilon^{\mu_1}\right)^2 \varepsilon^{2\gamma+3}.$$

Combining the obtained estimates, we get

$$\|f;V^0_{\gamma,\delta}(G(\varepsilon))\| \leqslant c\left(|a_2^-(\varepsilon)|\varepsilon^{-\mu_1-1}+|a_2^+(\varepsilon)|\varepsilon^{\mu_1}\right)\left(\varepsilon^{\kappa_2+1}+\varepsilon^{\gamma+3/2}\right). \tag{3.51}$$

Now, apply Proposition 3.6 to the function $u-\widetilde{u}$. In (3.43) the u and A_j^- must be replaced by $u-\widetilde{u}$ and $s_{1j}-\widetilde{s}_{1j}$, respectively. From (3.51) and $a_2^{\pm}=O(\varepsilon^{\varkappa_1+\mu_1+1})$, we obtain

$$|s_{11}(\varepsilon)-\widetilde{s}_{11}(\varepsilon)|+|s_{12}(\varepsilon)-\widetilde{s}_{12}(\varepsilon)| \leqslant \|u-\widetilde{u};V^2_{\gamma,\delta,-}(G(\varepsilon))\| \leqslant$$
$$\leqslant c(\varepsilon^{\kappa_2+1}+\varepsilon^{\gamma+3/2})\varepsilon^{\varkappa_1}. \tag{3.52}$$

Moreover, since the norm $\|u-\widetilde{u};V^2_{\gamma,\delta,-}(G(\varepsilon))\|$ is bounded, the function

$$v_{\gamma,\delta}\left(u-\widetilde{u}-(s_{11}-\widetilde{s}_{11})\eta_1U_1^- -(s_{12}-\widetilde{s}_{12})\eta_2U_2^-\right)$$

with $v_{\gamma,\delta}=\Theta^2\,(r^2+\varepsilon^2)^{\gamma-l+|\alpha|}+\sum_{j=1}^2\eta_j^2\exp(2\delta x_1^j)$ belongs to the Sobolev space $W_2^2(G(\varepsilon))$ and

$$\left\|v_\gamma\left(u-\widetilde{u}-(s_{11}-\widetilde{s}_{11})\eta_1U_1^- -(s_{12}-\widetilde{s}_{12})\eta_2U_2^-);W_2^2(G(\varepsilon)\right)\right\| \leqslant$$
$$\leqslant \|u-\widetilde{u};V^2_{\gamma,-}(G(\varepsilon))\|.$$

According to the known Sobolev imbedding theorem, any element u of $W_2^2(D)$ is a continuous function and $\|u;C(D)\|\leqslant\|u;W_2^2(D)\|$, $D\subset\mathbb{R}^3$. Hence,

$$\sup_{x\in G(\varepsilon)}|u(x;\varepsilon)-\widetilde{u}(x;\varepsilon)-(s_{11}(\varepsilon)-\widetilde{s}_{11}(\varepsilon))(\eta_1U_1^-)(x)-$$
$$-(s_{12}(\varepsilon)-\widetilde{s}_{12}(\varepsilon))(\eta_2U_2^-)(x)|\leqslant c(\varepsilon^{\kappa_2+1}+\varepsilon^{\gamma+3/2})\varepsilon^{\varkappa_1}.$$

Owing to (3.52), this gives

$$\sup_{x\in G(\varepsilon)}|u(x;\varepsilon)-\widetilde{u}(x;\varepsilon)|\leqslant c(\varepsilon^{\kappa_2+1}+\varepsilon^{\gamma+3/2})\varepsilon^{\varkappa_1},$$

and we arrive at (3.49). □

3.4 Tunneling in a waveguide with two narrows

In this section, we consider the problem in the waveguide $G(\varepsilon_1, \varepsilon_2)$ with two narrows. The limit domain $G(0,0)$ consists of three parts: infinite domains G_1, G_2 and the bounded "resonator" G_0. For k^2 in any interval, where the first kind limit problem in the resonator has no eigenvalues, all the results of the preceding section can be obtained by the same arguing for the waveguide with two narrows. Some new difficulties arise, when k^2 changes in a small neighborhood of an eigenvalue of the limit problem in the resonator. For the sake of simplicity, suppose that the eigenvalue is simple.

3.4.1 Special solution to the problem in resonator

Denote by k_e^2 a simple eigenvalue of the operator $-\triangle$ in the domain G_0 and by v_e an eigenfunction corresponding to k_e^2 and normalized by the condition $\int_{G_0} |v_e|^2 dx = 1$. According to Proposition 3.1, we have

$$v_e(x) \sim \begin{cases} b_1 \frac{1}{\sqrt{r_1}} \widetilde{J}_{\mu_{11}+1/2}(k_e r_1) \Phi_1^{L_1}(\varphi_1) & \text{near } O_1, \\ b_2 \frac{1}{\sqrt{r_2}} \widetilde{J}_{\mu_{21}+1/2}(k_e r_2) \Phi_1^{L_2}(\varphi_2) & \text{near } O_2, \end{cases} \tag{3.53}$$

where, as before, (r_j, φ_j) are polar coordinates with center at O_j ($j = 1,2$); μ_{j1} is a number such that $\mu_{j1}(\mu_{j1}+1)$ is the first eigenvalue of the Laplace–Beltrami operator on the base of L_j; $\Phi_1^{L_j}$ denotes an eigenfunction corresponding to $\mu_{j1}(\mu_{j1}+1)$ and normalized by $(2\mu_{j1}+1)\int |\Phi_1^{L_2}|^2 d\varphi = 1$. For k^2 in a small punctured neighborhood of the number k_e^2, introduce solutions v^{O_j}, $j = 1,2$, to the homogeneous first kind limit problem in G_0 by

$$v^{O_j}(x) = \Theta(r_j) v_1^{L_j}(r_j, \varphi_j) + \widetilde{v}^{O_j}(x), \tag{3.54}$$

where $v_1^{L_j}$, $j = 1,2$, are defined by (3.25) with K and $\varkappa$ replaced by L_j and μ_{j1}; $\widetilde{v}^{O_j}$ are the bounded solutions to the problem of the form (3.7) in the resonator for $f_j = -[\triangle, \Theta] v_j^{L_1}$. It is clear, that, as $k = k_e$, the problems to find $\widetilde{v}_j$ are, generally speaking, unsolvable and the functions v^{O_j} are defined incorrectly.

We set $\mathbf{v}_{01} = (k^2 - k_e^2) v^{O_1}$ and $\mathbf{v}_{02} = b_2 v^{O_1} - b_1 v^{O_2}$. As follows from Lemma 3.3 below, such the linear combinations are correct even at $k = k_e$. Owing to Proposition 3.1,

$$\mathbf{v}_{01}(x) \sim$$
$$\sim \begin{cases} \frac{1}{\sqrt{r_1}} \left((k^2 - k_e^2) \widetilde{N}_{\mu_{11}+1/2}(k r_1) + c_1(k) \widetilde{J}_{\mu_{11}+1/2}(k r_1) \right) \Phi_1^{L_1}(\varphi_1), & r_1 \to 0; \\ c_2(k) \frac{1}{\sqrt{r_2}} \widetilde{J}_{\mu_{21}+1/2}(k r_2) \Phi_1^{L_2}(\varphi_2), & r_2 \to 0; \end{cases} \tag{3.55}$$

$$\mathbf{v}_{02}(x) \sim \begin{cases} \frac{1}{\sqrt{r_1}} \left(b_2 \widetilde{N}_{\mu_{11}+1/2}(kr_1) + d_1(k) \widetilde{J}_{\mu_{11}+1/2}(kr_1) \right) \Phi_1^{L_1}(\varphi_1), & r_1 \to 0; \\ \frac{1}{\sqrt{r_2}} \left(-b_1 \widetilde{N}_{\mu_{11}+1/2}(kr_2) + d_2(k) \widetilde{J}_{\mu_{21}+1/2}(kr_2) \right) \Phi_1^{L_2}(\varphi_2), & r_2 \to 0. \end{cases} \tag{3.56}$$

Lemma 3.3. *In a punctured neighborhood of $k = k_e$ containing no eigenvalues of the problem in the resonator, the relations*

$$\widehat{v}^{O_j} = -\frac{b_j}{k^2 - k_e^2} v_e + \widehat{v}^{O_j}$$

are valid, where b_j are the coefficients in (3.53), $\widehat{v}^{O_j}$ are some functions analytic in k^2 in the mentioned neighborhood.

