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Sergey Matyukevich

The Nonstationary Maxwell System in Domains with Edges and Conical Points

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ABSTRACT

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We study the nonstationary Maxwell system in domains with conical points and edges. The system is endowed with two types of boundary conditions: conductive and impedance. We consider both homogeneous and inhomogeneous boundary conditions. Our main purpose is to study the behavior of solutions near the singularities. We derive the asymptotic expansions of solutions near the edges and conical points and obtain the explicit formulas for the coefficients in the asymptotics. The asymptotics of solutions and the formulas for the coefficients turn out to be of use in many applications. In particular, the results can be applied in mathematical and numerical treatment of the problems of electrodynamics in nonsmooth domains.

Keywords: nonstationary Maxwell system, conical points, edges, asymptotics of solutions near the singularities, energy estimates.

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CONTENTS

1 INTRODUCTION

1.1 Preliminaries

We study the nonstationary Maxwell system

$$\begin{cases} \partial\vec{E}/\partial t - \text{rot } \vec{B} = -\vec{J}, \\ \partial\vec{B}/\partial t + \text{rot } \vec{E} = -\vec{G}, \\ \qquad \text{div}\vec{E} = \rho, \\ \qquad \text{div}\vec{B} = \mu \end{cases} \tag{1.1}$$

in domains with conical points and edges. Two types of boundary conditions are considered: conductive

$$[\vec{E} \times \vec{\nu}] = [\vec{\Phi} \times \vec{\nu}], \langle \vec{B}, \vec{\nu} \rangle = \delta \tag{1.2}$$

and impedance

$$\vec{\nu} \times [\vec{B} \times \vec{\nu}] + \psi[\vec{\nu} \times \vec{E}] = [\vec{\Phi} \times \vec{\nu}]. \tag{1.3}$$

Here by $\vec{\nu}$ we denote the unit outward normal and by ψ we denote an impedance (a complex-valued function describing the conductive properties of the boundary).

Our main purpose is to study behavior of solutions to the problems (1.1), (1.2) and (1.1), (1.3) near the conical points and edges. We obtain and justify the asymptotics of solutions near the singularities. Besides, we derive the formulas for the coefficients in the asymptotics and study their properties.

Elliptic boundary value problem in domains with piecewise smooth boundary were thoroughly investigated in the works of V. A. Kondrat'ev, V. G. Maz'ya, B.A. Plamenevskii, and others (see, e.g., [28] and the extensive bibliography in this book). Since solutions to elliptic problems are not smooth in singular points of the boundary, it is important to describe the properties of the solutions near such points. The asymptotics of solutions near the singularities of the boundary is described in terms of eigenvalues and eigenfunctions of operator pencil (polynomial with operator coefficients) corresponding to the elliptic problem under consideration. These spectral characteristics of the pencil depend on the properties of the problem and the boundary near the singular point. The coefficients in the asymptotics are calculated through the right-hand side of the problem and certain singular solutions to the homogeneous adjoint problem. Note that the form of the asymptotics is defined by the local properties of the problem near the singular point, while the coefficients depend on the whole problem.

The "elliptic" results can be applied to other problems. For instance, the electromagnetic oscillations are governed by the Maxwell system, which is not elliptic. However, in certain cases the study of the singularities of solutions to the Maxwell system can be reduced to the singularities of solutions to second-order scalar elliptic equations. In the case of empty resonator with perfectly conducting boundary the Maxwell singularities are described in terms of the singularities of the Dirichlet and Neumann problem for the Laplace operator.

The results concerning the asymptotics of solutions near the singularities have numerous applications. Here we indicate one example that is about the Maxwell system in domains with reentrant edges or corners. It turns out that the application of nodal finite elements in numerical treatment of this problem leads to an error in calculations (see [5]). The discrete iterations converge to a wrong solution since principal Maxwell singularities can not be approximated by nodal elements. Different methods were suggested to avoid this "Maxwell bug" (see [12, 7]). In particular, some methods make use of the explicit asymptotic expansions of solutions near the singular points on the boundary. Namely, the Maxwell singularities are added to the standard finite element spaces.

Hyperbolic problems in nonsmooth domains are still poorly understood. Here we mention some papers, in which the behavior of solutions near the singularities is studied. G. I. Eskin in [9] studied the wave equation in a wedge with edge of codimension 2. On the boundary, the author considered homogeneous differential operators with constant coefficients satisfying the uniform Lopatinskii condition. The main result is an explicit formula for solutions. However the method (reduction to the Riemann – Hilbert problem) can not be generalized to edges of higher codimensions.

The Cauchy – Dirichlet problem for the wave equation in domains with conical points was studied by V. A. Kondrat'ev, O. A. Oleinik , and I. I. Mel'nikov in [14, 26]. The authors suggested the following method. They proved the solvability results for generalized solutions to the problem and estimated the derivatives of solutions with respect to time by some Sobolev norms of the right-hand side. Then all the derivatives with respect to time were rearranged to the right-hand side of the equation. This problem was treated as an elliptic one in order to obtain the asymptotics of solutions near the conical points by well-known "elliptic" technique (see [28] or [25]). Later Nguen Man' Khung in [29] applied this method to study strictly hyperbolic systems in domains with conical points. However the method suggested in [14], [26] does not lead to the formulas for the coefficients in the asymptotics.

B. A. Plamenevskii in [33], then B. A. Plamenevskii and A. Yu. Kokotov in [15, 16] investigated different initial boundary value problems for the wave equation and strictly hyperbolic systems in domains with edges and conical points. They proved the solvability results in scales of weighted spaces, derived the asymptotics of solutions near the singularities and obtained the formulas for the coefficients in the asymptotics. It turns out that the principal term of the asymptotics is of the form $u(x,t) \sim \sum c_j(t)u_j(x)$, where u_j are some special solutions to the homogeneous problem for the spatial part of the system under consideration. The hyperbolic properties of the asymptotics can be observed in the coefficients c_j expressed by the explicit formulas through the right-hand side of the system and some singular solutions to the homogeneous problem.

Further, we discuss some papers related to the stationary Maxwell system in nonsmooth domains. J. Saranen in [35] described the singularities of solutions to the Maxwell system in cones with smooth basis.

N. Weck in [38] investigated the generalized Maxwell acting on differential forms on Riemannian manifold. The author developed the solution theory for a certain class of manifolds with piecewise smooth boundary.

M. Sh. Birman and M. Z. Solomyak in [4] defined and studied the Maxwell operator in domains with much more relaxed restrictions on the boundary. They considered the perfectly conductive boundary conditions. For Lipschits domains the principal part of the asymptotics was obtained via the singularities of solutions to second-order elliptic equations.

M. Costabel and M. Dauge in [6] investigated the singularities of solutions to the time-harmonic Maxwell system in domains with edges, conical points, and polyhedral corners. Applying the classical Mellin analysis, the authors derived the asymptotic formulas based on Dirichlet and Neumann singularities for the Laplace operator.

We modify the approach suggested in [33, 15] to study the nonstationary Maxwell system in domains with conical points. In these papers, the results concerning elliptic boundary value problems are applied to obtain the asymptotics of solutions near the conical points, and the ellipticity of the spatial part of the system under consideration is crucial. However the spatial part of the Maxwell system is not elliptic. That is why we consider the augmented Maxwell system

$$\begin{cases} \partial\vec{E}/\partial t - \operatorname{rot}\vec{B} + \nabla h = -\vec{J}, \\ \partial\vec{B}/\partial t + \operatorname{rot}\vec{E} + \nabla q = -\vec{G}, \\ \qquad \partial h/\partial t + \operatorname{div}\vec{E} = \rho, \\ \qquad \partial q/\partial t + \operatorname{div}\vec{B} = \mu, \end{cases} \tag{1.4}$$

which has the elliptic spatial part (see, e.g., [4, 21]). For brevity we rewrite the system (1.4) in the form

$$\partial u/\partial t + A(\partial)u = f,$$

where $u = (\vec{E},\ \vec{B},\ h,\ q)$, $f = (-\vec{J},\ -\vec{G},\ \rho,\ \mu)$, and $\partial = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$. The augmented system (1.4) is endowed with the augmented boundary conditions

$$[\vec{E}\times\vec{\nu}] = [\vec{\Phi}\times\vec{\nu}],\ \langle\vec{B}\cdot\vec{\nu}\rangle = \varphi,\ h = H \tag{1.5}$$

or

$$\vec{\nu}\times[\vec{B}\times\vec{\nu}] + \psi[\vec{\nu}\times\vec{E}] = [\vec{\Phi}\times\vec{\nu}],\ h = H,\ q = Q \tag{1.6}$$

such that the boundary value problems (1.4), (1.5) and (1.4), (1.6) are elliptic.

Further, we give a detailed description of the method to be used. For the sake of simplicity we consider the system (1.4) with homogeneous boundary conditions (1.5) in the cone $\mathcal{K}$ smooth outside the vertex. By $\mathcal{F}_{t\to\tau}$ we denote the Fourier transformation defined by the formula

$$\widehat{u}(\tau) = \Big(\mathcal{F}_{t\to\tau}u\Big)(\tau) = \int\limits_{\mathbb{R}} e^{-it\tau}u(t)\,\mathrm{d}t,$$

where $\tau = \sigma - i\gamma$ $(\sigma \in \mathbb{R},\ \gamma > 0)$. Applying the Fourier transformation to the system under consideration we arrive at a problem with parameter in the cone. This problem is elliptic for a fixed τ, but the dependence on τ is "hyperbolic". As in [33, 15], the method is based on certain *a priori* estimates of solutions. The first estimate follows from simple energy arguments, where for the energy we take the expression

$$\int\limits_{\mathcal{K}} (|\vec{E}(x,t)|^2 + |\vec{B}(x,t)|^2 + |h(x,t)|^2 + |q(x,t)|^2)\,\mathrm{d}x.$$

Note that if $h \equiv 0$ and $q \equiv 0$, then it is the usual formula for the energy of electromagnetic field. The first estimate is of the form

$$\gamma^2 \int\limits_{\mathcal{K}} |v(x)|^2\,\mathrm{d}x \le c \int\limits_{\mathcal{K}} |(\tau + A(D_x))v(x)|^2\,\mathrm{d}x,$$

where the constant c is independent of τ. It is called the energy estimate. We apply it to prove the solvability results in the class of solutions with finite energy (energy solutions).

When proving the energy estimate for the augmented Maxwell system, we face the necessity to consider the spatial part $A(\partial)$ as a symmetric operator. Therefore one has to choose asymptotics near singularities of the boundary for the functions in the domain of $A(\partial)$. This gives rise to a family of self-adjoint extensions of $A(\partial)$. The possibility of coming back to the initial non-augmented Maxwell system depends on the choice of a self-adjoint extension. In particular, in a bounded domain with conical point, the passage to the initial system is possible for the only self-adjoint extension. To implement this passage, it suffices to take a right-hand side of the form $(-\vec{J},\ -\vec{G},\ \rho,\ \mu)$ for the augmented system, where $\vec{J},\ \vec{G},\ \rho,\ \mu$ are subject to the equations div $\vec{J} + \partial\rho/\partial t = 0$, div $\vec{G} + \partial\mu/\partial t = 0$, and to the boundary condition $\langle\vec{G}, \vec{\nu}\rangle = 0$. Then h and q vanish and the solution of the augmented system satisfies the usual Maxwell system. It turns out that the mentioned self-adjoint extension coincides with the Maxwell operator studied in [4] for the stationary situation.

We need a more informative estimate to investigate the behavior of energy solutions near the conical points. To prove it, we split the cone into zones. In a small zone near the vertex we can apply the weighted elliptic estimate of solutions. The volume of this zone is inversely related to the parameter. Far from the vertex we use a localized energy estimate. "Gluing" these inequalities together we obtain the so called weighted combined estimate

$$\begin{aligned}\gamma^2\|v; H^0_\beta(\mathcal{K}, |\tau|)\|^2 + \|\chi_\tau v; H^1_\beta(\mathcal{K}, |\tau|)\|^2 \le \\ \le c\Big\{ \int\limits_{\mathcal{K}} |(\tau + A(D_x))v(x)|^2 r^{2\beta}(x)\, \mathrm{d}x + \\ + \Big(|\tau|^{2(1-\beta)}/\gamma^2\Big) \int\limits_{\mathcal{K}} |(\tau + A(D_x))v(x)|^2\, \mathrm{d}x \Big\},\end{aligned}$$

where the constant c is independent of τ. Here by $r(x)$ we denote the distance between the point $x \in \mathcal{K}$ and the vertex of the cone, by χ we denote a cut-off function such that $\chi = 1$ for $r < 1$, and $\chi = 0$ for $r > 2$; $\chi_\tau(r) = \chi(|\tau|r)$. The function spaces $H^s_\beta(\mathcal{K}, |\tau|)$ are defined below, in the next section. The weighted combined estimate is proved for all $\beta \le 1$, $\beta \notin \{\beta_k\}_{k\ge1}$, where $\{\beta_k\}_{k\ge1}$ is a sequence such that $1/2 > \beta_1 > \beta_2 \cdots > \beta_k \to -\infty$. In spaces related to the weighted combined estimate there is a closed operator corresponding to the problem with parameter in the cone. The kernel and cokernel of this operator are trivial if $\beta \in\,]\beta_1,\ 1]$. As β decreases, the dimension of the cokernel increases (when β crosses β_k) but remains finite. The elements of a basis of the cokernel are uniquely determined by their asymptotics near the vertex. Along with the "elliptic" results on the asymptotics, this allows us to obtain the asymptotics of the solutions near the vertex of the cone, including formulas for the coefficients in the asymptotics. The inverse Fourier transformation carries the theory over to the nonstationary problem.

Now we discuss the structure of the asymptotics of solutions near the vertex of the cone. For simplicity we consider the asymptotics containing the principal part only. If the right-hand side of the system decays with a certain rate near the vertex, then for the solution the representation holds

$$u(x,t) = \sum_j c_j(t) u_j(x) + R(x,t).$$

Here by u_j we denote some special solutions to the homogeneous problem for the spatial part of the system under consideration, by R we denote the remainder. The coefficients are calculated by the formula

$$c_j(t) = \int\limits_{\mathcal{K}} \int\limits_{\mathbb{R}} \langle f(x, t-s),\, W_j(x,s)\rangle\, \mathrm{d}x\, \mathrm{d}s,$$

where f is a right-hand side of the system. By W_j we denote some singular solutions to the adjoint nonstationary problem determined by their growing asymptotics near the vertex of the cone. For model domains (a cone and a wedge) we can obtain explicit formulas for W_j. Applying these formulas, it is possible to observe some phenomena related to the finite propagation speed of the electromagnetic waves. Suppose that the singular support of the right-hand side f is located in the set $\{(x_1, x_2, x_3, t) : R_1 < r < R_2,\ 0 < t < t_0\}$. Then $c_j(t)$ are smooth for $t > t_0 + R_2$.

In other words, there is the phenomenon of "back edge": the coefficients have been smooth after the perturbation from the singular support of the right-hand side has left the vertex of $\mathcal{K}$. Assume now that supp $f \subset \{(x_1, x_2, x_3, t) \in \mathcal{K} \times \mathbb{R} : R_1 < r < R_2,\ t > 0\}$. Then $c_j(t) = 0$ for $t < R_1$. Thus we observe the phenomenon of "forward edge" in the coefficients: the coefficients have been equal to zero before the perturbation from the support of the right-hand side arrives at the vertex.

Now we turn to the augmented Maxwell system with inhomogeneous impedance boundary conditions. Actually the same method is applied as for the homogeneous boundary conditions. Namely, we prove certain *a priori* estimates. Using these estimates, we study the problem in scales of weighted spaces and prove the solvability results. Taking into account well-known elliptic results, we derive the asymptotics of solutions near the vertex of the cone and obtain the formulas for the coefficients in this asymptotics. However in the case of inhomogeneous boundary conditions, as distinct from the homogeneous boundary conditions, there is a considerable difficulty in proving the energy estimate.

In smooth domains there is a general approach for the proof of the energy estimates for hyperbolic problems satisfying the uniform Lopatinskii condition. This approach was suggested by S. Agmon in [1] for higher order scalar hyperbolic operators. Then R. Sakamoto in [34] improved the results of Agmon, and H.-O. Kreiss in [22] investigated hyperbolic systems. These authors proved the energy estimates in smooth domains considering model problems in $\mathbb{R}^n_+$ and $\mathbb{R}^n$. It turns out that their approach can not be applied to domains with conical points due to the difficulties arising from the model problem in a cone. Besides, the uniform Lopatinskii condition is not satisfied for the wave equation and the Lamé system with Neumann boundary conditions, which are important for the applications in elasticity theory.

For second-order scalar hyperbolic equations in smooth domains L. Gårding and L. Hörmander suggested another proof of the energy estimates. The equation was multiplied by a certain first-order differential operator applied to the solution and integrated over the domain, then integrated by parts. Using this method, L. Gårding in [11] investigated the oblique derivative problem for the wave equation and L. Hörmander in [13] studied the Cauchy – Dirichlet problem for second-order scalar hyperbolic equations.

B. A. Plamenevskii and A. Yu. Kokotov in [15] modified the method of Gårding and Hörmander to study hyperbolic systems in nonsmooth domains. Second - order strictly hyperbolic systems with inhomogeneous Dirichlet boundary conditions were considered in domains with edges and conical points. In particular, the authors proved the energy estimates and applied them for investigation of the problem. The suggested proof of the energy estimates is valid only for cones and wedges satisfying a certain admissibility condition. Namely, the wedge $\mathbb{D}$ is called admissible if there exists a constant vector $\vec{f}$ such that $\langle \vec{f},\ \vec{\nu} \rangle \geq c_0 > 0$ for all outward normals to $\partial\mathbb{D} \setminus M$, where M is the edge of the wedge. It was also shown that the energy estimates may fail for nonadmissible wedges.

In this work we prove the energy estimate for the augmented Maxwell system with boundary conditions (1.5) and (1.6) modifying the method of L. Gårding and L. Hörmander for first-order systems and taking into account certain specific features of the system (1.4). The energy estimate in the case of inhomogeneous conductive boundary conditions (1.5) is of the form

$$\gamma^2 \|v; L_2(\mathcal{K})\|^2 + \gamma \|Tv; L_2(\partial\mathcal{K})\|^2 \leq$$
$$\leq c\{\|(\tau + A(D_x))v; L_2(\mathcal{K})\|^2 + \gamma \|\Gamma v; L_2(\partial\mathcal{K})\|^2\},$$

where $v = (\vec{E},\ \vec{B},\ h,\ q)$, $\Gamma v = ([\vec{\nu} \times \vec{E}],\ \ \langle \vec{B}, \vec{\nu}\rangle,\ h)^T$, and $Tu = (\vec{\nu} \times [\vec{B} \times \vec{\nu}],\ q,\ \langle \vec{E},\ \vec{\nu}\rangle)^T$. As in [15], this estimate is valid for admissible cones.

The energy estimate in the case of inhomogeneous impedance boundary conditions (1.6) is of the form

$$\begin{gathered}\gamma^2 \|v; L_2(\mathcal{K})\|^2 + \gamma \|T_1 v; L_2(\partial\mathcal{K})\|^2 \leq \\ \leq c\{\|(\tau + A(D_x))v; L_2(\mathcal{K})\|^2 + \gamma \|\Gamma_1 v; L_2(\partial\mathcal{K})\|^2\},\end{gathered} \tag{1.7}$$

where $v = (\vec{E},\ \vec{B},\ h,\ q)$, $\Gamma_1 v = (\vec{\nu} \times [\vec{B} \times \vec{\nu}] + \psi[\vec{\nu} \times \vec{E}],\ h,\ q)$, and $T_1 u = ((1/\overline{\psi})\ \vec{\nu} \times [\vec{B} \times \vec{\nu}],\ \langle \vec{E}, \vec{\nu}\rangle,\ \langle \vec{B}, \vec{\nu}\rangle)$. This estimate is also valid for admissible cones.

It will be shown that the boundary conditions $H = 0,\ Q = 0$ in (1.6) are set when we "return" from the augmented system with impedance boundary conditions to the usual Maxwell system. Thus if we are mainly interested in the results for the usual Maxwell system, then we can consider only these boundary conditions when studying the augmented system. For the boundary conditions (1.6) with $H = 0,\ Q = 0$ we prove another energy estimate

$$\begin{gathered}\gamma^2 \|v; L_2(\mathcal{K})\|^2 + (\gamma |\mathrm{Re}\,\psi|/|\psi|) \cdot \|\vec{B} \times \vec{\nu}; L_2(\partial\mathcal{K})\|^2 \leq \\ \leq c\big\{\|M(D_x, \tau)v; L_2(\mathcal{K})\|^2 + (\gamma |\psi|/|\mathrm{Re}\,\psi|) \cdot \|\vec{\Phi} \times \vec{\nu}; L_2(\partial\mathcal{K})\|^2\big\},\end{gathered} \tag{1.8}$$

which is valid for arbitrary cones. Applying the energy estimate (1.8) instead of (1.7), we can obtain the results for the usual Maxwell system in arbitrary cones. However, the estimate (1.7) is more informative: its left-hand side contains also the normal boundary components of the fields $\vec{E},\ \vec{B}$.

The results for the usual Maxwell system (1.1) with boundary conditions (1.2) and (1.3) follow from those for the augmented problems (1.4), (1.5) and (1.4), (1.6). It turns out that if the right-hand side of the augmented problem satisfies some natural conditions (the compatibility conditions for the ordinary Maxwell system), then in the solution $(\vec{E},\ \vec{B},\ h,\ q)$ the functions $h,\ q$ vanish, and $(\vec{E},\ \vec{B})$ satisfies the ordinary Maxwell system.

1.2 Review of results

Here we briefly describe the main results and give the exact references on the corresponding theorems and propositions in this work.

In the second chapter we study the augmented Maxwell system (1.4) with homogeneous boundary conditions (1.5) in a cone and in a domain with conical point. Applying Fourier transformation we arrive at the problem with parameter. We prove *a priori* estimate taking into account simple energy arguments (Proposition 2.8), then we prove a more informative combined weighted estimate (Proposition 2.20). By means of these estimates the operator of the problem under consideration is investigated in a scale of weighted spaces (Theorems 2.16 and 2.24). Applying the properties of the operator and well-known elliptic results we derive the asymptotics of solutions near the conical point and obtain the explicit formulas for the coefficients in the asymptotics (Theorem 2.26). Finally, we examine the connection between the augmented and usual Maxwell systems in Theorem 2.34. Due to this theorem, all the results concerning the solvability and asymptotics of the usual Maxwell system can be obtained from those for the augmented Maxwell system. The inverse Fourier transformation carries the theory over to the nonstationary problem (Theorems 2.29, 2.31, and 2.32).

In the third chapter we study the augmented Maxwell system (1.4) with homogeneous boundary conditions (1.5) in a wedge and in a waveguide with edges. In comparison with the results obtained in [33, 15], we weaken the requirements on the right-hand side which are necessary to obtain the asymptotics of the solution near the edge. Namely, we do not require extra smoothness of the right-hand side along the edge. This reduction in requirements is achieved by the accurate consideration of the properties of the spatial part for the Maxwell system in a wedge (Theorem 3.2). The scheme of investigation is the same as in Chapter 2. In Proposition 3.3 and Proposition 3.8 we prove *a priori* estimates. By means of these estimates we study the operator of the problem with parameter (Theorem 3.14 and Theorem 3.15). Then we formulate Theorem 3.24 about the connection between the usual and the augmented Maxwell systems. Finally, applying the inverse Fourier transformation, we obtain the results for the nonstationary problem (see Theorems 3.19, 3.21, and 3.22).

In the fourth chapter we study the augmented Maxwell system (1.4) with inhomogeneous boundary conditions (1.5) in an admissible cone and in a domain with admissible conical points. The method is basically the same as in Chapters 2 and 3. However, the proof of the energy estimates presents a considerable difficulties due to the inhomogeneous boundary conditions (see Propositions 4.3 and 4.6). As soon as we have proved the energy estimate, we follow the same scheme as in Chapters 2 and 3. We prove the combined weighted estimate in Proposition 4.14, investigate the operator of the problem (Theorems 4.10 and 4.18), and study the asymptotics of solutions in Theorem 4.20. The connection between the augmented Maxwell system and the usual Maxwell system is examined in Theorem 4.28. Applying the inverse Fourier transformation, we obtain the results for the nonstationary problem (see Theorems 4.23, 4.25, and 4.26).

In the fifth chapter we study the augmented Maxwell system (1.4) with inhomogeneous impedance boundary conditions (1.6) in a cone and in a domain with conical points. Here we prove two types of energy estimates. The proof of the first energy estimate is quite complicated (see Propositions 5.8, 5.10). It follows the scheme suggested in Chapter 4. This energy estimate is valid only for admissible conical points. The second estimate is proved by simple energy arguments similar to those in Chapters 2 and 3 (see Proposition 5.13). This estimate is valid for arbitrary conical points, however it is less informative than the first one. The further study of the problem can be based on any of these two energy estimates. Here the more informative estimate is chosen. Again, as soon as we proved the energy estimate, we follow the same scheme as in Chapters 2 and 3. We prove the combined weighted estimate in Proposition 5.23, investigate the operator of the problem (Theorems 5.17 and 5.28), and study the asymptotics of solutions in Proposition 5.30. The connection between the augmented Maxwell system and the usual Maxwell system is examined in Theorem 5.39. Applying the inverse Fourier transformation, we obtain the results for the nonstationary problem (see Theorems 5.34, 5.36, and 5.37).

1.3 Basic definitions

Now we introduce notations to be used in the text. We describe domains, function spaces and present some known results concerning the augmented Maxwell system.

Let $\mathcal{K} \subset \mathbb{R}^3$ be a cone with vertex at the origin of coordinates $\mathcal{O}$ such that $\mathcal{K} \cap \mathcal{S}^2$ is one-connected. Let $G \subset \mathbb{R}^3$ be a bounded domain with only one conical point $\mathcal{O}$ such that G coincides with $\mathcal{K}$ in a neighborhood of $\mathcal{O}$. The boundaries of $\mathcal{K}$ and G are smooth outside $\mathcal{O}$.

We introduce function spaces in the domains. Let $s \in \mathbb{N}$, $\beta \in \mathbb{R}$, and $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. By $H^s_\beta(\mathcal{K})$ denote the completion of the set $\mathcal{C}^\infty_c(\overline{\mathcal{K}} \setminus \mathcal{O})$ with respect to the norm

$$\|u; H^s_\beta(\mathcal{K})\| = \Big(\sum_{|\alpha|\le s} \int_{\mathcal{K}} r^{2(\beta+|\alpha|-s)} |D^\alpha_x u(x)|^2 \,\mathrm{d}x\Big)^{1/2},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, $x = (x_1, x_2, x_3)$, $\mathrm{d}x = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$, $D^\alpha_x = D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} D^{\alpha_3}_{x_3}$, and $D_{x_k} = -i\partial/\partial x_k$. The space $H^s_\beta(\mathcal{K}, q)$ with $q > 0$ is endowed with the norm

$$\|u; H^s_\beta(\mathcal{K}, q)\| = \Big(\sum_{k=0}^{s} q^{2k} \|u; H^{s-k}_\beta(\mathcal{K})\|^2\Big)^{1/2}.$$

By $H^{s-1/2}_\beta(\partial\mathcal{K}, q)$ we denote the space of traces on $\partial\mathcal{K}$ of the functions belonging to $H^s_\beta(\mathcal{K}, q)$. Changing $\mathcal{K}$ for G, we analogously define the spaces $H^s_\beta(G)$, $H^s_\beta(G, q)$, and $H^{s-1/2}_\beta(\partial G, q)$.

In the cylinder $\mathcal{Q} = \mathcal{K} \times \mathbb{R}$ we introduce the space $H^s_\beta(\mathcal{Q})$ by completing $\mathcal{C}^\infty_c((\overline{\mathcal{K}} \setminus \mathcal{O}) \times \mathbb{R})$ in the norm

$$\|w; H^s_\beta(\mathcal{Q})\| = \Big(\sum_{|\alpha|+k\le s} \int_{\mathcal{K}} \int_{\mathbb{R}} r^{2(\beta+|\alpha|+k-s)} \big|D^\alpha_x D^k_t w(x,t)\big|^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}.$$

The space $H^s_\beta(\mathcal{Q}, q)$ with $q > 0$ is endowed with the norm

$$\|u; H^s_\beta(\mathcal{Q}, q)\| = \Big(\sum_{k=0}^{s} q^{2k} \|u; H^{s-k}_\beta(\mathcal{Q})\|^2\Big)^{1/2}.$$

Changing $\mathcal{K}$ for G, we define the spaces $H^s_\beta(\mathbf{Q})$ and $H^s_\beta(\mathbf{Q},q)$ in the cylinder $\mathbf{Q} = G \times \mathbb{R}$. Finally, we denote by $V^s_\beta(\mathcal{Q},\gamma)$ and $V^s_\beta(\mathbf{Q},\gamma)$ with $\gamma > 0$ the spaces with the norms

$$\|w; V^s_\beta(\mathcal{Q},\gamma)\| = \|w^\gamma; H^s_\beta(\mathcal{Q},\gamma)\| \quad \text{and} \quad \|w; V^s_\beta(\mathbf{Q},\gamma)\| = \|w^\gamma; H^s_\beta(\mathbf{Q},\gamma)\|,$$

respectively, where $w^\gamma = e^{-\gamma t} w$.

Let $\mathbb{K} = \{(r,\varphi) \ : \ r > 0, \ |\varphi| < \alpha\}$ be an angle in the plane $\mathbb{R}^2_{x_1,x_2}$ with opening 2α, where (r,φ) are polar coordinates centered at $\mathcal{O}$. Denote by $\mathbb{D} = \mathbb{K} \times \mathbb{R}$ the wedge with edge $M = \mathcal{O} \times \mathbb{R}$. We introduce function spaces. Let $s \in \mathbb{N}$, $\beta \in \mathbb{R}$, $r = (x_1^2 + x_2^2)^{1/2}$. Denote by $H^s_\beta(\mathbb{K})$ and $H^s_\beta(\mathbb{D})$ the completion of $\mathcal{C}^\infty_c(\overline{\mathbb{K}} \setminus \mathcal{O})$ and $\mathcal{C}^\infty_c(\overline{\mathbb{D}} \setminus M)$ with respect to the norms

$$\|u; H^s_\beta(\mathbb{K})\| = \Big(\sum_{k_1+k_2 \le s} \int_{\mathbb{K}} r^{2(\beta+k_1+k_2-s)} \big|D^{k_1}_{x_1} D^{k_2}_{x_2} u(x_1,x_2)\big|^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \Big)^{1/2},$$

$$\|v; H^s_\beta(\mathbb{D})\| = \Big(\sum_{|\alpha| \le s} \int_{\mathbb{D}} r^{2(\beta+|\alpha|-s)} \big|D^\alpha_x v(x)\big|^2 \, \mathrm{d}x \Big)^{1/2}.$$

The spaces $H^s_\beta(\mathbb{K},q)$, $H^s_\beta(\mathbb{D},q)$ for $q > 0$ are equipped with the norms

$$\|u; H^s_\beta(\mathbb{K},q)\| = \Big(\sum_{k=0}^s q^{2k} \|u; H^{s-k}_\beta(\mathbb{K})\|^2 \Big)^{1/2},$$

$$\|u; H^s_\beta(\mathbb{D},q)\| = \Big(\sum_{k=0}^s q^{2k} \|u; H^{s-k}_\beta(\mathbb{D})\|^2 \Big)^{1/2}.$$

By $H^{s-1/2}_\beta(\partial\mathbb{K},q)$ and $H^{s-1/2}_\beta(\partial\mathbb{D},q)$ we denote the spaces of traces of the functions belonging to $H^s_\beta(\mathbb{K},q)$ and $H^s_\beta(\mathbb{D},q)$ respectively.

We define functional spaces in the cylinder $\mathcal{T} = \mathbb{D} \times \mathbb{R}$. Let $H^s_\beta(\mathcal{T})$ stand for the completion of $\mathcal{C}^\infty_c((\overline{\mathbb{D}} \setminus M) \times \mathbb{R})$ in the norm

$$\|w; H^s_\beta(\mathcal{T})\| = \Big(\sum_{|\alpha|+k \le s} \int_{\mathbb{D}} \int_{\mathbb{R}} r^{2(\beta+|\alpha|+k-s)} \big|D^\alpha_x D^k_t w(x,t)\big|^2 \, \mathrm{d}x \, \mathrm{d}t \Big)^{1/2}.$$

The space $H^s_\beta(\mathcal{T},q)$ for $q > 0$ is endowed with the norm

$$\|u; H^s_\beta(\mathcal{T},q)\| = \Big(\sum_{k=0}^s q^{2k} \|u; H^{s-k}_\beta(\mathcal{T})\|^2 \Big)^{1/2}.$$

Finally, $V^s_\beta(\mathcal{T},\gamma)$ for $\gamma > 0$ denotes the space with norm

$$\|w; V^s_\beta(\mathcal{T},\gamma)\| = \|w^\gamma; H^s_\beta(\mathcal{T},\gamma)\|,$$

where $w^\gamma = e^{-\gamma t} w$.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with corner point $\mathcal{O}$. We assume that, in a neighborhood of $\mathcal{O}$, the domain Ω coincides with $\mathbb{K}$. Outside $\mathcal{O}$, the boundary of

Ω is smooth. Denote by Σ the waveguide $\Omega \times \mathbb{R}$ with edge $\mathcal{O} \times \mathbb{R}$, and by $\mathbf{T}$ the cylinder $\Sigma \times \mathbb{R}$. In Ω, Σ, and $\mathbf{T}$, we introduce function spaces similar those defined in $\mathbb{K}$, $\mathbb{D}$, and $\mathcal{T}$.

We also need the function space $E^l_\beta(\mathbb{K})$ which is the completion of $\mathcal{C}^\infty_c(\overline{\mathbb{K}} \setminus \mathcal{O})$ in the norm

$$\|u; E^l_\beta(\mathbb{K})\| = \Big(\sum_{k_1+k_2<l} \|r^\beta (1 + r^{k_1+k_2-l}) D^{k_1}_{x_1} D^{k_2}_{x_2} U; L_2(\mathbb{K})\|^2 \Big)^{1/2}.$$

By $E^{l-1/2}_\beta(\partial\mathbb{K})$ we denote the space of traces on $\partial\mathbb{K}$ of the functions belonging to $E^l_\beta(\mathbb{K})$. In fact, the norms $\|\cdot; E^l_\beta(\mathbb{K})\|$ and $\|\cdot; H^l_\beta(\mathbb{K}, 1)\|$ are equivalent. The notation $E^l_\beta(\mathbb{K})$ was used in [28, Chapter 8]. Here we introduced this notation for the convenience of references. The next assertion was proved in [28, Lemma 8.1.2]

Proposition 1.1. *The norms* $\|w; H^l_\beta(\mathbb{D})\|$ *and*

$$\Big(\int_{\mathbb{R}} |\xi|^{2(l-\beta)-2} \|W(\cdot, \xi); E^l_\beta(\mathbb{K})\|^2 \, \mathrm{d}\xi \Big)^{1/2}$$

are equivalent, where $W(\zeta, \xi) = (\mathcal{F}_{x_3 \to \xi} w)(|\xi|^{-1}\zeta, \xi)$ *and* $\zeta = (|\xi| x_1, |\xi| x_2)$.

In the following proposition we summarize some known properties of the augmented Maxwell system.

Proposition 1.2. 1) *The operator* $A(\partial)$ *is elliptic, but it is not strongly elliptic.*
2a) *In a domain* V *with smooth boundary* ∂V *the Green formula*

$$\begin{aligned} &\int_V \langle A(D)u, v\rangle_8 \, \mathrm{d}V + \int_{\partial V} \langle \Gamma u, -iT_0 v\rangle_5 \, \mathrm{d}S \\ &= \int_V \langle u, A(D)v\rangle_8 \, \mathrm{d}V + \int_{\partial V} \langle -iT_0 u, \Gamma v\rangle_5 \, \mathrm{d}S, \end{aligned} \tag{1.9}$$

holds, where $u, v \in \mathcal{C}^\infty(\overline{V}, \mathbb{C}^8)$. *By* $\langle\ ,\ \rangle_k$ *we denote the inner product in* $\mathbb{C}^k$. *By* Γu *and* $T_0 u$ *we denote* $(\vec{\nu} \times \vec{u},\ \langle \vec{v}, \vec{\nu}\rangle_3,\ h)^T$ *and* $(\vec{\nu} \times [\vec{v} \times \vec{\nu}],\ q,\ \langle \vec{u}, \vec{\nu}\rangle_3)^T$ *respectively, where* $u = (\vec{u},\ \vec{v},\ h,\ q)^T$.
2b) *The system* $A(\partial)u = f$ *with boundary conditions* $\Gamma u = g$ *is an elliptic boundary value problem self-adjoint with respect to the Green formula* (1.9).
3a) *In a domain* V *with smooth boundary* ∂V *the Green formula*

$$\begin{aligned} &\int_V \langle A(D)u, v\rangle_8 \, \mathrm{d}V + \int_{\partial V} \langle \Gamma_1 u, -iT_1 v\rangle_5 \, \mathrm{d}S \\ &= \int_V \langle u, A(D)v\rangle_8 \, \mathrm{d}V + \int_{\partial V} \langle -iT_2 u, \Gamma_2 v\rangle_5 \, \mathrm{d}S \end{aligned} \tag{1.10}$$

holds, where $u, v \in \mathcal{C}^\infty(\overline{V}, \mathbb{C}^8)$ *and* $\langle\ ,\ \rangle_k$ *stands for the inner product in* $\mathbb{C}^k$*; for* $u = (\vec{u},\ \vec{v},\ h,\ q)^T$ *we have* $\Gamma_1 u = (\vec{\nu} \times [\vec{v} \times \vec{\nu}] + \psi[\vec{\nu} \times \vec{u}],\ h,\ q)^T$, $T_1 u = ((1/\overline{\psi})\ \vec{\nu} \times [\vec{v} \times \vec{\nu}],\ \langle\vec{u}, \vec{\nu}\rangle,\ \langle\vec{v}, \vec{\nu}\rangle)^T$, $\Gamma_2 u = (-\vec{\nu} \times [\vec{v} \times \vec{\nu}] + \overline{\psi}[\vec{\nu} \times \vec{u}],\ h,\ q)^T$, *and* $T_2 u = ((1/\psi)\ \vec{\nu} \times [\vec{v} \times \vec{\nu}],\ \langle\vec{u}, \vec{\nu}\rangle,\ \langle\vec{v}, \vec{\nu}\rangle)^T$.
3b) *The boundary value problems* $\{A(\partial), \Gamma_1\}$ *and* $\{A(\partial), \Gamma_2\}$ *are elliptic and adjoint with respect to the Green formula (1.10).*

The proof is a straightforward check of the definitions of elliptic operators and elliptic boundary value problems. The Green formulas are obtained from the integration by parts and the Gauss divergency theorem.

2 THE PROBLEM IN A CONE AND IN A BOUNDED DOMAIN WITH CONICAL POINT

2.1 Preliminaries

In this chapter we study the augmented Maxwell system

$$\begin{cases} \partial\vec{E}/\partial t - \text{rot }\vec{B} + \nabla h = -\vec{J}, \\ \partial\vec{B}/\partial t + \text{rot }\vec{E} + \nabla q = -\vec{G}, \\ \quad \partial h/\partial t + \text{div }\vec{E} = \rho, \\ \quad \partial q/\partial t + \text{div }\vec{B} = \mu \end{cases} \tag{2.1}$$

in a model cone and in a bounded domain with conical point. The system (2.1) is endowed with boundary conditions corresponding to ideal conductive boundary:

$$\vec{\nu} \times \vec{E} = 0,\ \langle \vec{B}, \vec{\nu}\rangle_3 = 0,\ h = 0, \tag{2.2}$$

where $\vec{\nu}$ is the unit outward normal. The system (2.1) will be written in the form

$$\partial\mathcal{U}/\partial t + A(\partial)\mathcal{U} = \mathcal{F},$$

where $\mathcal{U} = (\vec{E},\ \vec{B},\ h,\ q)^T,\ \partial = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$.

Let $\tau = \sigma - i\gamma,\ \sigma \in \mathbb{R},\ \gamma > 0$. Applying the Fourier transform $\mathcal{F}_{t\to\tau}$ to the problem (2.1), (2.2), we obtain the problem with parameter τ in the cone $\mathcal{K}$ (in the domain G)

$$\begin{cases} \tau\widehat{u} + A(D_x)\widehat{u} = -i\widehat{f}, \\ \qquad \Gamma\widehat{u} = 0. \end{cases}$$

Rewrite this problem as follows

$$M(D_x, \tau)u = f, \tag{2.3}$$

$$\Gamma u = 0, \tag{2.4}$$

where $M(D_x, \tau) = \tau + A(D_x)$. The Green formula (1.9) for the problem (2.3), (2.4) takes the form

$$\Big(M(D_x,\tau)u, v\Big)_{\mathcal{K}} + \Big(\Gamma u, Tv\Big)_{\partial\mathcal{K}} = \Big(u, M(D_x,\overline{\tau})v\Big)_{\mathcal{K}} + \Big(Tu, \Gamma v\Big)_{\partial\mathcal{K}}, \tag{2.5}$$

where $T = -iT_0$.