Proof. First, we prove that $(v^{O_j}, v_e)_{G_O} = -b_j/(k^2 - k_e^2)$, where v^{O_j} are defined by (3.54). Consider

$$(\triangle v^{O_j} + k^2 v^{O_j}, v_e)_{G_\delta} - (v^{O_j}, \triangle v_e + k^2 v_e)_{G_\delta},$$

in the domain G_δ obtained from G_0 by cutting out the balls of radius δ centered at O_j. The expression equals $-(k^2 - k_e^2)(v^{O_j}, v_e)_{G_\delta}$. Applying the Green formula as in the proof of Lemma 3.1, we obtain that it equals $b_j + o(1)$. It remains to let δ go to zero.

Remembering that k_e^2 is a simple eigenvalue, obtain that

$$\widehat{v}^{O_j} = \frac{B_j(k^2)}{k^2 - k_e^2} v_e + \widehat{v}^{O_j}, \tag{3.57}$$

where $B_j(k^2)$ is independent of x; $\widehat{v}^{O_j}$ are some functions analytic in k^2 near the point $k^2 = k_e^2$. Multiplying (3.54) by v_e, taking account of (3.57), the formula for $(v^{O_j}, v_e)_{G_O}$, and the normalization condition $(v_e, v_e)_{G_0} = 1$, we find that $B_j(k^2) = -b_j + (k^2 - k_e^2)\widetilde{B}_j(k^2)$, $\widetilde{B}_j$ being an analytic function. Together with (3.57) this leads to the required statement. □

Consider a solution v_0 of the homogeneous first kind limit problem in the resonator G_0, which admits the expansions

$$v_0(x) \sim \begin{cases} \frac{1}{\sqrt{r_1}} \left(b_1^-(k) \widetilde{N}_{\mu_{11}+1/2}(kr_1) + b_1^+(k) \widetilde{J}_{\mu_{11}+1/2}(kr_1) \right) \Phi_1^{L_1}(\varphi_1), & r_1 \to 0; \\ \frac{1}{\sqrt{r_2}} \left(b_1^-(k) \widetilde{N}_{\mu_{11}+1/2}(kr_2) + b_2^+(k) \widetilde{J}_{\mu_{21}+1/2}(kr_2) \right) \Phi_1^{L_2}(\varphi_2), & r_2 \to 0. \end{cases} \tag{3.58}$$

Comparing the asymptotics near O_2, one can see that

$$v_0(x) = \frac{1}{b_1 c_2(k)} \left(b_1 b_2^+(k) + d_2(k) b_2^-(k) \right) \mathbf{v}_{01}(x) - \frac{b_2^-(k)}{b_1} \mathbf{v}_{02}(x). \tag{3.59}$$

This equation and the expansions (3.55) and (3.56) near O_1 give the following relation between the coefficients $b_1^\pm$ and $b_2^\pm$:

$$\begin{pmatrix} b_1^- \\ b_1^+ \end{pmatrix} = \frac{1}{b_1 c_2} \begin{pmatrix} (k^2 - k_e^2) d_2 - b_2 c_2 & (k^2 - k_e^2) b_1 \\ c_1 d_2 - c_2 d_1 & b_1 c_1 \end{pmatrix} \begin{pmatrix} b_2^- \\ b_2^+ \end{pmatrix}, \tag{3.60}$$

$$\begin{pmatrix} b_2^- \\ b_2^+ \end{pmatrix} = \frac{1}{(k^2-k_e^2)d_1 - b_2c_1} \begin{pmatrix} b_1c_1 & -(k^2-k_e^2)b_1 \\ -c_1d_2 + c_2d_1 & (k^2-k_e^2)d_2 - b_2c_2 \end{pmatrix} \begin{pmatrix} b_1^- \\ b_1^+ \end{pmatrix},$$

where, for the sake of implicity, the dependence of the coefficients $b_{1,2}^{\pm}$, $c_{1,2}$, $d_{1,2}$ on k is not shown.

Lemma 3.4. *Assume that the functions $\mathbf{v}_{01}$ and $\mathbf{v}_{02}$ make sense for a fixed k. Then, at this k, $(k^2-k_e^2)d_1(k) = b_2c_1(k) - b_1c_2(k)$.*

Proof. Let the domain G_δ be the same as in the proof of Lemma 3.3. Applying the Green formula to $\mathbf{v}_{01}$ and $\mathbf{v}_{02}$ in G_δ, using the expansions (3.55) and (3.56), and letting $\delta \to 0$, we complete the proof. □

3.4.2 Formal asymptotic expansions

In the waveguide $G(\varepsilon_1, \varepsilon_2)$, consider the wave function u_1 such that

$$u_1(x;k,\varepsilon_1,\varepsilon_2) \sim$$
$$\sim \begin{cases} U_1^+(x^1;k) + s_{11}(k,\varepsilon_1,\varepsilon_2)\, U_1^-(x^1;k), & x_1^1 \to +\infty, \\ s_{12}(k,\varepsilon_1,\varepsilon_2)\, U_2^-(x^2;k), & x_1^2 \to +\infty. \end{cases}$$

As in the case of a waveguide with one narrow, we derive approximations for the coefficients s_{11} and s_{12} by using only solutions to the first kind limit problems and the relation between coefficients in the asymptotics of these solutions near the opposite sides of a narrow (cf. Subsection 3.3.2). At the end of the subsection, we write out the first order approximation to u_1, which, besides the mentioned solutions to the first kind limit problems, contains solutions to the second kind limit problems supported near the narrows.

In the domains G_j, $j = 1, 2$, u_1 is approximated by the solutions v_j to the first kind limit problems in G_j, which admit the asymptotic expansions

$$v_j(x^j;k,\varepsilon_1,\varepsilon_2) \sim$$
$$\begin{cases} \frac{1}{\sqrt{r_j}}\left(a_j^-(k,\varepsilon_1,\varepsilon_2)\widetilde{N}_{\varkappa_{j1}+1/2}(kr_j) + a_j^+(k,\varepsilon_1,\varepsilon_2)\widetilde{J}_{\varkappa_{j1}+1/2}(kr_j)\right)\Phi_1^{K_j}(\varphi_j;k),\ r_j \to 0; \\ A_j^-(k,\varepsilon_1,\varepsilon_2)U_j^-(x^j;k) + A_j^+(k,\varepsilon_1,\varepsilon_2)U_j^+(x^j;k), \quad x_1^j \to +\infty, \end{cases} \tag{3.61}$$

where x^j are the coordinates with center at O_j introduced in Subsection 3.1.1. As was shown in Subsection 3.3.1, the coefficients $a_j^{\pm}$ and $A_j^{\pm}$ are connected by the relation

$$\begin{pmatrix} a_j^-(k,\varepsilon_1,\varepsilon_2) \\ a_j^+(k,\varepsilon_1,\varepsilon_2) \end{pmatrix} = \begin{pmatrix} \frac{1}{A_j(k)} & \frac{1}{\overline{A_j}(k)} \\ \frac{a_j(k)}{A_j(k)} & \frac{\overline{a}_j(k)}{\overline{A}_j(k)} \end{pmatrix} \begin{pmatrix} A_j^-(k,\varepsilon_1,\varepsilon_2) \\ A_j^+(k,\varepsilon_1,\varepsilon_2) \end{pmatrix}; \tag{3.62}$$

here A_j, a_j are the coefficients in asymptotics of special solutions $\mathbf{v}_j$ of the homogeneous first kind limit problems in G_j; the solutions $\mathbf{v}_j$ admit asymptotics of the form (3.61) and are uniquely determined by the conditions $a_j^- = 1$, $A_j^+ = 0$; at the same time, $a_j^+ = a_j$, $A_j^- = A_j$ (cf. (3.27)).

In the resonator G_0, u_1 is approximated by a solution v_0 of the first kind limit problem in G_0, which satisfies (3.58). Let $\{\mathbf{w}_j^\pm, H_{j1}^\pm, H_{j2}^\pm\}$ be special solutions to the problem (3.21)—(3.23) in Ω_j analogous to the solutions introduced in Subsection 3.3.2. Then the coefficients $b_j^\pm$ in (3.58) are connected with $a_j^\pm$ by the equality

$$\begin{pmatrix} b_j^-(k,\varepsilon_1,\varepsilon_2) \\ b_j^+(k,\varepsilon_1,\varepsilon_2) \end{pmatrix} = \begin{pmatrix} -H_{j2}^+\varepsilon_j^{\mu_{j1}-\varkappa_{j1}} & H_{j1}^+\varepsilon_j^{\varkappa_{j1}+\mu_{j1}+1} \\ H_{j2}^-\varepsilon_j^{-\mu_{j1}-\varkappa_{j1}-1} & -H_{j1}^-\varepsilon_j^{\varkappa_{j1}-\mu_{j1}} \end{pmatrix} \begin{pmatrix} a_j^-(k,\varepsilon_1,\varepsilon_2) \\ a_j^+(k,\varepsilon_1,\varepsilon_2) \end{pmatrix}.$$

Emphasize that $a_j^\pm$ are the coefficients in the asymptotics near the vertex of the cone K_j, and $b_j^\pm$ are those near the vertex of L_j; that is why, we use a relation similar to (3.35) rather then (3.34).

The last two equalities result in

$$\begin{pmatrix} b_j^-(k,\varepsilon_1,\varepsilon_2) \\ b_j^+(k,\varepsilon_1,\varepsilon_2) \end{pmatrix} = \begin{pmatrix} -\beta_j(k,\varepsilon_j) & -\overline{\beta}_j(k,\varepsilon_j) \\ \alpha_j(k,\varepsilon_j) & \overline{\alpha}_j(k,\varepsilon_j) \end{pmatrix} \begin{pmatrix} A_j^-(k,\varepsilon_1,\varepsilon_2) \\ A_j^+(k,\varepsilon_1,\varepsilon_2) \end{pmatrix}, \tag{3.63}$$

where

$$\begin{aligned} \alpha_j(k,\varepsilon_j) &= \frac{1}{A_j(k)}\left(H_{j2}^-\varepsilon_j^{-\varkappa_{j1}-\mu_{j1}-1} - a_j(k)H_{j1}^-\varepsilon_j^{\varkappa_{j1}-\mu_{j1}}\right), \\ \beta_j(k,\varepsilon_j) &= \frac{1}{A_j(k)}\left(H_{j2}^+\varepsilon_j^{-\varkappa_{j1}+\mu_{j1}} - a_j(k)H_{j1}^+\varepsilon_j^{\varkappa_{j1}+\mu_{j1}+1}\right). \end{aligned} \tag{3.64}$$

To make formulas shorter, denote the entries of the matrix connecting $b_1^\pm$ and $b_2^\pm$ (cf. (3.60)) by $B_{lm}(k)$. Then

$$\begin{aligned} \begin{pmatrix} A_1^-(k,\varepsilon_1,\varepsilon_2) \\ A_1^+(k,\varepsilon_1,\varepsilon_2) \end{pmatrix} &= \begin{pmatrix} -\beta_1(k,\varepsilon_1) & -\overline{\beta}_1(k,\varepsilon_1) \\ \alpha_1(k,\varepsilon_1) & \overline{\alpha}_1(k,\varepsilon_1) \end{pmatrix}^{-1} \times \\ &\times \begin{pmatrix} B_{11}(k) & B_{12}(k) \\ B_{21}(k) & B_{22}(k) \end{pmatrix} \begin{pmatrix} -\beta_2(k,\varepsilon_2) & -\overline{\beta}_2(k,\varepsilon_2) \\ \alpha_2(k,\varepsilon_2) & \overline{\alpha}_2(k,\varepsilon_2) \end{pmatrix} \begin{pmatrix} A_2^-(k,\varepsilon_1,\varepsilon_2) \\ A_2^+(k,\varepsilon_1,\varepsilon_2) \end{pmatrix}. \end{aligned}$$

Taking into account the relation

$$\begin{pmatrix} -\beta_1(k,\varepsilon_1) & -\overline{\beta}_1(k,\varepsilon_1) \\ \alpha_1(k,\varepsilon_1) & \overline{\alpha}_1(k,\varepsilon_1) \end{pmatrix}^{-1} = \frac{1}{2i}\begin{pmatrix} \overline{\alpha}_1(k,\varepsilon_1) & \overline{\beta}_1(k,\varepsilon_1) \\ -\alpha_1(k,\varepsilon_1) & -\beta_1(k,\varepsilon_1) \end{pmatrix}$$

and assuming that $A_1^+ = 1$, $A_1^- = \widetilde{s}_{11}$, $A_2^+ = 0$, $A_2^- = \widetilde{s}_{12}$, we obtain formulas for the first order approximations $\widetilde{s}_{11}$ and $\widetilde{s}_{12}$ to the amplitudes s_{11} and s_{12},

$$\begin{aligned} \widetilde{s}_{11} &= \frac{1}{2i}\left(-B_{11}\overline{\alpha}_1\beta_2 + B_{12}\overline{\alpha}_1\alpha_2 - B_{21}\overline{\beta}_1\beta_2 + B_{22}\overline{\beta}_1\alpha_2\right)\widetilde{s}_{12}, \\ 1 &= \frac{1}{2i}\left(B_{11}\alpha_1\beta_2 - B_{12}\alpha_1\alpha_2 + B_{21}\beta_1\beta_2 - B_{22}\beta_1\alpha_2\right)\widetilde{s}_{12}, \end{aligned}$$

where the arguments of all the coefficients are omitted. By a direct calculation (using Lemmas 3.1, 3.2, and 3.4), we find

$$|\widetilde{s}_{11}|^2-1=-\operatorname{Im}(\alpha_1\overline{\beta}_1)\operatorname{Im}(\alpha_2\overline{\beta}_2)\left(B_{11}B_{22}-B_{12}B_{21}\right)|\widetilde{s}_{12}|^2=$$
$$=\frac{\operatorname{Im}a_1}{|A_1|^2}\left(H_{11}^-H_{12}^+-H_{11}^+H_{12}^-\right)\frac{\operatorname{Im}a_2}{|A_2|^2}\left(H_{21}^-H_{22}^+-H_{21}^+H_{22}^-\right)|\widetilde{s}_{12}|^2=|\widetilde{s}_{12}|^2.$$

Let k^2 run over a set separated from eigenvalues of the problem in the resonator. Then $B_{12}\neq 0$ and the terms $B_{12}\overline{\alpha}_1\alpha_2$ and $B_{12}\alpha_1\alpha_2$ are leading in the formulas for $\widetilde{s}_{11}$ and $\widetilde{s}_{12}$. Using (3.64) and omitting the summands of higher order, we get

$$\widetilde{s}_{12}(k,\varepsilon_1,\varepsilon_2)=-\frac{2i}{B_{12}\alpha_1\alpha_2}\left(1+O\left(\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1}\right)\right)=$$
$$=-\frac{2iA_1A_2}{H_{12}^-H_{22}^-}\varepsilon_1^{\varkappa_{11}+\mu_{11}+1}\varepsilon_2^{\varkappa_{21}+\mu_{21}+1}\left(1+O\left(\varepsilon_1^{2\kappa_{11}+1}+\varepsilon_2^{2\kappa_{21}+1}\right)\right),$$
$$\widetilde{s}_{11}(k,\varepsilon_1,\varepsilon_2)=-\frac{\overline{\alpha}_1}{\alpha_1}\left(1+O\left(\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1}\right)\right)=-\frac{A_1}{\overline{A}_1}+O\left(\varepsilon_1^{2\kappa_{11}+1}+\varepsilon_2^{2\kappa_{21}+1}\right),$$

where $\kappa_{j1}=\min\{\varkappa_{j1},\mu_{j1}\}$, $j=1,2$.

Now, suppose that k^2 coincides with an eigenvalue k_e^2 of the problem in the resonator. In this case, $B_{12}=0$ and the expressions for $\widetilde{s}_{11}$ and $\widetilde{s}_{12}$ given above make no sense. To derive correct formulas as k^2 is near to k_e^2, one should consider higher order terms. We have

$$\frac{2iA_1A_2}{\widetilde{s}_{12}}=A_1A_2\left(-B_{12}\alpha_1\alpha_2+B_{11}\alpha_1\beta_2-B_{22}\beta_1\alpha_2+O\left(\varepsilon_1^{\mu_{11}-\varkappa_{11}}\varepsilon_2^{\mu_{21}-\varkappa_{21}}\right)\right)=$$
$$\frac{H_{12}^-H_{22}^-}{\varepsilon_1^{\varkappa_{11}+\mu_{11}+1}\varepsilon_2^{\varkappa_{21}+\mu_{21}+1}}\left[-B_{12}+B_{11}\frac{H_{22}^+}{H_{22}^-}\varepsilon_2^{2\mu_{21}+1}-B_{22}\frac{H_{12}^+}{H_{12}^-}\varepsilon_1^{2\mu_{11}+1}+O\left(\ldots\right)\right]+$$
$$\frac{iH_{11}^-H_{22}^-\operatorname{Im}a_1}{\varepsilon_1^{-\varkappa_{11}+\mu_{11}}\varepsilon_2^{\varkappa_{21}+\mu_{21}+1}}\left[B_{12}-B_{11}\frac{H_{22}^+}{H_{22}^-}\varepsilon_2^{2\mu_{21}+1}+B_{22}\frac{H_{11}^+}{H_{11}^-}\varepsilon_1^{2\mu_{11}+1}+O\left(\ldots\right)\right]+$$
$$\frac{iH_{12}^-H_{21}^-\operatorname{Im}a_2}{\varepsilon_1^{\varkappa_{11}+\mu_{11}+1}\varepsilon_2^{-\varkappa_{21}+\mu_{21}}}\left[B_{12}-B_{11}\frac{H_{21}^+}{H_{21}^-}\varepsilon_2^{2\mu_{21}+1}+B_{22}\frac{H_{12}^+}{H_{12}^-}\varepsilon_1^{2\mu_{11}+1}+O\left(\ldots\right)\right];$$

dots in brakets stand for $\varepsilon_1^{2\kappa_{11}+1}\varepsilon_2^{2\kappa_{21}+1}$. Denote by k_r^2 the value of k^2, at which the real part of $2iA_1A_2\widetilde{s}_{12}^{-1}$ equals zero. Substitute $B_{12}(k)=(k^2-k_e^2)/c_2(k)$ into $\operatorname{Re}2iA_1A_2\widetilde{s}_{12}^{-1}=0$ and rewrite the resulting equality it in the $k^2-k_e^2$-resolved form:

$$k^2-k_e^2=c_2(k)B_{11}(k)\frac{H_{22}^+}{H_{22}^-}\varepsilon_2^{2\mu_{21}+1}-c_2(k)B_{22}(k)\frac{H_{12}^+}{H_{12}^-}\varepsilon_1^{2\mu_{11}+1}+O\left(\varepsilon_1^{2\kappa_{11}+1}\varepsilon_2^{2\kappa_{21}+1}\right).$$