When considering the problem in $\mathcal{K}$, one can change the variables $\eta = (|\tau|x_1, |\tau|x_2, |\tau|x_3)$. Denote $\tau/|\tau|$ by θ , put $M(D_\eta, \theta) = \theta + A(D_\eta)$, $U(\eta,\tau) = \widehat{\mathcal{U}}(|\tau|^{-1}\eta, \tau)$, $F(\eta,\tau) = |\tau|^{-1}\widehat{\mathcal{F}}(|\tau|^{-1}\eta, \tau)$, and rewrite the problem (2.3), (2.4) in the form

$$M(D_\eta, \theta)U = F, \tag{2.6}$$

$$\Gamma U = 0. \tag{2.7}$$

2.2 Operator pencil

We introduce the operator pencil

$$\mathfrak{A}(\lambda)\Phi(\varphi,\vartheta) = r^{1-i\lambda}A(D_{x_1}, D_{x_2}, D_{x_3})r^{i\lambda}\Phi(\varphi,\vartheta) \tag{2.8}$$

for functions $\Phi \in H^1(\Xi)$ such that $r^{i\lambda}\Phi(\varphi,\vartheta)$ satisfy (2.2) on $\partial\mathcal{K}$. Here (r,ϑ,φ) are spherical coordinate centered at $\mathcal{O}$ and $\Xi = \mathcal{K} \cap S^2$. We will write the boundary conditions for Φ in an explicit form. Assume $(\vec{e}_r,\ \vec{e}_\vartheta,\ \vec{e}_\varphi)$ to be the basis vectors in the spherical coordinate system. Let $\vec{\sigma}$ be a tangent vector to $\partial\Xi$ and let $\vec{\Phi} = (\vec{U},\ \vec{V},\ H,\ Q)$. Then the boundary condition on $\partial\Xi$ take the form

$$\langle\vec{U},\ \vec{e}_r\rangle_3 = 0,\ \langle\vec{U},\ \vec{\sigma}\rangle_3 = 0,\ \langle\vec{V},\ \vec{\nu}\rangle_3 = 0,\ H = 0,$$

where $\vec{\nu}$ stands for the unit outward normal to the boundary of $\mathcal{K}$. Since $\mathfrak{A}$ is an elliptic pencil, its spectrum consists of normal eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$.

Proposition 2.1. *All the eigenvalues of $\mathfrak{A}$ belong to the imaginary axis. To every eigenvalue λ_j there corresponds a finite collection of linearly independent eigenvectors $\Phi_{s,j}$, $s = 1, .., N_j$. There are no associated vectors.*

Proof. The operator $A(D)$ in the spherical coordinates take the form

$$A(D) = A_1(\varphi,\vartheta)D_r + (1/r)A_2(\varphi,\vartheta)D_\vartheta + (1/r)A_3(\varphi,\vartheta)D_\varphi, \tag{2.9}$$

where $A_i(\varphi,\vartheta)$ are 8×8-matrices that can easily be calculated. The matrices satisfy

$$\begin{gathered} A_1\cdot A_1 = I,\ A_2\cdot A_2 = I,\ \sin^2\!\vartheta\cdot A_3\cdot A_3 = I,\\ A_1\cdot A_2 + A_2\cdot A_1 = 0,\ \sin\vartheta\cdot(A_1\cdot A_3 + A_3\cdot A_1) = 0,\\ \partial A_1/\partial\vartheta = A_2,\ \partial A_2/\partial\vartheta = -A_1,\ \partial A_1/\partial\varphi = \sin^2\vartheta\cdot A_3. \end{gathered} \tag{2.10}$$

From (2.9) it follows that

$$\mathfrak{A}(\lambda) = \lambda A_1 + A_2 D_\vartheta + A_3 D_\varphi. \tag{2.11}$$

Let λ_m be an eigenvalue of the operator pencil and Φ_m an eigenvector, i.e., $\mathfrak{A}(\lambda_m)\Phi_m = 0$. Then $B\Phi_m = \lambda_m\Phi_m$, where B is defined by $B = -A_1 \cdot A_2 D_\vartheta - A_1 \cdot A_3 D_\varphi$ for functions in the domain of $\mathfrak{A}$. Using the Green formula (2.5), it is not hard to verify that the operator iB is symmetric on $L_2(\Xi)$. Since the problem $\{A(D),\ \Gamma\}$ is elliptic (see Proposition 1.2), the operator iB with such a domain is self-adjoint. Its domain is compactly embedded into $L_2(\Xi)$, so the spectrum of B is discrete. Obviously, the spectrum of B coincides with that of the pencil $\mathfrak{A}$. Therefore, the eigenvalues of the pencil belong to the imaginary axis. We show there are no associated vectors. Let λ_m, Φ_m be an eigenvalue and an eigenvector of $\mathfrak{A}$ and let $\widetilde{\Phi}$ an associated vector. In other words, $\mathfrak{A}(\lambda_m)\widetilde{\Phi} = \partial_\lambda \mathfrak{A}(\lambda_m)\Phi_m$ or, equivalently, $(B - \lambda_m)\widetilde{\Phi} = \Phi_m$. However, Φ_m is not orthogonal to the subspace $\ker(B - \lambda_m)$. Therefore, $\widetilde{\Phi}$ does not exists. □

Consider the pencil $\mathfrak{A}^*(\lambda)$, defined by $\mathfrak{A}^*(\lambda) = (\mathfrak{A}(\overline{\lambda}))^*$. It is known that if μ is an eigenvalue of $\mathfrak{A}$, then $\overline{\mu}$ is an eigenvalue of $\mathfrak{A}^*$ while their multiplicities coincide. Eigenfunctions $\{\Phi_s\}_{s=1,..,N}$ and $\{\Psi_s\}_{s=1,..,N}$ can be chosen to satisfy the orthogonality and normalization conditions (see [28], Chapter 1, §2):

$$\int\limits_{\Xi} \langle \partial_\mu \mathfrak{A}(\mu)\Phi_s, \Psi_p\rangle_8 \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi = \delta_{s,p}. \tag{2.12}$$

In the next proposition we obtain the formula for $\mathfrak{A}^*(\lambda)$.

Proposition 2.2.

$$\mathfrak{A}^*(\lambda) = \mathfrak{A}(\lambda + 2i).$$

Proof. We verify this statement applying the definition of $\mathfrak{A}$ and the Green formula (1.9). Consider the inner product

$$\Big(\mathfrak{A}(\lambda)\Phi, \Psi\Big)_\Xi = \int\limits_\Xi \langle \mathfrak{A}(\lambda)\Phi(\omega), \Psi(\omega)\rangle_8 \, \mathrm{d}\omega, \tag{2.13}$$

where $\Phi, \Psi \in H^1(\Xi)$ such that $\Gamma(r^{i\mu}\Phi) = 0$ and $\Gamma(r^{i\mu}\Psi) = 0$ on $\partial\mathcal{K}$. Multiply both sides of (2.13) by $1/r$ and integrate with respect to r from $r = 1$ to $r = 2$. Then we get

$$\int\limits_1^2 \mathrm{d}r \frac{1}{r}\Big(\mathfrak{A}(\lambda)\Phi, \Psi\Big)_\Xi = \int\limits_1^2 \mathrm{d}r \frac{r^2}{r^2 \cdot r} \int\limits_\Xi \langle r^{1-i\lambda} A(D) r^{i\lambda}\Phi(\omega), \Psi(\omega)\rangle_8 \, \mathrm{d}\omega =$$

$$= \int\limits_{\mathcal{K}_{12}} \langle r^{-2-i\lambda} A(D) r^{i\lambda}\Phi(\omega), \Psi(\omega)\rangle_8 \, \mathrm{d}x = \Big(A(D) r^{i\lambda}\Phi, r^{i\overline{\lambda}-2}\Psi\Big)_{\mathcal{K}_{12}},$$

where $\mathcal{K}_{12} = \{x \in \mathcal{K} \ : \ 1 < r < 2\}$. Applying the Green formula (1.9), we arrive at

$$\begin{aligned} \Big(A(D) r^{i\lambda}\Phi, r^{i\overline{\lambda}-2}\Psi\Big)_{\mathcal{K}_{12}} &= \Big(r^{i\lambda}\Phi, A(D) r^{i\overline{\lambda}-2}\Psi\Big)_{\mathcal{K}_{12}} + \\ \Big(\Gamma r^{i\lambda}\Phi, T r^{i\overline{\lambda}-2}\Psi\Big)_{\partial\mathcal{K}_{12}} &+ \Big(T r^{i\lambda}\Phi, \Gamma r^{i\overline{\lambda}-2}\Psi\Big)_{\partial\mathcal{K}_{12}}. \end{aligned} \tag{2.14}$$

The integrals over $\{x \in \partial\mathcal{K} \,:\, 1 < |x| < 2\}$ are equal to zero due to the boundary conditions $\Gamma(r^{i\mu}\Phi) = 0$ and $\Gamma(r^{i\mu}\Psi) = 0$. Let us verify that the integrals over $\mathcal{K} \cap \{r = 1\}$ and $\mathcal{K} \cap \{r = 2\}$ are also equal to zero. We have

$$\Big(\Gamma r^{i\lambda}\Phi, T r^{i\overline{\lambda}-2}\Psi\Big)_{\mathcal{K}\cap\{r=1\}} + \Big(\Gamma r^{i\lambda}\Phi, T r^{i\overline{\lambda}-2}\Psi\Big)_{\mathcal{K}\cap\{r=2\}} =$$

$$= \int\limits_{\mathcal{K}\cap\{r=1\}} \mathrm{d}\omega\, r^2 \langle \Gamma(-\vec{\nu}) r^{i\lambda}\Phi(\omega), T(-\vec{\nu}) r^{i\overline{\lambda}-2}\Psi(\omega)\rangle_8 +$$

$$+ \int\limits_{\mathcal{K}\cap\{r=2\}} \mathrm{d}\omega\, r^2 \langle \Gamma(\vec{\nu}) r^{i\lambda}\Phi(\omega), T(\vec{\nu}) r^{i\overline{\lambda}-2}\Psi(\omega)\rangle_8 =$$

$$= \int\limits_{\Xi} \mathrm{d}\omega\, \langle \Gamma(-\vec{\nu})\Phi(\omega), T(-\vec{\nu})\Psi(\omega)\rangle_8 + \int\limits_{\Xi} \mathrm{d}\omega\, \langle \Gamma(\vec{\nu})\Phi(\omega), T(\vec{\nu})\Psi(\omega)\rangle_8,$$

where $\vec{\nu}$ is the outward unit normal to $\mathbb{S}^2$. Recall that $\Gamma(\vec{\nu})u = (\vec{\nu} \times \vec{u},\ \langle \vec{v} \cdot \vec{\nu}\rangle_3,\ h)$ and $T(\vec{\nu})u = -i(\vec{\nu} \times [\vec{v} \times \vec{\nu}],\ q,\ \langle \vec{u} \cdot \vec{\nu}\rangle_3)$, where $u = (\vec{u},\ \vec{v},\ h,\ q)$. From these formulas it follows that $\langle \Gamma(-\vec{\nu})\Phi, T(-\vec{\nu})\Psi\rangle_8 + \langle \Gamma(\vec{\nu})\Phi, T(\vec{\nu})\Psi\rangle_8 = 0$. In the same way we verify that the last integral on the right-hand side of (2.14) equals to zero. Therefore we get

$$\Big(\mathfrak{A}(\lambda)\Phi, \Psi\Big)_{\Xi} = \frac{1}{\ln 2}\Big(A(D) r^{i\lambda}\Phi, r^{i\overline{\lambda}-2}\Psi\Big)_{\mathcal{K}_{12}} = \frac{1}{\ln 2}\Big(r^{i\lambda}\Phi, A(D) r^{i\overline{\lambda}-2}\Psi\Big)_{\mathcal{K}_{12}} =$$

$$= \frac{1}{\ln 2}\int\limits_1^2 \mathrm{d}r \frac{r^2}{r^3} \int\limits_{\Xi} \mathrm{d}\omega\, \langle r^{3+i\lambda}\Phi(\omega), A(D) r^{i\overline{\lambda}-2}\Psi(\omega)\rangle_8 =$$

$$= \int\limits_{\Xi} \mathrm{d}\omega\, \langle \Phi, r^{3-i\overline{\lambda}} A(D) r^{i\overline{\lambda}-2}\Psi\rangle_8 = \Big(\Phi, \mathfrak{A}(\overline{\lambda} + 2i)\Psi\Big)_{\Xi}.$$

Then we have $\mathfrak{A}(\lambda)^* = \mathfrak{A}(\overline{\lambda} + 2i)$. To complete the proof it remains to recall the definition of $\mathfrak{A}^*(\lambda)$. □

Applying this proposition, we reformulate the above results as follows. If μ is an eigenvalue of $\mathfrak{A}$, then $\overline{\mu} + 2i$ is an eigenvalue of $\mathfrak{A}$ as well, their multiplicities coincide and the eigenfunctions can be chosen to satisfy the orthogonality and normalization conditions (2.12). The numbers μ and $\overline{\mu} + 2i$ are symmetric about the point i. In the following proposition we check that if Φ is an eigenvector of the pencil $\mathfrak{A}$ corresponding to an eigenvalue λ, then $A_1\Phi$ is an eigenvector of $\mathfrak{A}$ as well and corresponds to the eigenvalue $\overline{\lambda} + 2i$.

Proposition 2.3. *Suppose that* $\mathfrak{A}(\lambda)\Phi = 0$. *Then* $\mathfrak{A}(\overline{\lambda} + 2i)A_1\Phi = 0$.

Proof. Applying the formulas (2.10), we obtain

$$\mathfrak{A}(\overline{\lambda}+2i)A_1\Phi = (\overline{\lambda}+2i)A_1\cdot A_1\Phi + A_2 D_\vartheta(A_1\Phi) + A_3 D_\varphi(A_1\Phi) =$$

$$= A_1\overline{\lambda}A_1\Phi + 2i\Phi + A_2\cdot A_1 D_\vartheta\Phi - iA_2\cdot A_2\Phi + A_3\cdot A_1 D_\varphi\Phi - i\sin^2\vartheta A_3\cdot A_3\Phi =$$

$$= A_1\cdot(\overline{\lambda}A_1\Phi - A_2 D_\vartheta\Phi - A_3 D_\varphi\Phi) = A_1\cdot\overline{(\lambda A_1\Phi + A_2 D_\vartheta\Phi + A_3 D_\varphi\Phi)} = 0.$$

□

The pencil is of a block structure since the operator $A(D)$ is of a block structure. One block acts on the components $(\vec{U},\ Q)$ and the other one on the components $(\vec{V},\ H)$ of the function $\Phi = (\vec{U},\ \vec{V},\ H,\ Q)$. The matrix A_1 acts on the components in the following way:

$$A_1\,(\vec{U},\ \vec{0},\ 0,\ Q) = (\vec{0},\ \vec{e_r}\times\vec{U}+Q\vec{e}_r,\ \langle\vec{e}_r,\ \vec{U}\rangle,\ 0)^T,$$
$$A_1\,(\vec{0},\ \vec{V},\ H,\ 0) = (-\vec{e}_r\times\vec{V}+H\vec{e}_r,\ \vec{0},\ 0,\ \langle\vec{e}_r,\ \vec{V}\rangle)^T.$$

Recall that the eigenvectors of the pencil are also eigenvectors of a self-adjoint operator (see Proposition 2.1). Therefore eigenvectors corresponding to distinct eigenvalues are orthogonal. Taking into account this fact, the block structure of $\mathfrak{A}$ and the equality $\partial_\lambda\mathfrak{A}(\lambda) = A_1$, we can choose all the eigenvectors of the pencil in the form $(\vec{U},\ \vec{0},\ 0,\ Q)$ or $(\vec{0},\ \vec{V},\ H,\ 0)$ satisfying the orthogonality and normalization conditions (2.12). In the following three lemmas, we describe in more detail the properties of eigenvalues and eigenvectors of the pencil.

Lemma 2.4. *If the set $\{i\alpha\ :\ \alpha\in\,]0,\ 1[\}$ contains some eigenvalues of $\mathfrak{A}$, then the components H, Q of the corresponding eigenvectors vanish.*

Proof. Let $\Phi = (\vec{U},\ \vec{0},\ 0,\ Q)$ and let (Φ,λ) be an eigenvector and an eigenvalue of the pencil $\mathfrak{A}$. Then $A(D_{x_1}, D_{x_2}, D_{x_3})r^{i\lambda}\Phi = 0$. We again apply the operator $A(D_{x_1}, D_{x_2}, D_{x_3})$ and obtain $\triangle r^{i\lambda}\Phi = 0$. Let us find out the boundary conditions on Q. Since $r^{i\lambda}\vec{U}\times\vec{\nu} = 0$, we have $\langle\mathrm{rot}(r^{i\lambda}\vec{U}),\ \vec{\nu}\rangle = 0$. From the relation $\mathrm{rot}(r^{i\lambda}\vec{U})+\nabla(r^{i\lambda}Q) = 0$ it follows that $\partial(r^{i\lambda}Q)/\partial n = 0$ on $\partial\mathcal{K}$. This means that (λ, Q) is an eigenvalue and an eigenvector of the pencil of the Neumann problem for the Laplace operator. Analogously, if $\Phi = (\vec{0},\ \vec{V},\ H,\ 0)$, then (λ, H) is an eigenvalue and an eigenvector of the pencil of the Dirichlet problem for the Laplace operator. However, for the cone $\mathcal{K}\subset\mathbb{R}^3$, the strip $\{\lambda\in\mathbb{C}\ :\ \mathrm{Im}\,\lambda\in\,]0,\ 1[\}$ contains no eigenvalues of the pencils (see [18, §1], [33, §3]). □

Lemma 2.5. *Let λ be an eigenvalue of the pencil $\mathfrak{A}$, $\mathrm{Im}\,\lambda\in\,]1,\ 2[$, and let Φ be an eigenvector corresponding to λ.*
If $\Phi = (\vec{U},\ \vec{0},\ 0,\ Q)$, then $Q\not\equiv 0$, and if $\Phi = (\vec{0},\ \vec{V},\ H,\ 0)$, then $H\not\equiv 0$.

Proof. We consider one of these cases, the other can be verified in a similar way. Assume that $\Phi = (\vec{U},\ \vec{0},\ 0,\ Q)$. Then $\Psi = A_1\Phi$ is an eigenvector of $\mathfrak{A}$ corresponding to the eigenvalue $\mu = \overline{\lambda}+2i$. Moreover, $\mathrm{Im}\,\mu\in\,]0,\ 1[$ and

$\Psi = (\vec{0},\ \vec{e}_r \times \vec{U} + Q\vec{e}_r,\ \langle \vec{e}_r,\ \vec{U}\rangle,\ 0)$. According to Lemma 2.4, $\langle \vec{e}_r,\ \vec{U}\rangle = 0$. We suppose that $Q = 0$ and rewrite the pencil $\mathrm{rot}(r^{i\lambda}\vec{U}) + \nabla(r^{i\lambda}Q) = 0,\ \mathrm{div}(r^{i\lambda}\vec{U}) = 0$ in the spherical coordinates

$$\begin{cases} \dfrac{\cos\vartheta}{\sin\vartheta}U_\varphi + \partial_\vartheta U_\varphi - \dfrac{1}{\sin\vartheta}\partial_\varphi U_\vartheta + i\lambda Q = 0, \\[2ex] -(i\lambda+1)U_\varphi + \dfrac{1}{\sin\vartheta}\partial_\varphi U_r + \partial_\vartheta Q = 0, \\[2ex] (i\lambda+1)U_\vartheta + \dfrac{1}{\sin\vartheta}\partial_\varphi Q - \partial_\vartheta U_r = 0, \\[2ex] (2+i\lambda)U_r + \dfrac{\cos\vartheta}{\sin\vartheta}U_\vartheta + \partial_\vartheta U_\vartheta + \dfrac{1}{\sin\vartheta}\partial_\varphi U_\varphi = 0, \end{cases}$$

where $\vec{U} = U_r\vec{e}_r + U_\vartheta\vec{e}_\vartheta + U_\varphi\vec{e}_\varphi$. Taking $U_r = Q = 0$ and $\lambda \neq i$ into account, we obtain $U_\varphi = U_\vartheta = 0$. Hence, $\Phi = 0$. This contradiction completes the proof. $\square$

LEMMA 2.6. *The number $\lambda = i$ is regular for the pencil $\mathfrak{A}$.*

Proof. Suppose that $\lambda = i$ is an eigenvalue of $\mathfrak{A}$. Let $\Phi = (\vec{U},\ \vec{0},\ 0,\ Q)$ be an eigenvector corresponding to the eigenvalue. Then $\Psi = A_1\Phi = (\vec{0},\ -\vec{U}\times\vec{e}_r + Q\vec{e}_r,\ \langle\vec{U},\ \vec{e}_r\rangle, 0)$, too, is an eigenvector for $\lambda = i$. Arguing as in the proof of Lemma 2.4, one can verify that $Q = \mathrm{const}$ and $H = \langle\vec{e}_r,\ \vec{U}\rangle = 0$. Indeed, $\lambda = i$ is a regular number for the operator pencil of the Dirichlet problem for the Laplace operator. In the case of Neumann problem, a constant is an eigenvector for $\lambda = i$. Then $\vec{V} = -\vec{U}\times\vec{e}_r + Q\vec{e}_r$ satisfies $\mathrm{rot}(r^{-1}\vec{V}) = 0,\ \mathrm{div}(r^{-1}\vec{V}) = 0$. Since the cone $\mathcal{K}$ is 1-connected, this implies that $r^{-1}\vec{V} = \nabla Z$, where $\triangle Z = 0,\ \partial Z/\partial n = 0$. We rewrite $r^{-1}\vec{V} = \nabla Z$ in the spherical coordinates $r^{-1}\vec{V} = (\partial Z/\partial r)\vec{e}_r + r^{-1}(\partial Z/\partial\vartheta)\vec{e}_\vartheta + (r\sin\vartheta)^{-1}(\partial Z/\partial\varphi)\vec{e}_\varphi$ and obtain $Z = Q\log r + A(\varphi,\vartheta)$. The function Z satisfies the homogeneous Neumann problem for the Laplace operator in $\mathcal{K}\subset\mathbb{R}^3$. Using results on the asymptotics of solutions to elliptic problems near singularities of the boundary [28, Chapters 3,4], we arrive at $Q = 0,\ A(\varphi,\vartheta) = \mathrm{const}$. It follows that $\vec{V} = 0$ and $\vec{U} = 0$. $\square$

Let λ_k with $k > 0$ stand for the eigenvalues of $\mathfrak{A}$ whose imaginary part is greater than 1, $\mathrm{Im}\,\lambda_k \leq \mathrm{Im}\,\lambda_{k+1}$. Denote by λ_{-k} with $k > 0$ the eigenvalues of the pencil symmetric to λ_k about the point $\lambda = i$. Let $\{\Phi_{s,k}\}_{s=1,..,N_k}$ be a basis in the eigenspace corresponding to λ_k. We list some of properties of the pencil $\mathfrak{A}$ in the following assertion.

PROPOSITION 2.7. *Eigenvectors $\{\Phi_{s,k}\}_{s=1,..,N_k}$ corresponding to the eigenvalues λ_k of the pencil $\mathfrak{A}$ can be taken in the form $(\vec{U},\ \vec{0},\ 0,\ Q)$ or $(\vec{0},\ \vec{V},\ H,\ 0)$. Moreover,*

$$\int\limits_{\Xi} \langle\partial_\lambda\mathfrak{A}(\lambda_k)\Phi_{s,k},\Phi_{m,p}\rangle_8 \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi = \delta_{k,-p}\cdot\delta_{s,m}, \tag{2.15}$$

where $k,p = \mp1,\mp2...$, and $\delta_{k,p}$ is the Kronecker symbol.

Introduce the notation

$$u_{s,k} = r^{i\lambda_k}\Phi_{s,k}. \tag{2.16}$$

The functions $u_{s,k}$ satisfy $A(D)u_{s,k} = 0$ in $\mathcal{K}$ and $\Gamma u_{s,k} = 0$ on $\partial\mathcal{K}$.

2.3 A global energy estimate

PROPOSITION 2.8. *Assume that* $v = (\vec{u},\ \vec{v},\ h,\ q)^T$ *is in* $\mathcal{C}_c^\infty(\overline{G}\setminus\mathcal{O}, \mathbb{C}^8)$ *and satisfies the boundary conditions* (2.2)*. Then*

$$\gamma\|v; L_2(G)\| \le \|M(D_x,\tau)v; L_2(G)\|, \tag{2.17}$$

where $M(D_x,\tau)v = \tau v + A(D_{x_1}, D_{x_2}, D_{x_3})v$ *and* $\tau = \sigma - i\gamma$ *with* $\sigma \in \mathbb{R},\ \gamma > 0$.

Proof. Let $u(x,t) = \psi(t)v(x)$, where $\psi,\ e^{-\gamma t}\psi \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ stands for the Schwartz space. Taking the Green formula (2.5) and the boundary conditions into account, we obtain

$$\mathrm{Re}\int\limits_G \langle A(\partial)u(x,t), u(x,t)\rangle_8 \,\mathrm{d}x = 0\ .$$

Then

$$\frac{d}{dt}\|u; L_2(G)\|^2 = 2\mathrm{Re}\int\limits_G \langle u, u_t + A(\partial)u\rangle\,\mathrm{d}x.$$

Therefore,

$$\frac{d}{dt}\|u(\cdot,\ t); L_2(G)\|^2 \le \|u; L_2(G)\|\cdot\|M(D_x, D_t)u; L_2(G)\|$$

and consequently

$$\|u(\cdot,\ t); L_2(G)\|^2 \le 2\int\limits_{-\infty}^{t} \|u(\cdot,\ s); L_2(G)\|\cdot\|M(D_x, D_s)u(\cdot,\ s); L_2(G)\|\,\mathrm{d}s.$$

Multiply by $e^{-2\gamma t}$ and integrate from $-\infty$ to $+\infty$. Changing the order of integration in the right-hand side, we obtain

$$\int\limits_{-\infty}^{+\infty} e^{-2\gamma t}\|u(\cdot,\ t); L_2(G)\|^2\,\mathrm{d}t \le \gamma^{-1}\int\limits_{-\infty}^{+\infty} e^{-2\gamma t}\|u(\cdot,\ t); L_2(G)\|\times$$
$$\times\|M(D_x, D_t)u(\cdot,\ t); L_2(G)\|\,\mathrm{d}t.$$

Let us apply the Cauchy inequality to the right-hand side

$$\gamma^2\int\limits_{-\infty}^{+\infty} e^{-2\gamma t}\|u(\cdot,\ t); L_2(G)\|^2\,\mathrm{d}t \le \int\limits_{-\infty}^{+\infty} e^{-2\gamma t}\|M(D_x, D_t)u(\cdot,\ t); L_2(G)\|^2\,\mathrm{d}t.$$

By Parseval's equality,

$$\gamma^2 \int_G \mathrm{d}x \int_{\mathbb{R}-i\gamma} \mathrm{d}\tau \, |\hat{\psi}(\tau)|^2 |v(x)|^2 \leq$$
$$\leq \int_G \mathrm{d}x \int_{\mathbb{R}-i\gamma} \mathrm{d}\tau \, |\hat{\psi}(\tau)|^2 |M(D_x,\tau)v(x)|^2.$$

Since ψ is arbitrary, this leads to the wanted estimate (2.17). □

REMARK 2.9. *The same estimate is valid for the functions in $\mathcal{C}_c^\infty(\overline{\mathcal{K}} \setminus \mathcal{O})$ subject to the boundary conditions (2.2).*

REMARK 2.10. *The inequality (2.17) remains true with τ changed for $\overline{\tau}$.*

In what follows we do not indicate that v is a function in $\mathcal{C}_c^\infty(\overline{G} \setminus \mathcal{O}, \mathbb{C}^8)$, $v = (\vec{u},\ \vec{v},\ h,\ q)^T$, and v satisfies the boundary conditions (2.2). As a rule, we only mention that a function belongs to $\mathcal{C}_c^\infty(\overline{G} \setminus \mathcal{O})$ (or another function class) and is subject to (2.2).

Our next goal is to expand (2.17) to the lineal consisting of all linear combinations of the form $u = v + \chi \sum_{k,s} u_{s,k}$. Here v is in $\mathcal{C}_c^\infty(\overline{G} \setminus \mathcal{O})$ and satisfies (2.2). By χ we denote a cut-off function that is equal to 1 near the point $\mathcal{O}$ and vanishes outside a neighborhood where G coincides with $\mathcal{K}$. The functions $u_{s,k}$ are defined by (2.16). Since the inclusion $\chi u_{s,k} \in L_2(G)$ is required, the lineal contains only $\chi u_{s,k}$ with $\operatorname{Im}\lambda_k < 3/2$. The right-hand side $M(D_x,\tau)\chi u_{s,k}$ belongs to $L_2(G)$ because $A(D_x)u_{s,k} = 0$ (see Section 1.2). Moreover, when proving (2.17) on $D(G)$, we suppose that

$$\operatorname{Re} \int_G \langle A(\partial)u,\ u\rangle_8 \,\mathrm{d}x = 0.$$

If there is an eigenvalue λ_k of the pencil $\mathfrak{A}$ such that $\operatorname{Im}\lambda_k \in [1,\ 3/2[$, then by (2.15) we have

$$\operatorname{Re} \int_G \langle A(\partial)w, w\rangle_8 \,\mathrm{d}x = 2\operatorname{Re}(\alpha\overline{\beta})$$

for $w = \chi(\alpha u_{s,k} + \beta u_{s,-k})$. Therefore, the lineal contains combinations of the form $\chi(\alpha_{s,k}u_{s,k} + \beta_{s,k}u_{s,-k})$ for all λ_k in the strip $\operatorname{Im}\lambda \in]1/2,\ 3/2[$, where $\alpha_{s,k},\ \beta_{s,k}$ are some fixed coefficients satisfying $\operatorname{Re}\alpha_{s,k}\overline{\beta_{s,k}} = 0$, $|\alpha_{s,k}| + |\beta_{s,k}| > 0$. The latter condition will be explained a little bit later. It is easy to see that the proof of (2.17) is still valid for the functions in the modified lineal. Let us summarize the obtained results.

DEFINITION 2.11. *Denote by $D(G)$ the lineal spanned by the functions in $\mathcal{C}_c^\infty(\overline{G}\setminus\mathcal{O})$ satisfying the boundary conditions (2.2) on ∂G; by functions $\chi u_{s,k}$ for $\operatorname{Im}\lambda_k \leq 1/2$; by functions $\chi(\alpha_{s,k}u_{s,k} + \beta_{s,k}u_{s,-k})$ for $\operatorname{Im}\lambda_k \in]1,\ 3/2[$, where $\alpha_{s,k},\ \beta_{s,k}$ are some fixed coefficients such that $\operatorname{Re}\alpha_{s,k}\overline{\beta_{s,k}} = 0$, $|\alpha_{s,k}| + |\beta_{s,k}| > 0$. The lineal $D(\mathcal{K})$ is defined in the same way with G replaced by $\mathcal{K}$. Note that the lineals $D(G)$, $D(\mathcal{K})$ depend on the special choice of the pairs $\{\alpha_{s,k},\ \beta_{s,k}\}$.*

PROPOSITION 2.12. *The estimate (2.17) holds for any function in $D(G)$.*

In what follows we consider only the problem in G. However, all results remain valid for the problem in $\mathcal{K}$ as well. At the end of the section we give the corresponding statements.

We associate with problem (2.3), (2.4) the unbounded operator $v \mapsto M(\tau)v := M(D_x, \tau)v$ on $L_2(G)$. As the domain $\mathcal{D}M(\tau)$, we take $D(G)$ (see Definition 2.11). The operator $M(\tau)$ admits closure. Indeed, let $\{v_k\} \subset \mathcal{D}M(\tau)$, $v_k \to 0$, and $M(\tau)v_k \to f$ in $L_2(G)$. Then $(M(\tau)v_k, w)_G = (v_k, M(\overline{\tau})w)_G$ for any w in $\mathcal{C}_c^\infty(G)$. Letting $k \to +\infty$, we obtain $(f, w)_G = 0$, hence $f = 0$. We further consider the closed operator only, keeping the notation $M(\tau)$ and $\mathcal{D}M(\tau)$ for the operator and its domain. Clearly, the estimate

$$\gamma\|v; L_2(G)\| \leq \|M(\tau)v; L_2(G)\| \tag{2.18}$$

holds for any $v \in \mathcal{D}M(\tau)$. The next assertion follows from (2.18).

Proposition 2.13. *Ker* $M(\tau) = 0$ *and the range* $\mathcal{R}M(\tau)$ *of* $M(\tau)$ *is closed in* $L_2(G)$.

Proposition 2.14. $\mathcal{R}M(\tau) = L_2(G)$.

Proof. It suffices to verify that Ker $M(\tau)^* = \{0\}$. Suppose that $w \in$ Ker $M(\tau)^*$. Then in view of local properties of solutions to elliptic problems (see [28, Chapter 1]) we have $w \in \mathcal{C}^\infty(\overline{G} \setminus \mathcal{O})$ while w satisfies the homogeneous problem adjoint with respect to the Green formula (2.5) :

$$M(D_x, \overline{\tau})w = 0, \; x \in G, \tag{2.19}$$

$$\Gamma w = 0, \; x \in \partial G \setminus \mathcal{O}. \tag{2.20}$$

In a neighborhood of $\mathcal{O}$, the function w admits the asymptotic representation

$$w \sim \chi \sum_k \sum_{s=1,..,N_k} c_{s,k} V_{s,k,T}.$$

Here χ stands for a cut-off function, equal to 1 near $\mathcal{O}$ and $V_{s,k,T}$ is the sum of first T terms of the formal series

$$V_{s,k}(x, \overline{\tau}) = r^{i\lambda_{-k}} \sum_{q=0}^{\infty} r^q \overline{\tau}^q \Psi_q^{(s,-k)}(\vartheta, \varphi), \tag{2.21}$$

(where $\Psi_0^{(s,-k)} = \Phi_{s,-k}$) satisfying (2.19) and (2.20); for more detail, we refer, e.g., to [15, §4.2] or [28, Chapters 3,4]. Since $w \in \mathcal{D}M(\tau)^* \subset L_2(G)$, the asymptotics of w may contain only $V_{s,k,T}$ such that $\chi V_{s,k,T} \in L_2(G)$. However, it is not the only restriction on the terms of asymptotics. Let us find out which functions $\chi u_{s,k}$ are actually in the domain of $M(\tau)^*$. It is easily seen that $\chi u_{s,k} \in \mathcal{D}M(\tau)^*$ for λ_k with $\operatorname{Im}\lambda_k \leq 1/2$. We now consider the eigenvalues of the pencil $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda \in$

$]1/2,\ 3/2[$. The domain $\mathcal{D}M(\tau)$ contains terms of the form $\chi(\alpha_{s,k}u_{s,k}+\beta_{s,k}u_{s,-k})$ with $\alpha_{s,k},\ \beta_{s,k}$ satisfying $\operatorname{Re}\alpha_{s,k}\overline{\beta_{s,k}}=0,\ |\alpha_{s,k}|+|\beta_{s,k}|>0$. In view of (2.15),

$$\Big(M(D_x,\tau)\chi(\alpha_{s,k}\,u_{s,k}+\beta_{s,k}\,u_{s,-k})\ ,\ \chi(c\,u_{s,k}+d\,u_{s,-k})\Big)_G=$$
$$=(1/i)(\alpha_{s,k}\overline{d}+\beta_{s,k}\overline{c})+$$
$$+\Big(\chi(\alpha_{s,k}\,u_{s,k}+\beta_{s,k}\,u_{s,-k})\ ,\ M(D_x,\overline{\tau})\chi(c\,u_{s,k}+d\,u_{s,-k})\Big)_G.$$

The condition $\alpha_{s,k}\overline{d}+\beta_{s,k}\overline{c}=0$ is necessary for the inclusion $\chi(c\,u_{s,k}+d\,u_{s,-k})\in\mathcal{D}M(\tau)^*$. Together with the equality Re $\alpha_{s,k}\overline{\beta_{s,k}}=0$ this leads to

$$\chi(c\,u_{s,k}+d\,u_{s,-k})=\frac{c}{\alpha_{s,k}}\chi(\alpha_{s,k}\,u_{s,k}+\beta_{s,k}\,u_{s,-k})$$

for $\alpha_{s,k}\neq 0$. In the case $\alpha_{s,k}=0$ we obtain $\beta_{s,k}\neq 0$. Then the equality $\beta_{s,k}\overline{c}=0$ implies that $c=0$. Thus, the domain $\mathcal{D}M(\tau)^*$ contains the same combinations of $\chi u_{s,k}$ as $D(G)$. Therefore, the functions $V_{s,k,T}$ corresponding to the eigenvalues of $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda\in\,]1/2,\ 3/2[$ are included in the asymptotics of w as combinations $\alpha_{s,k}\ V_{s,-k,T}+\beta_{s,k}\ V_{s,k,T}$. To prove that $w=0$ it remains to take into account Remark 2.10 and Proposition 2.12. □

Let us discuss the obtained results. We have proved that the operator $M(\tau)$ with domain $D(G)$ admits closure; here $\tau=\sigma-i\gamma,\ \sigma\in\mathbb{R},\ \gamma>0$. The closed operator has trivial kernel, and its range coincides with $L_2(G)$. The inverse operator is bounded in virtue of (2.17). The same is true for $M(\overline{\tau})$. The operator $A(D_x)$ with domain $D(G)$is symmetric. It follows that for the closed operator $A=\overline{A(D)}$ there exists $(A-\lambda)^{-1}$ for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$, while A is self-adjoint. Note that for $D(G)$ one can choose various collections $\{\alpha_{s,k},\beta_{s,k}\}$ satisfying Re $\alpha_{s,k}\overline{\beta_{s,k}}=0$ and $|\alpha_{s,k}|+|\beta_{s,k}|>0$. This gives rise to various self-adjoint extensions A of $A(D)$. In what follows as A is taken any of the extensions, unless otherwise indicated.

Definition 2.15. *A solution of the equation $(\tau+A)u=f$ with $f\in L_2(G)$ is called a strong solution of the problem (2.3), (2.4).*

The next assertion summarizes the results of this section.

Theorem 2.16. *For any f in $L_2(G)$ and every $\tau=\sigma-i\gamma$ ($\sigma\in\mathbb{R},\ \gamma>0$) there exists a unique strong solution v to the problem* (2.3), (2.4) *with right-hand side f. The solution satisfies*

$$\gamma\|v;\ L_2(G)\|\leq\|f;\ L_2(G)\|.$$

Remark 2.17. *Theorem 2.16 is true for the problem (2.3), (2.4) in $\mathcal{K}$ as well.*

Remark 2.18. *Theorem 2.16 is valid for the problem (2.3), (2.4) in $\mathcal{K}$ and in G with τ replaced by $\overline{\tau}$.*

To complete the section, we discuss the condition $|\alpha_{s,k}|+|\beta_{s,k}|>0$ (see Definition 2.11). Assume that $|\alpha_{s_0,k_0}|+|\beta_{s_0,k_0}|=0$ holds for some $s_0,\ k_0$. Clearly, (2.17) is true for the functions in such a lineal, which will be denoted by $D_1(G)$. However the range of the corresponding closed operator does not coincide with $L_2(G)$. The point is that $\mathcal{D}M(\tau)^*$ contains the linear combination $\chi(\alpha u_{s_0,k_0} + \beta u_{s_0,-k_0})$ with arbitrary coefficients α, β not connected by $\operatorname{Re}\alpha\overline{\beta} = 0$. Therefore, (2.17) cannot be applied to w in the kernel of $M(\tau)^*$ to prove $w = 0$ (cf. the proof of Proposition 2.14). In this case the operator $M(\tau)^*$ has a kernel of dimension 1. We construct an element in the kernel. Assume that $f = M(D_x, \overline{\tau})\chi u_{s_0,k_0}$. Introduce the closure $M_0(\overline{\tau})$ of the operator $M(D_x, \overline{\tau})$ with domain $D_0(G)$, where $\alpha_{s_0,k_0} = 0$, $\beta_{s_0,k_0} = 1$ and the rest $\alpha_{s,k}$, $\beta_{s,k}$ are the same as in $D_1(G)$. Let v be a solution to the equation $M_0(\overline{\tau})v = f$. It is obvious that $w = v - \chi u_{s_0,k_0}$ is a wanted element in the kernel of $M(\tau)^*$. We show that any element in the kernel differs from w by a constant factor. Let $\widetilde{w} \in \operatorname{K}er\, M(\tau)^*$, $\widetilde{w} \neq w$. The asymptotics of $\widetilde{w}$ near $\mathcal{O}$ contains the term $\chi(cu_{s_0,k_0} + du_{s_0,-k_0})$ with some c, d. Then $\widetilde{\widetilde{w}} = \widetilde{w} + cw$ is in $\mathcal{D}M_0(\overline{\tau})$. Since $M_0(\overline{\tau})\widetilde{\widetilde{w}} = 0$, we obtain $\widetilde{\widetilde{w}} = 0$.

2.4 A combined weighted estimate

In this section, we prove a more informative estimate on solutions to problem (2.3), (2.4) in a bounded domain and in a cone which will be used in the study of the asymptotics of solutions near the point $\mathcal{O}$.

DEFINITION 2.19. *Let $D_\beta(G)$ with $\beta \leq 1$ stand for the lineal spanned by functions of the following three types:*

1) the functions in $\mathcal{C}_c^\infty(\overline{G} \setminus \mathcal{O})$, satisfying the boundary conditions (2.2) on $\partial G \setminus \mathcal{O}$.

2) the functions $\chi u_{s,-k}$ with $k > 0$ for the eigenvalues λ_{-k} of the pencil $\mathfrak{A}$ such that $\operatorname{Im}\lambda_{-k} < \beta + 1/2$ *and* $\operatorname{Im}\lambda_k \geq \beta + 1/2$.

3) the functions $\chi(\alpha_{s,k}u_{s,k} + \beta_{s,k}u_{s,-k})$ for the eigenvalues λ_k $(k > 0)$ of the pencil $\mathfrak{A}$ such that $\operatorname{Im}\lambda_{\mp k} < \beta + 1/2$, *where $\{\alpha_{s,k},\ \beta_{s,k}\}$ are fixed pairs satisfying* $\operatorname{Re}\alpha_{s,k}\overline{\beta_{s,k}} = 0$, $|\alpha_{s,k}| + |\beta_{s,k}| > 0$.

The lineal $D_\beta(\mathcal{K})$ is defined in a similar way.

PROPOSITION 2.20. *Let $\beta \leq 1$ and let the number $\lambda = i(\beta + 1/2)$ be regular for the pencil $\mathfrak{A}$. Then for $v \in D_\beta(\mathcal{K})$ the inequality*

$$\begin{gathered}\gamma^2\|v; H_\beta^0(\mathcal{K})\|^2 + \|\chi_\tau v; H_\beta^1(\mathcal{K}, |\tau|)\|^2 \leq \\ \leq c\{\|f; H_\beta^0(\mathcal{K})\|^2 + (|\tau|^{1-\beta}/\gamma)^2\|f; L_2(\mathcal{K})\|^2\}\end{gathered} \tag{2.22}$$

holds, where $f = (\tau + A(D_x))v$, $\chi_\tau(r) = \chi(|\tau|r)$ and χ is any fixed cut-off function in $\mathcal{C}_c^\infty(\overline{\mathcal{K}})$ equal to 1 near the vertex. The constant c is independent of v and τ.