Since ε_1 and ε_2 are small, this equation can be solved by the step-by-step method. Taking only the leading terms in the power series (in $\varepsilon_1,\varepsilon_2$) for $k_r^2-k_e^2$, changing B_{11} and B_{22} for their expressions (3.60), and using the equalities $c_j(k_e)=-b_1b_j$,

$j = 1, 2$, which follow from (3.54) and Lemma 3.3, we obtain the principal term in the asymptotics of k_r^2:

$$k_r^2 = k_e^2 + b_1^2 \frac{H_{12}^+}{H_{12}^-} \varepsilon_1^{2\mu_{11}+1} + b_2^2 \frac{H_{22}^+}{H_{22}^-} \varepsilon_2^{2\mu_{21}+1} + O\left(\varepsilon_1^{4\kappa_{11}+2} + \varepsilon_2^{4\kappa_{21}+2}\right). \tag{3.65}$$

Suppose that $k^2 - k_r^2 = O\left(\varepsilon_1^{2\kappa_{11}+1} + \varepsilon_2^{2\kappa_{21}+1}\right)$, then

$$\operatorname{Re} 2iA_1A_2\widetilde{s}_{12}^{-1} = \frac{H_{12}^- H_{22}^-}{b_1 b_2} \left(k^2 - k_r^2 + O\left(\varepsilon_1^{4\kappa_{11}+2} + \varepsilon_2^{4\kappa_{21}+2}\right)\right).$$

Under the same assumptions, $\operatorname{Im} 2iA_1A_2\widetilde{s}_{12}^{-1}$ can be rewritten in the form

$$(\operatorname{Im} a_1) \frac{\varepsilon_1^{\varkappa_{11}-\mu_{11}}}{\varepsilon_2^{\varkappa_{21}+\mu_{21}+1}} \left(\frac{b_1 H_{22}^-}{b_2 H_{11}^-} \varepsilon_1^{2\mu_{11}+1} - \frac{H_{11}^- H_{22}^-}{b_1 b_2} (k^2 - k_r^2) + O\left(\varepsilon_1^{4\kappa_{11}+2} + \varepsilon_2^{4\kappa_{21}+2}\right)\right) +$$
$$(\operatorname{Im} a_2) \frac{\varepsilon_2^{\varkappa_{21}-\mu_{21}}}{\varepsilon_1^{\varkappa_{11}+\mu_{11}+1}} \left(\frac{b_2 H_{12}^-}{b_1 H_{22}^-} \varepsilon_2^{2\mu_{21}+1} - \frac{H_{12}^- H_{21}^-}{b_1 b_2} (k^2 - k_r^2) + O\left(\varepsilon_1^{4\kappa_{11}+2} + \varepsilon_2^{4\kappa_{21}+2}\right)\right).$$

Now we take a more narrow interval for k^2,

$$k^2 - k_r^2 = O\left(\varepsilon_1^{2\mu_{11}+2+p_1} + \varepsilon_2^{2\mu_{21}+2+p_2}\right),$$

p_1, p_2 being small positive numbers (as it turns out below, the resonant peak lies inside this interval). Then, in the expression for the principal part of $\operatorname{Im} 2iA_1A_2 s_{12}^{-1}$, one can neglect the summands with $k^2 - k_r^2$. The coefficients A_j depend on k; we get rid of this dependence using the equalities $A_j(k) = A_j(k_e) + O(k^2 - k_e^2)$. As a result, we obtain

$$\widetilde{s}_{12}(k, \varepsilon_1, \varepsilon_2) = \frac{i \dfrac{A_1(k_e)}{|A_1(k_e)|} \dfrac{A_2(k_e)}{|A_2(k_e)|}}{\dfrac{i}{2}\left(z + \dfrac{1}{z}\right) + P \dfrac{k^2 - k_r^2}{\varepsilon_1^{\varkappa_{11}+\mu_{11}+1} \varepsilon_2^{\varkappa_{21}+\mu_{21}+1}}} \left(1 + O(\varepsilon_1^{p_1} + \varepsilon_2^{p_2})\right), \tag{3.66}$$

where

$$z = \frac{b_1 H_{12}^- |A_1(k_e)| \varepsilon_1^{\varkappa_{11}+\mu_{11}+1}}{b_2 H_{22}^- |A_2(k_e)| \varepsilon_2^{\varkappa_{21}+\mu_{21}+1}}, \qquad P = \frac{H_{12}^- H_{22}^-}{2 b_1 b_2 |A_1(k_e)| |A_2(k_e)|}.$$

Similarly,

$$\widetilde{s}_{11}(k, \varepsilon_1, \varepsilon_2) = \frac{A_1(k_e)}{\overline{A}_1(k_e)|} \frac{\dfrac{i}{2}\left(z - \dfrac{1}{z}\right) - P \dfrac{k^2 - k_r^2}{\varepsilon_1^{\varkappa_{11}+\mu_{11}+1} \varepsilon_2^{\varkappa_{21}+\mu_{21}+1}}}{\dfrac{i}{2}\left(z + \dfrac{1}{z}\right) + P \dfrac{k^2 - k_r^2}{\varepsilon_1^{\varkappa_{11}+\mu_{11}+1} \varepsilon_2^{\varkappa_{21}+\mu_{21}+1}}} \left(1 + O(\varepsilon_1^{p_1} + \varepsilon_2^{p_2})\right).$$

Now, we find approximations to the transmission and reflection coefficients:

$$\widetilde{T}(k,\varepsilon_1,\varepsilon_2) = \frac{1}{\dfrac{1}{4}\left(z+\dfrac{1}{z}\right)^2 + P^2\left(\dfrac{k^2-k_r^2}{\varepsilon_1^{\varkappa_{11}+\mu_{11}+1}\varepsilon_2^{\varkappa_{21}+\mu_{21}+1}}\right)^2}\left(1+O(\varepsilon_1^{p_1}+\varepsilon_2^{p_2})\right), \quad (3.67)$$

$$\widetilde{R}(k,\varepsilon_1,\varepsilon_2) = \frac{\dfrac{1}{4}\left(z-\dfrac{1}{z}\right)^2 + P^2\left(\dfrac{k^2-k_r^2}{\varepsilon_1^{\varkappa_{11}+\mu_{11}+1}\varepsilon_2^{\varkappa_{21}+\mu_{21}+1}}\right)^2}{\dfrac{1}{4}\left(z+\dfrac{1}{z}\right)^2 + P^2\left(\dfrac{k^2-k_r^2}{\varepsilon_1^{\varkappa_{11}+\mu_{11}+1}\varepsilon_2^{\varkappa_{21}+\mu_{21}+1}}\right)^2}\left(1+O(\varepsilon_1^{p_1}+\varepsilon_2^{p_2})\right).$$

It is easy to see that $\widetilde{T}$ has a peak at $k^2 = k_r^2$. The width of the peak at its half-height is

$$\Delta(\varepsilon_1,\varepsilon_2) = \frac{|z+z^{-1}|}{\sqrt{P}}\varepsilon_1^{\varkappa_{11}+\mu_{11}+1}\varepsilon_2^{\varkappa_{21}+\mu_{21}+1}. \quad (3.68)$$

Finally, present the asymptotics of the wave function. Let the cut-off functions $t \mapsto \Theta(t)$ and $x^j \mapsto \chi_{\varepsilon_j,j}(x^j)$, $j = 1,2$, be the same as in Subsections 3.3.2 and 3.3.3. Introduce one more cut-off function $x \mapsto \chi_{\varepsilon_1,\varepsilon_2}(x)$ by

$$\chi_{\varepsilon_1,\varepsilon_2}(x) = \mathbf{1}_{G_0}(x)\left(1-\Theta(r_1/\varepsilon_1)\right)\left(1-\Theta(r_2/\varepsilon_2)\right),$$

where $\mathbf{1}_{G_0}$ is the characteristic function of G_0. The leading term of the asymptotics of the wave function is of the form

$$\begin{aligned}\widetilde{u}(x;k,\varepsilon_1,\varepsilon_2) &= \chi_{1,\varepsilon_1}(x^1)v_1(x^1;k,\varepsilon_1,\varepsilon_2)+\\ &+\Theta(r_1)w_1(\varepsilon_1^{-1}x^1;k,\varepsilon_1,\varepsilon_2)+\chi_{\varepsilon_1,\varepsilon_2}(x)v_0(x;k,\varepsilon_1,\varepsilon_2)+\\ &+\Theta(r_2)w_2(\varepsilon_2^{-1}x^2;k,\varepsilon_1,\varepsilon_2)+\chi_{2,\varepsilon_2}(x^2)v_2(x^2;k,\varepsilon_1,\varepsilon_2),\end{aligned} \quad (3.69)$$

where, similarly to (3.29), (3.33), and (3.59),

$$\begin{aligned}
v_1(x^1;k,\varepsilon_1,\varepsilon_2) =& \frac{\widetilde{s}_{11}(k,\varepsilon_1,\varepsilon_2)}{A_1(k)}\mathbf{v}_1(x^1;k) + \frac{1}{\overline{A_1}(k)}\overline{\mathbf{v}}_1(x^1;k),\\
w_1(\xi^1;k,\varepsilon_1,\varepsilon_2) =& a_1^-(k,\varepsilon_1,\varepsilon_2)\varepsilon_1^{-\varkappa_{11}-1}\mathbf{w}_1^-(\xi^1) + a_1^+(k,\varepsilon_1,\varepsilon_2)\varepsilon_1^{\varkappa_{11}}\mathbf{w}_1^+(\xi^1),\\
v_0(x;k,\varepsilon_1,\varepsilon_2) =& \frac{1}{b_1c_2(k)}(b_1b_2^+(k,\varepsilon_1,\varepsilon_2) + d_2(k)b_2^-(k,\varepsilon_1,\varepsilon_2))\mathbf{v}_{01}(x;k) -\\
&-\frac{1}{b_1}b_2^-(k,\varepsilon_1,\varepsilon_2)\mathbf{v}_{02}(x;k),\\
w_2(\xi^2;k,\varepsilon_1,\varepsilon_2) =& a_2^-(k,\varepsilon_1,\varepsilon_2)\varepsilon_2^{-\varkappa_{21}-1}\mathbf{w}_2^-(\xi^2) + a_2^+(k,\varepsilon_1,\varepsilon_2)\varepsilon_2^{\varkappa_{21}}\mathbf{w}_2^+(\xi^2),\\
v_2(x^2;k,\varepsilon_1,\varepsilon_2) =& \frac{\widetilde{s}_{12}(k,\varepsilon_1,\varepsilon_2)}{A_2(k)}\mathbf{v}_2(x^2;k).
\end{aligned}$$

Here, owing to (3.62), (3.63), and our choice of the coefficients $A_j^{\pm}$,

$$\begin{pmatrix} a_1^- \\ a_1^+ \end{pmatrix} = \begin{pmatrix} \dfrac{1}{A_1} & \dfrac{1}{\overline{A}_1} \\ \dfrac{a_1}{A_1} & \dfrac{\overline{a}_1}{\overline{A}_1} \end{pmatrix} \begin{pmatrix} \widetilde{s}_{11} \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{\widetilde{s}_{11}}{A_1} + \dfrac{1}{\overline{A}_1} \\ \dfrac{a_1 \widetilde{s}_{11}}{A_1} + \dfrac{\overline{a}_1}{\overline{A}_1} \end{pmatrix}, \tag{3.70}$$

$$\begin{pmatrix} a_2^- \\ a_2^+ \end{pmatrix} = \begin{pmatrix} \dfrac{1}{A_2} & \dfrac{1}{\overline{A}_2} \\ \dfrac{a_1}{A_2} & \dfrac{\overline{a}_2}{\overline{A}_2} \end{pmatrix} \begin{pmatrix} \widetilde{s}_{12} \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{\widetilde{s}_{12}}{A_2} \\ \dfrac{a_2 \widetilde{s}_{12}}{A_2} \end{pmatrix}, \tag{3.71}$$

$$\begin{pmatrix} b_2^- \\ b_2^+ \end{pmatrix} = \begin{pmatrix} -\beta_2 & -\overline{\beta}_2 \\ \alpha_2 & \overline{\alpha}_2 \end{pmatrix} \begin{pmatrix} \widetilde{s}_{12} \\ 0 \end{pmatrix} = \begin{pmatrix} -\beta_2 \widetilde{s}_{12} \\ \alpha_2 \widetilde{s}_{12} \end{pmatrix}.$$

3.4.3 Estimate of remainders

Introduce function spaces for the problem

$$\triangle u + k^2 u = f \quad \text{in } G(\varepsilon_1, \varepsilon_2), \qquad u = 0 \quad \text{on } \partial G(\varepsilon_1, \varepsilon_2). \tag{3.72}$$

Let Θ be the same as was introduced before (3.26) and let η_j, $j = 0, 1, 2$, be supported by G_j and satisfy $\eta_1(x) + \Theta(r_1) + \eta_0(x) + \Theta(r_2) + \eta_2(x) = 1$ in $G(\varepsilon_1, \varepsilon_2)$. For $\gamma_1, \gamma_2 \in \mathbb{R}$, $\delta > 0$, and $l = 0, 1, \dots$, the space $V^l_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))$ is the completion in the norm

$$\|u; V^l_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\| = \Big(\int_{G(\varepsilon_1,\varepsilon_2)} \sum_{|\alpha|=0}^{l} \Big(\sum_{j=1}^{2} \Theta^2(r_j)\, (r_j^2 + \varepsilon_j^2)^{\gamma_j - l + |\alpha|} + \sum_{j=1}^{2} \eta_j^2 e^{2\delta x_1^j} + \eta_0 \Big) |\partial^\alpha v|^2 \, dx \Big)^{1/2} \tag{3.73}$$

of the set of smooth functions in $\overline{G(\varepsilon_1, \varepsilon_2)}$ having compact supports.

Proposition 3.7. *Let k_r be a resonance and let $|k^2 - k_r^2| = O(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{\mu_{21}+1})$. Assume that the first eigenvalues $\varkappa_{j1}(\varkappa_{j1}+1)$ and $\mu_{j1}(\mu_{j1}+1)$ of the Beltrami operator on the bases of the cones K_j and L_j, $j = 1, 2$, are subject to the condition $\mu_{j1} < \varkappa_{j1} + 2$, γ_1, γ_2 satisfy the inequalities $\mu_{j1} - 3/2 < \gamma_j - 1 < \min\{\varkappa_{j1}, \mu_{j1}\} + 1/2$, and $f \in V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))$. Suppose that u is a solution to the problem (3.72) that admits the representation*

$$u = \widetilde{u} + \eta_1 A_1^- U_1^- + \eta_2 A_2^- U_2^-,$$

where $A_j^- = const$ and $\widetilde{u} \in V^2_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))$, δ being a small positive number. Then the estimate

$$\|\widetilde{u}; V^2_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\| + |A_1^-| + |A_2^-| \leqslant C(\varepsilon_1, \varepsilon_2, k) \|f; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\| \tag{3.74}$$

holds, where

$$C(\varepsilon_1, \varepsilon_2, k) = \frac{c}{\varepsilon_1^{2\varkappa_{11}+2\mu_{11}+2} + \varepsilon_2^{2\varkappa_{12}+2\mu_{12}+2} + |k - k_r|},$$

c being independent of f and ε_1, ε_2.

Proof. Divide the proof into several steps.