Proof. Step 1. An estimate near the vertex of a cone

We consider problem (2.6), (2.7) in $\mathcal{K}$. According to Proposition 1.2, the problem $\{A(D_\eta),\ \Gamma\}$ is elliptic. Therefore, if the line $\operatorname{Im}\lambda = \beta + 1/2$ contains no eigenvalues of the pencil corresponding to the problem under consideration (the pencil $\mathfrak{A}$ in this case), then a function $U \in H^1_\beta(\mathcal{K},1)$ such that $\Gamma U = 0$ satisfies (see [28, Chapter 3, §5])

$$\|\chi U; H^1_\beta(\mathcal{K})\|^2 \le c\|A(D_\eta)\chi U; H^0_\beta(\mathcal{K})\|^2.$$

Since $A\chi U = \chi AU + [A,\chi]U$ and $M(D_\eta,\theta) = \theta + A(D_\eta)$, the inequality can be rewritten in the form

$$\|\chi U; H^1_\beta(\mathcal{K},1)\|^2 \le c\{\|\chi M(D_\eta,\theta)U; H^0_\beta(\mathcal{K})\|^2 + \|\psi U; H^0_\beta(\mathcal{K})\|^2\}, \tag{2.23}$$

where $\psi \in \mathcal{C}^\infty_c(\overline{\mathcal{K}}),\ \chi\psi = \chi$.

Step 2. An estimate far from the vertex

At this step we prove the inequality

$$\begin{gathered}(\gamma/|\tau|)^2\|\kappa_\infty U; H^0_\beta(\mathcal{K})\|^2 \le \\ \le c\{\|\kappa_\infty M(D_\eta,\theta)U; H^0_\beta(\mathcal{K})\|^2 + \|\psi_\infty U; H^0_{\beta-1}(\mathcal{K})\|^2\},\end{gathered} \tag{2.24}$$

for any $\beta \in \mathbb{R}$ and every $U \in H^1_\beta(\mathcal{K},1)$ satisfying the boundary conditions $\Gamma U = 0$, where the constant c is independent of U and τ, κ_∞ and ψ_∞ are smooth functions in $\mathcal{K}$, equal to 0 near the vertex and 1 in a neighborhood of infinity, while $\kappa_\infty\psi_\infty = \kappa_\infty$.

Assume that $\kappa,\psi \in \mathcal{C}^\infty(\overline{\mathcal{K}}),\ \kappa\psi = \kappa$, supp $\kappa \subset \{x \in \mathcal{K} : 1/2 < r < 2\}$, supp $\psi \subset \{x \in \mathcal{K} : 1/4 < r < 4\}$. According to (2.17),

$$\gamma^2\|\kappa U; L_2(\mathcal{K})\|^2 \le \|M(D_x,\tau)\kappa U; L_2(\mathcal{K})\|^2.$$

Since $M\kappa U = \kappa MU + [M,\kappa]U$, we have

$$\gamma^2\|\kappa U; L_2(\mathcal{K})\|^2 \le c\{\|\kappa M(D_x,\tau)U; L_2(\mathcal{K})\|^2 + \|\psi U; L_2(\mathcal{K})\|^2\}.$$

As U, we take the function $x \mapsto U^\varepsilon(x) = U(x_1/\varepsilon, x_2/\varepsilon, x_3/\varepsilon)$, and change τ for $\tau/(|\tau|\varepsilon)$, where $\varepsilon > 0$. Then the last inequality takes the form

$$\begin{gathered}(\gamma/|\tau|\varepsilon)^2\|\kappa U^\varepsilon; L_2(\mathcal{K})\|^2 \le \\ \le c\{\|\kappa M(D_x,\tau/|\tau|\varepsilon)U^\varepsilon; L_2(\mathcal{K})\|^2 + \|\psi U^\varepsilon; L_2(\mathcal{K})\|^2\}.\end{gathered}$$

After the change of variables $x \mapsto \eta = (x_1/\varepsilon, x_2/\varepsilon, x_3/\varepsilon)$ we arrive at

$$\begin{gathered}(\gamma/|\tau|)^2\|\kappa_\varepsilon U; L_2(\mathcal{K})\|^2 \le c\{\|\kappa_\varepsilon M(D_\eta,\theta)U; L_2(\mathcal{K})\|^2 + \\ +\varepsilon^2\|\psi_\varepsilon U; L_2(\mathcal{K})\|^2\},\end{gathered}$$

with $\kappa_\varepsilon(\eta) = \kappa(\varepsilon\eta)$. Multiplying the inequality by $\varepsilon^{-2\beta}$, putting $\varepsilon = 2^{-j}, j = 1,2,3,...$, and adding all these inequalities, we obtain (2.24).

Step 3. An estimate in intermediate zone

Add the inequalities (2.23), (2.24). Let $\kappa_\infty = 1$ outside the support of χ. Then κ_∞ can be dropped from the left-hand side because

$$(\gamma/|\tau|)\|\chi U; H^0_\beta(\mathcal{K})\| \le \|\chi U; H^0_\beta(\mathcal{K})\| \le \|\chi U; H^1_\beta(\mathcal{K},1)\|.$$

The inequality takes the form

$$\begin{gathered}(\gamma/|\tau|)^2\|U; H^0_\beta(\mathcal{K})\|^2 + \|\chi U; H^1_\beta(\mathcal{K},1)\|^2 \le \\ \le c\{\|M(D_\eta,\theta)U; H^0_\beta(\mathcal{K})\|^2 + \|\psi_\infty U; H^0_{\beta-1}(\mathcal{K})\|^2 + \|\psi U; H^0_\beta(\mathcal{K})\|^2\}.\end{gathered}$$

We transform the last term

$$\|\psi U; H^0_\beta(\mathcal{K})\|^2 \le \int\limits_{|\eta|<a} |\eta|^{2\beta}|U|^2\, d\eta =$$
$$= (\int\limits_{0\le|\eta|\le\varepsilon} + \int\limits_{\varepsilon\le|\eta|\le a})|\eta|^{2\beta}|U|^2\, d\eta.$$

The first integral does not exceed $c\varepsilon^2\|\chi U; H^1_\beta(\mathcal{K})\|^2$. Therefore by choosing a small enough ε, one can rearrange the integral to the left side of the inequality. The second integral is majorized by $c\|\psi_\infty U; H^0_{\beta-1}(\mathcal{K})\|^2$. Now the estimate can be rewritten in the form

$$(\gamma/|\tau|)^2\|U; H^0_\beta(\mathcal{K})\|^2 + \|\chi U; H^1_\beta(\mathcal{K},1)\|^2 \le$$
$$\le c\{\|M(D_\eta,\theta)U; H^0_\beta(\mathcal{K})\|^2 + \|\psi_\infty U; H^0_{\beta-1}(\mathcal{K})\|^2\}.$$

After the change of variables $x = |\tau|^{-1}\eta$, we obtain

$$\gamma^2\|v; H^0_\beta(\mathcal{K})\|^2 + \|\chi_\tau v; H^1_\beta(\mathcal{K},|\tau|)\|^2 \le$$
$$\le c\{\|M(D_x,\tau)v; H^0_\beta(\mathcal{K})\|^2 + \|\psi_{\infty,\tau} v; H^0_{\beta-1}(\mathcal{K})\|^2\},$$

where $\psi_{\infty,\tau}(r) = \psi_\infty(|\tau|r)$, $\chi_\tau(r) = \chi(|\tau|r)$, $v(x) = U(|\tau|x)$. Taking the inequalities (2.17) and $\beta \le 1$ into account, we have

$$\|\psi_{\infty,\tau} v; H^0_{\beta-1}(\mathcal{K})\|^2 \le \int\limits_{b/|\tau|<r} r^{2(\beta-1)}|v(x)|^2\, dx \le$$
$$\le c\,|\tau|^{2(1-\beta)} \int\limits_{\mathcal{K}} |v(x)|^2\, dx \le c\,|\tau|^{2(1-\beta)}\gamma^{-2}\|M(D_x,\tau)v; L_2(\mathcal{K})\|^2,$$

which leads to (2.22). $\square$

Introduce the spaces $\mathcal{D}H_\beta(G,\tau)$, $\mathcal{R}H_\beta(G,\tau)$ by completing the set $\mathcal{C}^\infty_c(\overline{G}\setminus\mathcal{O})$ in the norms

$$\|v; \mathcal{D}H_\beta(G,\tau)\| = \Big(\gamma^2\|v; H^0_\beta(G)\|^2 + \|\chi_\tau v; H^1_\beta(G,|\tau|)\|^2\Big)^{1/2},$$
$$\|f; \mathcal{R}H_\beta(G,\tau)\| = \Big(\|f; H^0_\beta(G)\|^2 + (|\tau|^{1-\beta}/\gamma)^2\|f; L_2(G)\|^2\Big)^{1/2},$$

while $\chi_\tau(x) = \chi(|\tau|x)$ and $\chi \in \mathcal{C}^\infty(\overline{G})$ is a cut-off function that is equal to 1 near the conical point $\mathcal{O}$ and vanishes outside the neighborhood where G coincides with $\mathcal{K}$. The spaces $\mathcal{D}H_\beta(\mathcal{K},\tau)$ and $\mathcal{R}H_\beta(\mathcal{K},\tau)$ are defined in a similar way. Now (2.22) takes the form

$$\|v; \mathcal{D}H_\beta(\mathcal{K},\tau)\| \le c\,\|M(D_x,\tau)v; \mathcal{R}H_\beta(\mathcal{K},\tau)\|. \qquad (2.25)$$

Using (2.25), we prove a similar estimate in the domain G.

PROPOSITION 2.21. *Let $\beta \le 1$ and let the number $\lambda = i(\beta + 1/2)$ be regular for the pencil $\mathfrak{A}$. Assume that $\gamma > \gamma_0$ with sufficiently large γ_0. Then the inequality*

$$\|v; \mathcal{D}H_\beta(G,\tau)\| \le c\|M(D_x,\tau)v; \mathcal{R}H_\beta(G,\tau)\|, \tag{2.26}$$

holds for any $v \in D_\beta(G)$ with a constant c independent of v and τ.

Proof. Let $\psi \in \mathcal{C}^\infty(\overline{G})$ be a cut-off function that is equal to 1 near $\mathcal{O}$ and vanishes outside the neighborhood where G coincides with $\mathcal{K}$. Since $v = \psi v + (1-\psi)v$, we have

$$\|v; \mathcal{D}H_\beta(G,\tau)\| \le \|\psi v; \mathcal{D}H_\beta(G,\tau)\| + \|(1-\psi)v; \mathcal{D}H_\beta(G,\tau)\|. \tag{2.27}$$

Estimate the first term on the right. Taking the inclusion $\psi v \in D_\beta(\mathcal{K})$ and the relation (2.25) into account, we arrive at

$$\begin{gathered}\|\psi v; \mathcal{D}H_\beta(G,\tau)\| \le \|M\psi v; \mathcal{R}H_\beta(\mathcal{K},\tau)\| \le \\ \le c\{\|\psi M v; \mathcal{R}H_\beta(\mathcal{K},\tau)\| + \|[M,\psi]v; \mathcal{R}H_\beta(\mathcal{K},\tau)\|\},\end{gathered}$$

where M denotes $M(D_x,\tau)$. Further,

$$\begin{gathered}\|[M,\psi]v; \mathcal{R}H_\beta(\mathcal{K},\tau)\| \le \{\|[M,\psi]v; H^0_\beta(\mathcal{K})\| + (|\tau|^{1-\beta}/\gamma)\|[M,\psi]v; L_2(\mathcal{K})\|\} \le \\ \le c\{\|v; H^0_\beta(G)\| + (|\tau|^{1-\beta}/\gamma)\|v; L_2(G)\|\},\end{gathered}$$

from whence, using (2.17), we conclude that

$$\begin{gathered}\|[M,\psi]v; \mathcal{R}H_\beta(\mathcal{K},\tau)\| \le c\{\|v; H^0_\beta(G)\| + (|\tau|^{1-\beta}/\gamma)\cdot(1/\gamma)\|Mv; L_2(G)\|\} \le \\ \le c\{\|v; H^0_\beta(G)\| + \|Mv; \mathcal{R}H_\beta(G,\tau)\|\}\end{gathered}$$

for $\gamma > 1$. In its turn,

$$\|\psi M v; \mathcal{R}H_\beta(\mathcal{K},\tau)\| \le c\|Mv; \mathcal{R}H_\beta(G,\tau)\|.$$

Thus, from (2.27) we obtain

$$\|\psi v; \mathcal{D}H_\beta(G,\tau)\| \le c\{\|Mv; \mathcal{R}H_\beta(G,\tau)\| + \|v; H^0_\beta(G)\|\}.$$

We now estimate the second term in the right-hand side of (2.27). From the definition of the norm in $\mathcal{D}H_\beta(G,\tau)$ it follows that

$$\|(1-\psi)v; \mathcal{D}H_\beta(G,\tau)\| \le \{\gamma\|(1-\psi)v; H^0_\beta(G)\| + \|\chi_\tau(1-\psi)v; H^1_\beta(G,|\tau|)\|\}.$$

For sufficiently large γ we have $\chi_\tau(1-\psi) \equiv 0$ because the supports of the factors do not overlap. In view of (2.17),

$$\begin{gathered}\gamma\|(1-\psi)v; H^0_\beta(G)\| \le c\gamma\|(1-\psi)v; L_2(G)\| \le c\|M(1-\psi)v; L_2(G)\| \le \\ \le c\{\|(1-\psi)Mv; L_2(G)\| + \|[M,(1-\psi)]v; L_2(G)\|\} \le \\ \le c\{\|Mv; H^0_\beta(G)\| + \|v; H^0_\beta(G)\|\} \le \\ \le c\{\|Mv; \mathcal{R}H_\beta(G,\tau)\| + \|v; H^0_\beta(G)\|\}.\end{gathered}$$

Summarizing the obtained estimates, we rewrite (2.27) in the form

$$\|v; \mathcal{D}H_\beta(G,\tau)\| \le c\{\|Mv; \mathcal{R}H_\beta(G,\tau)\| + \|v; H^0_\beta(G)\|\}.$$

We take account of the definition of the norm in $\mathcal{D}H_\beta(G,\tau)$, choose a large enough γ, and rearrange the second term to the left side. As a result, we obtain (2.26). $\square$

2.5 The operator of problem in a scale of weighted spaces

In the section, we study the operator of problem (2.3), (2.4) in spaces related to (2.25), (2.26). We consider the problem in a bounded domain G. The corresponding statements for the problem in $\mathcal{K}$ are given at the end of this section.

With the problem (2.3), (2.4) in a bounded domain G with conical point $\mathcal{O}$ we associate the operator $v \mapsto M_\beta(\tau)v := M(D_x,\tau)v$ with domain $D_\beta(G)$ acting from $\mathcal{D}H_\beta(G,\tau)$ to $\mathcal{R}H_\beta(G,\tau)$. It is easy to see that the operator $M_\beta(\tau)$ admits closure. Further we keep the same notation for the closed operator. If the number $\lambda = i(\beta+1/2)$ is regular for the pencil $\mathfrak{A}$ and $\beta \le 1$, then the estimate

$$\|v, \mathcal{D}H_\beta(G,\tau)\| \le c\,\|M_\beta(\tau)v, \mathcal{R}H_\beta(G,\tau)\| \tag{2.28}$$

holds for the functions in the domain $\mathcal{D}M_\beta(\tau)$ of the closed operator. The next proposition immediately follows from (2.28).

PROPOSITION 2.22. *Let the number $\lambda = i(\beta+1/2)$ be regular for the pencil $\mathfrak{A}$ and $\beta \le 1$. Then the kernel of the operator* $\operatorname{Ker} M_\beta(\tau)$ *is trivial and the range $\mathcal{R}M_\beta(\tau)$ is closed in $\mathcal{R}H_\beta(G,\tau)$.*

Let $1/2 > \beta_1 > \beta_2 > \dots$ be all numbers in the interval $]-\infty, 1/2[$ such that the number $\lambda = i(\beta_k+1/2)$ is an eigenvalue of the pencil $\mathfrak{A}$. Denote by S_m the sum of the multiplicities of all the eigenvalues of $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda \in [\beta_m+1/2;\ \beta_1+1/2]$.

DEFINITION 2.23. *A solution to the equation $M_\beta(\tau)v = f$, where $f \in \mathcal{R}H_\beta(G,\tau)$, is called a strong β-solution to the problem (2.3), (2.4) with right-hand side f.*

THEOREM 2.24. A) *Let $\beta \in [\beta_1,\ 1]$ and let the number $\lambda = i(\beta+1/2)$ be regular for the pencil $\mathfrak{A}$. Assume that $\gamma > \gamma_0$ with sufficiently large γ_0. Then for any $f \in \mathcal{R}H_\beta(G,\tau)$ there exists a unique strong β-solution v of the problem* (2.3), (2.4) *with right-hand side f and*

$$\|v; \mathcal{D}H_\beta(G,\tau)\| \le c\|f; \mathcal{R}H_\beta(G,\tau)\|.$$

B) *Assume that $\beta \in]\beta_{m+1}, \beta_m[$. A strong β-solution of the problem* (2.3), (2.4) *with right-hand side $f \in \mathcal{R}H_\beta(G,\tau)$ exists under the S_m conditions $(f, w_{s,k})_G = 0$, where $\{w_{s,k}\}_{k=-1,..,-m}^{s=1,..,N_k}$ is a basis in* $\operatorname{Ker} M_\beta(\tau)^*$. *Such a solution satisfies the estimate in* A).

Proof. A) Suppose that $w \in \operatorname{Ker} M_\beta(\tau)^*$, where $M_\beta(\tau)^*$ is the adjoint operator for $M_\beta(\tau)$ with respect to the extension of the inner product on $L_2(G)$. According to local properties of solutions to elliptic problems (e.g., see [28, Chapter 1]), w belongs to $\mathcal{C}^\infty(\overline{G}\setminus\mathcal{O})$ and satisfies (2.19), (2.20). Moreover, in a neighborhood of $\mathcal{O}$ the asymptotic representation

$$w \sim \chi \sum_k \sum_{s=1,..,N_k} c_{s,k} V_{s,k,T}$$

holds, where $V_{s,k,T}$ is the sum of the first T terms in the formal series (2.21). For a fixed τ,

$$\Big(\int\limits_G |v|^2(1+r^{2\beta})^{-1}\,\mathrm{d}x\Big)^{1/2}$$

is an equivalent norm in $\mathcal{R}H_\beta(G,\tau)^*$. If $\beta \geq 0$, then $w \in \mathcal{R}H_\beta(G,\tau)^* \subset L_2(G)$. In view of the equality $(M_\beta(\tau)u, v)_G = (u, M_\beta(\tau)^* v)_G$, the asymptotics near $\mathcal{O}$ of the functions in the domain $\mathcal{D}M_\beta(\tau)^*$ may contain the terms $V_{s,k,T}$ corresponding to the eigenvalues in the strip $\operatorname{Im}\lambda \in]1/2,\ 3/2[$ only in the combinations $\alpha_{s,k}V_{s,-k,T} + \beta_{s,k}V_{s,k,T}$ (see the proof of Proposition 2.14). The coefficients $\{\alpha_{s,k},\ \beta_{s,k}\}$ are the same as in the lineal $D_\beta(G)$, where the operator $M_\beta(\tau)$ was initially given and then extended by closing. Then w is in $L_2(G)$ and has asymptotics such that the inclusion $w \in \mathcal{D}M(\overline{\tau})$ holds, while the operator $M(\overline{\tau})$ extended by closing was initially given on the lineal $D(G)$, where the coefficients $\{\alpha_{s,k},\ \beta_{s,k}\}$ are the same as in $D_\beta(G)$. Taking account of estimate (2.17) and Remark 2.10, we obtain $w = 0$.

Let $\beta \in [\beta_1,\ 1]$ and $\beta < 0$. This means the strip $\operatorname{Im}\lambda \in [\beta_1 + 1/2,\ 3/2]$ contains no eigenvalues of the pencil $\mathfrak{A}$. Using results on the asymptotics of solutions to elliptic problems from [28, Chapter 4, §2], we deduce for w the asymptotic representation

$$w = \chi \sum_{s,k} c_{s,k}V_{s,k,T} + v.$$

Here $v \in \mathcal{R}H_{\beta'}(G,\tau)^*$, $\beta' \in [0,\ 1[$, the sum contains $V_{s,k,T}$ corresponding to the eigenvalues of $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda \in [\beta + 1/2,\ \beta' + 1/2]$. Since the strip is free from the spectrum, we have $w = v$ and $w \in \mathcal{R}H_{\beta'}(G,\tau)^* \subset L_2(G)$. The above argument for the case $\beta \geq 0$ leads to $w = 0$.

B) Assume that $\beta \in]\beta_{m+1},\ \beta_m[$. We construct a collection consisting of S_m function and prove that it is a basis in $\mathrm{K}er\, M_\beta(\tau)^*$. Thus we prove the theorem because the range of $M_\beta(\tau)$ is closed in $\mathcal{R}H_\beta(G,\tau)$ and the kernel is trivial. Let $M(\tau)$ and $M(\overline{\tau})$ be the closure of $M(D_x,\tau)$ and $M(D_x,\overline{\tau})$, respectively, given on the lineal $D(G)$ where $\alpha_{s,k} = 0$ and $\beta_{s,k} = 1$. Recall that $V_{s,k,T}$ is the sum of first T terms in (2.21). It is easy to see that $M(D_x,\overline{\tau})\chi V_{s,k,T} = O(r^{\operatorname{Im}\lambda_k - 2 + T})$ near the point $\mathcal{O}$. Choose a sufficiently large T to obtain the inclusion $M(D_x,\overline{\tau})$ $\chi V_{s,k,T} := F_{s,k,T} \in L_2(G)$. For the eigenvalues of $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda \in]1/2,\ 1[$, it suffices to take the first term $V_{s,k,1}(r,\vartheta,\varphi) = r^{i\lambda_{-k}}\Phi_{s,-k}(\vartheta,\varphi) = u_{s,-k}(r,\vartheta,\varphi)$. For the eigenvalues with imaginary part not exceeding $1/2$ one has to take more terms. The strip $\operatorname{Im}\lambda \in [\beta_m + 1/2,\ \beta_1 + 1/2]$ contains precisely m eigenvalues of $\mathfrak{A}$. Let λ_k be in the strip. The corresponding function $F_{s,k,T}$ is in $L_2(G)$. According to Theorem 2.16, there exists a solution to the equation $M(\overline{\tau})w_{s,k,T} = F_{s,k,T}$. Put $w_{s,k} := \chi V_{s,k,T} - w_{s,k,T}$. We construct such functions for all eigenvalues of $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda \in [\beta_m + 1/2,\ \beta_1 + 1/2]$. For λ_p there are N_p such functions. It is not hard to see that $w_{s,k}$ satisfies (2.19), (2.20) and belongs to $\mathcal{R}H_\beta(G,\tau)^*$. We show that $w_{s,k}$ are in the kernel of $M_\beta(\tau)^*$. Let $f \in \mathcal{C}^\infty(G) \cap \mathcal{R}H_\beta(G,\tau)$. From the inclusion $f \in \mathcal{R}H_\beta(G,\tau) \subset L_2(G)$ and Theorem 2.16 it follows there exists $v \in L_2(G)$ satisfying $M(\tau)v = f$. Near the point $\mathcal{O}$, the function v admits the asymptotic representation (see [15, §4.2] or [28, Chapters 3,4]):

$$v = \chi \sum d_{s,k} U_{s,k,T} + h,$$

where $U_{s,k,T}$ stands for the sum of first T terms of the formal series

$$U_{s,k}(x,\tau) = r^{i\lambda_k} \sum_{q=0}^{\infty} r^q \tau^q \Psi_q^{(s,k)}(\vartheta,\varphi), \tag{2.29}$$

while $\Psi_0^{(s,k)} = \Phi_{s,k}$. In $D(G)$, all the coefficients $\alpha_{s,k}$ are equal to 0, therefore v contains the terms corresponding to the eigenvalues in the strip $\operatorname{Im}\lambda \in [\beta_m + 1/2,\ \beta_1 + 1/2]$. The remainder h is $o(r^{i\lambda_m})$. The number T is taken sufficiently large so that $\chi r^{i\lambda_k + T + 1}\Psi_T$ decays more rapidly than $r^{i\lambda_m}$ as $r \to 0$. The coefficients $d_{s,k}$ are defined by $d_{s,k} = i(f, w_{s,k})_G$ (see [28, Chapter 4, §3]). The conditions $(f, w_{s,k}) = 0$ with all the constructed $w_{s,k}$ are necessary for the inclusion $v \in \mathcal{D}H_\beta(G,\tau)$. Thus $w_{s,k}$ are in the kernel of $M_\beta(\tau)^*$. We show that they form a basis in $\mathrm{K}er\, M_\beta(\tau)^*$. Let $w \in \mathrm{K}er\, M_\beta(\tau)^*$. Then the representation

$$w = \chi \sum c_{s,k} V_{s,k,T} + h$$

holds near the point $\mathcal{O}$, the sum contains the functions corresponding to the eigenvalues of $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda \in [\beta_m + 1/2,\ \beta_1 + 1/2]$, and the remainder h is in $L_2(G)$. We put $z = w - \sum c_{s,k} w_{s,k}$. The function z belongs to $L_2(G)$. The asymptotics contains the terms corresponding to the eigenvalues satisfying $\operatorname{Im}\lambda \le 1$. Therefore $z \in \mathcal{D}M(\overline{\tau})$. Since $M(\overline{\tau})z = 0$, we have $z = 0$ and $w = \sum c_{s,k} w_{s,k}$. □

Remark 2.25. *Theorem 2.24 is valid for the problem (2.3), (2.4) in the cone $\mathcal{K}$.*

The above proof goes almost without changes for the problem in $\mathcal{K}$. There is only one distinction. For the problem in $\mathcal{K}$, there arises a question on the behavior of functions in the kernel at infinity. Using (2.24), one can check that the functions decay more rapidly than any power of r.

2.6 The asymptotics of solutions

Let $f \in \mathcal{R}H_\beta(G,\tau)$ and $\beta \in]\beta_{m+1},\ \beta_m[$. Since $\mathcal{R}H_\beta(G,\tau) \subset L_2(G)$, there exists a unique strong solution u to problem (2.3), (2.4) (Theorem 2.16). According to Theorem 2.24, this solution is in $\mathcal{D}H_\beta(G,\tau)$ provided $\Big(f, w_{s,k}\Big)_G = 0$ with $k = -1, .., -m$ and $s = 1, .., N_k$, where $\{w_{s,k}\}_{k=-1,..,-m}^{s=1,..,N_k}$ is the basis in $\mathrm{K}er\, M_\beta(\tau)^*$ constructed in the proof of Theorem 2.24. For any f, we obtain an asymptotic formula for the strong solution with remainder in $\mathcal{D}H_\beta(G,\tau)$.

Theorem 2.26. *Assume that $f \in \mathcal{R}H_\beta(G,\tau)$, $\beta \in]\beta_{m+1},\ \beta_m[$, and $\gamma > \gamma_0$ with sufficiently large γ_0. Then the strong solution u to the problem* (2.3), (2.4) *with right-hand side f admits the representation*

$$u = \chi_\tau \sum c_{s,k} U_{s,k,T} + w. \tag{2.30}$$

Here $U_{s,k,T}$ denotes the sum of first T terms of the formal series (2.29), $w \in \mathcal{D}H_\beta(G,\tau)$, *and* $\chi_\tau(r) = \chi(|\tau|r)$, *while* χ *is a cut-off function that is equal to 1 near the point* $\mathcal{O}$ *and vanishes outside the neighborhood where the domain* G *coincides with the cone* $\mathcal{K}$. *The sum consists of the terms corresponding to the eigenvalues of the pencil* $\mathfrak{A}$ *in the strip* $\operatorname{Im}\lambda \in [\beta_m + 1/2,\ \beta_1 + 1/2]$. *The coefficients* $c_{s,k}$ *are defined by*

$$c_{s,k} = i\Big(f, w_{s,k}\Big)_G.$$

The estimates

$$|c_{s,k}| \le c\,|\tau|^{\beta+1/2-\operatorname{Im}\lambda_k}\|f; \mathcal{R}H_\beta(G,\tau)\|,$$
$$\|w; \mathcal{D}H_\beta(G,\tau)\| \le c\,(|\tau|/\gamma)\|f; \mathcal{R}H_\beta(G,\tau)\|,$$

hold with a constant c *independent of* τ.

Proof. As in the proof of Theorem 2.24, let $M(\tau, G)$ and $M(\overline{\tau}, G)$ be the closure of $M(D_x, \tau)$ and $M(D_x, \overline{\tau})$, respectively, given on the lineal $D(G)$ where $\alpha_{s,k} = 0,\ \beta_{s,k} = 1$. Denote by $\{w_{s,k}\}$ the basis in $\mathrm{K}er\, M_\beta(\tau, G)^*$ constructed in the proof of Theorem 2.24 with the help of the operator $M(\overline{\tau}, G)$. We will denote by $M(\theta, \mathcal{K})$ and $M(\overline{\theta}, \mathcal{K})$ the corresponding operators for the problem (2.6), (2.7) in the cone $\mathcal{K}$. The coefficients $\{\alpha_{s,k},\ \beta_{s,k}\}$ in $D(\mathcal{K})$ are the same as in $D(G)$. We write $\{\mathcal{W}_{s,k}\}$ for the basis in $\mathrm{K}er\, M_\beta(\theta, \mathcal{K})^*$ constructed by means of the operator $M(\overline{\theta}, \mathcal{K})$ in the same way as the basis $\{w_{s,k}\}$ in $\mathrm{K}er\, M_\beta(\tau, G)^*$ by means of $M(\overline{\tau}, G)$. Let $\mathcal{U}_{s,k}$ stand for the formal series similar to that in (2.29) satisfying (2.6), (2.7).

Note that the values of $\{\alpha_{s,k},\ \beta_{s,k}\}$ indicated above were chosen for the sake of simplicity. One can take any collection obeying $\operatorname{Re}\alpha_{s,k}\overline{\beta_{s,k}} = 0,\ |\alpha_{s,k}| + |\beta_{s,k}| > 0$. This gives rise to new operators $M(\tau, G),\ M(\overline{\tau}, G)$ and another basis $\{\widetilde{w}_{s,k}\}$ in $\mathrm{K}er\, M_\beta(\tau, G)^*$. The new formula (2.30) would contain the terms $c_{s,k}(\alpha_{s,k}U_{s,-k,T} + \beta_{s,k}U_{s,k,T})$ corresponding to the eigenvalues of $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda \in]1/2,\ 3/2[$. The formulas for the coefficients $c_{s,k}$ and the estimates in the statement of theorem do not change.

Let u be a solution to the equation $M(\tau, G)u = f$ and let $h := M(\tau, G)\chi u$. Denote by U a solution to the equation $M(\theta, \mathcal{K})U = H$ with $H(\eta) = (1/|\tau|)h(\eta/|\tau|)$, $\eta = |\tau|x$. Since a strong solution is unique, we have $U(\eta) = \chi(\eta/|\tau|)u(\eta/|\tau|)$. In view of the properties of solutions to elliptic problems in domains with singularities, the representation

$$U(\eta) = \zeta(\eta)\sum d_{s,k}\mathcal{U}_{s,k,T}(\eta) + \mathcal{V}(\eta) \tag{2.31}$$

holds in a neighborhood of $\mathcal{O}$. The sum contains $\mathcal{U}_{s,k,T}$ corresponding to the eigenvalues of $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda \in [\beta_m + 1/2,\ 1[$, while T is taken large enough in order for the inclusions $\chi r^{i\lambda_k+T+1}\Psi_{T+1} \in H^1_\beta(\mathcal{K})$ to be valid. The coefficients $d_{s,k}$ are defined by $d_{s,k} = i\Big(H, \mathcal{W}_{s,k}\Big)_{\mathcal{K}}$. The function $\zeta\mathcal{V}$ is in $H^1_\beta(\mathcal{K})$. Now we describe in more detail the properties of $\mathcal{V}$. To this end we consider $M(\theta, \mathcal{K})\widetilde{\mathcal{V}} = \widetilde{H}$, where $\widetilde{H} = H - M(D_\eta, \theta)(\zeta\sum d_{s,k}\mathcal{U}_{s,k,T})$ and the sum is the same as in (2.31). Taking into account the equality $\big(\widetilde{H}, \mathcal{W}_{s,k}\big)_{\mathcal{K}} = 0$, Theorem 2.24, and Remark 2.25, we obtain $\widetilde{\mathcal{V}} \in \mathcal{D}H_\beta(\mathcal{K}, 1)$. Since $M(\theta, \mathcal{K})\mathcal{V} = \widetilde{H}$, we see that $\mathcal{V} = \widetilde{\mathcal{V}}$ and $\mathcal{V} \in \mathcal{D}H_\beta(\mathcal{K}, 1)$.

Let us majorize the coefficients $d_{s,k}$. The representation $d_{s,k} = i\big(H, \mathcal{W}_{s,k}\big)_{\mathcal{K}}$ implies that

$$|d_{s,k}| \le c\|H; \mathcal{R}H_\beta(\mathcal{K},1)\| \le c|\tau|^{\beta+1/2}\|h; \mathcal{R}H_\beta(\mathcal{K},|\tau|)\|.$$

Using $h = \chi f + [M,\chi]u$, we obtain

$$\|h; \mathcal{R}H_\beta(\mathcal{K},\tau)\| \le \|f; \mathcal{R}H_\beta(G,\tau)\| + \|[M,\chi]u; \mathcal{R}H_\beta(\mathcal{K},\tau)\|.$$

The inequality $\gamma > \gamma_0$ with large γ_0 leads to the estimate

$$\begin{gathered}\|[M,\chi]u; \mathcal{R}H_\beta(\mathcal{K},\tau)\| \le \|[M,\chi]u; \mathcal{R}H_\beta(G,\tau)\| \le \\ \le c\{\|u; L_2(G)\| + (|\tau|^{1-\beta}/\gamma)\|u; L_2(G)\|\} \le \\ \le c\{(1/\gamma)\|f; L_2(G)\| + (|\tau|^{1-\beta}/\gamma^2)\|f; L_2(G)\|\} \le \\ \le c\|f; \mathcal{R}H_\beta(G,\tau)\|.\end{gathered}$$

Thus,

$$|d_{s,k}| \le c|\tau|^{\beta+1/2}\|f; \mathcal{R}H_\beta(G,\tau)\|$$

with a constant c independent of τ. Since $U(\eta) = \chi(r)u(x)$, the representation

$$\chi(x)u(x) = \zeta(|\tau|r)\sum d_{s,k}\mathcal{U}_{s,k,T}(|\tau|x, \tau/|\tau|) + \mathcal{V}(|\tau|x)$$

holds near the point $\mathcal{O}$. Taking into account the equality

$$\begin{gathered}\mathcal{U}_{s,k,T}(|\tau|x, \tau/|\tau|) = \sum_{q=0}^{T}(|\tau|r)^{i\lambda_k+q}(\tau/|\tau|)^q\Psi_s(\vartheta,\varphi) = \\ = |\tau|^{i\lambda_k}U_{s,k,T}(r,\vartheta,\varphi,\tau),\end{gathered}$$

we finally obtain

$$u(x) = \zeta(|\tau|r)\sum c_{s,k}U_{s,k,T}(x,\tau) + w(x)$$

with $c_{s,k} = |\tau|^{i\lambda_k}d_{s,k}$. It is not hard to verify that $\chi w \in H^1_\beta(\mathcal{K})$ and $c_{s,k} = i\Big(f, w_{s,k}\Big)_G$. Using the estimate on $d_{s,k}$, we have:

$$|c_{s,k}| \le c|\tau|^{\beta+1/2-\operatorname{Im}\lambda_k}\|\mathcal{R}H_\beta(G,\tau)\|.$$

Consider the remainder w. Since $M(\tau,G)w = \widetilde{f}$ with

$$\widetilde{f} = f - M(D_x,\tau)(\zeta_\tau\sum c_{s,k}U_{s,k,T})$$

and $\Big(\widetilde{f}, w_{s,k}\Big)_G = 0$, Theorem 2.24 leads to $w \in \mathcal{D}H_\beta(G,\tau)$ and

$$\begin{gathered}\|w; \mathcal{D}H_\beta(G,\tau)\| \le c\{\|f; \mathcal{R}H_\beta(G,\tau)\| + \\ + \|M(D_x,\tau)(\zeta_\tau\textstyle\sum c_{s,k}U_{s,k,T}); \mathcal{R}H_\beta(G,\tau)\|\}.\end{gathered}$$

Making use of the estimate on $c_{s,k}$ and the explicit form of $M(D_x,\tau)U_{s,k,T}$, we majorize the last term and obtain

$$\|w; \mathcal{D}H_\beta(G,\tau)\| \le c(|\tau|/\gamma)\|f; \mathcal{R}H_\beta(G,\tau)\|.$$

$\square$

Remark 2.27. *Theorem 2.26 is valid for the problem (2.3), (2.4) in* $\mathcal{K}$.

2.7 Nonstationary problem in the cylinders $\mathcal{Q}$ and $\mathbf{Q}$

Applying the inverse Fourier transform $\mathcal{F}^{-1}_{\tau\to t}$, we pass from problem (2.3), (2.4) to problem (2.1), (2.2).

DEFINITION 2.28. *Let $f \in V^0_0(\mathbf{Q},\gamma)$ and let $\widehat{u}(x,\tau)$ be the strong solution to the problem (2.3), (2.4) in G with right-hand side $-i\widehat{f}$, where $\widehat{f}(x,\tau) = \mathcal{F}_{t\to\tau} f(x,t)$. The function u, defined by $u(x,t) = \mathcal{F}^{-1}_{\tau\to t}\widehat{u}(x,\tau)$, is called a strong solution to the problem (2.1), (2.2) in the cylinder $\mathbf{Q}$ with right-hand side f.*

The next result follows from Theorem 2.16.

THEOREM 2.29. *For every $f \in V^0_0(\mathbf{Q},\gamma)$ and for any $\gamma > 0$ there exists a strong solution v to the problem* (2.1), (2.2) *with right-hand side f. Moreover,*

$$\gamma\|v; V^0_0(\mathbf{Q},\gamma)\| \le \|f; V^0_0(\mathbf{Q},\gamma)\|.$$

Let us fix a cut-off function $\chi \in \mathcal{C}^\infty(\overline{G})$ that equals 1 near the point $\mathcal{O}$ and vanishes outside the neighborhood where the domain G coincides with the cone $\mathcal{K}$. We put

$$Xu(x,t) = \mathcal{F}^{-1}_{\tau\to t}\chi(|\tau|r)\mathcal{F}_{t'\to\tau}u(x,t'),$$
$$\Lambda^\mu u(x,t) = \mathcal{F}^{-1}_{\tau\to t}|\tau|^\mu\mathcal{F}_{t'\to\tau}u(x,t').$$

Introduce the spaces $\mathcal{D}V_\beta(\mathbf{Q},\gamma)$, $\mathcal{R}V_\beta(\mathbf{Q},\gamma)$ equipped with the norms

$$\|u; \mathcal{D}V_\beta(\mathbf{Q},\gamma)\| = \left(\gamma^2\|u; V^0_\beta(\mathbf{Q},\gamma)\|^2 + \|Xu; V^1_\beta(\mathbf{Q},\gamma)\|^2\right)^{1/2},$$
$$\|f; \mathcal{R}V_\beta(\mathbf{Q},\gamma)\| = \left(\|f; V^0_\beta(\mathbf{Q},\gamma)\|^2 + (1/\gamma^2)\|\Lambda^{1-\beta} f; V^0_0(\mathbf{Q},\gamma)\|^2\right)^{1/2}.$$

DEFINITION 2.30. *Let $f \in \mathcal{R}V_\beta(\mathbf{Q},\gamma)$ and let $\widehat{u}(x,\tau)$ be the strong β-solution to the problem (2.3), (2.4) in G with right-hand side $-i\widehat{f}$, where $\widehat{f}(x,\tau) = \mathcal{F}_{t\to\tau} f(x,t)$. The function u, defined by $u(x,t) = \mathcal{F}^{-1}_{\tau\to t}\widehat{u}(x,\tau)$, is called a strong β-solution to the problem (2.1), (2.2) in the cylinder $\mathbf{Q}$ with right-hand side f.*

The next result follows from Theorem 2.24.

THEOREM 2.31. 1) *Let $\beta \in [\beta_1,\ 1]$ and let the number $\lambda = i(\beta + 1/2)$ be regular for the pencil $\mathfrak{A}$. Let $\gamma > \gamma_0$ with sufficiently large γ_0. Then there exists a unique strong β-solution v to the problem* (2.1), (2.2) *with any right-hand side $f \in \mathcal{R}V_\beta(\mathbf{Q},\gamma)$. Moreover,*

$$\|v; \mathcal{D}V_\beta(\mathbf{Q},\gamma)\| \le c\|f; \mathcal{R}V_\beta(\mathbf{Q},\gamma)\|.$$

2) *Let $\beta \in]\beta_{m+1},\ \beta_m[$. A strong β-solution to the problem* (2.1), (2.2) *with right-hand side $f \in \mathcal{R}V_\beta(\mathbf{Q},\gamma)$ exists (and is unique) if for all $\tau = \sigma - i\gamma$ $(\sigma \in \mathbb{R},\ \gamma > 0)$ the conditions $(\widehat{f}(\cdot,\tau), w_{s,k}(\cdot,\overline{\tau}))_G = 0$ holds, where $\{w_{s,k}\}^{s=1,..,N_k}_{k=-1,..,-m}$ is a basis in* $\operatorname{Ker} M_\beta(\tau)^*$. *If such a solution exists, it satisfies the inequality in* 1).

We now formulate the theorem obtained deduced from Theorem 2.26 by the inverse Fourier transform. As the spatial part of the system (2.1), we take the operator A which is the closure of the operator $A(D_x)$ with domain $D(G)$ where $\alpha_{s,k} = 0$.