Step A First, we construct an auxiliary function u_p. To this end, turn to the asymptotics of the wave function exposed at the end of the preceding subsection. All the solutions of the first kind limit problems that are used in the asymptotics, are defined as $k^2 \in \mathbb{R}$; nevertheless, they make sense for complex k^2, too. As follows from (3.66), the coefficient $\widetilde{s}_{12}$ has a pole k_p^2 in the lower complex half-plane, and

$$\begin{aligned} k_p^2 = k_r^2 - \frac{i}{2P}\left(z+\frac{1}{z}\right)\left(1+O\left(\varepsilon_1^2+\varepsilon_2^2\right)\right) = \\ = k_r^2 - i\left[\left(\frac{b_1|A_1(k_0)|}{H_{12}^-}\right)^2 \varepsilon_1^{2\varkappa_{11}+2\mu_{11}+2} + \left(\frac{b_2|A_2(k_0)|}{H_{22}^-}\right)^2 \varepsilon_2^{2\varkappa_{21}+2\mu_{21}+2}\right] \times \\ \times \left(1+O\left(\varepsilon_1^2+\varepsilon_2^2\right)\right). \end{aligned}$$

Multiply all the solutions of the limit problems in the expression for $\widetilde{u}_1$ by

$$(A_2(k)b_2/H_{22}^-\widetilde{s}_{12}(\varepsilon_1,\varepsilon_2,k))\varepsilon_2^{\varkappa_{21}+\mu_{21}+1},$$

substitute $k = k_p$, and denote the resulting functions by the same symbols with addition index "p". Then

$$\begin{aligned} v_{jp}(x;\varepsilon_1,\varepsilon_2) &= b_j H_{j2}^- \varepsilon_j^{\varkappa_{j1}+\mu_{j1}+1}\mathbf{v}_j(x;k_p), \\ v_{0p}(x;\varepsilon_1,\varepsilon_2) &= \left(-\frac{1}{b_1}+O\left(\varepsilon_1^{2\kappa_{11}+1}+\varepsilon_2^{2\kappa_{21}+1}\right)\right)\mathbf{v}_{01}(x;k_p) + \\ &+ \varepsilon_2^{2\mu_2+1}\left(\frac{b_2H_{22}^+}{b_1H_{22}^-}+O\left(\varepsilon_1^{2\kappa_{11}+1}+\varepsilon_2^{2\kappa_{21}+1}\right)\right)\mathbf{v}_{02}(x;k_p), \\ w_{jp}(\xi^j;\varepsilon_1,\varepsilon_2) &= \frac{b_j}{H_{j2}^-}\varepsilon_j^{\mu_{j1}}\left(\mathbf{w}_j^-(\xi^j)+a_j(k_p)\varepsilon_j^{2\varkappa_{j1}+1}\mathbf{w}_j^+(\xi^j)\right), \end{aligned}$$

where $j = 1, 2$; the dependence of k_p on $\varepsilon_1, \varepsilon_2$ is not shown. We set

$$\begin{aligned} &u_p(x;\varepsilon_1,\varepsilon_2) = \chi_{1,\varepsilon_1}(x^1)v_{1p}(x^1;\varepsilon_1,\varepsilon_2) + \\ &\Theta(\varepsilon_1^{-3/4}r_1)w_{1p}(\varepsilon_1^{-1}x^1;\varepsilon_1,\varepsilon_2) + \chi_{\varepsilon_1,\varepsilon_2}(x)v_{0p}(x;\varepsilon_1,\varepsilon_2) + \\ &\Theta(\varepsilon_2^{-3/4}r_2)w_2(\varepsilon_2^{-1}x^2;k,\varepsilon_1,\varepsilon_2) + \chi_{2,\varepsilon_2}(x^2)v_2(x^2;k,\varepsilon_1,\varepsilon_2), \end{aligned}$$

where the cut-off functions Θ, $\chi_{\varepsilon_j,j}$, and $\chi_{\varepsilon_1,\varepsilon_2}$ are the same as in the previous subsection. Obviously, the principal part of the norm of u_p is given by $\chi_{\varepsilon_1,\varepsilon_2}v_{0p}$. Taking into account the formula for v_{0p}, the definition of $\mathbf{v}_{01}$ (cf. Subsection 3.4.1), and Lemma 3.3, we obtain $\|\chi_{\varepsilon_1,\varepsilon_2}v_{0p}\| = \|v_e\| + o(1)$. For later use, note that the function $(\triangle + k_p^2)u_p$ is nonzero only in the region $\{r_1 < c_1\varepsilon_1^{3/4}\} \cup \{\{r_2 < c_2\varepsilon_2^{3/4}\}$. Arguing as in the proof of Theorem 3.1, we obtain the estimate

$$\begin{aligned} &\|(\triangle + k_p^2)u_p; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\| \leqslant \\ &\leqslant c\left[\varepsilon_1^{\mu_{11}}\left(\varepsilon_1^{\kappa_{12}+1}+\varepsilon_1^{\gamma_1+3/2}\right)+\varepsilon_2^{\mu_{21}}\left(\varepsilon_2^{\kappa_{22}+1}+\varepsilon_2^{\gamma_2+3/2}\right)\right]. \end{aligned} \tag{3.75}$$

Step B This part contains somewhat modified arguments from the proof of Proposition 3.6. Denote the left-hand sides of the inequalities (3.11) and (3.74) by

$$\|v; V^2_{\gamma_j,\delta,-}(G_j)\|^2, \quad \|u; V^2_{\gamma_1,\gamma_2,\delta,-}(G(\varepsilon_1,\varepsilon_2))\|^2,$$

respectively. Rewrite the right-hand side of the problem (3.72) in the form

$$\begin{aligned} f(x) &= f_1(x;\varepsilon_1) + f_0(x;\varepsilon_1,\varepsilon_2) + f_2(x;\varepsilon_2)+ \\ &+ \varepsilon_1^{-\gamma_1-3/2}F_1(\varepsilon_1^{-1}x^1;\varepsilon_1) + \varepsilon_2^{-\gamma_2-3/2}F_2(\varepsilon_2^{-1}x^2;\varepsilon_2), \end{aligned} \tag{3.76}$$

where

$$\begin{aligned} f_0(x;\varepsilon_1,\varepsilon_2) &= \chi_{\sqrt{\varepsilon_1},\sqrt{\varepsilon_2}}(x)f(x), \\ f_j(x;\varepsilon_j) &= \chi_{\sqrt{\varepsilon_j},j}(x)f(x), \\ F_j(\xi^j;\varepsilon_j) &= \varepsilon_j^{\gamma_j+3/2}\Theta(\sqrt{\varepsilon_j}\rho_j)f(x_{O_j}+\varepsilon_j\xi^j); \end{aligned}$$

x are arbitrary Cartesian coordinates; x_{O_j} denote the coordinates of the points O_j in the coordinate system x; x^j are introduced in Subsection 3.1.1. From the definition of the norms it follows that

$$\begin{aligned} &\|f_0; V^0_{\gamma_1,\gamma_2}(G_0)\| \leqslant c_0\|f; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\|, \\ &\|f_j; V^0_{\gamma_j,\delta}(G_j)\| \leqslant c_j\|f; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\|, \\ &\|F_j; V^0_{\gamma}(\Omega_j)\| \leqslant C_j\|f; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\|, \end{aligned} \tag{3.77}$$

where c_i and C_j are independent of ε_1, ε_2. Consider the solutions v_0, v_j, and w_j of the problems

$$\begin{aligned} \triangle v + k^2 v = f_0, &\quad \text{in } G_0, \qquad v = 0 \quad \text{on } \partial G_0; \\ \triangle v + k^2 v = f_j, &\quad \text{in } G_j, \qquad v = 0 \quad \text{on } \partial G_j; \\ \triangle w = F_j, &\quad \text{in } \Omega_j, \qquad w = 0 \quad \text{on } \partial \Omega_j, \end{aligned}$$

respectively; moreover, v_j satisfy the natural radiation conditions at infinity. Owing to Propositions 3.2 and 3.3, the problems in G_j and Ω_j, $j = 1, 2$, are uniquely solvable, and the following estimates

$$\begin{aligned} &\|v_j; V^2_{\gamma_j,-}(G_j)\| \leqslant \widetilde{c}_j\|f_j; V^0_{\gamma_j,\delta}(G_j)\|, \\ &\|w_j; V^2_{\gamma_j}(\Omega_j)\| \leqslant \widetilde{C}_j\|F_j; V^0_{\gamma_j}(\Omega_j)\| \end{aligned} \tag{3.78}$$

hold, where $\widetilde{c}_j$ and $\widetilde{C}_j$ are independent of ε_1, ε_2.

Step C Suppose that f in (3.72) analytically depends in k, takes values in the space $V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))$, and, for $k = k_e$, satisfies the condition $(\chi_{\sqrt{\varepsilon_1},\sqrt{\varepsilon_2}}f, v_e)_{G_0} = 0$ (the subspace of such functions is denoted by $V^{0,\perp}_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))$). Then the problem in G_0 is solvable for any k in a neighborhood of k_e. Assume the solution v_0 to be subject to the condition $(v_0, v_e)_{G_0} = 0$ as $k = k_e$. According to Proposition 3.1,

$$\|v_0; V^2_{\gamma_1,\gamma_2}(G_0)\| \leqslant \widetilde{c}_0\|f_0; V^0_{\gamma_1,\gamma_2}(G_0)\|, \tag{3.79}$$

where $\widetilde{c}_0$ does not depend on k. We set

$$U(x;\varepsilon_1,\varepsilon_2) = \chi_{\varepsilon_1,1}(x)v_1(x;\varepsilon_1) + \varepsilon_1^{-\gamma_1+1/2}\Theta(r_1)w_1(\varepsilon_1^{-1}x^1;\varepsilon_1)+$$
$$+\chi_{\varepsilon_1,\varepsilon_2}(x)v_0(x;\varepsilon_1,\varepsilon_2) + \varepsilon_2^{-\gamma_2+1/2}\Theta(r_2)w_2(\varepsilon_2^{-1}x^2;\varepsilon_2) + \chi_{\varepsilon_2,2}(x)v_2(x;\varepsilon).$$

The estimates (3.77)—(3.79) result in

$$\|U; V^2_{\gamma_1,\gamma_2,-}(G(\varepsilon_1,\varepsilon_2))\| \leqslant \widetilde{\widetilde{c}}\|f; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\|, \tag{3.80}$$

where $\widetilde{\widetilde{c}}$ is independent of ε_1, ε_2. This means that the mapping $R_{\varepsilon_1,\varepsilon_2} : f \mapsto U(f)$ is uniformly bounded with respect to ε_1, ε_2. Arguing as in the proof of Proposition 3.6, one can verify that $(\triangle + k^2)R_{\varepsilon_1,\varepsilon_2} = I + S_{\varepsilon_1,\varepsilon_2}$, where $S_{\varepsilon_1,\varepsilon_2}$ is an operator in $V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))$ of small norm.

Remind that the operator $S_{\varepsilon_1,\varepsilon_2}$ is defined on the subspace $V^{0,\perp}_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))$. We need the image of the operator $S_{\varepsilon_1,\varepsilon_2}$ be included in $V^{0,\perp}_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))$, too. To this end, replace the mapping $R_{\varepsilon_1,\varepsilon_2}$ by $\widetilde{R}_{\varepsilon_1,\varepsilon_2} : f \mapsto U(f) + a(f)u_p$, where u_p is constructed in **A**, $a(f)$ being a constant. Then $(\triangle + k^2)\widetilde{R}_{\varepsilon_1,\varepsilon_2} = I + \widetilde{S}_{\varepsilon_1,\varepsilon_2}$, with $\widetilde{S}_{\varepsilon_1,\varepsilon_2} = S_{\varepsilon_1,\varepsilon_2} + a(\cdot)(\triangle + k^2)u_p$. The condition $(\chi_{\sqrt{\varepsilon_1},\sqrt{\varepsilon_2}}\widetilde{S}_{\varepsilon_1,\varepsilon_2}f, v_e)_{G_0} = 0$ as $k = k_e$ gives $a(f) = -(\chi_{\sqrt{\varepsilon_1},\sqrt{\varepsilon_2}}S_{\varepsilon_1,\varepsilon_2}f, v_e)_{G_0}/(\chi_{\sqrt{\varepsilon_1},\sqrt{\varepsilon_2}}(\triangle + k_0^2)u_p, v_e)_{G_0}$.

Prove that $\|\widetilde{S}_{\varepsilon_1,\varepsilon_2}\| \leqslant c\|S_{\varepsilon_1,\varepsilon_2}\|$, c being independent of $\varepsilon_1, \varepsilon_2, k$. We have

$$\|\widetilde{S}_{\varepsilon_1,\varepsilon_2}f\| \leqslant \|S_{\varepsilon_1,\varepsilon_2}f\| + |a(f)| \cdot \|(\triangle + k^2)u_p\|.$$

The estimate (3.75), the formula for k_p, the condition $k^2 - k_0^2 = O(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1})$, and the inequalities $\gamma_j > \mu_{j1} - 1/2$ result in

$$\|(\triangle + k^2)u_p; V^0_{\gamma_1,\gamma_2,\delta}\| \leqslant |k^2 - k_p^2|\|u_p; V^0_{\gamma_1,\gamma_2,\delta}\| + \|(\triangle + k_p^2)u_p; V^0_{\gamma_1,\gamma_2,\delta}\| \leqslant$$
$$\leqslant c\left(|k^2 - k_p^2| + \left[\varepsilon_1^{\mu_{11}}\left(\varepsilon_1^{\kappa_{12}+1} + \varepsilon_1^{\gamma_1+3/2}\right) + \varepsilon_2^{\mu_{21}}\left(\varepsilon_2^{\kappa_{22}+1} + \varepsilon_2^{\gamma_2+3/2}\right)\right]\right) \leqslant$$
$$\leqslant c(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1}).$$

Since the supports of the functions $(\triangle + k_p^2)u_p$ and $\chi_{\sqrt{\varepsilon_1},\sqrt{\varepsilon_2}}$ do not intersect,

$$|(\chi_{\sqrt{\varepsilon_1},\sqrt{\varepsilon_2}}(\triangle + k_0^2)u_p, v_e)_{G_0}| = |(k_0^2 - k_p^2)(u_p, v_e)_{G_0}| \geqslant c(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1}),$$

and, since $\gamma_j - 1 < \min\{\varkappa_{j1}, \mu_{j1}\} + 1/2)$,

$$|(\chi_{\sqrt{\varepsilon_1},\sqrt{\varepsilon_2}}S_{\varepsilon_1,\varepsilon_2}f, v_e)_{G_0}| \leqslant \|S_{\varepsilon_1,\varepsilon_2}f; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\| \times$$
$$\times \|\chi_{\sqrt{\varepsilon_1},\sqrt{\varepsilon_2}}v_e; V^0_{-\gamma_1,-\gamma_2}(G_0)\| \leqslant c\|S_{\varepsilon_1,\varepsilon_2}f; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\|.$$

Hence,

$$|a(f)| \leqslant c\|S_{\varepsilon_1,\varepsilon_2}f; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\|/(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1})$$

and $\|\widetilde{S}_{\varepsilon_1,\varepsilon_2}f\| \leqslant c\|S_{\varepsilon_1,\varepsilon_2}f\|$.

Thus the operator $I+\widetilde{S}_{\varepsilon_1,\varepsilon_2}$ in $V^{0,\perp}_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))$ is invertible, which is also true for the operator of the problem (3.40)

$$A_{\varepsilon_1,\varepsilon_2}: u\mapsto \triangle u+k^2u : \mathring{V}^{2,\perp}_{\gamma_1,\gamma_2,-}(G(\varepsilon_1,\varepsilon_2))\mapsto V^{0,\perp}_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2));$$

here $\mathring{V}^{2,\perp}_{\gamma_1,\gamma_2,-}(G(\varepsilon_1,\varepsilon_2))$ denotes the space of elements of $V^{2}_{\gamma_1,\gamma_2,-}(G(\varepsilon_1,\varepsilon_2))$ that vanish on $\partial G(\varepsilon_1,\varepsilon_2)$ and are sent by the operator $\triangle+k^2$ to $V^{0,\perp}_{\gamma_1,\gamma_2,\delta}$. The inverse operator $A^{-1}_{\varepsilon_1,\varepsilon_2}=\widetilde{R}_{\varepsilon_1,\varepsilon_2}(I+\widetilde{S}_{\varepsilon_1,\varepsilon_2})^{-1}$ is bounded uniformly with respect to $\varepsilon_1,\varepsilon_2,k$. Hence, the inequality (3.74) holds with C independent of $\varepsilon_1,\varepsilon_2,k$.

Step D Consider now a solution u of the problem (3.72) with arbitrary f in $C^\infty_c(G(\varepsilon_1,\varepsilon_2))$. Rewrite this solution in the form $(u-b(f)u_p)+b(f)u_p$, where

$$b(f)=(f,\chi_{\sqrt{\varepsilon_1},\sqrt{\varepsilon_2}}v_e)_{G_0}/((\triangle+k^2)u_p,\chi_{\sqrt{\varepsilon_1}\sqrt{\varepsilon_2}}v_e)_{G_0}.$$

The difference $u-b(f)u_p$ is a solution of the problem (3.72) with right-hand side $f-b(f)(\triangle+k^2)u_p$ in $V^{0,\perp}_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))$. From **C** and the obvious inequality $|b(f)|\leqslant c|k^2-k_p^2|^{-1}$, we obtain

$$\|u-b(f)u_p;V^2_{\gamma_1,\gamma_2,-}(G(\varepsilon_1,\varepsilon_2))\|\leqslant c\|f-b(f)(\triangle+k^2)u_p;V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\|\leqslant$$
$$\leqslant c\frac{\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1}}{|k^2-k_p^2|}\|f;V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\|.$$

Hence,

$$\|u;V^2_{\gamma_1,\gamma_2,-}(G(\varepsilon_1,\varepsilon_2))\|\leqslant\|u-b(f)u_p;V^2_{\gamma_1,\gamma_2,-}(G(\varepsilon_1,\varepsilon_2))\|+$$
$$+|b(f)|\|u_p;V^2_{\gamma_1,\gamma_2,-}(G(\varepsilon_1,\varepsilon_2))\|\leqslant\frac{c}{|k^2-k_p^2|}\|f;V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\|.$$

It remains to take account of

$$|k^2-k_p^2|\geqslant c\left(|k^2-k_r^2|+\varepsilon_1^{2\varkappa_{11}+2\mu_{11}+2}+\varepsilon_2^{2\varkappa_{21}+2\mu_{21}+2}\right)\qquad\text{as}\qquad k\in\mathbb{R}.$$

□

Remark 3.1. In the formulation of the theorem, the requirement

$$|k^2-k_r^2|=O(\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1})$$

and the conditions on $\varkappa_{j1},\mu_{j1}$ are not essential. They can be eliminated by using for proof an auxiliary function u_p, such that

$$\|(\triangle+k^2)u_p\|=O(\varepsilon_1^{2\varkappa_{11}+2\mu_{11}+2}+\varepsilon_2^{2\varkappa_{21}+2\mu_{21}+2}).$$

One can construct such u_p with the help of an higher order approximation to the wave function u_1.