THEOREM 2.32. *Assume that* $f \in \mathcal{R}V_\beta(\mathbf{Q}, \gamma)$, $\gamma > \gamma_0$ *with sufficiently large* γ_0, *and* $\beta \in]\beta_{m+1}, \beta_m[$. *Then the strong solution to the problem* (2.1), (2.2) *admits the representation*

$$u(x,t) = \sum U_{s,k,T}(r, \varphi, \vartheta, D_t)(X\check{c}_{s,k})(x,t) + w(x,t),$$

where $w \in \mathcal{D}V_\beta(\mathbf{Q}, \gamma)$. *The sum consists of the terms corresponding to the eigenvalues of the pencil* $\mathfrak{A}$ *in the strip* $\operatorname{Im}\lambda \in [\beta_m + 1/2,\ \beta_1 + 1/2]$. *The coefficients are defined by* $\check{c}_{s,k}(t) = \mathcal{F}^{-1}_{\tau\to t} c_{s,k}(\tau)$ *with* $c_{s,k} = \big(\widehat{f}(\cdot,\tau), w_{s,k}(\cdot, \overline{\tau})\big)_G$ *or, equivalently,*

$$\check{c}_{s,k}(t) = \int\limits_G \mathrm{d}x \int\limits_{\mathbb{R}} \mathrm{d}s\, \langle f(x, t-s), W_{s,k}(x,s)\rangle_{\mathbb{R}^8}$$

with $W_{s,k}(x,t) = \mathcal{F}^{-1}_{\tau\to t}\overline{w_{s,k}(x,\overline{\tau})}$. *Moreover,*

$$\|e^{-\gamma t}\check{c}_{s,k}(\cdot); H^{\operatorname{Im}\lambda_k - \beta - 1/2}(\mathbb{R})\| \le c\|f; \mathcal{R}V_\beta(\mathbf{Q},\gamma)\|,$$
$$\|w; \mathcal{D}V_\beta(\mathbf{Q},\gamma)\| \le (c/\gamma)\|\Lambda f; \mathcal{R}V_\beta(\mathbf{Q},\gamma)\|.$$

Strong solutions and strong β-solutions to the problem (2.1), (2.2) in $\mathcal{Q}$ can be defined in the same way as those to the problem in the cylinder $\mathbf{Q}$ in (2.28), (2.30).

REMARK 2.33. *All the theorems in this section are still valid for the problem (2.1), (2.2) in* $\mathcal{Q}$.

2.8 Explicit formulas for the functions $w_{s,k}$ and $W_{s,k}$ for the problem in the cone $\mathcal{K}$

According to Theorem 2.26 and Remark 2.27, the strong solution u to the problem (2.3), (2.4) in $\mathcal{K}$ with right-hand side $f \in \mathcal{R}H_\beta(\mathcal{K}, \tau)$ admits the asymptotic representation

$$u(x) = \chi_\tau(r) \sum c_{s,k} U_{s,k,T}(x,\tau) + w(x).$$

The coefficients $c_{s,k}$ are defined by

$$c_{s,k}(\tau) = \int\limits_{\mathcal{K}} \mathrm{d}x\, \langle f(x,\tau), w_{s,k}(x, \overline{\tau})\rangle_8,$$

where $\{w_{s,k}\}_{k=-1,..,-m}^{s=1,..,N_k}$ is the same basis in $\mathrm{K}er\, M_\beta(\tau, \mathcal{K})^*$ as before. Recall some properties of the functions $w_{s,k}$. They satisfy the homogeneous problem (2.3), (2.4) with $\overline{\tau}$ instead of τ. In a neighborhood of the conical point

$$w_{s,k} = r^{i\lambda_{-k}}\Phi_{s,-k}(\vartheta, \varphi) + o(r^{i\lambda_{-k}}).$$

In this section, we obtain explicit formulas for the functions $w_{s,k}$ and their Fourier transforms. Denote $\overline{w_{s,k}(x,\overline{\tau})}$ by $h_{s,k}(x,\tau)$. Since $w_{s,k}$ satisfies $M(D_x,\overline{\tau})w_{s,k}=0$, we have $M(D_x,-\tau)h_{s,k}=0$. In spherical coordinates (r,ϑ,ϕ) the operator A takes the form

$$A(\partial)=A_1(\vartheta,\varphi)\frac{\partial}{\partial r}+\frac{1}{r}A_2(\vartheta,\varphi)\frac{\partial}{\partial\vartheta}+\frac{1}{r}A_3(\vartheta,\varphi)\frac{\partial}{\partial\varphi}.$$

We write $h_{s,k}$ in the form

$$h_{s,k}(x,\tau)=r^{i\lambda_{-k}}(i\tau\xi(r)I+\xi'(r)A_1(\vartheta,\varphi))\Phi_{s,-k}(\vartheta,\varphi),$$

ξ being a scalar function. Such a representation of $h_{s,k}$ is motivated by the corresponding argument for the Helmholtz equation in [17, §3] and by the equality

$$M(D_x,\tau)M(-D_x,\tau)=\triangle+\tau^2.$$

We substitute the above expression for $h_{s,k}$ in $(-\tau+A(D))h_{s,k}=0$ and, using (2.10), arrive at

$$\xi''+\frac{2(i\lambda_{-k}+1)}{r}\xi'+\tau^2\xi=0.$$

Choose the solution $\xi(r)=cr^{\nu}K_{\nu}(i\tau r)$ with $\nu=-(2i\lambda_k+1)/2$. Here by $K_\nu(z)$ we denote the Macdonald functions, which are also called the modified bessel functions of the second kind. The coefficient

$$c=(i\tau)^{\nu-1}2^{1-\nu}/\Gamma(\nu).$$

is determined by the behavior of $h_{s,k}$ near the vertex of $\mathcal{K}$. Then

$$h_{s,k}(x,t)=r^{i\lambda_{-k}}\frac{2^{1-\nu}}{\Gamma(\nu)}\{(i\tau r)^{\nu}K_{\nu}(i\tau r)I-(i\tau r)^{\nu}K_{\nu-1}(i\tau r)A_1(\vartheta,\varphi)\}\Phi_{s,-k}(\vartheta,\varphi).$$

We apply the inverse Fourier transform. It is known (see [3]) that

$$2^{2\mu}\Gamma(2\mu+1)(\frac{p}{r})^{-2\mu}K_{2\nu}(rp)=\int\limits_{\mathbb{R}}exp(-pt)P(t)\;\mathrm{d}t\;\;\text{for }\operatorname{Re}\mu>-1/4,$$

where

$$P(t)=\theta(t-r)\;\pi^{1/2}(t^2-r^2)^{(4\mu-1)/2}F(\mu-\nu,\mu+\nu,2\mu+1/2,1-t^2/r^2),$$

and $F(a,b,c,z)$ is the hypergeometric function. Then

$$(i\tau)^{\nu}K_{\nu}(i\tau r)=\frac{2^{-\mu}}{\Gamma(\mu+1/2)}r^{\nu-N}\mathcal{F}_{t\to\tau}(d/dt)^N\mathcal{T}_N(r,t,\mu,\nu),$$

where

$$\mathcal{T}_N(r,t,\mu,\nu)=\theta(t-r)\pi^{1/2}(t^2-r^2)^{\mu-1/2}\times$$
$$F((\mu-\nu)/2,(\mu+\nu)/2,\mu+1/2,1-t^2/r^2)$$

with $\mu = [\nu] - \nu + m,\ N = [\nu] + m$, and an arbitrary positive integer m. The Fourier transform and differentiation are understood in the sense of distribution theory.

Thus, according to Theorem 2.32 and Remark 2.33, the strong solution u to the problem (2.1), (2.2) in $\mathcal{Q}$ with right-hand side $f \in \mathcal{RV}_\beta(\mathcal{Q}, \gamma)$ admits the asymptotic representation

$$u(x,t) = \sum U_{s,k,T}(r, \varphi, \vartheta, D_t)(X\check{c}_{s,k})(x,t) + w(x,t),$$

where

$$\check{c}_{s,k}(t) = \int\limits_{\mathcal{K}} \mathrm{d}x \int\limits_{\mathbb{R}} \mathrm{d}s \, \langle f(x, t-s), W_{s,k}(x,s) \rangle_{\mathbb{R}^8}.$$

Moreover,

$$W_{s,k}(x,t) = \mathcal{F}^{-1}_{\tau\to t} h_{s,k}(x,\tau) = r^{i\lambda - k} \frac{2^{1-\nu-\mu}}{\Gamma(\nu)\Gamma(\mu+1/2)} r^{\nu-\mu} \times$$

$$\{(d/dt)^N \mathcal{T}_N(r,t,\mu,\nu) - (d/dt)^N \mathcal{T}_{N-1}(r,t,\mu,\nu-1) A_1(\vartheta,\varphi)\} \Phi_{s,-k}(\vartheta,\varphi).$$

Let us discuss some properties of the coefficients $\check{c}_{s,k}$ which follow from those of $W_{s,k}$ and from the formula for $\check{c}_{s,k}$. Note that supp $W_{s,k} = \{(x,t) \in \mathcal{K} \times \mathbb{R} \ : \ r \le t\}$ and sing supp $W_{s,k} = \{(x,t) \in \mathcal{K} \times \mathbb{R} \ : \ r = t\}$. Assume that f is a smooth function and supp $f \subset \{(x,t) \in \mathcal{K} \times \mathbb{R} : R_1 < r < R_2,\ t > 0\}$. Then $\check{c}_{s,k}$ are smooth while $\check{c}_{s,k}(t) = 0$ for $t < R_1$. Thus we observe the phenomenon of "forward edge" in the coefficients. Suppose now that the singular support of the right-hand side f is located in the set $\{(x,t) : R_1 < r < R_2,\ 0 < t < t_0\}$. Then $\check{c}_{s,k}(t)$ vanish for $t < R_1$ and are smooth for $t > t_0 + R_2$. In other words, there is the phenomenon of "back edge": the coefficients have been smooth after the perturbation from the singular support of the right-hand side has left the vertex of $\mathcal{K}$.

2.9 Connection between the augmented and non-augmented Maxwell systems

Up to this point the discussion was related to the augmented Maxwell system. We are now going to prove that under some conditions on the right-hand side of such a system, its solutions satisfy the usual (non-augmented) Maxwell system. For sufficiently smooth vector fields $\vec{E},\ \vec{B}$, the right-hand side $(-\vec{J}, -\vec{G}, \rho, \mu)$ of the usual Maxwell system is subject to the compatibility conditions $\partial\rho/\partial t + \operatorname{div}\ \vec{J} = 0,\ \partial\mu/\partial t + \operatorname{div}\ \vec{G} = 0$ and the boundary condition $\langle \vec{G}, \vec{\nu} \rangle = 0$. We show, for a certain self-adjoint extension of $A(\partial)$ taken as the spatial part of the augmented system, that if a right-hand $(\vec{f_1}, \vec{f_2}, g_1, g_2)$ satisfies $\partial g_k/\partial t - \operatorname{div}\ \vec{f_k} = 0$ $(k = 1, 2)$ inside the domain and the boundary condition $\langle \vec{f_2}, \vec{\nu} \rangle = 0$, then in the corresponding strong solution $u = (\vec{u}, \vec{v}, h, q)$ we have $h \equiv 0,\ q \equiv 0$.

Consider the problem in a bounded domain $G \subset \mathbb{R}^3$ with conical point $\mathcal{O}$. Applying the Fourier transform, we rewrite the condition on right-hand side in the form $i\tau\widehat{g_k} - \operatorname{div}\,\widehat{\vec{f_k}} = 0$ $(k = 1, 2)$, $\langle\widehat{\vec{f_2}}, \vec{\nu}\rangle = 0$. Since the right-hand side is an element in $L_2(G)$, the conditions should be understood in a proper way. Let $\operatorname{div}\,\widehat{\vec{f_1}}$, $\operatorname{div}\,\widehat{\vec{f_2}} \in L_2(G)$, where div is understood in the sense of distributions. The boundary condition $\langle\widehat{\vec{f_2}}, \vec{\nu}\rangle = 0$ means that

$$\Big(\widehat{\vec{f_2}},\ \nabla\psi\Big)_G = -\Big(\operatorname{div}\,\widehat{\vec{f_2}},\ \psi\Big)_G \quad \forall\, \psi \in H^1(G).$$

Let us rewrite the indicated properties of the vector fields $\widehat{\vec{f_k}}$ in another form. To this end introduce the space

$$H(\operatorname{div}, G) = \{\vec{u} \in L_2(G) :\ \operatorname{div}\,\vec{u} \in L_2(G)\}$$

with norm $\|\vec{u};\ H(\operatorname{div}, G)\| = \Big(\|\vec{u};\ L_2(G)\|^2 + \|\operatorname{div}\,\vec{u};\ L_2(G)\|^2\Big)^{1/2}$ and its closed subspace

$$\overset{\circ}{H}(\operatorname{div}, G) = \{\vec{u} \in H(\operatorname{div}, G)\ :\ \langle\vec{u}, \vec{\nu}\rangle = 0\}.$$

Then $\widehat{\vec{f_1}} \in H(\operatorname{div}, G)$, $\widehat{\vec{f_2}} \in \overset{\circ}{H}(\operatorname{div}, G)$, $\widehat{g_1}, \widehat{g_2} \in L_2(G)$, $i\tau\widehat{g_k} = \operatorname{div}\,\widehat{\vec{f_k}}$.

We consider $M(\tau) = \tau + A$, while A is the self-adjoint extension of the differential operator $A(D_x)$ given on the lineal $D(G)$, where all the $\alpha_{s,k}$ vanish. In other words, the functions in the domain of A increase near the conical point $\mathcal{O}$ slower than the functions in the domains of all the other self-adjoint extensions.

For the proof of theorem on connection between solutions to the augmented and usual Maxwell systems, we need some properties of the Helmholtz operator in the domain G. Here we recall all the necessary definitions. For details and proofs we refer the reader to the articles [18] and [33]. Let $\Xi = \mathcal{K} \cap \mathcal{S}^2$, where the cone $\mathcal{K}$ coincides with the domain G near the conical point $\mathcal{O}$. For the Laplace operator we introduce the operator pencil $\mathfrak{C}$ in the domain Ξ by the formula

$$\mathfrak{C}(\lambda) = (i\lambda)^2 + i\lambda - \delta,$$

where δ is the Laplace-Beltrami operator. In the case of the Dirichlet problem the pencil $\mathfrak{C}$ is defined on the functions $u \in \mathrm{H}^2(\Xi)$ such that $u|_{\partial\Xi} = 0$. In the case of the Neumann problem the pencil $\mathfrak{C}$ is defined on the functions $u \in \mathrm{H}^2(\Xi)$ such that $\partial_\nu|_{\partial\Xi} = 0$. Let $\{\mu_k,\ w_k\}$ and $\{\widetilde{\mu}_k, \widetilde{w}_k\}$ be the sets of eigenvalues and eigenfunctions of the Dirichlet and Neumann problems for the operator pencil $\mathfrak{C}$. Denote by L_D the lineal spanned by the functions in $\mathcal{C}_c^\infty(G)$ and by functions of the form $\chi r^{i\mu_k} w_k$ with $\operatorname{Im}\mu_k < 0$, where χ is a cut-off function equal to 1 near the conical point. We also introduce the lineal L_N spanned by the functions in $\mathcal{C}_c^\infty(\overline{G} \setminus \mathcal{O})$ with normal derivative vanishing on $\partial G \setminus \mathcal{O}$ and by functions of the form $\chi r^{i\widetilde{\mu}_k}\widetilde{w}_k$, where $\operatorname{Im}\widetilde{\mu}_k \le 0$. According to [18, §4] and [33, §3], the range of the Helmholtz operator $\tau^2 + \triangle$ with $\tau = \sigma - i\gamma$ $(\gamma \ne 0)$ given on L_D or L_N is dense in $L_2(G)$.

Theorem 2.34. 1) *Suppose that A is the self-adjoint extension of the differential operator $A(D_x)$ given on the lineal $D(G)$, where all the coefficients $\alpha_{s,k}$ are zeros.*

Let the operator A be taken as the spatial part of the system (2.3), (2.4). *Assume that the right-hand side $f = (-\vec{J}, -\vec{G}, \rho, \mu)$ of the system is subject to the conditions $\rho, \mu \in L_2(G)$, $\vec{J} \in H(\mathrm{div}\ , G)$, $\vec{G} \in \overset{\circ}{H}(\mathrm{div}\ , G)$ while $i\tau\rho + \mathrm{div}\ \vec{J} = 0$ and $i\tau\mu + \mathrm{div}\ \vec{G} = 0$. Then the corresponding strong solution u is of the form $u = (\vec{u}, \vec{v}, 0, 0)$.* 2) *If the role of spatial part is played by a self-adjoint extension distinct from that in* 1), *then there exist right-hand sides subject to the conditions in* 1) *such that the corresponding strong solutions has nonzero components h, q.*

Proof. Prove the first part of the theorem. Verify the equality $h = 0$. Since A is the closure of the differential operator $A(D_x)$ given on $D(G)$, there exists a sequence $\{u_k\} \subset D(G)$ such that $u_k \to u = (\vec{u}, \vec{v}, h, q)$ and $f_k := M(\tau)u_k \to f$ (the convergence in $L_2(G)$). We have $u_k \in C^\infty(\overline{G} \setminus \mathcal{O})$ so the system (2.3), (2.4) can be understood as usual. In particular,

$$\begin{gathered} i\tau\vec{u}_k - \mathrm{rot}\ \vec{v}_k + \nabla h_k = -\vec{J}_k, \\ i\tau h_k + \mathrm{div}\ \vec{u}_k = \rho_k. \end{gathered}$$

Moreover, u_k satisfies the boundary conditions (2.4) on $\partial G \setminus \mathcal{O}$. Assume that $\phi \in L_D$. Multiply the first equality by $\nabla\phi$, the second one by ϕ, and integrate over G. Then

$$i\tau\big(\vec{u}_k, \nabla\phi\big)_G - \big(\mathrm{rot}\ \vec{v}_k, \nabla\phi\big)_G + \big(\nabla h_k, \nabla\phi\big)_G = -\big(\vec{J}_k, \nabla\phi\big)_G\ ,$$

$$-\tau^2\big(h_k, \phi\big)_G + i\tau\big(\mathrm{div}\ \vec{u}_k, \phi\big)_G = i\tau\big(\rho_k, \phi\big)_G.$$

We integrate by parts in the two first terms of the first equality, then add the second line and obtain

$$-\tau^2\big(h_k, \phi\big)_G + \big(\nabla h_k, \nabla\phi\big)_G = i\tau\big(\rho_k, \phi\big)_G - \big(\vec{J}_k, \nabla\phi\big)_G.$$

We integrate the second term by parts, let $k \to +\infty$, and take into account that $-(\vec{J}, \nabla\phi)_G = (\mathrm{div}\vec{J}, \phi)_G$ and $\mathrm{div}\vec{J} + i\tau\rho = 0$. Then

$$\Big(h, (\overline{\tau}^2 + \triangle)\phi\Big)_G = 0.$$

Therefore $h = 0$ because the range of the operator $\overline{\tau}^2 + \triangle$ given on L_D is dense in $L_2(G)$.

Verify the equality $q = 0$. We have

$$\begin{gathered} i\tau\vec{v}_k + \mathrm{rot}\ \vec{u}_k + \nabla q_k = -\vec{G}_k, \\ i\tau q_k + \mathrm{div}\ \vec{v}_k = \mu_k. \end{gathered}$$

Assume that $\phi \in L_N$. Then

$$i\tau\big(\vec{v}_k, \nabla\phi\big)_G + \big(\mathrm{rot}\ \vec{u}_k, \nabla\phi\big)_G + \big(\nabla q_k, \nabla\phi\big)_G = -\big(\vec{G}_k, \nabla\phi\big)_G\ ,$$

$$-\tau^2\big(q_k, \phi\big)_G + i\tau\big(\mathrm{div}\ \vec{v}_k, \phi\big)_G = i\tau\big(\mu_k, \phi\big)_G.$$

We integrate by parts all the terms in the left-hand side of the first line, add the second line, and let $k \to \infty$:

$$\big(q, (\overline{\tau}^2 + \triangle)\phi\big)_G = i\tau\big(\mu, \phi\big)_G - \big(\vec{G}, \nabla\phi\big)_G.$$

Using the inclusion $\vec{G} \in \overset{\circ}{\mathrm{H}}(\mathrm{div}\ , G)$, $\phi \in L_N \subset H^1(G)$ and the condition $\mathrm{div}\,\vec{G} + i\tau\mu = 0$, we obtain

$$\Big(q, (\overline{\tau}^2 + \triangle)\phi\Big)_G = 0.$$

Since the range of the operator $\overline{\tau}^2 + \triangle$ given on L_N is dense in $L_2(G)$, we arrive at $q = 0$.

Let us turn to the second part of the theorem. The domain of any other self-adjoint extension contains at least one function $\chi r^{i\lambda_k}\Phi_{s,k}$ with $\mathrm{Im}\,\lambda_k \in]1,\ 3/2[$. In view of the properties of the pencil $\mathfrak{A}$, the eigenfunction $\Phi_{s,k}$ is of the form $(\vec{U},\ \vec{0},\ 0,\ Q)$ or $(\vec{0},\ \vec{V},\ H,\ 0)$. For instance, assume that $\Phi_{s,k} = (\vec{U},\ \vec{0},\ 0,\ Q)$. According to [18, §4], there exists a solution q_0 to the homogeneous Neumann problem for the equation $\triangle q_0 + \tau^2 q_0 = 0$ having the asymptotics $q_0 \sim r^{i\lambda_k}Q$ near the conical point $\mathcal{O}$ while $q_0 \in C^\infty(\overline{G} \setminus \mathcal{O})$. We set $u_0 := (\chi r^{i\lambda_k}\vec{U},\ \vec{0},\ 0,\ q_0)$ è $f_0 := M(\tau)u_0 = (i\tau\chi r^{i\lambda_k}\vec{U},\ \mathrm{rot}\ (\chi r^{i\lambda_k}\vec{U}) + \nabla q_0,\ \mathrm{div}\ (\chi r^{i\lambda_k}\vec{U}),\ i\tau q_0)$. It is not hard to see that $f_0 \in L_2(G) \cap C^\infty(\overline{G} \setminus \mathcal{O})$ and f_0 satisfies the conditions in part 1) of the theorem. However, the component q_0 of the solution u_0 differs from zero. We now assume that $\Phi_{s,k} = (\vec{0},\ \vec{V},\ H,\ 0)$. According to [33, Proposition 5.2], there exists a solution h_0 to the homogeneous Dirichlet problem for the equation $\triangle h_0 + \tau^2 h_0 = 0$ having the asymptotics $h_0 \sim r^{i\lambda_k}H$ while $h_0 \in C^\infty(\overline{G} \setminus \mathcal{O})$. We set $u_1 = (\vec{0},\ \chi r^{i\lambda_k}\vec{V},\ h_0,\ 0)$ and $f_1 := M(\tau)u_1 = (-\mathrm{rot}\ (\chi r^{i\lambda_k}\vec{V}) + \nabla h_0,\ i\tau\chi r^{i\lambda_k}\vec{V},\ i\tau h_0,\ \mathrm{div}\ (\chi r^{i\lambda_k}\vec{V}))$. The function f_1 satisfies the conditions in 1). However the component h_1 of u_1 differs from zero. $\square$

The extension A chosen in part 1 of Theorem 2.34 coincides with the operator investigated in [4, §2.2]. Before proving this, we recall some definitions and statements in [4] taking into account that, in the case under consideration, the dielectric and magnetic permittivity matrices are equal to the identity matrix. Set

$$F(G) = \{\vec{u} \in L_2(G) :\ \mathrm{div}\ \vec{u} \in L_2(G),\ \mathrm{rot}\ \vec{u} \in L_2(G)\}.$$

The class $F(G)$ is a complete Hilbert space with inner product defined by the norm

$$\|\vec{u};\ F(G)\| = \Big(\|\mathrm{div}\ \vec{u};\ L_2(G)\|^2 + \|\mathrm{rot}\ \vec{u};\ L_2(G)\|^2 + \|\vec{u};\ L_2(G)\|^2\Big)^{1/2}.$$

Introduce the closed subspaces

$$F(\tau, G) = \{\vec{u} \in F(G) :\ \vec{u} \times \vec{\nu} = 0\},\ F(\nu, G) = \{\vec{u} \in F(G) :\ \langle\vec{u},\ \vec{\nu}\rangle = 0\}.$$

The condition $\vec{u} \times \vec{\nu} = 0$ is understood in the following sense:

$$\Big(\vec{u},\ \mathrm{rot}\ \vec{z}\Big)_G = \Big(\mathrm{rot}\ \vec{u},\ \vec{z}\Big)_G\ \ \forall \vec{z} \in L_2(G)\ \text{ such that }\ \mathrm{rot}\ \vec{z} \in L_2(G).$$

The dense subsets $D(\nu, G) = F(\nu, G) \oplus \overset{\circ}{H}{}^1(G)$ and $D(\tau, G) = F(\tau, G) \oplus H^1(G)$ in the Hilbert space $B(G) = L_2(G, \mathbb{C}^3) \oplus L_2(G)$ are taken as domains of the operators $L(\nu)\{\vec{v},\ q\} = \{\mathrm{rot}\ \vec{v} + \nabla q,\ -\mathrm{div}\ \vec{v}\}$ and $L(\tau)\{\vec{u},\ h\} = \{\mathrm{rot}\ \vec{u} + \nabla h,\ -\mathrm{div}\ \vec{u}\}$. It turns out (see [4, Lemma 2.4]) that the block operator

$$\mathfrak{L} = \begin{pmatrix} 0 & iL(\nu) \\ -iL(\tau) & 0 \end{pmatrix}.$$

is self-adjoint in $B(G) \oplus B(G)$. Let P be the matrix defined by the equality $P(\vec{u},\ \vec{v},\ h, q)^T = (\vec{u},\ -q,\ \vec{v},\ h)^T$. We prove that $PAP^{-1} = \mathfrak{L}$. The properties of the pencil $\mathfrak{A}$ and the asymptotics of the functions in $D(G)$ near the conical point $\mathcal{O}$ imply that $P(D(G)) \subset\ D(\nu, G) \oplus D(\tau, G)$. The self-adjoint operator A is the closure of the differential operator $A(D_x)$ defined on $D(G)$. Since the operators $A(D_x)$ and $P^{-1}AP$ coincide on $D(G)$, we obtain $A = P^{-1}\mathfrak{L}P$.

3 THE PROBLEM IN A WEDGE AND IN A WAVEGUIDE WITH EDGE

3.1 Preliminaries

In this chapter we study the augmented Maxwell system

$$\begin{cases} \partial\vec{E}/\partial t - \mathrm{rot}\ \vec{B} + \nabla h = -\vec{J}, \\ \partial\vec{B}/\partial t + \mathrm{rot}\ \vec{E} + \nabla q = -\vec{G}, \\ \partial h/\partial t + \mathrm{div}\ \vec{E} = \rho, \\ \partial q/\partial t + \mathrm{div}\ \vec{B} = \mu \end{cases} \tag{3.1}$$

with boundary conditions

$$\vec{\nu} \times \vec{E} = 0,\ \langle \vec{B},\ \vec{\nu} \rangle_3 = 0,\ h = 0, \tag{3.2}$$

where $\vec{\nu}$ is the unit outward normal. The problem (3.1), (3.2) is considered in a wedge $\mathbb{D} = \mathbb{K} \times \mathbb{R}$ and in a waveguide $\Sigma = \Omega \times \mathbb{R}$. Rewrite the system (3.1) in the form

$$\partial u/\partial t + A(\partial)u = f,$$

where $u = (\vec{E},\ \vec{B},\ h,\ q)^T,\ \partial = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$.

Assume that $\tau = \sigma - i\gamma$, $\sigma \in \mathbb{R}$, $\gamma > 0$, and $\xi \in \mathbb{R}$. Applying the Fourier transform $\mathcal{F}_{(x_3,t)\to(\xi,\tau)}$ to (3.1),(3.2), we obtain the problems in $\mathbb{K}$ (in Ω) with parameter (ξ, τ):

$$\begin{cases} \tau\widehat{u} + A(D_{x_1}, D_{x_2}, \xi)\widehat{u} = -i\widehat{f}, \\ \Gamma\widehat{u} = 0. \end{cases}$$

Rewrite this problem as follows

$$M(D_x, \xi, \tau)u = f, \tag{3.3}$$

$$\Gamma u = 0, \tag{3.4}$$

where $M(D_x,\xi,\tau)=\tau+A(D_{x_1},D_{x_2},\xi)$. The Green formula (1.9) for the problem (3.3), (3.4) takes the form

$$\Big(M(D_x,\xi,\tau)u,v\Big)_{\mathbb{K}}+\Big(\Gamma u,Tv\Big)_{\partial\mathbb{K}}=\Big(u,M(D_x,\xi,\overline{\tau})v\Big)_{\mathbb{K}}+\Big(Tu,\Gamma v\Big)_{\partial\mathbb{K}}, \qquad (3.5)$$

where $T=-iT_0$. When considering the problem in $\mathbb{K}$, we may change the variables $\eta=(px_1,px_2)$, where $p=(|\tau|^2+|\xi|^2)^{1/2}$. Denote $(\xi/p,\tau/p)$ by θ, introduce the notation $M(D_\eta,\theta)=\tau/p+A(D_\eta,\xi/p)$ and $U(\eta,\xi,\tau)=\widehat{u}(p^{-1}\eta,\xi,\tau)$, $F(\eta,,\xi,\tau)=p^{-1}\widehat{f}(p^{-1}\eta,\xi,\tau)$ and rewrite (3.3), (3.4) in the form

$$M(D_\eta,\theta)U=F, \qquad (3.6)$$

$$\Gamma U=0. \qquad (3.7)$$

3.2 Operator pencil

We define the operator pencil

$$\mathfrak{B}(\lambda)\Phi(\varphi)=r^{1-i\lambda}A(D_{x_1},D_{x_2},0)r^{i\lambda}\Phi(\varphi)$$

for $\Phi\in H^1[-\alpha,\ \alpha]$ such that $r^{i\lambda}\Phi(\varphi)$ satisfy (3.4) on $\partial\mathbb{K}$, where (r,φ) are the polar coordinates. If $\Phi(\varphi)=(\vec{u}(\varphi),\ \vec{v}(\varphi),\ h(\varphi),\ q(\varphi))$, then the boundary conditions can be written as

$$\begin{cases} h(\alpha)=h(-\alpha)=u_3(\alpha)=u_3(-\alpha)=0,\\ u_1(\alpha)\cos\alpha+u_2(\alpha)\sin\alpha=0,\\ u_1(-\alpha)\cos\alpha-u_2(-\alpha)\sin\alpha=0,\\ v_1(\alpha)\sin\alpha-v_2(\alpha)\cos\alpha=0,\\ v_1(-\alpha)\sin\alpha+v_2(-\alpha)\cos\alpha=0.\end{cases}$$

In polar coordinates, the operator $A(D_{x_1},D_{x_2},0)$ is of the form

$$A(D_{x_1},D_{x_2},0)=A_1(\varphi)D_r+r^{-1}A_2(\varphi)D_\varphi, \qquad (3.8)$$

where A_1 and A_2 are 8×8-matrices. Let G stand for $A(0,0,1)$. We have

$$\begin{gathered} A_1\cdot A_1=I,\ A_2\cdot A_2=I,\ A_1\cdot A_2+A_2\cdot A_1=0,\\ \frac{dA_1}{d\varphi}=A_2,\ \frac{dA_2}{d\varphi}=-A_1,\\ G\cdot G=I,\ A_1\cdot G+G\cdot A_1=0,\ G\cdot A_2+A_2\cdot G=0.\end{gathered} \qquad (3.9)$$

The eigenfunctions and eigenvalues of the pencil $\mathfrak{B}$ satisfy $\mathfrak{B}(\lambda)\Phi=0$. Taking (3.8) into account, we obtain

$$\lambda A_1(\varphi)\Phi(\varphi)+A_2(\varphi)D_\varphi\Phi(\varphi)=0. \qquad (3.10)$$

From the formulas for A_1 and A_2 and the boundary conditions for Φ, it follows that the eigenfunctions and eigenvalues of $\mathfrak{B}$ can be found by solving the two Sturm-Liouville problems for the system

$$\begin{cases} da/d\varphi+i\lambda b=0,\\ db/d\varphi-i\lambda a=0\end{cases}$$

with boundary conditions

$$1)\ a(\alpha) = a(-\alpha) = 0,$$
$$2)\ a(\alpha)\sin\alpha - b(\alpha)\cos\alpha = 0,\ a(-\alpha)\sin\alpha + b(-\alpha)\cos\alpha = 0.$$

Thus the spectrum of $\mathfrak{B}$ consists of the two sequences of eigenvalues

$$\lambda_{k,1} = i\frac{\pi k}{2\alpha},\ \lambda_{k,2} = i(\frac{\pi k}{2\alpha} + 1),\ k \in \mathbb{Z}.$$

Using the link of the Sturm-Liouville problems with (3.10), one can show that two linearly independent eigenfunctions correspond to every eigenvalues while there are no associated functions. Moreover, the eigenvalues $\lambda_{k,1}$ correspond to the components $h,\ q,\ u_3,\ v_3$ while $\lambda_{k,2}$ to $u_1,\ u_2,\ v_1,\ v_2$. The eigenvalues of $\mathfrak{B}$ are symmetrically located about the point $i/2$. If $i/2$ is an eigenvalue, we denote it by λ_0. Let λ_k with $k > 0$ stand for the eigenvalues of $\mathfrak{B}$ such that $\operatorname{Im}\lambda_k > 1/2$ numbered in order of increasing imaginary part. By λ_{-k} with $k > 0$ we denote the eigenvalue symmetric to λ_k about the point $i/2$. The same argument as in Section 2.2 shows that the eigenfunctions $\{\Phi_{s,\mp k}\}_{s=1,2}$ corresponding to $\lambda_{\mp k}$ can be chosen to satisfy the orthogonality and normalization conditions

$$\int_{-\alpha}^{\alpha} \langle \partial_\lambda \mathfrak{B}(\lambda_k)\Phi_{s,k}, \Phi_{p,-k}\rangle_8 \, \mathrm{d}\varphi = \delta_{s,p}.$$

If Φ is an eigenfunction of $\mathfrak{B}$ corresponding to λ_k, then $G\Phi$, too, is an eigenfunction corresponding to the same eigenvalue λ_k.

The properties of the pencil $\mathfrak{B}$ discussed in this section are summarized in the next proposition.

PROPOSITION 3.1. 1) *To every eigenvalue λ_k of $\mathfrak{B}$ with $k \neq 0$ there correspond two linearly independent eigenfunctions $\{\Phi_{s,k}\}_{s=1,2}$ satisfying $G\Phi_{1,k} = \Phi_{2,k}$. There are no associated functions. The eigenfunctions can be chosen so that*

$$\int_{-\alpha}^{\alpha} \langle A_1(\varphi)\Phi_{s,k}(\varphi), \Phi_{p,m}(\varphi)\rangle_8 \, \mathrm{d}\varphi = \delta_{s,p}\cdot\delta_{k,-m}. \tag{3.11}$$

2) *If $\alpha = \pi$, then $\lambda_0 = i/2$ is an eigenvalue of the pencil $\mathfrak{B}$. There are four linearly independent functions $\{\Phi_{s,\mp 0}\}_{s=1,2}$ corresponding to this eigenvalue. The functions can be chosen to satisfy the orthogonality and normalization conditions*

$$\int_{-\alpha}^{\alpha} \langle A_1(\varphi)\Phi_{s,\mp 0}(\varphi), \Phi_{p,\pm 0}(\varphi)\rangle_8 \, \mathrm{d}\varphi = \delta_{s,p},\ \int_{-\alpha}^{\alpha} \langle A_1(\varphi)\Phi_{s,\pm 0}(\varphi), \Phi_{p,\pm 0}(\varphi)\rangle_8 \, \mathrm{d}\varphi = 0,$$
$$\int_{-\alpha}^{\alpha} \langle A_1(\varphi)\Phi_{s,k}(\varphi), \Phi_{p,\pm 0}(\varphi)\rangle_8 \, \mathrm{d}\varphi = 0$$

for $k \neq 0$.

We put

$$v_{s,k} = r^{i\lambda_k}\Phi_{s,k},\ v_{s,\mp 0} = r^{i\lambda_0}\Phi_{s,\mp 0}. \tag{3.12}$$

The functions $v_{s,k}$ satisfy the boundary value problem $A(D_{x_1}, D_{x_2}, 0)v_{s,k} = 0$ in $\mathbb{K}$, $\Gamma v_{s,k} = 0$ on $\partial\mathbb{K}$.

3.3 On properties of the operator $A(D)$

Let us consider the elliptic problem in a wedge

$$\begin{cases} A(D_{x_1}, D_{x_2}, D_{x_3})\mathcal{U}(x_1, x_2, x_3) = \mathcal{F}(x_1, x_2, x_3),\ (x_1, x_2, x_3) \in \mathbb{D}, \\ \Gamma\mathcal{U}(x_1, x_2, x_3) = \mathcal{H}(x_1, x_2, x_3),\ (x_1, x_2, x_3) \in \partial\mathbb{D} \setminus M. \end{cases} \tag{3.13}$$

In this section, we study the operator $\mathcal{A}_\beta$ of problem (3.13),

$$\mathcal{A}_\beta = \{A(D), \Gamma\}\ :\ H^1_\beta(\mathbb{D}) \to H^0_\beta(\mathbb{D}) \times H^{1/2}_\beta(\partial\mathbb{D}),$$

and find out the numbers β for which the operator is an isomorphism. We apply the Fourier transform $\mathcal{F}_{x_3\to\xi}$ to problem (3.13) and obtain the family of problems

$$\begin{cases} A(D_{x_1}, D_{x_2}, \xi)\widehat{\mathcal{U}}(x_1, x_2, \xi) = \widehat{\mathcal{F}}(x_1, x_2, \xi),\ (x_1, x_2) \in \mathbb{K}, \\ \Gamma\widehat{\mathcal{U}}(x_1, x_2, \xi) = \widehat{\mathcal{H}}(x_1, x_2, \xi),\ (x_1, x_2) \in \partial\mathbb{K} \setminus \mathcal{O}. \end{cases} \tag{3.14}$$

We introduce the new variables

$$\eta = |\xi|(x_1, x_2),\ \omega = \xi/|\xi|,\ U(\eta, \xi) = \widehat{\mathcal{U}}(\eta/|\xi|, \xi),$$
$$F(\eta, \xi) = |\xi|^{-1}\widehat{\mathcal{F}}(\eta/|\xi|, \xi),\ H(\eta, \xi) = \widehat{\mathcal{H}}(\eta/|\xi|, \xi),$$

and rewrite (3.14) in the form

$$\begin{cases} A(D_\eta, \omega)U(\eta, \xi) = F(\eta, \xi),\ \eta \in \mathbb{K}, \\ \Gamma U(\eta, \xi) = H(\eta, \xi),\ \eta \in \partial\mathbb{K} \setminus \mathcal{O}. \end{cases} \tag{3.15}$$

Consider the operator

$$\mathcal{A}_\beta(\omega) = \{A(D_\eta, \omega), \Gamma\}\ :\ E^1_\beta(\mathbb{K}) \to E^0_\beta(\mathbb{K}) \times E^{1/2}_\beta(\partial\mathbb{K}).$$

THEOREM 3.2. *If $\alpha < \pi$, then the operator $\mathcal{A}_\beta$ is an isomorphism for* $\beta \in]\max\{0, 1 - \pi/2\alpha\},\ \min\{1, \pi/2\alpha\}[$.

Proof. According to [28, Theorem 8.2.1], the operator $\mathcal{A}_\beta$ implements isomorphism if and only if $\mathcal{A}_\beta(\omega)$ is an isomorphism for $\omega = \mp 1$. The operator $\mathcal{A}_\beta(\omega)$ is Fredholm if and only if the line $\operatorname{Im}\lambda = \beta$ is free from the spectrum of the pencil $\mathfrak{B}$ (see [28, Theorem 8.2.3]). We show that the kernel and cokernel of $\mathcal{A}_\beta(\omega)$ are trivial in some interval $\operatorname{Im}\lambda_k < \beta < \operatorname{Im}\lambda_{k+1}$. Hence $\mathcal{A}_\beta(\omega)$ is an isomorphism for such β. Outside the interval, the operator $\mathcal{A}_\beta(\omega)$ is not an isomorphism (see [28, §8.3]).