Consider the solution u_1 of the homogeneous problem (3.1) defined by (3.4). Let s_{11} and s_{12} be the entries of scattering matrix determined by this solution. The function $\widetilde{u}_1$ and the numbers $\widetilde{s}_{11}$, $\widetilde{s}_{12}$ are constructed in Subsection 3.4.2.

Theorem 3.2. *Let the hypotheses of Propositions 3.2 and 3.7 be fulfilled and assume that the coefficients A_j introduced below the relations (3.61) are nonzero. Then the estimate*

$$\sup_{x\in G(\varepsilon_1,\varepsilon_2)} |u_1(x) - \widetilde{u}_1(x)| + |s_{11} - \widetilde{s}_{11}| + |s_{12} - \widetilde{s}_{12}| \leqslant$$
$$\leqslant c\varepsilon_1^{\varkappa_{11}+\mu_{11}+1} \frac{\varepsilon_1^{\mu_{11}}(\varepsilon_1^{\kappa_{12}+1} + \varepsilon_1^{\gamma_1+3/2}) + \varepsilon_2^{\mu_{21}}(\varepsilon_2^{\kappa_{22}+1} + \varepsilon_2^{\gamma_2+3/2})}{\left(|k^2 - k_r^2| + \varepsilon_1^{2\varkappa_{11}+2\mu_{11}+2} + \varepsilon_2^{2\varkappa_{21}+2\mu_{21}+2}\right)^2}$$

holds, where $\kappa_{j2} = \min\{\varkappa_{j2}, \mu_{j2}\}$, $j = 1, 2$; c is independent of $\varepsilon_1, \varepsilon_2, k$.

Proof. The difference $u_1 - \widetilde{u}_1$ is in the space $V^2_{\gamma_1,\gamma_2,-}(G(\varepsilon_1, \varepsilon_2))$, and $f_1 := (\triangle + k^2)u_1 - \widetilde{u}_1$ is in $V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))$. By Proposition 3.7,

$$\|u_1 - \widetilde{u}_1; V^2_{\gamma_1,\gamma_2,-}(G(\varepsilon_1, \varepsilon_2))\| \leqslant c\, \frac{\|f_1; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\|}{|k^2 - k_r^2| + \varepsilon_1^{2\varkappa_{11}+2\mu_{11}+2} + \varepsilon_2^{2\varkappa_{21}+2\mu_{21}+2}}.$$

Arguing as in the proof of Theorem 3.1, we obtain that

$$\|f_1; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\| \leqslant c\left((|a_1^-|\varepsilon_1^{-\varkappa_{11}-1} + |a_1^+|\varepsilon_1^{\varkappa_{11}})(\varepsilon_1^{\kappa_{12}+1} + \varepsilon_1^{\gamma_1+3/2}) + \right.$$
$$\left. + (|a_2^-|\varepsilon_2^{-\varkappa_{21}-1} + |a_2^+|\varepsilon_2^{\varkappa_{21}})(\varepsilon_2^{\kappa_{22}+1} + \varepsilon_2^{\gamma_2+3/2})\right).$$

From (3.70)-(3.71), it follows that

$$|a_j^-|\varepsilon_j^{-\varkappa_{j1}-1} + |a_j^+|\varepsilon_j^{\varkappa_{j1}} \leqslant c\varepsilon_1^{\varkappa_{11}+\mu_{11}+1}\varepsilon_j^{\mu_{j1}}(\varepsilon_j^{\kappa_{j2}+1} + \varepsilon_j^{\gamma_j+3/2}).$$

The required estimate follows from the last three inequalities by the arguing used at the end of the proof of Theorem 3.1. □

4 CONCLUSION

We considered an infinite waveguide with two cylindric ends and two narrows of diameters ε_1 and ε_2. We have given an asymptotic description of the electron wave propagation in such a waveguide as ε_1 and ε_2 tend to zero. The wave number k is assumed to be between the first and the second thresholds, so only one incoming and one outgoing wave may propagate in every outlet of the waveguide to infinity.

The asymptotic formulas depend on the shape of narrows (in the limit as $\varepsilon_1, \varepsilon_2 \to 0$) of the waveguide. In the 2D case, assume that a neighborhood of each narrow, in the limit as $\varepsilon_1, \varepsilon_2 \to 0$, coincides with a neighborhood of the vertex of two vertical angles of opening ω_1 and ω_2, respectively.

While the diameters and openings of the narrows are the same ($\varepsilon_1 = \varepsilon_2 = \varepsilon$, $\omega_1 = \omega_2 = \omega$) and the resonator is symmetric in a sense, the resonant energies (i.e., the energies k^2 at which the resonant tunneling occurs) are given by (2.17):

$$k_r^2(\varepsilon) = k_0^2 + k_1 \varepsilon^{2\pi/\omega} + O(\varepsilon^{2\pi/\omega+2}),$$

where k_0^2 is an eigenvalue of the resonator (part of the waveguide between two narrows obtained after passing to the limit as $\varepsilon \to 0$), k_1 is a constant independent of ε, which can be found numerically. Near $k = k_r$, the transition coefficient is of the form (see (2.21))

$$T(k, \varepsilon) = \frac{1}{1 + Q\left(\dfrac{k^2 - k_r^2}{\varepsilon^{4\pi/\omega}}\right)^2},$$

Q being a positive constant independent of ε. From here we find the width of the resonant peak at half-height:

$$\Delta(\varepsilon) = \frac{2}{\sqrt{Q}} \varepsilon^{4\pi/\omega}.$$

To obtain these formulas we first construct the asymptotic representation (2.8) of the corresponding wave function using the method of "compound" asymptotics.

Analogous formulas (2.22)—(2.24) (becoming more sophisticated) are valid for asymmetric waveguides where the resonator is not symmetric and (or) the narrows have distinct openings and diameters. In particular, in contrast to the symmetric case, the principal part of the asymptotics already shows that the maximum of the transmission coefficient is less than 1.

In the 3D case we assume that a neighborhood of each narrow, in the limit as $\varepsilon_j \to 0$, coincides with a neighborhood of the vertex of a double cone. Asymptotic formulas (2.25)—(2.27), (3.65)—(3.68) for the basic characteristics of the process are analogous to those in the 2D case. The exponent $2\pi/\omega_j$ of ε_j must be replaced by the number μ such that $\mu(\mu+1)$ is the first eigenvalue of the Dirichlet problem for the Beltrami operator on the base of the cone.

Various electronic devices (transistors, key devices, electron energy monochromators, amplifiers) can be based on waveguides of variable cross-section. The formulas obtained can be useful to calculate characteristics of these devices and to provide optimal regimes of theirs operation.

YHTEENVETO (FINNISH SUMMARY)

Halkaisijaltaan muunnellun kvanttiaaltojohtimen kapenemat toimivat tehokkaina potentiaalivalleina elektronin pitkittäissuuntaiselle liikkeelle. Kaksi kapenemaa muodostaa kvanttiresonaattorin, jossa resonoiva tunnelointi voi tapahtua. Tämä tarkoittaa sitä, että elektronit, joiden energia on lähellä resonanssia, läpäisevät resonaattorin todennäköisyydellä lähellä yhtä. Kuvailemme asymptoottisesti elektroniaallon etenemistä kvanttiaaltojohtimessa, jossa on kaksi kapenemaa. Aaltoluvun k oletetaan olevan ensimmäisen ja toisen kynnysluvun välissä, jolloin vain sisään tuleva ja poismenevä aalto voivat edetä jokaisessa aaltojohtimen päässä äärettömyydessä. Esitämme asymptoottiset kehitelmät aaltofunktioille, ja heijastus- ja siirtymäkertoimille, kun kapenemien halkaisijat menevät nollaan. Lisäksi esitämme asymptoottiset kaavat resonaatiotaajuuksille ja analysoimme kertoimien käyttäytymistä lähellä resonanssipistettä.

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