Let U be a solution in $E^1_\beta(\mathbb{K})$ to the homogeneous problem (3.15). Then ([28, Proposition 8.2.6]) $\chi U \in E^l_\gamma(\mathbb{K})$, where χ is any function in $C^\infty(\overline{\mathbb{K}})$ vanishing near the vertex, the numbers $l \geq 1$ and $\gamma \in \mathbb{R}$ being arbitrary. We apply $A(-D_\eta, \omega)$ to the equality $A(D_\eta,\omega)U=0$ and obtain $(-\triangle_\eta+1)U=0$. Denote by $(\vec{u},\ \vec{v},\ h,\ q)$ the components of U. We are now looking for β such that $h=0$. The function h satisfies

$$\begin{cases}(-\triangle_\eta+1)h=0,\ \eta\in\mathbb{K},\\ \quad h=0,\ \eta\in\partial\mathbb{K}\setminus\mathcal{O}.\end{cases}$$

Multiply $(-\triangle_\eta+1)h=0$ by h and integrate over $\mathbb{K}_\varepsilon=\{\eta\in\mathbb{K}\ :\ |\eta|>\varepsilon\}$. Integrating by parts and taking into account the boundary condition, we arrive at

$$\int\limits_{\mathbb{K}_\varepsilon}(|h|^2+|\nabla h|^2)d\eta_1 d\eta_2=\varepsilon\int\limits_{-\alpha}^{\alpha}h\cdot\frac{\partial h}{\partial\rho}\,\mathrm{d}\phi, \tag{3.16}$$

where (ρ,ϕ) are polar coordinates in the plane $\mathbb{R}^2_\eta$. Let χ be a cut-off function equal to 1 near the vertex. We have $\chi h\in H^1_\beta(\mathbb{K})$. According to [28, Theorem 4.2.1], the representation

$$h=\sum_k c_{s,k}\rho^{i\lambda_k}H_{s,k}+R \tag{3.17}$$

holds in a neighborhood of the vertex, where $\chi R\in H^1_\gamma(\mathbb{K})$, $1>\beta-\gamma>0$, and λ_k is an eigenvalue of $\mathfrak{B}$ which is an element of the sequence $\{\pi m/2\alpha\}_{m\in\mathbb{Z}\setminus 0}$. The sum consists of the terms for the indicated eigenvalues in the strip $\operatorname{Im}\lambda\in]\gamma,\ \beta[$, while $H_{s,k}$ are components of the eigenfunctions $\Phi_{s,k}$ of $\mathfrak{B}$ corresponding to h. In order that the right-hand side of (3.16) tends to 0 as $\varepsilon\to 0$, the inequality $\operatorname{Im}\lambda_k<0$ is necessary for all the terms in (3.17). Since $h\in E^1_\beta(\mathbb{K})$ we have $\operatorname{Im}\lambda_k<\beta$. Assume that $\beta<\pi/2\alpha$. Then $\operatorname{Im}\lambda_k\leq-\pi/2\alpha$ for all the terms in (3.17). Therefore $h=0$ in $\mathbb{K}$. Similarly, u_3 vanishes in $\mathbb{K}$ for the same β. Now the equality $A(D_\eta,\omega)U=0$ implies that

$$D_{\eta_1}u_1+D_{\eta_2}u_2=-\omega u_3=0,\ \eta\in\mathbb{K},$$
$$D_{\eta_2}v_1-D_{\eta_1}v_2=-\omega h=0,\ \eta\in\mathbb{K}.$$

We consider $u_1,\ u_2$ and put $\vec{w}=(u_1,u_2,0)$. Rewrite $(-\triangle_\eta+1)\vec{w}=0$ in the form

$$\operatorname{rot}\operatorname{rot}\vec{w}-\nabla\operatorname{div}\vec{w}+\vec{w}=0.$$

We calculate the inner product of this expression and $\vec{w}$ and integrate it over $\mathbb{K}_\varepsilon\times I_{\eta_3}$, where $I=[0,\ 1]$. Taking account of the boundary conditions $\vec{u}\times\vec{\nu}=0,\ D_{\eta_1}u_1+D_{\eta_2}u_2=0$ on $\partial\mathbb{D}$ and applying the Stokes formula, we obtain

$$\int\limits_{\mathbb{K}_\varepsilon}(|\operatorname{rot}\vec{w}|^2+|\operatorname{div}\vec{w}|^2+|\vec{w}|^2)d\eta_1 d\eta_2=\varepsilon\int\limits_{-\alpha}^{\alpha}\Big(\langle\operatorname{rot}\vec{w},\ [\vec{\nu}\times\vec{w}]\rangle+\operatorname{div}\vec{w}\cdot\langle\vec{\nu},\ \vec{w}\rangle\Big)\,\mathrm{d}\phi, \tag{3.18}$$

where $\vec{\nu}=(\cos\phi,\ \sin\phi,\ 0)$. Let λ_{k_0} be the only eigenvalue of $\mathfrak{B}$ in the sequence $\{i(\pi m/2\alpha+1)\}_{m\in\mathbb{Z}}$ located in the strip $\operatorname{Im}\lambda\in\,]\operatorname{Im}\lambda_{k_0},\ \beta[$. The asymptotic representation

$$\vec{w}=\sum_{s=1}^{N_{k_0}}c_{s,k_0}(r^{i\lambda_{k_0}}\vec{W}_{s,k_0}+r^{i\lambda_{k_0}+1}\tau\vec{\Psi}_s)+\sum_k c_{s,k}r^{i\lambda_k}\vec{W}_{s,k}+R \tag{3.19}$$

holds near the vertex. The third component of every term in the first sum vanishes, while the two first components are equal to those of the partial sum $U_{s,k_0,2}$ (consisting of two terms) of the formal series (2.29). The two first components of $\vec{W}_{s,k}$ are equal to those of $\Phi_{s,k}$, and the third component vanishes. The second sum consists of the terms corresponding to the eigenvalues λ_k of the pencil $\mathfrak{B}$ which are elements of the sequence $\{i(\pi m/2\alpha+1)\}_{m\in\mathbb{Z}}$ located in the strip $\operatorname{Im}\lambda \in]\operatorname{Im}\lambda_{k_0}-1,\ \operatorname{Im}\lambda_{k_0}[$. The function R is a remainder. Using the definition of $\mathfrak{B}$, one can check that $\operatorname{rot} r^{i\lambda_k}\vec{W}_{s,k}=0,\ \operatorname{div} r^{i\lambda_k}\vec{W}_{s,k}=0$. Therefore the right-hand side in (3.18) behaves as $\varepsilon^{1-2\operatorname{Im}\lambda_k}$, i.e., under the condition $\operatorname{Im}\lambda_{k_0}<1/2$, it tends to zero as $\varepsilon\to 0$. Such a condition is fulfilled in the case $\beta<1$. Therefore, if $\beta<1$, then $u_1=u_2=0$ in $\mathbb{K}$. The same argument shows that if $\beta<1$, then $v_1=v_2=0$. The equality $A(D_\eta,\omega)U=0$ implies that

$$\begin{aligned}-D_{\eta_2}u_1+D_{\eta_1}u_2&=-\omega q=0,\\ D_{\eta_1}v_1+D_{\eta_2}v_2&=-\omega v_3=0.\end{aligned}$$

Thus, if $\beta<\min\{1,\pi/2\alpha\}$, then the kernel of $\mathcal{A}_\beta(\omega)$ is trivial. Consider the adjoint operator $\mathcal{A}_\beta(\omega)^*$. Let $V\in \operatorname{Ker}\mathcal{A}_\beta(\omega)^*$. According to [28, Theorem 8.3.3], $V\in\operatorname{Ker}\mathcal{A}_{1-\beta}(\omega)$. Hence, $V\equiv 0$ provided $1-\beta<\min\{1,\pi/2\alpha\}$. Thus, if

$$\max\{0,1-\pi/2\alpha\}<\beta<\min\{1,\pi/2\alpha\},$$

then the operator $\mathcal{A}_\beta(\omega)$ is an isomorphism. The operator $\mathcal{A}_\beta$ implements an isomorphism under the same condition. □

Note that if $2\alpha<\pi$, then $\mathcal{A}_\beta$ is an isomorphism for $\beta\in]0,\ 1[$. If $\pi\le 2\alpha<2\pi$, then $\mathcal{A}_\beta$ is an isomorphism for $\beta\in]1-\pi/2\alpha,\ \pi/2\alpha[$. In the case $2\alpha=2\pi$, the operator $\mathcal{A}_\beta$ is not an isomorphism for any β.

3.4 Estimates on solutions to problems in a wedge and in an angle

In this section, we prove a global energy estimate and a weighted combined estimate on solutions to the problem (3.3), (3.4) in $\mathbb{K}$ and in Ω. We drop some proofs similar to those for the problems in $\mathcal{K}$ and in G.

Proposition 3.3. *Let v be in $\mathcal{C}_c^\infty(\overline{\mathbb{K}}\setminus\mathcal{O})$ satisfying the boundary conditions* (3.4). *Then*

$$\gamma\|v;L_2(\mathbb{K})\|\le\|M(D_{x_1},D_{x_2},\xi,\tau)v;L_2(\mathbb{K})\|,\tag{3.20}$$

where $M(D_{x_1},D_{x_2},\xi,\tau)=\tau+A(D_{x_1},D_{x_2},\xi),\ \tau=\sigma-i\gamma\ (\sigma\in\mathbb{R},\ \gamma>0),\ \xi\in\mathbb{R}$.

Remark 3.4. *The estimate (3.20) is still valid with τ replaced by $\overline{\tau}$.*

We define lineals similar to $D(\mathcal{K})$, $D_\beta(\mathcal{K})$ in Sections 2.3, 2.4. Since the spectrum of the pencil $\mathfrak{B}$ is known, we can describe the lineals $D(\mathbb{K})$ and $D_\beta(\mathbb{K})$ in more detail.

DEFINITION 3.5. *Define the lineal $D(\mathbb{K})$ by the following conditions:*

A) If $2\alpha \le \pi$, then $D(\mathbb{K})$ is spanned by the functions in $\mathcal{C}_c^\infty(\overline{\mathbb{K}} \setminus \mathcal{O})$ satisfying the boundary conditions (3.4) and by the functions $\chi v_{s,k}$ corresponding to the eigenvalues of $\mathfrak{B}$ subject to the inequality $\operatorname{Im}\lambda_k < 1$; here χ stands for a cut-off function equal to 1 near the vertex.

B) If $2\alpha \in \,]\pi,\ 2\pi[$, then $D(\mathbb{K})$ is spanned by the functions in $\mathcal{C}_c^\infty(\overline{\mathbb{K}} \setminus \mathcal{O})$ satisfying (3.4), by the functions $\chi v_{s,k}$ corresponding to the eigenvalues of $\mathfrak{B}$ subject to $\operatorname{Im}\lambda_k \le 0$, and by the linear combinations $\chi(\alpha_s v_{s,1} + \beta_s v_{s,-1})$, where α_s, β_s $(s = 1, 2)$ are fixed coefficients such that $\operatorname{Re}\alpha_s\overline{\beta_s} = 0$ and $|\alpha_s| + |\beta_s| > 0$.

C) If $2\alpha = 2\pi$, then $D(\mathbb{K})$ is spanned by the functions in $\mathcal{C}_c^\infty(\overline{\mathbb{K}} \setminus \mathcal{O})$, subject to (3.4), by the functions $\chi v_{s,k}$ corresponding to the eigenvalues of $\mathfrak{B}$ subject to $\operatorname{Im}\lambda_k \le 0$, and by the linear combinations $\chi(\alpha_s v_{s,+0} + \beta_s v_{s,-0})$, where α_s, β_s $(s = 1, 2)$ are fixed coefficients satisfying $\operatorname{Re}\alpha_s\overline{\beta_s} = 0$ and $|\alpha_s| + |\beta_s| > 0$.

The next assertion is analogous to Proposition 2.12.

PROPOSITION 3.6. *The estimate* (3.20) *holds for any function in $D(\mathbb{K})$.*

We now define a lineal $D_\beta(\mathbb{K})$ and prove a combined estimate in $\mathbb{K}$.

DEFINITION 3.7. *Let $\beta < \min\{1, \pi/2\alpha\}$. Introduce the lineal $D_\beta(\mathbb{K})$ spanned by the functions in $\mathcal{C}_c^\infty(\overline{\mathbb{K}}\setminus\mathcal{O})$ satisfying the boundary conditions (3.4) and by the functions $\chi v_{s,k}$ corresponding to the eigenvalues of $\mathfrak{B}$ subject to the inequality $\operatorname{Im}\lambda < \beta$.*

PROPOSITION 3.8. *Let $\beta < \min\{1, \pi/2\alpha\}$ and let the number $\lambda = i\beta$ be regular for the pencil $\mathfrak{B}$. Then the inequality*

$$\begin{aligned} &\gamma^2\|v; H^0_\beta(\mathbb{K})\|^2 + \|\chi_\tau v; H^1_\beta(\mathbb{K}, p)\|^2 \\ &\le c\{\|f; H^0_\beta(\mathbb{K})\|^2 + (|\tau|^{2(1-\beta)}/\gamma^2)\|f; L_2(\mathbb{K})\|^2\} \end{aligned} \tag{3.21}$$

holds for any $v \in D_\beta(\mathbb{K})$, where $f = (\tau + A(D_{x_1}, D_{x_2}, \xi))v$, $\chi_\tau(r) = \chi(|\tau|r)$, and χ is a cut-off function in $\mathcal{C}_c^\infty(\overline{\mathbb{K}})$ equal to 1 near the vertex. The constant c is independent of ξ and τ.

Proof. Step 1. Estimating in a neighborhood of the edge. Applying the Fourier transform $\mathcal{F}_{t\to\tau}$ to the problem (3.1), (3.2) in the cylinder $\mathcal{T}$, we arrive at the problem in the wedge $\mathbb{D}$

$$\begin{cases} \tau\widehat{\mathcal{U}} + A(D_x)\widehat{\mathcal{U}} = \widehat{\mathcal{F}},\ x \in \mathbb{D}, \\ \Gamma\widehat{\mathcal{U}} = 0,\ x \in \partial\mathbb{D} \setminus M, \end{cases}$$

with parameter τ. Introduce the new variables

$$\zeta = |\tau|x,\ U(\zeta,\tau) = \widehat{\mathcal{U}}(\zeta/|\tau|,\tau),\ F(\zeta,\tau) = |\tau|^{-1}\widehat{\mathcal{F}}(\zeta/|\tau|,\tau),$$

and rewrite the problem in the form

$$\begin{cases} (\tau/|\tau|)U + A(D_\zeta)U = F,\ \zeta = (\zeta_1,\zeta_2,\zeta_3) \in \mathbb{D}, \\ \Gamma U = 0,\ \zeta \in \partial\mathbb{D} \setminus M. \end{cases}$$

If $\beta < \min\{1, \pi/2\alpha\}$ and the number $\lambda = i\beta$ is regular for the pencil $\mathfrak{B}$, then the operator $\mathcal{A}_\beta(\omega)$ is Fredholm with trivial kernel (see the proof of Theorem 3.2). The inequality

$$\|u; E^1_\beta(\mathbb{K})\| \leq c\|\mathcal{A}_\beta(\omega)u; E^0_\beta(\mathbb{K})\|$$

holds for any function u in $E^1_\beta(\mathbb{K})$ satisfying the boundary conditions $\Gamma u = 0$. Using Proposition 1.1, one can show that the estimate

$$\|U; H^1_\beta(\mathbb{D})\| \leq c\|A(D_\zeta)U; H^0_\beta(\mathbb{D})\|$$

is valid for U in $H^1_\beta(\mathbb{D})$ subject to $\Gamma U = 0$. Hence

$$\|\chi U; H^1_\beta(\mathbb{D}, 1)\| \leq c\{\|\chi A(D_\zeta)U; H^0_\beta(\mathbb{D})\| + \|\psi U; H^0_\beta(\mathbb{D})\|\},$$

where $\chi = \chi(r)$ and $\psi = \psi(r)$ are smooth cut-off functions equal to 1 near the vertex such that $\chi\psi = \chi$. Rewrite the inequality in the form

$$\|\chi U; H^1_\beta(\mathbb{D}, 1)\| \leq c\{\|\chi M(D_\zeta, \tau/|\tau|)U; H^0_\beta(\mathbb{D})\| + \|\psi U; H^0_\beta(\mathbb{D})\|\}. \tag{3.22}$$

Step 2. Estimating far from the edge. We prove that, for any $\beta \in \mathbb{R}$ and each $U \in H^1_\beta(\mathbb{D}, 1)$ satisfying the boundary condition $\Gamma U = 0$, the inequality

$$\begin{aligned} &(\gamma/|\tau|)^2\|\kappa_\infty U; H^0_\beta(\mathbb{D})\|^2 \leq \\ &\leq c\{\|\kappa_\infty M(D_\zeta, \tau/|\tau|)U; H^0_\beta(\mathbb{D})\|^2 + \|\psi_\infty U; H^0_{\beta-1}(\mathbb{D})\|^2\} \end{aligned} \tag{3.23}$$

holds with constant c independent of U and τ; here the functions $\kappa_\infty = \kappa_\infty(r)$, $\psi_\infty = \psi_\infty(r)$ are smooth in $\overline{\mathbb{K}}$, vanish near the vertex, and equal 1 in a neighborhood of infinity while $\kappa_\infty\psi_\infty = \kappa_\infty$.

Let $\kappa, \psi \in C^\infty(\overline{\mathbb{K}})$, $\kappa\psi = \kappa$, supp $\kappa \subset \{(x_1, x_2) \in \mathbb{K} : 1/2 < r < 2\}$, supp $\psi \subset \{(x_1, x_2) \in \mathbb{K} : 1/4 < r < 4\}$. The estimate (3.20) and Parseval's equality imply that

$$\gamma^2\|\kappa U; L_2(\mathbb{D})\|^2 \leq \|M(D_\zeta, \tau)\kappa U; L_2(\mathbb{D})\|^2.$$

Since $M\kappa U = \kappa MU + [M, \kappa]U$, we have

$$\gamma^2\|\kappa U; L_2(\mathbb{D})\|^2 \leq c\{\|\kappa M(D_\zeta, \tau)U; L_2(\mathbb{D})\|^2 + \|\psi U; L_2(\mathbb{D})\|^2\}.$$

As U, we take $\zeta \mapsto U^\varepsilon(\zeta) = U(\zeta/\varepsilon)$ and change τ for $\tau/|\tau|\varepsilon$, where $\varepsilon > 0$. Then

$$\begin{aligned} &(\gamma/|\tau|\varepsilon)^2\|\kappa U^\varepsilon; L_2(\mathbb{D})\|^2 \leq \\ &\leq c\{\|\kappa M(D_\zeta, \tau/|\tau|\varepsilon)U^\varepsilon, L_2(\mathbb{D})\|^2 + \|\psi U^\varepsilon; L_2(\mathbb{D})\|^2\}. \end{aligned}$$

After the change of variables $\zeta \mapsto \zeta/\varepsilon$ we obtain

$$\begin{aligned} &(\gamma/|\tau|)^2\|\kappa_\varepsilon U; L_2(\mathbb{D})\|^2 \leq \\ &\leq c\{\|\kappa_\varepsilon M(D_\zeta, \tau/|\tau|)U; L_2(\mathbb{D})\|^2 + \varepsilon^2\|\psi_\varepsilon U; L_2(\mathbb{D})\|^2\}, \end{aligned}$$

where $\kappa_\varepsilon(\eta) = \kappa(\varepsilon\eta)$. Multiplying the estimate by $\varepsilon^{-2\beta}$, putting $\varepsilon = 2^{-j}, j = 1, 2, 3, ...$, and adding the inequalities, we arrive at (3.23).

Step 3. Estimating in an intermediate zone. As in the proof of Proposition 2.20 (Step 3), from (3.22) and (3.23) we deduce the inequality

$$\begin{aligned}\gamma^2\|u; H^0_\beta(\mathbb{D})\|^2 + \|\chi_\tau u; H^1_\beta(\mathbb{D}, |\tau|)\|^2 \le \\ \le c\{\|M(D_x,\tau)u; H^0_\beta(\mathbb{D})\|^2 + \\ (|\tau|^{1-\beta}/\gamma)^2\|M(D_x,\tau)u; L_2(\mathbb{D})\|^2\}.\end{aligned} \tag{3.24}$$

Step 4 Estimating in the angle $\mathbb{K}$. Let $u(x_1,x_2,x_3) = v(x_1,x_2)\psi(x_3)$, where $v \in D_\beta(\mathbb{K})$ and ψ, $e^{-\gamma t}\psi \in \mathcal{S}(\mathbb{R})$. Since ψ is arbitrary, the estimate (3.24) and Parseval's equality lead to (3.21). □

The lineals $D(\Omega)$ and $D_\beta(\Omega)$ are defined like $D(\mathbb{K})$ and $D_\beta(\mathbb{K})$.

PROPOSITION 3.9. *The estimate*

$$\gamma\|v; L_2(\Omega)\| \le \|M(D_{x_1}, D_{x_2}, \xi, \tau); L_2(\Omega)\|$$

holds for any v in $D(\Omega)$

PROPOSITION 3.10. *Let $\beta < \min\{1, \pi/2\alpha\}$ and let the number $\lambda = i\beta$ be regular for the pencil $\mathfrak{B}$. Assume that $\gamma > \gamma_0$ with sufficiently large γ_0. Then*

$$\begin{aligned}\gamma^2\|v; H^0_\beta(\Omega)\|^2 + \|\chi_\tau v; H^1_\beta(\Omega, p)\|^2 \le \\ \le c(\gamma_0)\{\|f; H^0_\beta(\Omega)\|^2 + (|\tau|^{2(1-\beta)}/\gamma^2)\|f; L_2(\Omega)\|^2\}\end{aligned}$$

for every v in $D_\beta(\Omega)$, where $f = (\tau + A(D_{x_1}, D_{x_2}, \xi))v$, $\chi_\tau(r) = \chi(|\tau|r)$, and χ is any fixed cut-off function in $\mathcal{C}^\infty_c(\overline{\Omega})$ equal to 1 near the point $\mathcal{O}$. The constant c is independent of (ξ, τ) and v.

We introduce the spaces $\mathcal{D}H_\beta(\mathbb{K}, \xi, \tau)$ and $\mathcal{R}H_\beta(\mathbb{K}, \xi, \tau)$ by completing $\mathcal{C}^\infty_c(\overline{\mathbb{K}} \setminus 0)$ with respect to the norms

$$\|v; \mathcal{D}H_\beta(\mathbb{K}, \xi, \tau)\| = \left(\gamma^2\|v; H^0_\beta(\mathbb{K})\|^2 + \|\chi_\tau v; H^1_\beta(\mathbb{K}, p)\|^2\right)^{1/2},$$
$$\|f; \mathcal{R}H_\beta(\mathbb{K}, \xi, \tau)\| = \left(\|f; H^0_\beta(\mathbb{K})\|^2 + (|\tau|^{1-\beta}/\gamma)^2\|f; L_2(\mathbb{K})\|^2\right)^{1/2}.$$

Changing $\mathbb{K}$ for Ω, we define $\mathcal{D}H_\beta(\Omega, \xi, \tau)$, $\mathcal{R}H_\beta(\Omega, \xi, \tau)$.

3.5 The operators of problems in $\mathbb{K}$ and Ω

We associate with problem (3.3), (3.4) in $\mathbb{K}$ the unbounded operator $v \mapsto M(\xi,\tau)v := M(D_{x_1}, D_{x_2}, \xi, \tau)v$ on $L_2(\mathbb{K})$ with domain $DM(\xi,\tau) := D(\mathbb{K})$. As for the operator $M(\tau)$ in the previous chapter, it is easy to show that $M(\xi,\tau)$ admits closure and the estimate

$$\gamma\|v; L_2(\mathbb{K})\| \le \|\overline{M(\xi,\tau)}v; L_2(\mathbb{K})\|$$

holds for the functions in the domains of the closed operator. In what follows, $M(\xi,\tau)$ and $DM(\xi,\tau)$ denote the closed operator and its domain. The proof of the next assertion is similar to those of Propositions 2.13 and Proposition 2.14.

Proposition 3.11. Ker $M(\xi, \tau) = 0$, $\mathcal{R}M(\xi, \tau) = L_2(\mathbb{K})$.

The operator $A(D_{x_1}, D_{x_2}, 0)$ with domain $D(\mathbb{K})$ is symmetric. It follows from Proposition 3.11 that the closure A of $A(D_{x_1}, D_{x_2}, 0)$ is a self-adjoint operator. If $2\alpha > \pi$, then $D(\mathbb{K})$ contains the combinations $\chi(\alpha_s v_{s,1} + \beta_s v_{s,-1})$ $(s = 1, 2)$, where α_s, β_s are fixed coefficients satisfying $\operatorname{Re} \alpha_s \overline{\beta}_s = 0$, $|\alpha_s| + |\beta_s| > 0$. Thus for $2\alpha > \pi$, one can define various self-adjoint extensions of $A(D_{x_1}, D_{x_2}, 0)$ by choosing the parameters $\{\alpha_s, \beta_s\}_{s=1,2}$. In what follows we take as A any of the extensions, unless otherwise indicated.
Let us turn to the problem in a scale of weighted spaces. For the problem (3.3), (3.4) we introduce the unbounded operator $v \mapsto M_\beta(\xi, \tau) := M(D_{x_1}, D_{x_2}, \xi, \tau)$ with domain $D_\beta(\mathbb{K})$, acting from $\mathcal{D}H_\beta(\mathbb{K}, \xi, \tau)$ to $\mathcal{R}H_\beta(\mathbb{K}, \xi, \tau)$. The operator $M_\beta(\xi, \tau)$ admits closure. We will denote by $M_\beta(\xi, \tau)$ and $DM_\beta(\xi, \tau)$ the closed operator and its domain. Let $1/2 > \beta_1 > \beta_2 > \dots$ be all the numbers in $]-\infty,\ 1/2[$ such that $\lambda = i\beta_k$ is an eigenvalue of the pencil $\mathfrak{B}$. Denote by S_m the total multiplicity of the eigenvalues of $\mathfrak{B}$ in the strip $\operatorname{Im} \lambda \in\ [\beta_m,\ \beta_1]$.

Definition 3.12. *A solution to the equation $M(\xi, \tau)v = f$ is called a strong solution to the problem (3.3), (3.4) with right-hand side $f \in L_2(\mathbb{K})$.*

Definition 3.13. *A solution to the equation $M_\beta(\xi, \tau)v = f$ is called a strong β-solution to the problem (3.3), (3.4) with right-hand side $f \in \mathcal{R}H_\beta(\mathbb{K}, \xi, \tau)$.*

The next result follows from Proposition 3.11.

Theorem 3.14. *For every f in $L_2(\mathbb{K})$ and any $\tau = \sigma - i\gamma$ $(\sigma \in \mathbb{R},\ \gamma > 0)$, and $\xi \in \mathbb{R}$, there exists a unique strong solution v to the problem* (3.3), (3.4) *with right-hand side f. Moreover,*

$$\gamma \|v; L_2(\mathbb{K})\| \le \|f; L_2(\mathbb{K})\|.$$

The next assertion can be verified in the same way as Theorem 2.24.

Theorem 3.15. *Assume that $\alpha < \pi$.*
A)*Let $\beta \in]\beta_1,\ \min\{1, \pi/2\alpha\}[$. Then for every $f \in \mathcal{R}H_\beta(\mathbb{K}, \xi, \tau)$ there exists a unique strong β-solution v to the problem* (3.3), (3.4) *in $\mathbb{K}$ with right-hand side f. Moreover,*

$$\|v; \mathcal{D}H_\beta(\mathbb{K}, \xi, \tau)\| \le c\|f; \mathcal{R}H_\beta(\mathbb{K}, \xi, \tau)\|.$$

B) *Let $\beta \in\ ]\beta_{m+1},\ \beta_m[$. A strong β-solution to the problem* (3.3), (3.4) *with right-hand side $f \in \mathcal{R}H_\beta(\mathbb{K}, \xi, \tau)$ exists and is unique under the following S_m conditions: $(f, w_{s,k})_{\mathbb{K}} = 0$, where $\{w_{s,k}\}_{k=-1,..,-m}^{s=1,2}$ is a basis in* Ker $M_\beta(\xi, \tau)^*$. *The solution satisfies the estimate in part* A) of the theorem.

Remark 3.16. *Theorem 3.14 is valid for the problem (3.3), (3.4) in Ω. Theorem 3.15 is still true for the problem (3.3), (3.4) in Ω under the supplementary condition $\gamma > \gamma_0$ with sufficiently large γ_0.*

Now we formulate the remark concerning the angle $\mathbb{K}$ of opening $2\alpha = 2\pi$. In this case the interval of unique solvability vanishes. For $\beta \in]\beta_1,\ 1/2[$ there exists the cokernel of dimension 2 corresponding to the eigenvalue $\lambda_0 = i/2$ of the pencil $\mathfrak{B}$. The elements $w_{1,-0},\ w_{2,-0}$ of the cokernel near the corner point admit the asymptotics $w_{s,-0} \sim \alpha_s v_{s,+0} + \beta_s v_{s,-0}$ with $\operatorname{Re} \alpha_s \overline{\beta}_s = 1,\ s = 1, 2$.

REMARK 3.17. *Assume that $\alpha = \pi$. Let $\beta \in]\beta_{m+1},\ \beta_m[$. A strong β-solution to the problem* (3.3), (3.4) *with right-hand side $f \in \mathcal{R}H_\beta(\mathbb{K}, \xi, \tau)$ exists and is unique under the following $S_m + 2$ conditions: $(f, w_{s,k})_{\mathbb{K}} = 0$, where $\{w_{s,k}\}_{k=-0,..,-m}^{s=1,2}$ is a basis in* $\operatorname{Ker} M_\beta(\xi, \tau)^*$. *The solution satisfies the following estimate*

$$\|v; \mathcal{D}H_\beta(\mathbb{K}, \xi, \tau)\| \le c\|f; \mathcal{R}H_\beta(\mathbb{K}, \xi, \tau)\|.$$

3.6 The problems in the cylinders T and $\mathcal{T}$

Applying the Fourier transform $\mathcal{F}^{-1}_{(\xi,\tau)\to(x_3,t)}$, we pass to the problem (3.1), (3.2) in $\mathcal{T}$ (or $\mathbf{T}$).

DEFINITION 3.18. *Let $f \in V_0^0(\mathcal{T}, \gamma)$ and let $\widehat{u}(x_1, x_2, \xi, \tau)$ be a strong solution to the problem (3.3), (3.4) in $\mathbb{K}$ with right-hand side $-i\widehat{f}$, where*

$$\widehat{f}(x_1, x_2, \xi, \tau) = \mathcal{F}_{(x_3,t)\to(\xi,\tau)} f(x_1, x_2, x_3, t).$$

The function u defined by $u(x_1, x_2, x_3, t) = \mathcal{F}^{-1}_{(\xi,\tau)\to(x_3,t)} \widehat{u}(x_1, x_2, \xi, \tau)$ is called a strong solution to the problem (3.1), (3.2) in $\mathcal{T}$ with right-hand side f.

Theorem 3.14 leads to the following result.

THEOREM 3.19. *For every $f \in V_0^0(\mathcal{T}, \gamma)$ and any $\gamma > 0$ there exists a strong solution v to the problem* (3.1), (3.2) *with right-hand side f. Moreover,*

$$\gamma\|v; V_0^0(\mathcal{T}, \gamma)\| \le \|f; V_0^0(\mathcal{T}, \gamma)\|.$$

We fix a cut-off function $\chi \in \mathcal{C}^\infty(\overline{\mathbb{K}})$ equal to 1 near the corner point $\mathcal{O}$. Put

$$\begin{aligned} Xu(x_1, x_2, x_3, t) &= \mathcal{F}^{-1}_{\tau\to t} \chi(|\tau| x_1, |\tau| x_2) \mathcal{F}_{t'\to\tau} u(x_1, x_2, x_3, t'), \\ \Lambda^\mu u(x_1, x_2, t) &= \mathcal{F}^{-1}_{\tau\to t} |\tau|^\mu \mathcal{F}_{t'\to\tau} u(x_1, x_2, x_3, t'), \\ P^\mu u(x_1, x_2, x_3, t) &= \mathcal{F}^{-1}_{(\xi,\tau)\to(x_3,t)} p^\mu \mathcal{F}_{(x_3',t')\to(\xi,\tau)} u(x_1, x_2, x_3', t'). \end{aligned}$$

Here Λ is the same operator as in Section 2.7 while X differs in that the cut-off function is independent of x_3. Introduce the spaces $\mathcal{D}V_\beta(\mathcal{T}, \gamma)$, $\mathcal{R}V_\beta(\mathcal{T}, \gamma)$ with norms

$$\begin{aligned} \|u; \mathcal{D}V_\beta(\mathcal{T}, \gamma)\| &= \big(\gamma^2 \|u; V_\beta^0(\mathcal{T}, \gamma)\|^2 + \|Xu; V_\beta^1(\mathcal{T}, \gamma)\|^2\big)^{1/2}, \\ \|f; \mathcal{R}V_\beta(\mathcal{T}, \gamma)\| &= \big(\|f; V_\beta^0(\mathcal{T}, \gamma)\|^2 + (1/\gamma^2)\|\Lambda^{1-\beta} f; V_0^0(\mathcal{T}, \gamma)\|^2\big)^{1/2}. \end{aligned}$$

Definition 3.20. *Let* $f \in \mathcal{R}V_\beta(\mathcal{T},\gamma)$ *and let* $\widehat{u}(x_1,x_2,\xi,\tau)$ *be a strong* β*-solution to the problem (3.3), (3.4) in* $\mathbb{K}$ *with right-hand side* $-i\widehat{f}$*, where*

$$\widehat{f}(x_1,x_2,\xi,\tau) = \mathcal{F}_{(x_3,t)\to(\xi,\tau)} f(x_1,x_2,x_3,t).$$

The function u *given by* $u(x_1,x_2,x_3,t) = \mathcal{F}^{-1}_{(\xi,\tau)\to(x_3,t)}\, \widehat{u}(x_1,x_2,\xi,\tau)$ *is called a strong* β*-solution to the problem (3.1), (3.2) in* $\mathcal{T}$ *with right-hand side* f*.*

The next assertion follows from Theorem 3.15.

Theorem 3.21. *Suppose that* $\alpha < \pi$*.*
A) Let $\beta \in]\beta_1,\ \min\{1,\pi/2\alpha\}[$. Then for every function $f \in \mathcal{R}V_\beta(\mathcal{T},\gamma)$ there exists a unique strong β-solution v to the problem (3.1) (3.2) with right-hand side f. Moreover,

$$\|v;\mathcal{D}V_\beta(\mathcal{T},\gamma)\| \le c\|f;\mathcal{R}V_\beta(\mathcal{T},\gamma)\|.$$

B) Let $\beta \in]\beta_{m+1},\ \beta_m[$. A strong β-solution to the problem (3.1), (3.2) with right-hand side $f \in \mathcal{R}V_\beta(\mathcal{T},\gamma)$ exists and is unique under the following S_m conditions $(\widehat{f}(\cdot,\xi,\tau), w_{s,k}(\cdot,\xi,\overline{\tau}))_{\mathbb{K}} = 0$ for all $\xi \in \mathbb{R}$ and $\tau \in \mathbb{R} - i\gamma$, where $\{w_{s,k}\}_{k=-1,..,-m}^{s=1,2}$ is a basis in Ker $M_\beta(\xi,\tau)^*$. Such a solution satisfies the estimate in part A) of the theorem.

The following theorem on the asymptotics of solutions near an edge is analogous to Theorem 2.32. One can prove it by modifying in an evident way the proof of Theorem 2.26 and by applying the Fourier transform $\mathcal{F}^{-1}_{(\xi,\tau)\to(x_3,t)}$. For the sake of simplicity we exclude the case of screen. Introduce the formal series

$$R_{s,k}(r,\varphi,\xi,\tau) = \sum_{q=0}^{\infty} r^{i\lambda_k+q}(M(0,0,\xi,\tau))^q \Psi_q^{(s,k)}(\varphi),\ \Psi_0^{(s,k)} = \Phi_{s,k},$$

satisfying the homogeneous problem (3.3), (3.4). Denote by $R_{s,k,T}$ the first T terms of $R_{s,k}$.

Theorem 3.22. *Let* $\alpha < \pi$*. Assume that* f *and* $\Lambda^{1-\beta}P^\beta f$ *are in* $\mathcal{R}V_\beta(\mathcal{T},\gamma)$ *with* $\beta \in]\beta_{m+1},\beta_m[$*. Then the strong solution to problem* (3.1), (3.2) *in* T *admits the asymptotic representation*

$$u(x,t) = \sum R_{s,k,T}(r,\varphi,D_{x_3},D_t)(X\check{c}_{s,k})(x,t) + w(x,t) \tag{3.25}$$

with $w \in \mathcal{D}V_\beta(\mathcal{T},\gamma)$*. The sum consists of the terms corresponding to the eigenvalues of* $\mathfrak{B}$ *in the strip* $\operatorname{Im}\lambda \in [\beta_m,\ \beta_1]$*. The coefficients are defined by* $\check{c}_{s,k}(x_3,t) = \mathcal{F}^{-1}_{(\xi,\tau)\to(x_3,t)} c_{s,k}(\xi,\tau)$ *and* $c_{s,k} = \big(\widehat{f}(\cdot,\xi,\tau), w_{s,k}(\cdot,\xi,\overline{\tau})\big)_{\mathbb{K}}$*, the collection* $\{w_{s,k}\}_{k=-1,..,-m}^{s=1,2}$ *being the basis in* Ker $M_\beta(\xi,\tau)^*$ *as before. Moreover,*

$$\|e^{-\gamma t}\check{c}_{s,k}(\cdot);H^{\operatorname{Im}\lambda_k-\beta}(\mathbb{R}^2)\| \le c\|f;\mathcal{R}V_\beta(\mathcal{T},\gamma)\|,$$
$$\|w;\mathcal{D}V_\beta(\mathcal{T},\gamma)\| \le (c/\gamma)\|\Lambda^{1-\beta}P^\beta f;\mathcal{R}V_\beta(\mathcal{T},\gamma)\|.$$

Remark 3.23. *Theorem 3.19 is valid for problem (3.1), (3.2) in* $\mathbf{T}$*. Theorem 3.21 and Theorem 3.22 remain true for the problem (3.1), (3.2) in* $\mathbf{T}$ *under the condition* $\gamma > \gamma_0$ *with sufficiently large* γ_0*.*

3.7 Explicit formulas for the coefficients in the asymptotics of solutions to the problem in $\mathcal{T}$

In Theorem 3.22, the coefficients were expressed in terms of the elements $\{w_{s,k}\}_{k=-1,..,-m}^{s=1,2}$ of a basis in the kernel of $M_\beta(\xi,\tau)^*$. In the section, we explicitly calculate the mentioned elements. Note that the elements satisfy the equation $M(D_x, D_y, \xi, \overline{\tau})w_{s,k} = 0$ and the boundary conditions (3.4). Near the vertex of $\mathbb{K}$, the functions $w_{s,k}$ admit the asymptotic representations

$$w_{s,k} = r^{i\lambda_{-k}}\Phi_{s,-k}(\varphi) + o(r^{i\lambda_{-k}}).$$

From (3.23) it follows that $w_{s,k}$ decay more rapidly than any power of the distance far from the vertex. Since

$$c_{s,k}(\xi,\tau) = \int_{\mathbb{K}} \langle \widehat{f}(x_1,x_2,\xi,\tau), w_{s,k}(x_1,x_2,\xi,\tau)\rangle_8 \, \mathrm{d}x_1\, \mathrm{d}x_2,$$

we have to find $h_{s,k} = \overline{w_{s,k}}$. It is clear that $M(D_{x_1}, D_{x_2}, -\xi, -\tau)h_{s,k} = 0$ because $M(D_{x_1}, D_{x_2}, \xi, \overline{\tau})w_{s,k} = 0$. In polar coordinates

$$M = \tau I + \xi G + A_1(\varphi)D_r + (1/r)A_2(\varphi)D_\varphi.$$

Introduce

$$\widetilde{h}_{s,k}(r,\varphi) = (i\tau\eta(r)I - iG\xi\eta(r) + A_1(\varphi)\eta'(r))r^{i\lambda_{-k}}\Phi_{s,-k}(\varphi),$$

where η is a scalar function. Such a representation of $\widetilde{h}_{s,k}$ is motivated by the corresponding argument for the Helmholtz equation in [17, §3], [18, §4.2] and by the fact that $M(D_{x_1}, D_{x_2}, \xi, \tau)\cdot M(-D_{x_1}, -D_{x_2}, -\xi, \tau) = \triangle + \tau^2 - \xi^2$. Denote $i\tau\eta I - i\xi\eta G + A_1(\varphi)\eta'$ by $S(r,\varphi,\xi,\tau)$. Then the equation $M(D_{x_1}, D_{x_2}, -\xi, -\tau)\widetilde{h}_{s,k} = 0$ can be written in the form

$$(-\tau - \xi G + A_1(\varphi)D_r + (1/r)A_2(\varphi)D_\varphi)r^{i\lambda_{-k}}S(r,\varphi,\xi,\tau)\Phi_{s,-k}(\varphi) = 0. \tag{3.26}$$

Being eigenfunctions of the pencil $\mathfrak{B}$, $\Phi_{s,-k}$ satisfy equation (3.10), therefore

$$\frac{d\Phi_{s,-k}(\varphi)}{d\varphi} = -i\lambda_{-k}A_2(\varphi)A_1(\varphi)\Phi_{s,-k}(\varphi). \tag{3.27}$$

We now transform (3.26) and write λ and Φ for λ_{-k} and $\Phi_{s,-k}$ to simplify notation.

$$\begin{gathered}-(i\tau + i\xi G)r^{i\lambda}S\Phi + i\lambda r^{i\lambda-1}A_1S\Phi + r^{i\lambda}A_1(dS/dr)\Phi \\ +r^{i\lambda-1}A_2(dS/d\varphi)\Phi + r^{i\lambda}S(d\Phi/d\varphi) = 0.\end{gathered}$$

Taking (3.27) into account, we obtain

$$\begin{gathered}(-i\tau - i\xi G)S + (i\lambda/r)A_1S + A_1(dS/dr) \\ +(1/r)A_2(dS/d\varphi) - (i\lambda/r)A_2SA_2A_1 = 0.\end{gathered} \tag{3.28}$$

Substituting S and its derivatives into (3.28) and making use of (3.9), we arrive at

$$\eta'' + \frac{2i\lambda_{-k}+1}{r}\eta' + (\tau^2-\xi^2)\eta = 0,$$

which reduces to Bessel's equation. As a solution, we take a function $\eta(r) = Cr^\nu K_\nu(i\sqrt{\tau^2-\xi^2}r)$, $\nu = i\lambda_{-k}$ assuming that $\eta(r) \to 0$ as $r \to +\infty$. Here by $K_\nu(z)$ we denote the Macdonald functions, which are also called the modified bessel functions of the second kind. The asymptotic representations

$$\eta(r) = \frac{C\pi 2^{\nu-1}}{i^\nu(\tau^2-\xi^2)^{\nu/2}\sin(\pi\nu)\Gamma(1-\nu)} + o(1),$$

$$\eta'(r) = o(1)$$

hold as $r \to 0$. This and the equality $G\Phi_{1,-k} = \Phi_{2,-k}$ (see Proposition 3.1) lead to

$$\begin{aligned}\widetilde{h}_{1,k} &= \eta(0)(i\tau\Phi_{1,-k}(\varphi) - i\xi\Phi_{2,-k}(\varphi))r^{i\lambda_{-k}} + o(r^{i\lambda_{-k}}),\\ \widetilde{h}_{2,k} &= \eta(0)(i\tau\Phi_{2,-k}(\varphi) - i\xi\Phi_{1,-k}(\varphi))r^{i\lambda_{-k}} + o(r^{i\lambda_{-k}}).\end{aligned}$$

We put

$$h_{1,k} = \tau\widetilde{h}_{1,k} + \xi\widetilde{h}_{2,k},\ h_{2,k} = \tau\widetilde{h}_{2,k} + \xi\widetilde{h}_{1,k}.$$

Then, in a neighborhood of the vertex,

$$\begin{aligned}h_{1,k} &= i\eta(0)(\tau^2-\xi^2)r^{i\lambda_{-k}}\Phi_{1,-k} + o(r^{i\lambda_{-k}}),\\ h_{2,k} &= i\eta(0)(\tau^2-\xi^2)r^{i\lambda_{-k}}\Phi_{2,-k} + o(r^{i\lambda_{-k}}).\end{aligned}$$

We fix the constant C assuming that $i\eta(0)(\tau^2-\xi^2) = 1$. Therefore,

$$C = \frac{\sin(\pi\nu)\Gamma(1-\nu)}{\pi 2^{\nu-1}} i^{\nu-1}(\tau^2-\xi^2)^{(\nu-2)/2}.$$

Finally, applying the formula $\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin(\pi\nu)$, we obtain

$$\eta(r) = \frac{1}{i2^{\nu-1}\Gamma(\nu)}\frac{1}{\tau^2-\xi^2}\big(ir\sqrt{\tau^2-\xi^2}\big)^\nu K_\nu\big(ir\sqrt{\tau^2-\xi^2}\big),$$

$$\eta'(r) = \frac{1}{i2^{\nu-1}\Gamma(\nu)} r\big(ir\sqrt{\tau^2-\xi^2}\big)^{\nu-1} K_{\nu-1}\big(ir\sqrt{\tau^2-\xi^2}\big).$$

Let B stand for $1/(2^{\nu-1}\Gamma(\nu))$. Recall that

$$h_{1,k} = i(\tau^2-\xi^2)\eta(r)r^{i\lambda_{-k}}\Phi_{1,-k} + \tau A_1\eta'(r)r^{i\lambda_{-k}}\Phi_{1,-k} + \xi A_1\eta'(r)r^{i\lambda_{-k}}\Phi_{2,-k},$$

$$h_{2,k} = i(\tau^2-\xi^2)\eta(r)r^{i\lambda_{-k}}\Phi_{2,-k} + \xi A_1\eta'(r)r^{i\lambda_{-k}}\Phi_{1,-k} + \tau A_1\eta'(r)r^{i\lambda_{-k}}\Phi_{2,-k}.$$

To find $W_{s,k} = \mathcal{F}^{-1}_{(\xi,\tau)\to(x_3,t)}h_{s,k}$, we have to calculate the Fourier transform $\mathcal{F}^{-1}_{(\xi,\tau)\to(x_3,t)}$ of the functions

$$\begin{gathered}\big(ir\sqrt{\tau^2-\xi^2}\big)^\nu K_\nu\big(ir\sqrt{\tau^2-\xi^2}\big)\ ,\\ \big(ir\sqrt{\tau^2-\xi^2}\big)^{\nu-1} K_{\nu-1}\big(ir\sqrt{\tau^2-\xi^2}\big).\end{gathered}$$

Denote these functions by $f_\nu(r,x_3,t)$ and $f_{\nu-1}(r,x_3,t)$, respectively. Then

$$\begin{aligned} W_{1,k}(x,t) = {} & Bf_\nu(r,x_3,t)r^{i\lambda_{-k}}\Phi_{1,-k} - BA_1(\varphi)\frac{\partial f_{\nu-1}}{\partial t}r^{i\lambda_{-k}}\Phi_{1,-k} \\ & -BA_1(\varphi)\frac{\partial f_{\nu-1}}{\partial x_3}r^{i\lambda_{-k}}\Phi_{2,-k}\ , \\ W_{2,k}(x,t) = {} & Bf_\nu(r,x_3,t)r^{i\lambda_{-k}}\Phi_{2,-k} - BA_1(\varphi)\frac{\partial f_{\nu-1}}{\partial t}r^{i\lambda_{-k}}\Phi_{2,-k} \\ & -BA_1(\varphi)\frac{\partial f_{\nu-1}}{\partial x_3}r^{i\lambda_{-k}}\Phi_{1,-k}\ . \end{aligned}$$

We now calculate $f_p,\ p=\nu,\nu-1,$

$$f_p(r,x_3,t) = \int_{\mathbb{R}} \mathrm{d}\xi \int_{\mathbb{R}-i\gamma} \mathrm{d}\tau\ e^{ix_3\xi+it\tau}\big(ir\sqrt{\tau^2-\xi^2}\big)^p K_p\big(ir\sqrt{\tau^2-\xi^2}\big).$$

After the change of variables $u=i(\tau+\xi)/2, s=i(\tau-\xi)/2$, we have

$$f_p(r,x_3,t) = 2\int_{\operatorname{Re}u=\gamma/2} \mathrm{d}u \int_{\operatorname{Re}s=\gamma/2} \mathrm{d}s\ e^{u(t+x_3)+s(t-x_3)}\big(2ru^{1/2}s^{1/2}\big)^p K_p(2ru^{1/2}s^{1/2}).$$

The equality (see [3])

$$\int_{\operatorname{Re}s=\gamma/2} \mathrm{d}s\ e^{st}s^{\nu/2}K_\nu(2\alpha^{1/2}s^{1/2}) = \frac{1}{2}\ \theta(t)\ \alpha^{\nu/2}e^{-\alpha/t}\frac{1}{t^{\nu+1}}$$

leads to

$$f_p(r,x_3,t) = 2^p r^{2p}\int_{\operatorname{Re}u=\gamma/2} \mathrm{d}u\ e^{u(t+x_3)}u^p\ \theta(t-x_3)\ e^{-r^2u/(t-x_3)}\frac{1}{(t-x_3)^{p+1}}\ .$$

We now employ the formula (see [37, Chapter 2, §9.3])

$$f_\alpha(t) = \int_{\operatorname{Re}p=\gamma/2} e^{pt}\frac{1}{p^\alpha}\,\mathrm{d}p\ ,\ f_\alpha(t) = \begin{cases} \dfrac{\theta(t)}{\Gamma(\alpha)}t^{\alpha-1}, \alpha>0, \\ \\ \dfrac{d^N f_{\alpha+N}(t)}{dt^N},\ \alpha\le 0,\ \alpha+N>0 \end{cases}$$

and obtain

$$f_p(r,x_3,t) = 2^p r^{2p}\theta(t-x_3)\frac{1}{(t-x_3)^{p+1}}\frac{d^N}{dv^N}f_{N-p}(v)|_{v=(t^2-x_3^2-r^2)/(t-x_3)},$$

where $p=\nu,\ \nu-1\ (\nu>0)$. Thus, we have got the explicit expressions for $h_{s,k}$ and $W_{s,k}$, which participate in the following representations of the coefficients in the asymptotic formula (3.25):

$$c_{s,k}(\xi,\tau) = \int_{\mathbb{K}} \langle\widehat{f}(x_1,x_2,\xi,\tau), h_{s,k}(x_1,x_2,\xi,\tau)\rangle_{\mathbb{R}^8}\ \mathrm{d}x_1\,\mathrm{d}x_2,$$

$$\check{c}_{s,k}(x_3,t) = \int_{\mathbb{K}}\int_{\mathbb{R}}\int_{\mathbb{R}} \langle f(x_1,x_2,x_3-s,t-u), W_{s,k}(x_1,x_2,s,u)\rangle_{\mathbb{R}^8}\ \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}s\,\mathrm{d}u.$$

3.8 Connection between the augmented and non-augmented Maxwell systems

In this section we briefly discuss the connection between the augmented and the usual Maxwell systems. Here the case of screen is excluded, in other words we assume that $\alpha < \pi$. The main result is essentially the same as in Chapter 2. Namely, if the right-hand side of the augmented system is subject to the compatibility conditions of the usual Maxwell system, then the strong solution is of the form $(\vec{u}, \vec{v}, 0, 0)$, hence $(\vec{u}, \vec{v})$ satisfies the usual Maxwell system. Here we consider the problem in Ω, though all the results are valid for the problem in $\mathbb{K}$.

Introduce the space

$$H(\mathrm{div}\ , \Omega) = \{\vec{u} \in L_2(\Omega, \mathbb{C}^2) :\ \mathrm{div}\ \vec{u} \in L_2(\Omega)\}$$

with norm $\|\vec{u};\ H(\mathrm{div}\ , \Omega)\| = \Big(\|\vec{u};\ L_2(\Omega)\|^2 + \|\mathrm{div}\ \vec{u};\ L_2(\Omega)\|^2\Big)^{1/2}$ and its closed subspace

$$\overset{\circ}{H}(\mathrm{div}\ , \Omega) = \{\vec{u} \in H(\mathrm{div}\ , \Omega)\ :\ \langle\vec{u},\ \vec{\nu}\rangle = 0\}.$$

Here div is understood in the sense of distributions and the boundary condition $\langle\vec{u},\ \vec{\nu}\rangle = 0$ means that

$$\int\limits_{\Omega} \langle\vec{w}, \nabla\psi\rangle_3\ \mathrm{d}x + \int\limits_{\Omega} \mathrm{div}\,\vec{w}\cdot\psi\ \mathrm{d}x = 0\ \ \forall\ \psi \in H^1(\Omega).$$

Recall that for $2\alpha > \pi$ one can define various self-adjoint extensions of $A(D_{x_1}, D_{x_2}, 0)$ by choosing the parameters $\{\alpha_s, \beta_s\}_{s=1,2}$ such that $\mathrm{Re}\,\alpha_s\overline{\beta_s} = 0,\ |\alpha_s| + |\beta_s| > 0$. In the following theorem we consider the self-adjoint extension A with $\alpha_1 = \alpha_2 = 0$. If $2\alpha < \pi$ then we have the only self-adjoin extension, which we consider in the following theorem.

THEOREM 3.24. *Assume that $\alpha < \pi$.*

1) *Suppose that A is the self-adjoint extension of the differential operator $A(D_{x_1}, D_{x_2}, 0)$ given on the lineal $D(\Omega)$ with $\alpha_1 = \alpha_2 = 0$. Let the operator A be taken as the spatial part of the system* (3.3), (3.4). *Let $f = (-\vec{J}, -\vec{G}, \rho, \mu)$ be the right-hand side of the system with $\vec{G} = (G_1, G_2, G_3)$, $\vec{J} = (J_1, J_2, J_3)$, where $J_3, G_3, \rho, \mu \in L_2(\Omega)$, $\vec{j} = (J_1, J_2) \in H(\mathrm{div}\ , \Omega)$, $\vec{g} = (G_1, G_2) \in \overset{\circ}{H}(\mathrm{div}\ , \Omega)$, and $i\tau\rho + i\xi J_3 + \mathrm{div}\ \vec{j} = 0$, $i\tau\mu + i\xi G_3 + \mathrm{div}\ \vec{g} = 0$. Then the strong solution is of the form $u = (\vec{u}, \vec{v}, 0, 0)$.*

2) *Let a self-adjoint extension A be distinct from that in* 1). *Then there exist right-hand sides satisfying the conditions in* 1) *such that the components h, q of the corresponding strong solutions do not vanish.*

This theorem is proved in the same way as Theorem 2.34. For the proof we need two lineals. Let $\{\mu_k,\ w_k\}$ and $\{\widetilde{\mu}_k, \widetilde{w}_k\}$ be the sets of eigenvalues and eigenfunctions of the operator pencils of the Dirichlet and Neumann problems for the Laplace equation in Ω. Denote by L_D the lineal spanned by the functions in

$\mathcal{C}_c^\infty(\Omega)$ and by functions of the form $\chi r^{i\mu_k} w_k$ with $\operatorname{Im}\mu_k < 0$, where χ is a cut-off function equal to 1 near the conical point. We also introduce the lineal L_N spanned by the functions in $\mathcal{C}_c^\infty(\overline{\Omega} \setminus \mathcal{O})$ with normal derivative vanishing on $\partial\Omega \setminus \mathcal{O}$ and by functions of the form $\chi r^{i\widetilde{\mu}_k} \widetilde{w}_k$, where $\operatorname{Im}\widetilde{\mu}_k \leq 0$. According to [18, §4] and [33, §3], the range of the Helmholtz operator $\tau^2 - \xi^2 + \triangle$ with $\tau = \sigma - i\gamma$ ($\gamma \neq 0$) given on L_D or L_N is dense in $L_2(\Omega)$.

Let us turn to the problem in the waveguide Σ. Consider the lineal $D(\Omega)$ with $\alpha_1 = \alpha_2 = 0$ (see Definition 3.5). Let $D(\Sigma)$ stand for the lineal spanned by the functions in $D(\Omega)$ with coefficients in the Schwartz space $\mathcal{S}(\mathbb{R}_{x_3})$. Denote by A the closure in $L_2(\Sigma)$ of the differential operator $A(D_x)$ given on $D(\Sigma)$. It is not difficult to prove that A is a self-adjoint operator. Let P be the matrix defined by the equality $P(\vec{u},\ \vec{v},\ h, q)^T = (\vec{u},\ -q,\ \vec{v},\ h)^T$. It turns out that the operator PAP^{-1} coincides with the augmented Maxwell operator studied in [4, §2.2]. The proof is basically the same as the proof of the similar fact in Section 2.9.

4 THE PROBLEM WITH INHOMOGENEOUS CONDUCTIVE BOUNDARY CONDITIONS

4.1 Preliminaries

In the fourth chapter we study the augmented Maxwell system

$$\begin{cases} \partial\vec{E}/\partial t - \operatorname{rot}\,\vec{B} + \nabla h = -\vec{J}, \\ \partial\vec{B}/\partial t + \operatorname{rot}\,\vec{E} + \nabla q = -\vec{G}, \\ \qquad \partial h/\partial t + \operatorname{div}\,\vec{E} = \rho, \\ \qquad \partial q/\partial t + \operatorname{div}\,\vec{B} = \mu \end{cases} \tag{4.1}$$

in a model cone and in a bounded domain with conical point. The system is endowed it with the inhomogeneous boundary conditions

$$\vec{\nu} \times \vec{E} = \vec{C},\ \langle \vec{B},\, \vec{\nu} \rangle = \delta,\ h = \omega, \tag{4.2}$$

where $\vec{\nu}$ is the unit outward normal and $\vec{C}$ is tangent to the boundary. The system (4.1) is rewritten in the short form

$$\partial u/\partial t + A(\partial)u = f, \tag{4.3}$$

where $u = (\vec{E},\ \vec{B},\ h,\ q)^T,\ f = (-\vec{J},\ -\vec{G},\ \rho,\ \mu)^T,$ and $\partial = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$.
Let $\tau = \sigma - i\gamma$, where $\sigma \in \mathbb{R},\ \gamma > 0$. Applying $\mathcal{F}_{t\to\tau}$ to the problem (4.1), (4.2), we obtain the problem with parameter in the cone $\mathcal{K}$ (in the domain G):

$$\begin{cases} \tau\widehat{u} + A(D_x)\widehat{u} = -i\widehat{f}, \\ \qquad \Gamma\widehat{u} = \widehat{g}. \end{cases}$$

Let us rewrite this problem as follows

$$M(D_x, \tau)u = f, \tag{4.4}$$

$$\Gamma u = g, \tag{4.5}$$

where $M(D_x,\tau)=\tau+A(D_x)$. Note that for the problem (4.4), (4.5) the Green formula (1.9) is rewritten in the form

$$\Big(M(D_x,\tau)u,v\Big)_{\mathcal{K}}+\Big(\Gamma u,Tv\Big)_{\partial\mathcal{K}}=\Big(u,M(D_x,\overline{\tau})v\Big)_{\mathcal{K}}+\Big(Tu,\Gamma v\Big)_{\partial\mathcal{K}}, \qquad (4.6)$$

where $T=-iT_0$. Considering the problem (4.4), (4.5) in the cone $\mathcal{K}$, we can change the variables $\eta=(|\tau|x_1,|\tau|x_2,|\tau|x_3)$. Denote $\tau/|\tau|$ by θ, put $U(\eta,\tau)=u(|\tau|^{-1}\eta,\tau)$, $F(\eta,\tau)=|\tau|^{-1}f(|\tau|^{-1}\eta,\tau)$, and $G(\eta,\tau)=g(|\tau|^{-1}\eta,\tau)$. Thus the problem (4.4), (4.5) can be rewritten in the form

$$M(D_\eta,\theta)U=F, \qquad (4.7)$$

$$\Gamma U=G. \qquad (4.8)$$

With the problem (4.4), (4.5) we associate the operator pencil $\mathfrak{A}$ considered in Chapter 2. In this chapter we make use of the functions $u_{s,k}$ defined by (2.16) and the formal series $V_{s,k}$, $U_{s,k}$ defined by (2.21), (2.29).

4.2 Energy estimate

In this section we obtain a global energy estimate for solutions to the problem (4.4), (4.5) in a domain with smooth boundary and in a class of cones in space.

4.2.1 The main identity

In the fixed cartesian coordinates the operator $A(\partial)$ can be written in the form $A(\partial)=\sum_{k=1}^{3}g^k\partial_k$, where $\{g^k\}$ are the constant real symmetric 8×8 – matrices. Let us rewrite the equation (4.3):

$$\sum_{\alpha=0}^{3}\sum_{j=1}^{8}g^\alpha_{ij}\partial_\alpha u_j=f_i,\ i=1,..,8;$$

by ∂_0 and g^0 denote ∂_t and I (the identity matrix) respectively. Consider the following identity

$$\partial_\alpha\{r_{ik}\overline{u_k}g^\alpha_{ij}u_j\}=\partial_\alpha(r_{ik}\overline{u_k})g^\alpha_{ij}u_j+r_{ik}\overline{u_k}g^\alpha_{ij}\partial_\alpha u_j,$$

where $R=\{r_{ik}\}$ is a real 8×8 – matrix. By $\overline{z}$ we denote the complex conjugate of z. Note that repeated indices imply summation; here $\alpha=0,..,3$ and $i,j,k=1,..,8$. Multiplying both sides by $e^{-2\gamma t}$, we get

$$\begin{aligned}&\partial_\alpha\{e^{-2\gamma t}r_{ik}\overline{u_k}g^\alpha_{ij}u_j\}+2\gamma e^{-2\gamma t}r_{ik}\overline{u_k}u_i\\&=e^{-2\gamma t}\partial_\alpha(r_{ik}\overline{u_k})g^\alpha_{ij}u_j+e^{-2\gamma t}r_{ik}\overline{u_k}g^\alpha_{ij}\partial_\alpha u_j.\end{aligned} \qquad (4.9)$$

Let $\Omega\subset\mathbb{R}^3$ be a domain with smooth boundary and let $u\in\mathcal{C}^\infty_c(\overline{\Omega}\times\mathbb{R},\mathbb{C}^8)$. We integrate (4.9) over $\Omega\times\mathbb{R}$, applying the divergence theorem of Gauss to the first

We see that

$$Pu = (\vec{\nu} \times [\vec{u} \times \vec{\nu}],\ \vec{\nu}\langle \vec{v}, \vec{\nu}\rangle_3,\ h,\ 0)^T.$$

Note that the vector Pu corresponds to the boundary conditions (4.2) for the vector u. Let $Q = I - P$, then we have

$$Qu = (\vec{\nu}\langle \vec{u}, \vec{\nu}\rangle_3,\ \vec{\nu} \times [\vec{v} \times \vec{\nu}],\ 0,\ q)^T.$$

Taking into account the formulas for Pu and Qu, we obtain

$$P + Q = I,\ \langle Pu, Qv\rangle_8 = 0,$$
$$P^2 = P,\ Q^2 = Q.$$

Therefore P, Q are orthogonal projections and we have the decomposition

$$\mathbb{C}^8 = \mathcal{R}P \oplus \mathcal{R}Q,$$

where by $\mathcal{R}L$ we denote the range of L. Using the explicit formulas for $A(\vec{\nu}), P, Q$, we get

$$A(\vec{\nu})Pu = (\vec{\nu}h,\ \vec{\nu} \times [\vec{\nu} \times [\vec{u} \times \vec{\nu}]],\ 0,\ \langle \vec{v}, \vec{\nu}\rangle_3)^T,$$
$$A(\vec{\nu})Qu = (-\vec{\nu} \times [\vec{\nu} \times [\vec{v} \times \vec{\nu}]],\ \vec{\nu}q,\ \langle \vec{u}, \vec{\nu}\rangle_3,\ 0)^T,$$
$$A(\vec{\nu})Pu \perp \mathcal{R}P,\ A(\vec{\nu})Qu \perp \mathcal{R}Q.$$

These formulas imply the following representation for the matrix $A(\vec{\nu})$ in a basis adapted to the decomposition $\mathbb{C}^8 = \mathcal{R}P \oplus \mathcal{R}Q$:

$$A(\vec{\nu})u = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}, \quad \text{where } u = \begin{pmatrix} U \\ V \end{pmatrix},$$
$$Pu = \begin{pmatrix} U \\ 0 \end{pmatrix} \in \mathcal{R}P, \quad Qu = \begin{pmatrix} 0 \\ V \end{pmatrix} \in \mathcal{R}Q.$$

Since $A(\vec{\nu})$ is a real symmetric matrix and $A(\vec{\nu})A(\vec{\nu}) = I$, it follows that

$$A(\vec{\nu}) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}, \quad BB^T = B^TB = I.$$

Now we choose the matrix R in (4.10). Let $R = I$, then we get

$$\langle A(\vec{\nu})u, Ru\rangle_8 = \langle \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}, \begin{pmatrix} U \\ V \end{pmatrix} \rangle_8 = 2\mathrm{Re}\, \langle BU, V\rangle_4 . \tag{4.11}$$

Let $R = R(\vec{\nu})$ such that

$$R(\vec{\nu}) = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix},$$

then we obtain

$$\langle A(\vec{\nu})u, Ru\rangle_8 = \left\langle \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}, \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \right\rangle_8 = \\ = |BV|^2 - |B^TU|^2. \tag{4.12}$$

Let us find the representation of the matrix $R(\vec{\nu})$ in the initial standard basis. In a basis adapted to the decomposition $\mathbb{C}^8 = \mathcal{R}P \oplus \mathcal{R}Q$ we have

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, A(\vec{\nu}) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}, R(\vec{\nu}) = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix},$$

$$A(\vec{\nu})P = \begin{pmatrix} 0 & 0 \\ B^T & 0 \end{pmatrix}, A(\vec{\nu})Q = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}.$$

It now follows that $R(\vec{\nu}) = -A(\vec{\nu})P + A(\vec{\nu})Q$. Finally, applying the formulas for $A(\vec{\nu}), P, Q$, we obtain

$$R(\vec{\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 & \nu_3 & -\nu_2 & -\nu_1 & 0 \\ 0 & 0 & 0 & -\nu_3 & 0 & \nu_1 & -\nu_2 & 0 \\ 0 & 0 & 0 & \nu_2 & -\nu_1 & 0 & -\nu_3 & 0 \\ 0 & \nu_3 & -\nu_2 & 0 & 0 & 0 & 0 & \nu_1 \\ -\nu_3 & 0 & \nu_1 & 0 & 0 & 0 & 0 & \nu_2 \\ \nu_2 & -\nu_1 & 0 & 0 & 0 & 0 & 0 & \nu_3 \\ \nu_1 & \nu_2 & \nu_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\nu_1 & -\nu_2 & -\nu_3 & 0 & 0 \end{pmatrix}. \tag{4.13}$$

Using the formulas for $A(\vec{\nu})$ and $R(\vec{\nu})$, it can be easily checked that

$$A(\vec{a})R(\vec{b}) = -R(\vec{b})A(\vec{a}) \tag{4.14}$$

for all $\vec{a}, \vec{b} \in \mathbb{C}^3$.

4.2.3 Energy estimate for the problem (4.4), (4.5) in Ω

If we replace R by I in (4.10) and recall (4.11), we get

$$2\gamma \int_\Omega \int_\mathbb{R} \mathrm{d}x \, \mathrm{d}t \; e^{-2\gamma t} |u|^2 =$$
$$= 2\mathrm{Re} \int_\Omega \int_\mathbb{R} \mathrm{d}x \, \mathrm{d}t \; e^{-2\gamma t} \langle u, Mu \rangle_8 - 2\mathrm{Re} \int_{\partial\Omega} \int_\mathbb{R} \mathrm{d}S \, \mathrm{d}t \; e^{-2\gamma t} \langle BV, U \rangle_4 \, .$$

Let us apply the Cauchy inequality to the first term on the right

$$\gamma^2 \int_\Omega \int_\mathbb{R} \mathrm{d}x \, \mathrm{d}t \; e^{-2\gamma t} |u|^2 \leq$$
$$\leq c \Big(\int_\Omega \int_\mathbb{R} \mathrm{d}x \, \mathrm{d}t \; e^{-2\gamma t} |Mu|^2 - 2\gamma \mathrm{Re} \int_{\partial\Omega} \int_\mathbb{R} \mathrm{d}S \, \mathrm{d}t \; e^{-2\gamma t} \langle BV, U \rangle_4 \Big) \, . \tag{4.15}$$

Let $\zeta \in \mathcal{C}^\infty(\overline{\Omega})$ such that $\zeta = 1$ on the set $\{y \in \overline{\Omega} \; : \; \mathrm{dist}(y, \partial\Omega) < \delta\}$ and $\zeta = 0$ on $\{y \in \Omega \; : \; \mathrm{dist}(y, \partial\Omega) > 2\delta\}$, where δ is a small positive number. Let $\vec{n} \in \mathcal{C}^\infty(\overline{\Omega})$ be a smooth vector field such that $\vec{n}|_{\partial\Omega} = \vec{\nu}$. In (4.10) we put term on the left. Then we obtain

$$\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t} r_{ik}\overline{u_k}(g^1_{ij}\nu_1 + g^2_{ij}\nu_2 + g^3_{ij}\nu_3)u_j + 2\gamma \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t} r_{ik}\overline{u_k}u_i$$
$$= \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\partial_\alpha(r_{ik}\overline{u_k})g^\alpha_{ij}u_j + \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t} r_{ik}\overline{u_k}g^\alpha_{ij}\partial_\alpha u_j,$$

where $\vec{\nu} = (\nu_1,\ \nu_2,\ \nu_3)$ is the unit outward normal to Ω. Introduce the notation

$$A(\vec{\nu}) = \sum_{i=1}^{3} g^i \nu_i \quad\text{and}\quad M = \sum_{\alpha=0}^{3} g^\alpha \partial_\alpha.$$

Using these notations we rewrite the last identity in the vector form

$$\begin{aligned}
&\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t}\langle A(\vec{\nu})u, Ru\rangle_8 + 2\gamma \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle u, Ru\rangle_8 \\
&= \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle u, MRu\rangle_8 + \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle Mu, Ru\rangle_8.
\end{aligned} \tag{4.10}$$

Our aim is to estimate the energy $\|u;\ L_2(\Omega)\|$ by appropriate norms of the right-hand side $\{Mu, \Gamma u\}$. For this purpose we investigate the structure of the matrix $A(\vec{\nu})$ and choose a matrix R. Then, applying the main integral identity (4.10), we arrive at the required energy estimate.

4.2.2 Structure of the matrix $A(\vec{\nu})$

For the matrix $A(\vec{\nu})$ we have the formula

$$A(\vec{\nu}) = \begin{pmatrix}
0 & 0 & 0 & 0 & \nu_3 & -\nu_2 & \nu_1 & 0 \\
0 & 0 & 0 & -\nu_3 & 0 & \nu_1 & \nu_2 & 0 \\
0 & 0 & 0 & \nu_2 & -\nu_1 & 0 & \nu_3 & 0 \\
0 & -\nu_3 & \nu_2 & 0 & 0 & 0 & 0 & \nu_1 \\
\nu_3 & 0 & -\nu_1 & 0 & 0 & 0 & 0 & \nu_2 \\
-\nu_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 & \nu_3 \\
\nu_1 & \nu_2 & \nu_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \nu_1 & \nu_2 & \nu_3 & 0 & 0
\end{pmatrix}.$$

Then $A(\vec{\nu})A(\vec{\nu}) = I$ and $A(\vec{\nu})$ takes the vector $u = (\vec{u},\ \vec{v},\ h,\ q)^T$ to

$$A(\vec{\nu})u = (-[\vec{\nu}\times\vec{v}] + \vec{\nu}h,\ [\vec{\nu}\times\vec{u}] + \vec{\nu}q,\ \langle\vec{u},\vec{\nu}\rangle_3,\ \langle\vec{v},\vec{\nu}\rangle_3)^T.$$

Consider the 8×8 matrix

$$P = \begin{pmatrix}
1-\nu_1\nu_1 & -\nu_1\nu_2 & -\nu_1\nu_3 & 0 & 0 & 0 & 0 & 0 \\
-\nu_2\nu_1 & 1-\nu_2\nu_2 & -\nu_2\nu_3 & 0 & 0 & 0 & 0 & 0 \\
-\nu_3\nu_1 & -\nu_3\nu_2 & 1-\nu_3\nu_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \nu_1\nu_1 & \nu_1\nu_2 & \nu_1\nu_3 & 0 & 0 \\
0 & 0 & 0 & \nu_2\nu_1 & \nu_2\nu_2 & \nu_2\nu_3 & 0 & 0 \\
0 & 0 & 0 & \nu_3\nu_1 & \nu_3\nu_2 & \nu_3\nu_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

$R = \zeta R(\vec{n})$, where $R(\vec{n})$ is defined by the formula (4.13) with $\vec{n}$ instead of $\vec{\nu}$. Recall that $A(\vec{\nu}) = g^1\nu_1 + g^2\nu_2 + g^3\nu_3$. Then $g^i = A(\vec{e}_i)$, where $\{\vec{e}_i\}_{i=1}^3$ is the standard basis in $\mathbb{R}^3$. Using (4.14), we get

$$\begin{aligned} MRu = R\partial_t u + \sum_{i=1}^3 g^i(\partial_i R)u + \sum_{i=1}^3 g^i R\partial_i u = \\ = R\partial_t u + \sum_{i=1}^3 g^i(\partial_i R)u - \sum_{i=1}^3 Rg^i\partial_i u = \\ = 2R\partial_t u + Su - RMu, \end{aligned}$$

where $S = \sum\limits_{i=1}^3 g^i(\partial_i R)$. Now if we recall (4.12), we rewrite (4.10) in the form

$$\begin{aligned} &\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t}(|BV|^2 - |B^T U|^2) + 2\gamma \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle u, Ru\rangle_8 = \\ &\quad = \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle Mu, Ru\rangle_8 - \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle u, RMu\rangle_8 \\ &\quad + \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle u, Su\rangle_8 + 2\int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle u, R\partial_t u\rangle_8. \end{aligned} \tag{4.16}$$

Let us take the real part of the identity (4.16) and apply the Cauchy inequality to the term with S. Then we have

$$\begin{aligned} &\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t}|BV|^2 \le c\Big(\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t}|B^T U|^2 + \\ &+2\mathrm{Re}\int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle Mu, Ru\rangle_8 + \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}|u|^2 + \\ &\quad +2\mathrm{Re}\int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle u, R\partial_t u\rangle_8\Big)\,. \end{aligned} \tag{4.17}$$

In (4.15) and (4.17) we take $u(x,t) = \psi(t)v(x)$ with ψ, $e^{-\gamma t}\psi \in \mathcal{S}(\mathbb{R})$ and $v \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{C}^8)$. Introduce the notations $\widetilde{U}, \widetilde{V}$ such that $U = \psi\widetilde{U}$, $V = \psi\widetilde{V}$. Applying the Fourier transform $\mathcal{F}_{t\to\tau}$, we obtain

$$\begin{aligned} &\gamma^2 \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}\sigma\; |\widehat{\psi}(\sigma - i\gamma)|^2 \cdot |v(x)|^2 \le \\ &\le c\Big(\int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}\sigma\; |\widehat{\psi}(\sigma - i\gamma)|^2 \cdot |M(D_x,\tau)v(x)|^2 + \\ &+2\gamma\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}\sigma\; |\widehat{\psi}(\sigma - i\gamma)|^2 \cdot |B\widetilde{V}(x)| \cdot |\widetilde{U}(x)|\Big) \end{aligned}$$

and

$$\int_{\partial\Omega}\int_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}\sigma\, |\widehat{\psi}(\sigma - i\gamma)|^2 \cdot |B\widetilde{V}(x)|^2 \leq c\Big(\int_{\partial\Omega}\int_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}\sigma\, |\widehat{\psi}(\sigma - i\gamma)|^2 \cdot |B^T\widetilde{U}(x)|^2 +$$
$$+\int_{\Omega}\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}\sigma\, |\widehat{\psi}(\sigma - i\gamma)|^2 \cdot |M(D_x,\tau)v(x)| \cdot |v(x)| +$$
$$+\int_{\Omega}\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}\sigma\, |\widehat{\psi}(\sigma - i\gamma)|^2 \cdot |v(x)|^2 + \int_{\Omega}\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}\sigma\, |\widehat{\psi}(\sigma - i\gamma)|^2 \cdot |\tau| \cdot |v(x)|^2\Big)\,.$$

Since ψ is arbitrary, we obtain the estimates

$$\gamma^2 \int_{\Omega} \mathrm{d}x\, |v(x)|^2 \leq c\Big(\int_{\Omega} \mathrm{d}x\, |M(D_x,\tau)v(x)|^2 + \gamma \int_{\partial\Omega} \mathrm{d}S\, |B\widetilde{V}(x)| \cdot |\widetilde{U}(x)|\Big) \tag{4.18}$$

and

$$\int_{\partial\Omega} \mathrm{d}S\, |B\widetilde{V}(x)|^2 \leq c\Big(\int_{\partial\Omega} \mathrm{d}S\, |B^T\widetilde{U}(x)|^2 + \int_{\Omega} \mathrm{d}x\, |M(D_x,\tau)v(x)| \cdot |v(x)| + \int_{\Omega} \mathrm{d}x\, |v(x)|^2 + |\tau| \int_{\Omega} \mathrm{d}x\, |v(x)|^2\Big)\,. \tag{4.19}$$

Let us remember that for $x \in \partial\Omega$ we have

$$v = \begin{pmatrix} \widetilde{U} \\ \widetilde{V} \end{pmatrix}, Pv = \begin{pmatrix} \widetilde{U} \\ 0 \end{pmatrix}, Qv = \begin{pmatrix} 0 \\ \widetilde{V} \end{pmatrix},$$

and Pv corresponds to the boundary conditions (4.2). This implies that

$$|\widetilde{U}| = |\Gamma v| \text{ and } \|\widetilde{U}; L_2(\partial\Omega)\| = \|\Gamma v; L_2(\partial\Omega)\|.$$

Similarly we obtain

$$|\widetilde{V}| = |Tv| \text{ and } \|\widetilde{V}; L_2(\partial\Omega)\| = \|Tv; L_2(\partial\Omega)\|.$$

Applying the Cauchy inequality to the surface integral in (4.18), we get

$$\gamma \int_{\partial\Omega} \mathrm{d}S\, |B\widetilde{V}(x)| \cdot |\widetilde{U}(x)| \leq \frac{|\tau|}{\varepsilon}\|\Gamma v; L_2(\partial\Omega)\|^2 + \frac{\varepsilon\gamma^2}{|\tau|}\|B\widetilde{V}; L_2(\partial\Omega)\|^2, \tag{4.20}$$

where ε is a sufficiently small positive number. Let $\gamma \geq 1$. Combining (4.18),(4.19), and (4.20) we obtain

$$\gamma^2\|v; L_2(\Omega)\|^2 + \frac{\gamma^2}{|\tau|}\|Tv; L_2(\partial\Omega)\|^2 \leq$$
$$\leq c\{\|M(D_x,\tau)v; L_2(\Omega)\|^2 + |\tau| \cdot \|\Gamma v; L_2(\partial\Omega)\|^2\}. \tag{4.21}$$

The inequality (4.21) is called the global energy estimate for the problem (4.4), (4.5) in Ω.

4.2.4 Energy estimate for the problem (4.4), (4.5) in $\mathcal{K}$ and in G

In this subsection we show that the global energy estimate (4.21) is valid in a certain class of cones in space. Now we introduce this class.

DEFINITION 4.1. *A cone $\mathcal{K} \subset \mathbb{R}^3$ is called admissible if there exists a constant vector $\vec{f} \in \mathbb{R}^3$ such that $\langle \vec{f}, \vec{\nu} \rangle \geq c_0 > 0$ for all outward normals to $\partial\mathcal{K}$.*

It is not hard to find a nonadmissible cone, considering nonconvex sets $\mathcal{K} \cap \mathcal{S}^2$. In what follows we only deal with admissible cones.

DEFINITION 4.2. *Let $\mathcal{D}(\mathcal{K})$ denote the lineal spanned by the functions $w \in \mathcal{C}_c^\infty(\overline{\mathcal{K}} \setminus \mathcal{O}, \mathbb{C}^8)$ and by the functions of the form $\chi u_{s,k}$ for $\operatorname{Im} \lambda_k < 1$ (see (2.16)). Here $\chi \in \mathcal{C}_c^\infty(\overline{\mathcal{K}})$ is a cut-off function such that $\chi = 1$ near the point $\mathcal{O}$.*

Note that for any function $u \in \mathcal{D}(\mathcal{K})$ we have $u \in L_2(\mathcal{K})$, $u|_{\partial\mathcal{K}} \in L_2(\partial\mathcal{K})$, and $A(D_x)u \in L_2(\mathcal{K})$. Now we prove the first main result of this section.

PROPOSITION 4.3. *For any function $v \in \mathcal{D}(\mathcal{K})$ the estimate*

$$\begin{gathered}\gamma^2 \|v; L_2(\mathcal{K})\|^2 + \frac{\gamma^2}{|\tau|} \|Tv; L_2(\partial\mathcal{K})\|^2 \leq \\ \leq c\{\|M(D_x, \tau)v; L_2(\mathcal{K})\|^2 + |\tau| \cdot \|\Gamma v; L_2(\partial\mathcal{K})\|^2\}\end{gathered} \tag{4.22}$$

holds, where $\tau = \sigma - i\gamma$ with $\sigma \in \mathbb{R}$, $\gamma > 0$. The constant c is independent of the parameter τ and of the function v.

Proof. Let $u(x,t) = \psi(t)v(x)$ with ψ, $e^{-\gamma t}\psi \in \mathcal{S}(\mathbb{R})$ and $v \in \mathcal{D}(\mathcal{K})$. We use the same notations $U, V, \widetilde{U}, \widetilde{V}$ as in the previous subsection. The estimate (4.18) in $\mathcal{K}$ is proved in the same way as in Ω. Now we turn to the estimate (4.19) in $\mathcal{K}$. Let $\vec{f} \in \mathbb{R}^3$ be a constant vector from the condition imposed on the cone $\mathcal{K}$ (see Definition 4.1). We put $R = R(\vec{f})$ in (4.10). Let us remark that $R(\vec{f})$ is a constant matrix, then $\sum\limits_{i=1}^{3} g^i(\partial_i R) = 0$. Since $MRu = 2R\partial_t u - RMu$, we get

$$\begin{aligned}&\int\limits_{\partial\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}S\, \mathrm{d}t\, e^{-2\gamma t} \langle A(\vec{\nu})u, R(\vec{f})u\rangle_8 + 2\gamma \int\limits_{\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}x\, \mathrm{d}t\, e^{-2\gamma t}\langle u, R(\vec{f})u\rangle_8 = \\ &\quad = \int\limits_{\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}x\, \mathrm{d}t\, e^{-2\gamma t}\langle Mu, R(\vec{f})u\rangle_8 - \int\limits_{\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}x\, \mathrm{d}t\, e^{-2\gamma t}\langle u, R(\vec{f})Mu\rangle_8 \\ &\qquad + 2\int\limits_{\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}x\, \mathrm{d}t\, e^{-2\gamma t}\langle u, R(\vec{f})\partial_t u\rangle_8.\end{aligned} \tag{4.23}$$

For the vector $\vec{f}$ we have $\vec{f} = s(x)\vec{\nu}(x) + \vec{\sigma}(x)$, $x \in \partial\mathcal{K} \setminus \mathcal{O}$, where $\vec{\nu}$ is the unit outward normal to $\partial\mathcal{K}$, $\vec{\sigma}$ is tangent to $\partial\mathcal{K}$, and the function s satisfies the inequalities $0 < c_0 \leq s(x) \leq C$ for $x \in \partial\mathcal{K}$. Then we have $R(\vec{f}) = sR(\vec{\nu}) + R(\vec{\sigma})$.

Using the explicit formulas for $A(\vec{\nu})$ and $R(\vec{\sigma})$, it is not hard to prove that

$$|\langle A(\vec{\nu})u, R(\vec{\sigma})u\rangle_8| \leq c|U|\cdot|V|.$$

Since $BB^T = B^TB = I$, we get

$$c_1|B^TU| \leq |U| \leq c_2|B^TU|, \ \ c_1|BV| \leq |V| \leq c_2|BV|,$$

and

$$|U|\cdot|V| \leq c(\varepsilon|BV|^2 + \frac{1}{\varepsilon}|B^TU|^2), \tag{4.24}$$

where $\varepsilon > 0$ is sufficiently small. Let us take the real part of (4.23) and apply (4.12), (4.24), then we obtain

$$\int\limits_{\partial\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t}|BV|^2 \leq c\Big(\int\limits_{\partial\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t}|B^TU|^2 +$$
$$+2\mathrm{Re}\int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle Mu, R(\vec{f})u\rangle_8 + 2\mathrm{Re}\int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle u, R(\vec{f})\partial_t u\rangle_8\Big)\,.$$

Since ψ is arbitrary, after the Fourier transform we arrive at the estimate

$$\int\limits_{\partial\mathcal{K}} \mathrm{d}S\;|B\widetilde{V}(x)|^2 \leq c\Big(\int\limits_{\partial\mathcal{K}} \mathrm{d}S\;|B^T\widetilde{U}(x)|^2 + \int\limits_{\mathcal{K}} \mathrm{d}x\;|M(D_x,\tau)v(x)|\cdot|v(x)| +$$
$$+|\tau|\int\limits_{\mathcal{K}} \mathrm{d}x\;|v(x)|^2\Big)\,. \tag{4.25}$$

Combining (4.18) for $\mathcal{K}$, (4.20), and (4.25), we obtain (3.13). □

Remark 4.4. *Proposition 4.3 remains valid with $\overline{\tau}$ instead of τ.*

To prove this we put $-\gamma$ instead of γ in (4.10). Then we repeat the proofs with obvious changes.

Finally, we consider the problem (4.4), (4.5) in G and we prove the result similar to Proposition 4.3. Let us remember that $G \subset \mathbb{R}^3$ is a domain such that G coincides with an admissible cone $\mathcal{K}$ in a neighborhood of the point $\mathcal{O}$.

Definition 4.5. *Let $\mathcal{D}(G)$ denote the lineal spanned by the functions $w \in \mathcal{C}_c^\infty(\overline{G}\setminus \mathcal{O})$ and by the functions of the form $\chi u_{s,k}$ for $\operatorname{Im}\lambda_k < 1$. Here $\chi \in \mathcal{C}^\infty(\overline{G})$ is a cut-off function such that $\chi = 1$ near the point $\mathcal{O}$ and $\chi = 0$ outside a neighborhood where G coincides with $\mathcal{K}$.*

Proposition 4.6. *For any function $v \in \mathcal{D}(G)$ the estimate*

$$\gamma^2\|v; L_2(G)\|^2 + \frac{\gamma^2}{|\tau|}\|Tv; L_2(\partial G)\|^2 \leq$$
$$\leq c\{\|M(D_x,\tau)v; L_2(G)\|^2 + |\tau|\cdot\|\Gamma v; L_2(\partial G)\|^2\} \tag{4.26}$$

holds. Here $\tau = \sigma - i\gamma$, $\sigma \in \mathbb{R}$, $\gamma > \gamma_0$ with sufficiently large γ_0. The constant c is independent of the parameter τ and of the function v.

Proof. Let $\chi + \zeta = 1$ be a partition of unity on G such that $\chi = 1$ near the point $\mathcal{O}$ and $\chi = 0$ outside a neighborhood where G coincides with $\mathcal{K}$. Then we have

$$\gamma^2\|v; L_2(G)\|^2 + \frac{\gamma^2}{|\tau|}\|Tv; L_2(\partial G)\|^2 \leq (\gamma^2\|\chi v; L_2(\mathcal{K})\|^2 + \frac{\gamma^2}{|\tau|}\|T(\chi v); L_2(\partial\mathcal{K})\|^2)+ \\ +(\gamma^2\|\zeta v; L_2(G)\|^2 + \frac{\gamma^2}{|\tau|}\|T(\zeta v); L_2(\partial G)\|^2).$$

For the first expression in brackets we use the estimate (4.22). The second expression in brackets is estimated by the inequality (4.21) for domains with smooth boundary. Therefore we obtain

$$\gamma^2\|v; L_2(G)\|^2 + \frac{\gamma^2}{|\tau|}\|Tv; L_2(\partial G)\|^2 \leq$$

$$\leq c\{\|M(D_x, \tau)(\chi v); L_2(\mathcal{K})\|^2 + |\tau| \cdot \|\Gamma(\chi v); L_2(\partial\mathcal{K})\|^2+ \\ +\|M(D_x, \tau)(\zeta v); L_2(G)\|^2 + |\tau| \cdot \|\Gamma(\zeta v); L_2(\partial G)\|^2\} \leq$$

$$\leq c\{\|M(D_x, \tau)v; L_2(G)\|^2 + |\tau| \cdot \|\Gamma v; L_2(\partial G)\|^2+ \\ +\|[A(D_x), \chi]v; L_2(G)\|^2 + \|[A(D_x), \zeta]v; L_2(G)\|^2\}.$$

For the commutators we get

$$\|[A(D_x), \chi]v; L_2(G)\|^2 \leq c\|v; L_2(G)\|^2, \quad \|[A(D_x), \zeta]v; L_2(G)\|^2 \leq c\|v; L_2(G)\|^2.$$

By choosing a large γ, we can rearrange the term $c\|v; L_2(G)\|^2$ to the left side. $\square$

We note that the obtained estimates do not contain the boundary Sobolev spaces of fractional order $H^s(\partial\mathcal{K})$ and $H^s(\partial G)$. One might expect the order $s = 1/2$ considering the elliptic problem or applying some kind of trace theorems. The order $s = -1/2$ could appear if the estimate was proved applying some "duality" technique. But the original problem is not elliptic, it is hyperbolic and we treated it applying completely distinct methods. In the following section these estimates will lead us to "weak" operator of the problem with parameter.

4.3 Operator of problem

In this section we investigate the operator of the problem (4.4), (4.5) in spaces related to (4.22). In what follows we consider only the problem in $\mathcal{K}$. However, all results remain valid for the problem in G as well. At the end of the section we give the necessary remarks.

Let us introduce the function space corresponding to the boundary operator Γ by

$$L_{2,T}(\partial\mathcal{K}) = \\ = \Big\{(\vec{u},\ v,\ h)\ :\ v, h \in L_2(\partial\mathcal{K}); \vec{u} \in L_2(\partial\mathcal{K}, \mathbb{C}^3) \text{ and } \langle\vec{u},\ \vec{\nu}\rangle_3 = 0 \text{ a.e. on } \partial\mathcal{K}\Big\}.$$

Further we often use function spaces of vector-valued functions, e.g., $L_2(\mathcal{K}, \mathbb{C}^8)$, $H^1_\beta(\mathcal{K}, |\tau|, \mathbb{C}^8)$. As a rule, we omit the symbol $\mathbb{C}^k$ and keep the simple notations $L_2(\mathcal{K})$, $H^1_\beta(\mathcal{K}, |\tau|)$.

We associate with the problem (4.4), (4.5) in $\mathcal{K}$ the unbounded operator $v \mapsto \mathcal{M}(\tau)v := \{M(D_x,\tau)v, \Gamma v\}$ with domain $\mathcal{D}(\mathcal{K})$ (see Definition 4.2) acting from $L_2(\mathcal{K})$ to $L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$. We claim that $\mathcal{M}(\tau)$ admits closure. Indeed, let $\{v_m\} \subset \mathcal{D}(\mathcal{K})$, $v_m \to 0$ in $L_2(\mathcal{K})$, and $\{M(D_x,\tau)v_m, \Gamma v_m\} \to \{f, g\}$ in $L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$ as $m \to \infty$. Then $(M(D,\tau)v_m, w)_{\mathcal{K}} = (v_m, M(D_x, \overline{\tau})w)_{\mathcal{K}}$ for any $w \in \mathcal{C}^\infty_c(\mathcal{K})$. Letting $m \to \infty$, we obtain $(f, w)_{\mathcal{K}} = 0$, hence $f = 0$. Now let $w \in \mathcal{C}^\infty_c(\overline{\mathcal{K}} \setminus \mathcal{O})$ such that $\Gamma w = 0$. Applying the Green formula (4.6), we get

$$(M(D_x,\tau)v_m, w)_{\mathcal{K}} + (\Gamma v_m, Tw)_{\partial\mathcal{K}} = (v_m, M(D_x,\overline{\tau})w)_{\mathcal{K}} + (Tv_m, \Gamma w)_{\mathcal{K}}.$$

As $m \to \infty$ we have $(g, Tw)_{\partial\mathcal{K}} = 0$, hence $g = 0$. In what follows we deal with the closed operator only, keeping the notations $\mathcal{M}(\tau)$ and $\mathcal{DM}(\tau)$ for the closed operator and its domain. Using (4.22) it is easy to prove that for any $v \in \mathcal{DM}(\tau)$ such that $\mathcal{M}(\tau)v = \{f, g\}$ we have

$$\gamma^2\|v; L_2(\mathcal{K})\|^2 \le c\{\|f; L_2(\mathcal{K})\|^2 + |\tau| \cdot \|g; L_{2,T}(\partial\mathcal{K})\|^2\}.$$

This estimate implies the following result.

Proposition 4.7. *A)* Ker $\mathcal{M}(\tau) = \{0\}$,
B) The range $\mathcal{RM}(\tau)$ is closed in $L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$.

Let us now prove that the range $\mathcal{RM}(\tau)$ of $\mathcal{M}(\tau)$ coincides with $L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$. To this end we investigate the kernel of adjoint operator.

Proposition 4.8. *$\mathcal{RM}(\tau) = L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$.*

Proof. It suffices to verify that K*er* $\mathcal{M}(\tau)^* = \{0\}$. Suppose that $\{w, z\} \in$ K*er* $\mathcal{M}(\tau)^*$. Now we apply local properties of solutions to elliptic problems and properties of adjoint operators to elliptic problems (see [28, Chapter 1]). Then we have $w \in \mathcal{C}^\infty(\overline{\mathcal{K}} \setminus \mathcal{O})$ and $z = Tw$ while w satisfies the homogeneous problem adjoint with respect to the Green formula (4.6):

$$M(D_x,\overline{\tau})w = 0,\ x \in \mathcal{K}, \tag{4.27}$$

$$\Gamma w = 0,\ x \in \partial\mathcal{K}. \tag{4.28}$$

Moreover, in a neighborhood of $\mathcal{O}$, the function w admits the asymptotic representation

$$w = \chi \sum_k \sum_{s=1,..,N_k} c_{s,k} V_{s,k,N} + h, \tag{4.29}$$

where $V_{s,k}$ is defined by (2.21). Since $\{w, Tw\} \in L_2(\mathcal{K}) \times L_2(\partial\mathcal{K})$, the asymptotics contains only $V_{s,k,N}$ such that $\operatorname{Im}\lambda_{-k} < 1$ (this condition implies that $\chi V_{s,k,N} \in L_2(\mathcal{K})$ and $\chi V_{s,k,N}|_{\partial\mathcal{K}} \in L_2(\partial\mathcal{K})$, where $\chi \in \mathcal{C}^\infty_c(\overline{\mathcal{K}})$ such that $\chi = 1$ near the point $\mathcal{O}$).

Let $\zeta_n \in \mathcal{C}^\infty(\overline{\mathcal{K}})$ such that $\zeta_n = 1$ for $r < n$ and $\zeta_n = 0$ for $r > n+1$. Applying the estimate (4.22) and Remark 4.4 to $\zeta_n w$, we obtain

$$\gamma^2 \|\zeta_n w; L_2(\mathcal{K})\|^2 \leq c\|[M(D_x, \overline{\tau}), \zeta_n]w; L_2(\mathcal{K})\|^2.$$

The commutator is estimated as follows

$$\|[M(D_x, \overline{\tau}), \zeta_n]w; L_2(\mathcal{K})\| \leq c\|w; L_2(\mathcal{K} \cap \{n < r < n+1\})\|.$$

Since $w \in L_2(\mathcal{K})$, then $\|w; L_2(\mathcal{K} \cap \{n < r < n+1\})\| \to 0$ as $n \to \infty$. Hence $w = 0$. $\square$

DEFINITION 4.9. *A solution of the equation $\mathcal{M}(\tau)v = \{f, g\}$ with $\{f, g\} \in L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$ is called a strong solution of the problem (4.4), (4.5) in $\mathcal{K}$.*

In the same way, we define strong solutions of the problem (4.4), (4.5) in G. The next assertion summarizes the results of this section.

THEOREM 4.10. *For any $\{f, g\} \in L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$ and every $\tau = \sigma - i\gamma$ ($\sigma \in \mathbb{R}$, $\gamma > 0$) there exists a unique strong solution v to the problem (4.4), (4.5) in $\mathcal{K}$ with right-hand side $\{f, g\}$. The solution satisfies*

$$\gamma^2\|v;\ L_2(\mathcal{K})\| \leq c\{\|f;\ L_2(\mathcal{K})\|^2 + |\tau| \cdot \|g;\ L_2(\partial\mathcal{K})\|^2\},$$

where c is independent of τ and of v.

REMARK 4.11. *Let $\gamma \geq \gamma_0$ with sufficiently large γ_0. Then Theorem 4.10 is true for the problem (4.4), (4.5) in G as well.*

REMARK 4.12. *Theorem 4.10 (for $\mathcal{K}$ and G) remains true with $\overline{\tau}$ instead of τ.*

4.4 Weighted combined estimate

In this section, we prove a more informative *a priori* estimate for the problem (4.4), (4.5). This estimate will be used in the study of the asymptotics of solutions near the point $\mathcal{O}$.

DEFINITION 4.13. *Let $\mathcal{D}_\beta(\mathcal{K})$ with $\beta \leq 1$ stand for the lineal spanned by the functions $w \in \mathcal{C}_c^\infty(\overline{\mathcal{K}} \setminus \mathcal{O}, \mathbb{C}^8)$ and by the functions of the form $\chi u_{s,k}$ such that* $\operatorname{Im}\lambda_k < min\{1, \beta + 1/2\}$. *Here $\chi \in \mathcal{C}_c^\infty(\overline{\mathcal{K}})$ is a cut-off function such that $\chi = 1$ near the point $\mathcal{O}$.*

The lineal $\mathcal{D}_\beta(G)$ is defined in a similar way.

4.4.1 The estimate in the cone $\mathcal{K}$

PROPOSITION 4.14. *Let $\beta \le 1$ and let the number $\lambda = i(\beta+1/2)$ be regular for the pencil $\mathfrak{A}$. Then for $v \in \mathcal{D}_\beta(\mathcal{K})$ the inequality*

$$\begin{aligned}\gamma^2\|v; H^0_\beta(\mathcal{K})\|^2 + \|\chi_\tau v; H^1_\beta(\mathcal{K}, |\tau|)\|^2 \le c\Big(\|M(D_x,\tau)v; H^0_\beta(\mathcal{K})\|^2 + \\ +|\tau|\cdot\|\Gamma v; H^0_\beta(\partial\mathcal{K})\|^2 + \|\chi_\tau\Gamma v; H^{1/2}_\beta(\partial\mathcal{K})\|^2 + \\ +(|\tau|^{1-\beta}/\gamma)^2\{\|M(D_x,\tau)v; L_2(\mathcal{K})\|^2 + |\tau|\cdot\|\Gamma v; L_2(\partial\mathcal{K})\|^2\}\Big)\end{aligned} \tag{4.30}$$

holds, where $\chi_\tau(r) = \chi(|\tau|r)$ and $\chi \in \mathcal{C}^\infty_c(\overline{\mathcal{K}})$ is a fixed cut-off function such that $\chi = 1$ near the point $\mathcal{O}$. The constant c is independent of v and τ.

Proof. Step 1. An estimate near the vertex of $\mathcal{K}$. We consider problem (4.7), (4.8) in $\mathcal{K}$. According to Proposition 1.2, the problem $\{A(D_\eta),\ \Gamma\}$ is elliptic. Then the following result holds (see [28, Chapter 3, §5.2]) : if the line $\operatorname{Im}\lambda = \beta+1/2$ contains no eigenvalues of the pencil corresponding to the problem under consideration (the pencil $\mathfrak{A}$ in this case), then a function $U \in H^1_\beta(\mathcal{K}, 1)$ satisfies the inequality

$$\|\chi U; H^1_\beta(\mathcal{K})\|^2 \le c\{\|A(D_\eta)\chi U; H^0_\beta(\mathcal{K})\|^2 + \|\Gamma\chi U; H^{1/2}_\beta(\partial\mathcal{K})\|^2\}.$$

Since $A(D_\eta)\chi U = \chi A(D_\eta)U + [A(D_\eta),\chi]U$ and $M(D_\eta,\theta) = \theta + A(D_\eta)$, the inequality can be rewritten in the form

$$\begin{aligned}\|\chi U; H^1_\beta(\mathcal{K},1)\|^2 \le c\{\|\chi M(D_\eta,\theta)U; H^0_\beta(\mathcal{K})\|^2 + \\ +\|\Gamma\chi U; H^{1/2}_\beta(\partial\mathcal{K})\|^2 + \|\psi U; H^0_\beta(\mathcal{K})\|^2\},\end{aligned} \tag{4.31}$$

where $\psi \in \mathcal{C}^\infty_c(\overline{\mathcal{K}})$, $\chi\psi = \chi$.

Step 2. An estimate far from the vertex. On this step we prove the inequality

$$\begin{aligned}\frac{\gamma^2}{|\tau|^2}\|\kappa_\infty U; H^0_\beta(\mathcal{K})\|^2 \le c\{\|\kappa_\infty M(D_\eta,\theta)U; H^0_\beta(\mathcal{K})\|^2 + \\ +\|\psi_\infty U; H^0_{\beta-1}(\mathcal{K})\|^2 + \|\kappa_\infty\Gamma U; H^0_\beta(\partial\mathcal{K})\|^2\}\end{aligned} \tag{4.32}$$

for any $\beta \in \mathbb{R}$ and every $U \in H^1_\beta(\mathcal{K},1)$, where the constant c is independent of U and τ. The functions κ_∞ and ψ_∞ are smooth in $\mathcal{K}$, equal to 0 near the vertex and 1 in a neighborhood of infinity, while $\kappa_\infty\psi_\infty = \kappa_\infty$.

Let $\kappa, \psi \in \mathcal{C}^\infty(\overline{\mathcal{K}})$ such that supp $\kappa \subset \{(x_1,x_2,x_3) \in \mathcal{K} : 1/2 < r < 2\}$, supp $\psi \subset \{(x_1,x_2,x_3) \in \mathcal{K} : 1/4 < r < 4\}$, and $\kappa\psi = \kappa$. The application of (4.22) yields

$$\gamma^2\|\kappa v; L_2(\mathcal{K})\|^2 \le \{\|\kappa M(D_x,\tau)v; L_2(\mathcal{K})\|^2 + \|\psi v; L_2(\mathcal{K})\|^2 + |\tau|\cdot\|\kappa\Gamma v; L_2(\partial\mathcal{K})\|^2\}.$$

If we replace v by the function $(x_1,x_2,x_3) \mapsto U^\varepsilon(x_1,x_2,x_3) = U(x_1/\varepsilon, x_2/\varepsilon, x_3/\varepsilon)$, and change τ for $\tau/(|\tau|\varepsilon)$ with $\varepsilon > 0$, we obtain

$$\begin{aligned}(\gamma/|\tau|\varepsilon)^2\|\kappa U^\varepsilon; L_2(\mathcal{K})\|^2 \le c\{\|\kappa M(D_x,\tau/|\tau|\varepsilon)U^\varepsilon; L_2(\mathcal{K})\|^2 + \\ +\|\psi U^\varepsilon; L_2(\mathcal{K})\|^2 + (1/\varepsilon)\|\kappa\Gamma U^\varepsilon; L_2(\partial\mathcal{K})\|^2\}.\end{aligned}$$

After the change of variables $(x_1, x_2, x_3) \mapsto (\eta_1, \eta_2, \eta_3) = (x_1/\varepsilon, x_2/\varepsilon, x_3/\varepsilon)$ we arrive at

$$\frac{\gamma^2}{|\tau|^2}\|\kappa_\varepsilon U; L_2(\mathcal{K})\|^2 \leq c\{\|\kappa_\varepsilon M(D_\eta,\theta)U; L_2(\mathcal{K})\|^2 + \\ +\varepsilon^2\|\psi_\varepsilon U; L_2(\mathcal{K})\|^2 + \|\kappa_\varepsilon \Gamma U; L_2(\partial\mathcal{K})\|^2\},$$

where $\kappa_\varepsilon(\eta) = \kappa(\varepsilon\eta)$, $\psi_\varepsilon(\eta) = \psi(\varepsilon\eta)$. Multiplying the inequality by $\varepsilon^{-2\beta}$, putting $\varepsilon = 2^{-j}$, $j = 1, 2, 3, \ldots$, and adding all these inequalities, we obtain (4.32).

Step 3. An estimate in intermediate zone. Let $\kappa_\infty = 1$ outside the support of χ. Since

$$(\gamma/|\tau|)\|\chi U; H^0_\beta(\mathcal{K})\| \leq \|\chi U; H^0_\beta(\mathcal{K})\| \leq \|\chi U; H^1_\beta(\mathcal{K}, 1)\|,$$

then, summing (4.31) and (4.32), we obtain the inequality

$$\frac{\gamma^2}{|\tau|^2}\|U; H^0_\beta(\mathcal{K})\|^2 + \|\chi U; H^1_\beta(\mathcal{K}, 1)\|^2 \leq c\{\|M(D_\eta,\theta)U; H^0_\beta(\mathcal{K})\|^2 + \\ +\|\psi_\infty U; H^0_{\beta-1}(\mathcal{K})\|^2 + \|\psi U; H^0_\beta(\mathcal{K})\|^2 + \|\kappa_\infty \Gamma U; H^0_\beta(\partial\mathcal{K})\|^2 + \|\chi\Gamma U; H^{1/2}_\beta(\partial\mathcal{K})\|^2\}.$$

Now we estimate the term on the right

$$\|\psi U; H^0_\beta(\mathcal{K})\|^2 \leq \int\limits_{|\eta|<a} |\eta|^{2\beta}|U|^2\, \mathrm{d}\eta = \\ = (\int\limits_{0\leq|\eta|\leq\varepsilon} + \int\limits_{\varepsilon\leq|\eta|\leq a})|\eta|^{2\beta}|U|^2\, \mathrm{d}\eta.$$

The first integral is majorized by $c\varepsilon^2\|\chi U; H^1_\beta(\mathcal{K})\|^2$. We can rearrange it to the left side of the inequality, choosing ε sufficiently small. The second integral does not exceed $c\|\psi_\infty U; H^0_{\beta-1}(\mathcal{K})\|^2$. Now the estimate can be rewritten in the form

$$\frac{\gamma^2}{|\tau|^2}\|U; H^0_\beta(\mathcal{K})\|^2 + \|\chi U; H^1_\beta(\mathcal{K}, 1)\|^2 \leq c\{\|M(D_\eta,\theta)U; H^0_\beta(\mathcal{K})\|^2 + \\ +\|\psi_\infty U; H^0_{\beta-1}(\mathcal{K})\|^2 + \|\kappa_\infty \Gamma U; H^0_\beta(\partial\mathcal{K})\|^2 + \|\chi\Gamma U; H^{1/2}_\beta(\partial\mathcal{K})\|^2\}.$$

After the change of variables $(\eta_1, \eta_2, \eta_3) \mapsto (x_1, x_2, x_3) = (|\tau|^{-1}\eta_1, |\tau|^{-1}\eta_2, |\tau|^{-1}\eta_3)$ we obtain

$$\gamma^2\|v; H^0_\beta(\mathcal{K})\|^2 + \|\chi_\tau v; H^1_\beta(\mathcal{K}, |\tau|)\|^2 \leq c\{\|M(D_x,\tau)v; H^0_\beta(\mathcal{K})\|^2 + \\ +\|\psi_{\infty,\tau} v; H^0_{\beta-1}(\mathcal{K})\|^2 + |\tau|\cdot\|\kappa_{\infty,\tau}\Gamma v; H^0_\beta(\partial\mathcal{K})\|^2 + \|\chi_\tau\Gamma v; H^{1/2}_\beta(\partial\mathcal{K})\|^2\},$$

where $\psi_{\infty,\tau}(x) = \psi_\infty(|\tau|x)$, $\chi_\tau(x) = \chi(|\tau|x), v(x_1, x_2, x_3) = U(|\tau|x_1, |\tau|x_2, |\tau|x_3)$. Note that

$$\|\kappa_{\infty,\tau}\Gamma v; H^0_\beta(\partial\mathcal{K})\| \leq \|\Gamma v; H^0_\beta(\partial\mathcal{K})\|.$$

Taking the inequalities (4.22) and $\beta \leq 1$ into account, we get

$$\|\psi_{\infty,\tau} v; H^0_{\beta-1}(\mathcal{K})\|^2 \leq \int\limits_{b/|\tau|<r} r^{2(\beta-1)}|v|^2\, \mathrm{d}x \leq$$

$$\leq c\,|\tau|^{2(1-\beta)}\int\limits_{\mathcal{K}} |v|^2\, \mathrm{d}x \leq c\frac{|\tau|^{2(1-\beta)}}{\gamma^2}\Big(\|M(D_x,\tau)v; L_2(\mathcal{K})\|^2 + |\tau|\cdot\|\Gamma v; L_2(\partial\mathcal{K})\|^2\Big),$$

which leads to (4.30). $\square$

By $\mathcal{D}H_\beta(\mathcal{K},\tau)$ we denote the completion of the set $\mathcal{C}_c^\infty(\overline{\mathcal{K}}\setminus\mathcal{O},\mathbb{C}^8)$ with respect to the norm

$$\|v;\mathcal{D}H_\beta(\mathcal{K},\tau)\| = \Big(\gamma^2\|v;H^0_\beta(\mathcal{K})\|^2 + \|\chi_\tau v;H^1_\beta(\mathcal{K},|\tau|)\|^2\Big)^{1/2}.$$

By $\mathcal{R}H_\beta(\mathcal{K},\tau)$ we denote the completion of the set $\mathcal{C}_c^\infty(\overline{\mathcal{K}}\setminus\mathcal{O},\mathbb{C}^8)\times\mathcal{C}_c^\infty(\overline{\mathcal{K}}\setminus\mathcal{O},\mathbb{C}^5)$ with respect to the norm

$$\begin{gathered}\|\{f,g\};\mathcal{R}H_\beta(\mathcal{K},\tau)\| = \Big(\|f;H^0_\beta(\mathcal{K})\|^2+\\ +|\tau|\cdot\|g;H^0_\beta(\partial\mathcal{K})\|^2+\|\chi_\tau g;H^{1/2}_\beta(\partial\mathcal{K})\|^2+\\ +(|\tau|^{1-\beta}/\gamma)^2(\|f;L_2(\mathcal{K})\|^2+|\tau|\cdot\|g;L_2(\partial\mathcal{K})\|^2)\Big)^{1/2}.\end{gathered}$$

Here $\chi_\tau(x_1,x_2,x_3)=\chi(|\tau|x_1,|\tau|x_2,|\tau|x_3)$ and $\chi\in\mathcal{C}_c^\infty(\overline{\mathcal{K}})$ is a cut-off function that is equal to 1 near the conical point $\mathcal{O}$. The spaces $\mathcal{D}H_\beta(G,\tau)$ and $\mathcal{R}H_\beta(G,\tau)$ are defined in a similar way. Using these new notations, one can rewrite the estimate (4.30) in the form

$$\|v;\mathcal{D}H_\beta(\mathcal{K},\tau)\|\le c\,\|\{M(D_x,\tau)v,\Gamma v\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|. \tag{4.33}$$

4.4.2 The estimate in G

PROPOSITION 4.15. *Let $\beta\le 1$ and let the number $\lambda=i(\beta+1/2)$ be regular for the pencil $\mathfrak{A}$. Assume that $\gamma>\gamma_0$ with sufficiently large γ_0. Then the inequality*

$$\|v;\mathcal{D}H_\beta(G,\tau)\|\le c\|\{M(D_x,\tau)v,\Gamma v\};\mathcal{R}H_\beta(G,\tau)\| \tag{4.34}$$

holds for any $v\in\mathcal{D}_\beta(G)$ with a constant c independent of v and τ.

Proof. Let $\psi\in\mathcal{C}^\infty(\overline{G})$ be a cut-off function that is equal to 1 near $\mathcal{O}$ and vanishes outside the neighborhood where G coincides with $\mathcal{K}$. Since $v=\psi v+(1-\psi)v$, we have

$$\|v;\mathcal{D}H_\beta(G,\tau)\|\le\|\psi v;\mathcal{D}H_\beta(\mathcal{K},\tau)\|+\|(1-\psi)v;\mathcal{D}H_\beta(G,\tau)\|. \tag{4.35}$$

Estimate the first term on the right. Applying (4.33), we obtain

$$\begin{gathered}\|\psi v;\mathcal{D}H_\beta(\mathcal{K},\tau)\|\le\|\{M\psi v,\Gamma\psi v\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|\le\\ \le c\{\|\{\psi Mv,\psi\Gamma v\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|+\|\{[M,\psi]v,0\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|\},\end{gathered}$$

where M denotes $M(D_x,\tau)$. Clearly

$$\|\{\psi Mv,\psi\Gamma v\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|\le\|\{Mv,\Gamma v\};\mathcal{R}H_\beta(G,\tau)\|.$$

For the commutator we have

$$\begin{gathered}\|\{[M,\psi]v,0\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|^2=\|[M,\psi]v;H^0_\beta(\mathcal{K})\|^2+(|\tau|^{1-\beta}/\gamma)^2\|[M,\psi]v;L_2(\mathcal{K})\|^2\\ \le c\{\|v;H^0_\beta(G)\|^2+(|\tau|^{1-\beta}/\gamma)^2\|v;L_2(G)\|^2\}.\end{gathered}$$

Taking the estimate (4.26) into account we arrive at

$$\begin{gathered}\|\{[M,\psi]v,0\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|^2\leq\\ \leq c\Big(\|v;H^0_\beta(G)\|^2+(|\tau|^{1-\beta}/\gamma)^2\cdot(1/\gamma^2)\{\|Mv;L_2(G)\|^2+|\tau|\cdot\|\Gamma v;L_2(\partial G)\|^2\}\Big)\\ \leq c\{\|v;H^0_\beta(G)\|^2+(1/\gamma^2)\|\{Mv,\Gamma v\};\mathcal{R}H_\beta(G,\tau)\|^2\}.\end{gathered}$$

Finally, for the first term in the right-hand side of (4.35) we get

$$\|\psi v;\mathcal{D}H_\beta(\mathcal{K},\tau)\|^2\leq c\Big((1+\ 1/\gamma^2)\|\{Mv,\Gamma v\};\mathcal{R}H_\beta(G,\tau)\|^2+\|v;H^0_\beta(G)\|^2\Big).$$

We now estimate the second term in the right-hand side of (4.35). From the definition of the norm in $\mathcal{D}H_\beta(G,\tau)$ it follows that

$$\|(1-\psi)v;\mathcal{D}H_\beta(G,\tau)\|^2=\gamma^2\|(1-\psi)v;H^0_\beta(G)\|^2+\|\chi_\tau(1-\psi)v;H^1_\beta(G,|\tau|)\|^2.$$

For sufficiently large γ we have $\chi_\tau(1-\psi)\equiv 0$ because the supports of the factors do not overlap. Then we obtain

$$\begin{gathered}\gamma^2\|(1-\psi)v;H^0_\beta(G)\|^2\leq c\gamma^2\|(1-\psi)v;L_2(G)\|^2\leq\\ \leq c\Big(\|M(1-\psi)v;L_2(G)\|^2+|\tau|\cdot\|(1-\psi)\Gamma v;L_2(\partial G)\|^2\Big)\leq\\ \leq c\Big(\|(1-\psi)Mv;H^0_\beta(G)\|^2+|\tau|\cdot\|(1-\psi)\Gamma v;H^0_\beta(\partial G)\|^2+\|[M,(1-\psi)]v;L_2(G)\|^2\Big)\\ \leq c\Big(\|\{Mv,\Gamma v\};\mathcal{R}H_\beta(G,\tau)\|^2+\|v;H^0_\beta(G)\|^2\Big).\end{gathered}$$

Summarizing the obtained estimates, we rewrite (4.35) in the form

$$\|v;\mathcal{D}H_\beta(G,\tau)\|^2\leq c\Big((1+1/\gamma^2)\|\{Mv,\Gamma v\};\mathcal{R}H_\beta(G,\tau)\|^2+\|v;H^0_\beta(G)\|^2\Big).$$

Since

$$\|v;H^0_\beta(G)\|\leq(1/\gamma)\|v;\mathcal{D}H_\beta(G,\tau)\|,$$

we choose a large enough γ and rearrange the term $(1/\gamma)\|v;\mathcal{D}H_\beta(G,\tau)\|$ to the left side. As a result, we obtain (4.34). $\square$

4.5 The operator of problem in a scale of weighted spaces

In the section, we study the operator of problem (4.4), (4.5) in spaces related to (4.33), (4.34). We consider the problem in an admissible cone $\mathcal{K}$. The corresponding statements for the problem in G are given at the end of this section.
Let us introduce a subspace of $\mathcal{R}H_\beta(\mathcal{K},\tau)$ similar to $L_{2,T}(\mathcal{K})$ by

$$\begin{gathered}\mathcal{R}H_{\beta,T}(\mathcal{K},\tau)=\Big\{\{f,g\}\in\mathcal{R}H_\beta(\mathcal{K},\tau)\text{ with }g=(\vec{u},v,h)\\ \text{such that }\langle\vec{u},\vec{\nu}\rangle_3=0\text{ a.e. on }\partial\mathcal{K}\Big\}.\end{gathered}$$

It is easy to see that for any $v\in\mathcal{D}_\beta(\mathcal{K})$ we have the inclusion $\{M(D_x,\tau)v,\Gamma v\}\in\mathcal{R}H_{\beta,T}(\mathcal{K},\tau)$.

We associate with the problem (4.4), (4.5) in the cone $\mathcal{K}$ the operator $v \mapsto \mathcal{M}(\tau,\beta)v := \{M(D_x,\tau)v, \Gamma v\}$ with domain $\mathcal{D}_\beta(\mathcal{K})$ such that

$$\mathcal{M}(\tau,\beta)\ :\ \mathcal{D}H_\beta(\mathcal{K},\tau) \mapsto \mathcal{R}H_{\beta,T}(\mathcal{K},\tau).$$

It can be proved that the operator $\mathcal{M}(\tau,\beta)$ admits closure. We keep the notations $\mathcal{M}(\tau,\beta)$ and $\mathcal{D}\mathcal{M}(\tau,\beta)$ for the closed operator and its domain. If the number $\lambda = i(\beta+1/2)$ is regular the pencil $\mathfrak{A}$ and $\beta \le 1$, then the estimate

$$\|v, \mathcal{D}H_\beta(\mathcal{K},\tau)\| \le c\,\|\mathcal{M}(\tau,\beta)v, \mathcal{R}H_{\beta,T}(\mathcal{K},\tau)\|$$

holds for the functions in $\mathcal{D}\mathcal{M}(\tau,\beta)$. The next proposition immediately follows from this inequality.

Proposition 4.16. *Let $\beta \le 1$ and let the number $\lambda = i(\beta+1/2)$ be regular for the pencil $\mathfrak{A}$. Then the kernel of the operator* $\operatorname{Ker}\mathcal{M}(\tau,\beta)$ *is trivial and the range $\mathcal{R}\mathcal{M}(\tau,\beta)$ is closed in $\mathcal{R}H_{\beta,T}(\mathcal{K},\tau)$.*

Let $1/2 > \beta_1 > \beta_2 > \dots$ be all numbers from $]-\infty,1/2[$ such that $\lambda = i(\beta_k+1/2)$ is an eigenvalue of $\mathfrak{A}$. Denote by S_m the sum of the multiplicities of all the eigenvalues of $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda \in [\beta_m+1/2,\ \beta_1+1/2]$.

Definition 4.17. *A solution to the equation $\mathcal{M}(\tau,\beta)v = \{f,g\}$, where $\{f,g\} \in \mathcal{R}H_{\beta,T}(\mathcal{K},\tau)$, is called a strong β-solution to the problem (4.4), (4.5) in $\mathcal{K}$ with right-hand side $\{f,g\}$.*

Theorem 4.18. A) *Let $\beta \in [\beta_1,\ 1]$ and let the number $\lambda = i(\beta+1/2)$ be regular for the pencil $\mathfrak{A}$. Then for any $\{f,g\} \in \mathcal{R}H_{\beta,T}(\mathcal{K},\tau)$ there exists a unique strong β-solution v of the problem (4.4), (4.5) with right-hand side $\{f,g\}$ and the estimate*

$$\|v; \mathcal{D}H_\beta(\mathcal{K},\tau)\| \le c\|\{f,g\}; \mathcal{R}H_{\beta,T}(\mathcal{K},\tau)\|$$

holds. The constant c is independent of v and of τ
B) *Assume that $\beta \in]\beta_{m+1},\beta_m[$. A strong β-solution of the problem (4.4), (4.5) with right-hand side $\{f,g\} \in \mathcal{R}H_{\beta,T}(\mathcal{K},\tau)$ exists under the S_m conditions*

$$(f, w_{s,k})_{\mathcal{K}} + (g, Tw_{s,k})_{\partial\mathcal{K}} = 0,$$

where $\{w_{s,k}, Tw_{s,k}\}_{k=-1,..,-m}^{s=1,..,N_k}$ is a basis in $\operatorname{Ker}\mathcal{M}(\tau,\beta)^*$. *Such a solution satisfies the estimate in* A).

Proof. A) Suppose that $\{w,z\} \in \operatorname{Ker}\mathcal{M}(\tau,\beta)^*$, where $\mathcal{M}(\tau,\beta)^*$ is the adjoint operator for $\mathcal{M}(\tau,\beta)$ with respect to the extension of the inner product on $L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$. Now we apply local properties of solutions to elliptic problems and properties of adjoint operators to elliptic problems, e.g., see [28, Chapter 1]. Then we have $w \in \mathcal{C}^\infty(\overline{\mathcal{K}} \setminus \mathcal{O})$ and $z = Tw$, while w satisfies (4.27), (4.28). Moreover, in a neighborhood of $\mathcal{O}$ the asymptotic representation of the form (4.29) holds. The estimate (4.32) with $\overline{\tau}$ instead of τ implies that w decays more rapidly

The formal series $U_{s,k}$ is defined by (2.29). The sum (4.37) contains the terms corresponding to the eigenvalues of $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda \in [\beta_m + 1/2,\ \beta_1 + 1/2]$. The remainder h decays more rapidly than $r^{i\lambda_m}$ as $r \to 0$. The number N for each term is taken sufficiently large so that $\chi r^{i\lambda_k+N+1}\Psi_N = o(r^{i\lambda_m})$ as $r \to 0$. The coefficients $d_{s,k}$ are defined by the formula (see [28, Chapter 3 §5 and Chapter 4 §3])

$$d_{s,k} = i(f, w_{s,k})_{\mathcal{K}} + i(g, Tw_{s,k})_{\partial\mathcal{K}}.$$

If $\{f, g\}$ is in the range $\mathcal{RM}(\tau, \beta)$ then the conditions $d_{s,k} = 0$ in the sum (4.37) are necessary for the inclusion $v \in \mathcal{D}H_\beta(\mathcal{K}, \tau)$. Thus we have $\{w_{s,k}, Tw_{s,k}\}_{k=-1,..,-m}^{s=1,..,N_k} \in$ $\mathrm{K}er\,\mathcal{M}(\tau, \beta)^*$. We show that they form a basis in $\mathrm{K}er\,\mathcal{M}(\tau, \beta)^*$. Let $\{w, Tw\} \in$ $\mathrm{K}er\,\mathcal{M}(\tau, \beta)^*$. Then the representation

$$w = \chi \sum c_{s,k} V_{s,k,N} + h$$

holds near the point $\mathcal{O}$, the sum contains $\chi V_{s,k,N}$ corresponding to the eigenvalues of $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda \in [\beta_m + 1/2,\ \beta_1 + 1/2]$. For the remainder h the inclusion $\{h, Th\} \in L_2(\mathcal{K}) \times L_2(\partial\mathcal{K})$. We put

$$z = w - \sum_{k=-1,..,-m}\ \sum_{s=1,..,N_k} c_{s,k} w_{s,k}.$$

We see that $\{z, Tz\} \in L_2(\mathcal{K}) \times L_2(\partial\mathcal{K})$, $M(D_x, \overline{\tau})z = 0$, and $\Gamma z = 0$. It is not hard to prove that $z \in \mathcal{DM}(\overline{\tau})$. Applying the estimate (4.22) with $\overline{\tau}$ instead of τ, we get $z = 0$. Then we obtain $w = \sum c_{s,k} w_{s,k}$. $\square$

Remark 4.19. *Assume that $\gamma > \gamma_0$ with sufficiently large γ_0. Then Theorem 4.18 is valid for the problem (4.4), (4.5) in G.*

4.6 Asymptotics of solution

Let $\{f, g\} \in \mathcal{R}H_{\beta,T}(G, \tau)$ and let $\beta \in]\beta_{m+1},\ \beta_m[$. Since $\mathcal{R}H_{\beta,T}(G, \tau) \subset L_2(G) \times L_{2,T}(\partial G)$, then there exists a unique strong solution u to the problem (4.4), (4.5) in G with right-hand side $\{f, g\}$ (Theorem 4.10). According to Theorem 4.18 and Remark 4.19, this solution is in $\mathcal{D}H_\beta(G, \tau)$ provided

$$\Big(f, w_{s,k}\Big)_G + \Big(g, Tw_{s,k}\Big)_{\partial G} = 0$$

with $k = -1, .., -m$ and $s = 1, .., N_k$, where $\{w_{s,k}, Tw_{s,k}\}_{k=-1,..,-m}^{s=1,..,N_k}$ is the basis in $\mathrm{Ker}\,\mathcal{M}(\tau, \beta)^*$ (see Theorem 4.18). For any $\{f, g\}$, we obtain an asymptotic formula for the strong solution with remainder in $\mathcal{D}H_\beta(G, \tau)$.

Theorem 4.20. *Assume that $\{f, g\} \in \mathcal{R}H_{\beta,T}(G, \tau)$, $\beta \in]\beta_{m+1},\ \beta_m[$, and $\gamma \geq \gamma_0$ with sufficiently large γ_0. Then the strong solution u to the problem (4.4), (4.5) in G with right-hand side $\{f, g\}$ admits the representation*

$$u = \chi_\tau \sum c_{s,k} U_{s,k,N} + w.$$

Here $U_{s,k,N}$ denotes the sum of first N terms of the formal series (2.29), $w \in \mathcal{D}H_\beta(G,\tau)$, and $\chi_\tau(r) = \chi(|\tau|r)$, while χ is a cut-off function that is equal to 1 near the point $\mathcal{O}$ and vanishes outside the neighborhood where the domain G coincides with the cone $\mathcal{K}$. The sum consists of the terms corresponding to the eigenvalues of the pencil $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda \in [\beta_m + 1/2,\ \beta_1 + 1/2]$. The number N in each term is taken large enough to provide the inclusion $\chi r^{i\lambda_k+N+1}\Psi^{(s,k)}_{N+1} \in \mathcal{D}H_\beta(G,\tau)$, where $\Psi^{(s,k)}_{N+1}$ are the functions from the definition of $U_{s,k}$ (see the formula (2.29)) . The coefficients $c_{s,k}$ are defined by

$$c_{s,k} = i\Big(f, w_{s,k}\Big)_G + i\Big(g, Tw_{s,k}\Big)_{\partial G}.$$

The estimates

$$|c_{s,k}| \le c\,|\tau|^{\beta+1/2-\operatorname{Im}\lambda_k}\|\{f,g\};\mathcal{R}H_{\beta,T}(G,\tau)\|,$$
$$\|w;\mathcal{D}H_\beta(G,\tau)\| \le c\,(|\tau|/\gamma)\|\{f,g\};\mathcal{R}H_{\beta,T}(G,\tau)\|,$$

hold with a constant c independent of τ.

Proof. In this proof we use the operators for the problems (4.4), (4.5) in $\mathcal{K}$ and G and for the problem (4.7), (4.8) in $\mathcal{K}$. In order to distinguish all these operators we slightly change the notations. Namely, we insert the domains into the notations: $\mathcal{M}(\tau,G)$, $\mathcal{M}(\tau,\mathcal{K}), \mathcal{M}_\beta(\tau,G)$, $\mathcal{M}_\beta(\theta,\mathcal{K})$, etc. By $\{\mathcal{W}_{s,k}, T\mathcal{W}_{s,k}\}$ we denote the basis in $\operatorname{Ker}\mathcal{M}_\beta(\theta,\mathcal{K})^*$. Let $\mathcal{U}_{s,k}$ stand for the formal series (2.29) with $\theta = \tau/|\tau|$ instead of τ.

Let u be a solution to the equation $\mathcal{M}(\tau,G)u = \{f,g\}$. We put $\{h,s\} = \{\mathcal{M}(\tau,G)\chi u, \Gamma\chi u\}$. Denote by U a solution to the equation $\mathcal{M}(\theta,\mathcal{K})U = \{H,S\}$ where $H(\eta) = (1/|\tau|)h(\eta/|\tau|)$, $S(\eta) = s(\eta/|\tau|)$, and new variables η are introduced by the formula $(\eta_1,\eta_2,\eta_3) = (|\tau|x_1, |\tau|x_2, |\tau|x_3)$. Since a strong solution is unique, we have $U(\eta) = \chi(\eta/|\tau|)u(\eta/|\tau|)$. In view of the properties of solutions to elliptic problems in domains with singularities, the representation

$$U(\eta) = \chi(\eta)\sum d_{s,k}\mathcal{U}_{s,k,N}(\eta) + \mathcal{V}(\eta)$$

holds. The sum contains $\mathcal{U}_{s,k,N}$ corresponding to the eigenvalues of $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda \in [\beta_m + 1/2,\ \beta_1 + 1/2]$, while N in each term is taken large enough in order for the inclusions $\chi r^{i\lambda_k+N+1}\Psi^{(s,k)}_{N+1} \in H^1_\beta(\mathcal{K})$ to be valid. For the remainder $\mathcal{V}$ we have $\chi\mathcal{V} \in H^1_\beta(\mathcal{K})$. The coefficients $d_{s,k}$ are defined by

$$d_{s,k} = i\Big(H, \mathcal{W}_{s,k}\Big)_{\mathcal{K}} + i\Big(S, T\mathcal{W}_{s,k}\Big)_{\mathcal{K}}. \tag{4.38}$$

Let us majorize the coefficients $d_{s,k}$. From (4.38) it follows that

$$|d_{s,k}| \le c\|\{H,S\};\mathcal{R}H_{\beta,T}(\mathcal{K},1)\| \le c|\tau|^{\beta+1/2}\|\{h,s\};\mathcal{R}H_{\beta,T}(\mathcal{K},|\tau|)\|.$$

Using $h = \chi f + [M,\chi]u$ and $s = \chi g$, we obtain

$$\|\{h,s\};\mathcal{R}H_{\beta,T}(\mathcal{K},\tau)\| \le \|\{f,g\};\mathcal{R}H_{\beta,T}(G,\tau)\| + \|\{[M,\chi]u,0\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|.$$

than any power of r as $r \to \infty$. Since $\{w, Tw\} \in \mathcal{R}H_\beta(\mathcal{K}, \tau)^*$, then for a fixed parameter τ we see that

$$\int\limits_{\mathcal{K}} \mathrm{d}x\, |w|^2 (1 + r^{2\beta})^{-1} \le c \|\{w, Tw\}; \mathcal{R}H_\beta(\mathcal{K}, \tau)^*\|^2. \tag{4.36}$$

Assume that $\beta > 0$. Since w decays as $r \to \infty$ and the estimate (4.36) holds, it follows that $w \in L_2(\mathcal{K})$. Then the asymptotic representation (4.29) contains $\chi V_{s,k,N}$ such that $\operatorname{Im} \lambda_{-k} < 3/2$ (this inequality implies the inclusion $\chi V_{s,k,N} \in L_2(\mathcal{K})$). Moreover, we claim that the asymptotic representation contains only $\chi V_{s,k,N}$ such that $\operatorname{Im} \lambda_{-k} < 1$. Indeed, the formula (2.15) implies that

$$(M(D_x, \tau)\chi u_{s,k}, \chi u_{s,-k})_{\mathcal{K}} = i + (\chi u_{s,k}, M(D_x, \overline{\tau})\chi u_{s,-k})_{\mathcal{K}}.$$

The functions $\chi u_{s,k}$ with $k < 0$ belong to $\mathcal{DM}(\tau, \beta)$ (see Definition 4.13). Hence the functions $\chi u_{s,k}$ with $k > 0$ can not be included into $\mathcal{DM}(\tau, \beta)^*$. It follows that the asymptotic representation (4.29) contains the terms $\chi V_{s,k,N}$ such that $\operatorname{Im} \lambda_{-k} < 1$. Note that the inequality $\operatorname{Im} \lambda_{-k} < 1$ implies the inclusions $\chi V_{s,k,N} \in L_2(\mathcal{K})$, $\chi V_{s,k,N}|_{\partial\mathcal{K}} \in L_2(\partial\mathcal{K})$. Now it is not hard to prove that $w \in \mathcal{DM}(\overline{\tau})$. Since $\mathcal{M}(\overline{\tau})w = \{0, 0\}$, we have $w = 0$.
Now assume that $\beta < 0$. The estimate (4.36) implies that the asymptotic representation (4.29) contains the terms $\chi V_{s,k,N}$ such that $\operatorname{Im} \lambda_{-k} < 3/2 - \beta$. Let us rewrite the asymptotics in the form

$$w = \chi \sum_{s,k} c_{s,k} V_{s,k,N} + v,$$

where the sum contains $\chi V_{s,k,N}$ such that $\operatorname{Im} \lambda_{-k} \in]1,\ 3/2 - \beta[$, while $v \in L_2(\mathcal{K})$ and $v|_{\partial\mathcal{K}} \in L_2(\partial\mathcal{K})$. Since the spectrum of $\mathfrak{A}$ is symmetric about the line $\operatorname{Im} \lambda = 1$, the strip $\operatorname{Im} \lambda \in]\beta_1 + 1/2,\ 3/2 - \beta_1[$ does not contain the eigenvalues of $\mathfrak{A}$. Thus the sum vanishes and we have $w = v$. Taking the inclusions of v into account, it is not hard to prove that $v \in \mathcal{DM}(\overline{\tau})$. Since $\mathcal{M}(\overline{\tau})w = \{0, 0\}$, we get $w = 0$.
B). Assume that $\beta \in]\beta_{m+1},\ \beta_m[$. We construct a collection consisting of S_m function and prove that it is a basis in $\mathrm{K}er\, \mathcal{M}(\tau, \beta)^*$. Thus we prove the theorem because the range of $\mathcal{M}(\tau, \beta)$ is closed in $\mathcal{R}H_{\beta,T}(\mathcal{K}, \tau)$ and the kernel $\mathrm{K}er\, \mathcal{M}(\tau, \beta)$ is trivial. The strip $\operatorname{Im} \lambda \in [\beta_m + 1/2,\ \beta_1 + 1/2]$ contains precisely m eigenvalues of $\mathfrak{A}$. Let λ_k be in the strip. We put $F_{s,k,N} := M(D_x, \overline{\tau})\chi V_{s,k,N}$. Since $M(D_x, \overline{\tau})\chi V_{s,k,N} = O(r^{\operatorname{Im} \lambda_k - 2 + N})$ as $r \to 0$, one can choose sufficiently large T such that $F_{s,k,N} \in L_2(\mathcal{K})$. According to Theorem 4.10 and Remark 4.12, there exists a solution to the equation $\mathcal{M}(\overline{\tau})w_{s,k,N} = \{F_{s,k,N}, 0\}$. Put $w_{s,k} := \chi V_{s,k,N} - w_{s,k,N}$. We construct such functions for all eigenvalues of $\mathfrak{A}$ in the strip $\operatorname{Im} \lambda \in [\beta_m + 1/2,\ \beta_1 + 1/2]$. For λ_p there are N_p such functions. It is not hard to see that $w_{s,k}$ satisfies (4.27), (4.28) and $\{w_{s,k}, Tw_{s,k}\} \in \mathcal{R}H_{\beta,T}(\mathcal{K}, \tau)^*$. We show that $\{w_{s,k}, Tw_{s,k}\} \in \mathrm{K}er\, \mathcal{M}(\tau, \beta)^*$. Let $\{f, g\} \in \mathcal{R}H_{\beta,T}(\mathcal{K}, \tau)$. From the inclusion $\mathcal{R}H_{\beta,T}(\mathcal{K}, \tau) \subset L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$ and Theorem 4.10 it follows that there exists $v \in L_2(\mathcal{K})$ satisfying $\mathcal{M}(\tau)v = \{f, g\}$. Near the point $\mathcal{O}$, the function v admits the asymptotic representation

$$v = \chi \sum d_{s,k} U_{s,k,N} + h. \tag{4.37}$$

The inequality $\gamma > \gamma_0$ with large γ_0 leads to the estimates

$$\begin{gathered}\|\{[M,\chi]u,0\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|^2 = \|\{[M,\chi]u,0\};\mathcal{R}H_\beta(G,\tau)\|^2 \le \\ \le c\Big\{\|u;L_2(G)\|^2 + (|\tau|^{2(1-\beta)}/\gamma^2)\|u;L_2(G)\|^2\Big\} \le \\ \le c(1/\gamma^2 + |\tau|^{2(1-\beta)}/\gamma^4)\Big\{\|f;L_2(G)\|^2 + |\tau|\cdot\|g;L_{2,T}(\partial G)\|^2\Big\} \le \\ \le c(|\tau|^{2(1-\beta)}/\gamma^2)\Big\{\|f;L_2(G)\|^2 + |\tau|\cdot\|g;L_{2,T}(\partial G)\|^2\Big\} \le c\|\{f,g\};\mathcal{R}H_{\beta,T}(G,\tau)\|^2.\end{gathered}$$

Thus, we arrive at the estimate

$$|d_{s,k}| \le c|\tau|^{\beta+1/2}\|\{f,g\};\mathcal{R}H_{\beta,T}(G,\tau)\| \tag{4.39}$$

with a constant c independent of τ. Since $U(\eta) = \chi(r)u(x_1,x_2,x_3)$, the representation

$$u(x_1,x_2,x_3) = \chi(|\tau|r)\sum d_{s,k}\mathcal{U}_{s,k,N}(|\tau|x_1,|\tau|x_2,|\tau|x_3,\theta) + w(x_1,x_2,x_3)$$

holds in G. Taking into account the equality

$$\begin{gathered}\mathcal{U}_{s,k,N}(|\tau|x_1,|\tau|x_2,|\tau|x_3,\theta) = \sum_{q=0}^{N}(|\tau|r)^{i\lambda_k+q}(\tau/|\tau|)^q\Psi_s(\vartheta,\varphi) = \\ = |\tau|^{i\lambda_k}U_{s,k,N}(x_1,x_2,x_3,\tau),\end{gathered}$$

we finally obtain

$$u(x_1,x_2,x_3) = \chi(|\tau|r)\sum c_{s,k}U_{s,k,N}(x_1,x_2,x_3,\tau) + w(x_1,x_2,x_3)$$

with $c_{s,k} = |\tau|^{i\lambda_k}d_{s,k}$ and $\chi w \in H^1_\beta(G)$. It is not hard to verify that

$$c_{s,k} = i\Big(f,w_{s,k}\Big)_G + i\Big(g,Tw_{s,k}\Big)_{\partial G},$$

where $\{w_{s,k},Tw_{s,k}\}$ is a basis in Ker $\mathcal{M}_\beta(\tau,G)^*$. Using the estimate (4.39), we have

$$|c_{s,k}| \le c|\tau|^{\beta+1/2-\operatorname{Im}\lambda_k}\|\{f,g\}\mathcal{R}H_{\beta,T}(G,\tau)\|.$$

Consider the remainder w. Since $M(\tau,G)w = \{\widetilde{f},g\}$ with

$$\widetilde{f} = f - M(D_x,\tau)(\chi_\tau\sum c_{s,k}U_{s,k,N})$$

and

$$\Big(\widetilde{f},w_{s,k}\Big)_G + \Big(g,Tw_{s,k}\Big)_{\partial G} = 0,$$

Theorem 4.18 and Remark 4.19 leads to the inclusion $w \in \mathcal{D}H_\beta(G,\tau)$ and the inequality

$$\|w;\mathcal{D}H_\beta(G,\tau)\| \le c\|\{\widetilde{f},g\};\mathcal{R}H_{\beta,T}(G,\tau)\|.$$

Making use of the estimate on $c_{s,k}$ and the explicit form of $M(D_x,\tau)U_{s,k,N}$, we arrive at the estimate

$$\|w;\mathcal{D}H_\beta(G,\tau)\| \le c(|\tau|/\gamma)\|\{f,g\};\mathcal{R}H_{\beta,T}(G,\tau)\|.$$

□

Remark 4.21. *Theorem 4.20 is valid for the problem (4.4), (4.5) in $\mathcal{K}$ for $\gamma > 0$.*

4.7 Nonstationary problem in the cylinders $\mathcal{Q}$ and $\mathbf{Q}$

Applying the inverse Fourier transform $\mathcal{F}^{-1}_{\tau\to t}$, we pass from problem (4.4), (4.5) to problem (4.1), (4.2). Let us fix a cut-off function $\chi \in \mathcal{C}^\infty_c(\overline{G})$ that equals 1 near the point $\mathcal{O}$. We put

$$Xu(x_1,x_2,x_3,t) = \mathcal{F}^{-1}_{\tau\to t}\chi(|\tau|r)\mathcal{F}_{t'\to\tau}u(x_1,x_2,x_3,t'),$$

$$\Lambda^\mu u(x_1,x_2,x_3,t) = \mathcal{F}^{-1}_{\tau\to t}|\tau|^\mu\mathcal{F}_{t'\to\tau}u(x_1,x_2,x_3,t').$$

Introduce the spaces $\mathcal{D}V_\beta(\mathcal{Q},\gamma)$, $\mathcal{R}V(\mathcal{Q},\gamma)$, and $\mathcal{R}V_\beta(\mathcal{Q},\gamma)$ equipped with the norms

$$\|u;\mathcal{D}V_\beta(\mathcal{Q},\gamma)\| = \Big(\gamma^2\|u;V^0_\beta(\mathcal{Q},\gamma)\|^2 + \|Xu;V^1_\beta(\mathcal{Q},\gamma)\|^2\Big)^{1/2},$$

$$\|\{f,g\};\mathcal{R}V(\mathcal{Q},\gamma)\| = \Big(\|f;V^0_0(\mathcal{Q},\gamma)\|^2 + \|\Lambda^{1/2}g;V^0_0(\partial\mathcal{Q},\gamma)\|^2\Big)^{1/2},$$

$$\|\{f,g\};\mathcal{R}V_\beta(\mathcal{Q},\gamma)\| = \Big(\|f;V^0_\beta(\mathcal{Q},\gamma)\|^2 + (1/\gamma^2)\|\Lambda^{1-\beta}f;V^0_0(\mathcal{Q},\gamma)\|^2$$
$$+\|\Lambda^{1/2}g;V^0_\beta(\partial\mathcal{Q},\gamma)\|^2 + (1/\gamma^2)\|\Lambda^{3/2-\beta}g;V^0_0(\partial\mathcal{Q},\gamma)\|^2 + \|Xg;V^{1/2}_\beta(\partial\mathcal{Q},\gamma)\|^2\Big)^{1/2}.$$

Denote by $\mathcal{R}V_T(\mathcal{Q},\gamma)$ and $\mathcal{R}V_{\beta,T}(\mathcal{Q},\gamma)$ the subspaces of $\mathcal{R}V(\mathcal{Q},\gamma)$ and $\mathcal{R}V_\beta(\mathcal{Q},\gamma)$ respectively such that for any $\{f,g\} \in \mathcal{R}V_T(\mathcal{Q},\gamma)$ (or $\mathcal{R}V_{\beta,T}(\mathcal{Q},\gamma)$) with $g = (\vec{u},v,h)$ we have $\langle\vec{u},\,\vec{\nu}\rangle_3 = 0$ a.e. on $\partial\mathcal{Q}$. The spaces in the cylinder $\mathbf{Q}$ are defined in a similar way.

Definition 4.22. *Let $\{f,g\} \in \mathcal{R}V_T(\mathcal{Q},\gamma)$ and let $\widehat{u}(x_1,x_2,x_3,\tau)$ be the strong solution to the problem (4.4), (4.5) in $\mathcal{K}$ with right-hand side $\{-i\widehat{f},\widehat{g}\}$, where $\widehat{f}(x_1,x_2,x_3,\tau) = \mathcal{F}_{t\to\tau}f(x_1,x_2,x_3,t)$, $\widehat{g}(x_1,x_2,x_3,\tau) = \mathcal{F}_{t\to\tau}g(x_1,x_2,x_3,t)$. The function u, defined by $u(x_1,x_2,x_3,t) = \mathcal{F}^{-1}_{\tau\to t}\widehat{u}(x_1,x_2,x_3,\tau)$, is called a strong solution to the problem (4.1), (4.2) in the cylinder $\mathcal{Q}$ with right-hand side $\{f,g\}$.*

The next result follows from Theorem 4.10.

Theorem 4.23. *For every $\{f,g\} \in \mathcal{R}V_T(\mathcal{Q},\gamma)$ and for any $\gamma > 0$ there exists a strong solution v to the problem (4.1), (4.2) in $\mathcal{Q}$ with right-hand side $\{f,g\}$. Moreover,*

$$\gamma\|v;V^0_0(\mathcal{Q},\gamma)\| \le \|\{f,g\};\mathcal{R}V_T(\mathcal{Q},\gamma)\|.$$

Definition 4.24. *Let $\{f,g\} \in \mathcal{R}V_{\beta,T}(\mathcal{Q},\gamma)$ and let $\widehat{u}(x_1,x_2,x_3,\tau)$ be the strong β-solution to the problem (4.4), (4.5) in $\mathcal{K}$ with right-hand side $\{-i\widehat{f},\widehat{g}\}$, where $\widehat{f}(x_1,x_2,x_3,\tau) = \mathcal{F}_{t\to\tau}f(x_1,x_2,x_3,t)$, $\widehat{g}(x_1,x_2,x_3,\tau) = \mathcal{F}_{t\to\tau}g(x_1,x_2,x_3,t)$. The function u, defined by $u(x_1,x_2,x_3,t) = \mathcal{F}^{-1}_{\tau\to t}\widehat{u}(x_1,x_2,x_3,\tau)$, is called a strong β-solution to the problem (4.1), (4.2) in the cylinder $\mathcal{Q}$ with right-hand side $\{f,g\}$.*

The next result follows from Theorem 4.18.

THEOREM 4.25. *1) Let $\beta \in [\beta_1,\ 1]$ and let the number $\lambda = i(\beta+1/2)$ be regular for the pencil $\mathfrak{A}$. Then there exists a unique strong β-solution v to the problem (4.1), (4.2) in $\mathcal{Q}$ with any right-hand side $\{f,g\} \in \mathcal{R}V_{\beta,T}(\mathcal{Q},\gamma)$. Moreover,*

$$\|v;\mathcal{D}V_\beta(\mathcal{Q},\gamma)\| \le c\|\{f,g\};\mathcal{R}V_{\beta,T}(\mathcal{Q},\gamma)\|.$$

2) Let $\beta \in]\beta_{m+1},\ \beta_m[$. A strong β-solution to the problem (4.1), (4.2) in $\mathcal{Q}$ with right-hand side $\{f,g\} \in \mathcal{R}V_{\beta,T}(\mathcal{Q},\gamma)$ exists (and is unique) if for all $\tau = \sigma - i\gamma$ ($\sigma \in \mathbb{R},\ \gamma > 0$) the conditions

$$\Big(-i\widehat{f}(\cdot,\tau), w_{s,k}(\cdot,\overline{\tau})\Big)_{\mathcal{K}} + \Big(\widehat{g}(\cdot,\tau), Tw_{s,k}(\cdot,\overline{\tau})\Big)_{\partial\mathcal{K}} = 0$$

hold, where $\{w_{s,k}, Tw_{s,k}\}_{k=-1,..,-m}^{s=1,..,N_k}$ is a basis in $\operatorname{Ker}\mathcal{M}(\tau,\beta)^*$. If such a solution exists, it satisfies the inequality in 1).

Next theorem follows from Theorem 4.38 and Remark 4.39

THEOREM 4.26. *Assume that $\{f,g\} \in \mathcal{R}V_{\beta,T}(\mathcal{Q},\gamma)$ and $\beta \in]\beta_{m+1},\beta_m[$. Then the strong solution to the problem (4.1), (4.2) admits the representation*

$$u(x_1,x_2,x_3,t) = \sum U_{s,k,N}(r,\varphi,\vartheta,D_t)(X\check{c}_{s,k})(x_1,x_2,x_3,t) + w(x_1,x_2,x_3,t),$$

where $w \in \mathcal{D}V_\beta(\mathcal{Q},\gamma)$. The sum consists of the terms corresponding to the eigenvalues of the pencil $\mathfrak{A}$ in the strip $\operatorname{Im}\lambda \in [\beta_m+1/2,\ \beta_1+1/2]$, while N in each term is taken large enough in order for the inclusions $\chi r^{i\lambda_k+N+1}\Psi_{N+1} \in \mathcal{D}H_\beta(G,\tau)$ to be valid. The coefficients are defined by

$$\check{c}_{s,k}(t) = \mathcal{F}^{-1}_{\tau\to t} c_{s,k}(\tau)$$

with

$$c_{s,k}(\tau) = i\big(-i\widehat{f}(\cdot,\tau), w_{s,k}(\cdot,\overline{\tau})\big)_{\mathcal{K}} + i\big(\widehat{g}(\cdot,\tau), Tw_{s,k}(\cdot,\overline{\tau})\big)_{\partial\mathcal{K}}.$$

Moreover, we have the estimates

$$\|e^{-\gamma t}\check{c}_{s,k}(\cdot); H^{\operatorname{Im}\lambda_k-\beta-1/2}(\mathbb{R})\| \le c\|\{f,g\};\mathcal{R}V_{\beta,T}(\mathcal{Q},\gamma)\|,$$
$$\|w;\mathcal{D}V_\beta(\mathcal{Q},\gamma)\| \le (c/\gamma)\|\{\Lambda f,\Lambda g\};\mathcal{R}V_{\beta,T}(\mathcal{Q},\gamma)\|.$$

Strong solutions and strong β-solutions to the problem (4.1), (4.2) in $\mathbf{Q}$ can be defined in the same way as those to the problem in the cylinder $\mathcal{Q}$ in Definition 4.22 and Definition 4.24.

REMARK 4.27. *All the theorems in this section are still valid for the problem (4.1), (4.2) in $\mathbf{Q}$ if $\gamma > \gamma_0$ with sufficiently large γ_0.*

Applying the Fourier transform to the formula for $c_{s,k}(\tau)$, we get

$$\check{c}_{s,k}(t) = \int\limits_{\mathcal{K}} \mathrm{d}x \int\limits_{\mathbb{R}} \mathrm{d}u\ \langle f(x,y,z,t-u), W_{s,k}(x,y,z,u)\rangle_{\mathbb{R}^8} -$$
$$-\int\limits_{\partial\mathcal{K}} \mathrm{d}S \int\limits_{\mathbb{R}} \mathrm{d}u\ \langle g(x,y,z,t-u), T_0 W_{s,k}(x,y,z,u)\rangle_{\mathbb{R}^5},$$

with $W_{s,k}(x_1, x_2, x_3, t) = \mathcal{F}^{-1}_{\tau \to t} \overline{w_{s,k}(x_1, x_2, x_3, \overline{\tau})}$. The explicit formulas for $W_{s,k}$ in $\mathcal{K}$ were derived in Chapter 2.

4.8 Connection between the augmented and non-augmented Maxwell systems

Up to this point the discussion has been related to the augmented Maxwell system. In this section we prove that under some conditions on the right-hand side of such a system, its solutions have the form $u = (\vec{u}, \vec{v}, 0, 0)$, therefore $(\vec{u}, \vec{v})$ satisfy the usual (non-augmented) Maxwell system. The mentioned conditions on the right-hand side are derived from the compatibility of the usual Maxwell system

$$\begin{cases} \partial \vec{E}/\partial t - \mathrm{rot}\ \vec{B} = -\vec{J}, \\ \partial \vec{B}/\partial t + \mathrm{rot}\ \vec{E} = -\vec{F}, \\ \mathrm{div}\ \vec{E} = \rho, \quad \mathrm{div}\ \vec{B} = \mu \end{cases}$$

with boundary conditions

$$[\nu \times \vec{E}] = \vec{\Phi}, \ \langle \vec{B}, \vec{\nu} \rangle = \delta.$$

Namely, for sufficiently smooth functions from the first and the third equations of this system we obtain

$$\partial \rho/\partial t + \mathrm{div}\ \vec{J} = 0. \tag{4.40}$$

Similarly, from the second and the forth equations we get

$$\partial \mu/\partial t + \mathrm{div}\ \vec{F} = 0. \tag{4.41}$$

Consider the inner product of the second equation with unit outward normal ν:

$$\frac{\partial}{\partial t} \langle \vec{B}, \vec{\nu} \rangle_3 + \langle \mathrm{rot}\ \vec{E}, \vec{\nu} \rangle_3 + \langle \vec{F}, \vec{\nu} \rangle_3 = 0. \tag{4.42}$$

Let us apply the formula

$$\langle \mathrm{rot}\ \vec{E}, \vec{\nu} \rangle_3 = -\mathrm{Div}\ [\vec{\nu} \times \vec{E}],$$

where Div is the surface divergency of a tangent vector field on the boundary (see, e.g., [8]). Then we rewrite (4.42) in the form

$$\frac{\partial}{\partial t} \delta - \mathrm{Div}\ \vec{\Phi} + \langle \vec{F}, \vec{\nu} \rangle_3 = 0. \tag{4.43}$$

Consider the problem (4.4), (4.5) in G with right-hand side $\{f, g\} \in L_2(G) \times L_{2,T}(\partial G)$, where $f = (\vec{A}, \vec{B}, \alpha, \beta)^T$ and $g = (\vec{\Phi}, \delta, \omega)^T$. Since the right-hand side is not smooth, the conditions should be understood in a proper way. For this purpose we recall the following definitions.
Let $\vec{u}$ and v be functions such that for any $\phi \in \mathcal{C}^\infty_c(G)$ we have

$$\int\limits_G \mathrm{d}x\ \langle \vec{u}, \nabla \phi \rangle_3 + \int\limits_G \mathrm{d}x\ v \cdot \overline{\phi} = 0.$$

Then we say that $v = \operatorname{div} \vec{u}$ in the distributional sense. By $H(\operatorname{div}, G)$ we denote the space of functions $\vec{u} \in L_2(G)$ such that $\operatorname{div} \vec{u} \in L_2(G)$, where the divergency is understood in the distributional sense. We endow this space with the norm

$$\|\vec{u}; H(\operatorname{div}, G)\| = \Big(\|\vec{u}; L_2(G)\|^2 + \|\operatorname{div} \vec{u}; L_2(G)\|^2\Big)^{1/2}.$$

With the obvious inner product, $H(\operatorname{div}, G)$ is a Hilbert space. The mapping γ_ν is defined by the formula $\gamma_\nu \vec{v} = \langle \vec{v}|_{\partial G}, \vec{\nu}\rangle_3$ on smooth vector-valued functions. This mapping can be extended by continuity to a continuous map $\gamma_\nu : H(\operatorname{div}, G) \mapsto H^{-1/2}(\partial G)$. Moreover, the following Green formula holds for functions $\vec{v} \in H(\operatorname{div}, G)$, $\phi \in H^1(G)$ (see [27, Chapter 3, Theorem 3.24]):

$$\int\limits_G \mathrm{d}x\, \langle \vec{v}, \nabla \phi\rangle_3 + \int\limits_G \mathrm{d}x\, \operatorname{div} \vec{v} \cdot \overline{\phi} = \int\limits_{\partial G} \mathrm{d}S\, \gamma_\nu(\vec{v}) \cdot \overline{\phi}. \tag{4.44}$$

Now we turn to the space of tangent vector fields on the boundary. Let $\vec{u} \in L_{2,T}(\partial G)$ and $v \in L_2(G)$ be functions such that for any $\phi \in H^1(\partial G)$ the equality

$$\int\limits_{\partial G} \mathrm{d}S\, \langle \vec{u}, \nabla_T \phi\rangle_3 + \int\limits_{\partial G} \mathrm{d}S\, v \cdot \overline{\phi} = 0 \tag{4.45}$$

holds, where ∇_T is the surface gradient. Then we write $v = \operatorname{Div} \vec{u}$ and we say that the function $\vec{u} \in L_{2,T}(\partial G)$ has the square-integrable surface divergency. By $\mathcal{H}(\operatorname{Div}, \partial G)$ we denote the space of all functions from $L_{2,T}(\partial G)$ with the square-integrable surface divergency. The space $\mathcal{H}(\operatorname{Div}, \partial G)$ is endowed with the norm

$$\|\vec{u}; \mathcal{H}(\operatorname{Div}, \partial G)\| = \Big(\|\vec{u}; L_2(\partial G)\|^2 + \|\operatorname{Div} \vec{u}; L_2(\partial G)\|^2\Big)^{1/2}.$$

With the obvious inner product, $\mathcal{H}(\operatorname{Div}, \partial G)$ is a Hilbert space.

Applying the Fourier transform $\mathcal{F}_{t\to\tau}$, we rewrite (4.40), (4.41) in the form

$$\vec{A} \in H(\operatorname{div}, G),\ \alpha \in L_2(G),\ \operatorname{div} \vec{A} = i\tau\alpha, \tag{4.46}$$

$$\vec{B} \in H(\operatorname{div}, G),\ \beta \in L_2(G),\ \operatorname{div} \vec{B} = i\tau\beta. \tag{4.47}$$

For (4.43) we have

$$\vec{\Phi} \in \mathcal{H}(\operatorname{Div}, \partial G),\ \delta \in L_2(\partial G),\ i\tau\delta - \operatorname{Div} \vec{\Phi} - \gamma_\nu(\vec{B}) = 0. \tag{4.48}$$

The last condition imposed on the right-hand side $\{f, g\}$ is the boundary condition $\omega = 0$.

For the proof of theorem on connection between solutions to the augmented and usual Maxwell systems, we need some properties of the Helmholtz operator in the domain G. Here we recall all the necessary definitions. For details and proofs we refer the reader to the articles [18] and [33]. Let $\Xi = \mathcal{K} \cap \mathcal{S}^2$, where the cone

$\mathcal{K}$ coincides with the domain G near the conical point $\mathcal{O}$. For the Laplace operator we introduce the operator pencil $\mathfrak{E}$ in the domain Ξ by the formula

$$\mathfrak{E}(\lambda) = (i\lambda)^2 + i\lambda - \delta,$$

where δ is the Laplace-Beltrami operator. In the case of the Dirichlet problem the pencil $\mathfrak{E}$ is defined on the functions $u \in \mathrm{H}^2(\Xi)$ such that $u|_{\partial\Xi} = 0$. In the case of the Neumann problem the pencil $\mathfrak{E}$ is defined on the functions $u \in \mathrm{H}^2(\Xi)$ such that $\partial_\nu|_{\partial\Xi} = 0$. Let $\{\mu_k,\ w_k\}$ and $\{\widetilde{\mu}_k, \widetilde{w}_k\}$ be the sets of eigenvalues and eigenfunctions of the Dirichlet and Neumann problems for the operator pencil $\mathfrak{E}$. Denote by L_D the lineal spanned by the functions in $\mathcal{C}_c^\infty(G)$ and by functions of the form $\chi r^{i\mu_k} w_k$ with $\operatorname{Im}\mu_k < 0$, where χ is a cut-off function equal to 1 near the point $\mathcal{O}$ and 0 outside the neighborhood where G coincides with a cone. We also introduce the lineal L_N spanned by the functions in $\mathcal{C}_c^\infty(\overline{G} \setminus \mathcal{O})$ with normal derivative vanishing on $\partial G \backslash \mathcal{O}$ and by functions of the form $\chi r^{i\widetilde{\mu}_k} \widetilde{w}_k$, where $\operatorname{Im}\widetilde{\mu}_k \le 0$. According to [18, §4] and [33, §3], the range of the Helmholtz operator $\tau^2 + \triangle$ with $\tau = \sigma - i\gamma$ ($\gamma \neq 0$) given on L_D or L_N is dense in $L_2(G)$.

THEOREM 4.28. *Consider the problem (4.4), (4.5) in the domain G with the right-hand side $\{f, g\}$, where the functions $f = (\vec{A}, \vec{B}, \alpha, \beta)^T$ and $g = (\vec{\Phi}, \delta, 0)^T$ are subject to the conditions (4.46), (4.47), and (4.48). Then the corresponding strong solution u is of the form $u = (\vec{u}, \vec{v}, 0, 0)$.*

Proof. Since $\mathcal{M}(\tau, G)$ is the closure of the differential operator $M(D_x, \tau)$ given on $\mathcal{D}(G)$, there exists a sequence $\Big\{u_k = (\vec{u}_k, \vec{v}_k, h_k, q_k)^T\Big\} \subset \mathcal{D}(G)$ such that $u_k \to u$ (the convergence in $L_2(G)$) and $\Big\{M(D_x, \tau)u_k, \Gamma u_k\Big\} \to \{f, g\}$ (the convergence in $L_2(G) \times L_{2,T}(\partial G)$). We have $u_k \in \mathcal{C}^\infty(\overline{G} \setminus \mathcal{O}))$ so the system (4.4), (4.5) can be understood as usual. In particular,

$$\begin{gathered} i\tau\vec{u}_k - \operatorname{rot}\vec{v}_k + \nabla h_k = \vec{A}_k, \\ i\tau h_k + \operatorname{div}\vec{u}_k = \alpha_k. \end{gathered}$$

We show that $h = 0$. Assume that $\phi \in L_D$. Multiply the first equality by $\nabla\overline{\phi}$, the second one by $i\tau\overline{\phi}$, and integrate over G. Then

$$i\tau \int\limits_G \mathrm{d}x\, \langle \vec{u}_k, \nabla\phi\rangle_3 - \int\limits_G \mathrm{d}x\, \langle \operatorname{rot}\vec{v}_k, \nabla\phi\rangle_3 + \int\limits_G \mathrm{d}x\, \langle \nabla h_k, \nabla\phi\rangle_3 = \int\limits_G \mathrm{d}x\, \langle \vec{A}_k, \nabla\phi\rangle_3\,,$$
$$-\tau^2 \int\limits_G \mathrm{d}x\; h_k \cdot \overline{\phi}\; + i\tau \int\limits_G \mathrm{d}x\, \operatorname{div}\vec{u}_k \cdot \overline{\phi} = i\tau \int\limits_G \mathrm{d}x\; \alpha_k \cdot \overline{\phi}.$$

We integrate by parts in the two first terms of the first equality, add the first and the second equalities, and obtain

$$-\tau^2 \int\limits_G \mathrm{d}x\; h_k \cdot \overline{\phi} + \int\limits_G \mathrm{d}x\, \langle \nabla h_k, \nabla\phi\rangle_3 = i\tau \int\limits_G \mathrm{d}x\; \alpha_k \cdot \overline{\phi} + \int\limits_G \mathrm{d}x\, \langle \vec{A}_k, \nabla\phi\rangle_3.$$

Integrate by parts the second term on the left. Then we get

$$-\tau^2 \int\limits_G \mathrm{d}x\ h_k \cdot \overline{\phi} - \int\limits_G \mathrm{d}x\ h_k \cdot \overline{\triangle \phi} + \int\limits_{\partial G} \mathrm{d}S\ h_k \cdot \overline{\partial \phi / \partial \nu} = i\tau \int\limits_G \mathrm{d}x\ \alpha_k \cdot \overline{\phi} + \int\limits_G \mathrm{d}x\ \langle \vec{A}_k, \nabla \phi \rangle_3.$$

As $k \to \infty$, taking into account the boundary condition for h and the property (4.46), we arrive at the formula

$$\int\limits_G \mathrm{d}x\ h \cdot \overline{(\overline{\tau}^2 + \triangle)\phi} = 0.$$

Therefore $h = 0$ because the range of the operator $\overline{\tau}^2 + \triangle$ given on L_D is dense in $L_2(G)$.
Verify the equality $q = 0$. We have

$$\begin{gathered} i\tau \vec{v}_k + \mathrm{rot}\ \vec{u}_k + \nabla q_k = \vec{B}_k, \\ i\tau q_k + \mathrm{div}\ \vec{v}_k = \beta_k. \end{gathered}$$

Assume that $\phi \in L_N$. Then

$$\begin{gathered} i\tau \int\limits_G \mathrm{d}x\ \langle \vec{v}_k, \nabla \phi \rangle_3 + \int\limits_G \mathrm{d}x\ \langle \mathrm{rot}\ \vec{u}_k, \nabla \phi \rangle_3 + \int\limits_G \mathrm{d}x\ \langle \nabla q_k, \nabla \phi \rangle_3 = \int\limits_G \mathrm{d}x\ \langle \vec{B}_k, \nabla \phi \rangle_3\ , \\ -\tau^2 \int\limits_G \mathrm{d}x\ q_k \cdot \overline{\phi} + i\tau \int\limits_G \mathrm{d}x\ \mathrm{div}\ \vec{v}_k \cdot \overline{\phi} = i\tau \int\limits_G \mathrm{d}x\ \beta_k \cdot \overline{\phi}. \end{gathered}$$

We integrate by parts all the terms in the left-hand side of the first line, add the second line, and let $k \to \infty$:

$$\begin{gathered} -\int\limits_G \mathrm{d}x\ q \cdot \overline{(\overline{\tau}^2 + \triangle)\phi} + i\tau \int\limits_{\partial G} \mathrm{d}S\ \delta \cdot \overline{\phi} + \int\limits_{\partial G} \mathrm{d}S\ \langle \vec{\Phi}, \nabla \phi \rangle_3 = \\ = i\tau \int\limits_G \mathrm{d}x\ \beta \cdot \overline{\phi} + \int\limits_G \mathrm{d}x\ \langle \vec{B}, \nabla \phi \rangle_3. \end{gathered}$$

Since $L_N \subset H^1(G)$ and $L_N|_{\partial G} \subset H^1(\partial G)$, we can apply the formulas (4.44), (4.45). Therefore we obtain

$$\begin{gathered} -\int\limits_G \mathrm{d}x\ q \cdot \overline{(\overline{\tau}^2 + \triangle)\phi} + \int\limits_{\partial G} \mathrm{d}S\ (i\tau\delta - \mathrm{Div}\vec{\Phi} - \gamma_\nu(\vec{B})) \cdot \overline{\phi} = \\ = i\tau \int\limits_G \mathrm{d}x\ \beta \cdot \overline{\phi} - \int\limits_G \mathrm{d}x\ \mathrm{div}\ \vec{B} \cdot \overline{\phi}. \end{gathered}$$

Finally, using (4.47), (4.48)

$$\int\limits_G \mathrm{d}x\ q \cdot \overline{(\overline{\tau}^2 + \triangle)\phi} = 0.$$

Since the range of the operator $\overline{\tau}^2 + \triangle$ given on L_N is dense in $L_2(G)$, we arrive at $q = 0$. $\square$

5 THE PROBLEM WITH INHOMOGENEOUS IMPEDANCE BOUNDARY CONDITIONS

5.1 Preliminaries

In the fifth chapter we study the augmented Maxwell system

$$\begin{cases} \partial \vec{E}/\partial t - \text{rot } \vec{B} + \nabla h = -\vec{J}, \\ \partial \vec{B}/\partial t + \text{rot } \vec{E} + \nabla q = -\vec{G}, \\ \partial h/\partial t + \text{div } \vec{E} = \rho, \\ \partial q/\partial t + \text{div } \vec{B} = \mu \end{cases} \tag{5.1}$$

in a model cone and in a bounded domain with conical point. On the boundary, we impose the inhomogeneous impedance conditions

$$\vec{\nu} \times [\vec{B} \times \vec{\nu}] + \psi[\vec{\nu} \times \vec{E}] = \vec{\Phi},\ h = H,\ q = Q, \tag{5.2}$$

where $\vec{\nu}$ is the unit outward normal and $\psi \in \mathbb{C}$ is an impedance. Suppose the impedance ψ to be constant such that $\operatorname{Re}\psi < 0$. We rewrite the system (5.1) in the form

$$\partial u/\partial t + A(\partial)u = f, \tag{5.3}$$

where $u = (\vec{E},\ \vec{B},\ h,\ q)^T,\ f = (-\vec{J},\ -\vec{G},\ \rho,\ \mu)^T,$ and $\partial = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$.
Let $\tau = \sigma - i\gamma$, where $\sigma \in \mathbb{R},\ \gamma > 0$. Applying the Fourier transform $\mathcal{F}_{t\to\tau}$ to the equations (5.1), (5.2), we obtain the problem with parameter in the cone $\mathcal{K}$ (in the domain G):

$$\begin{cases} \tau\widehat{u} + A(D_x)\widehat{u} = -i\widehat{f}, \\ \Gamma_1\widehat{u} = \widehat{g}. \end{cases}$$

Rewrite this problem as follows

$$M(D_x, \tau)u = f, \tag{5.4}$$

$$\Gamma_1 u = g, \tag{5.5}$$

where $M(D_x, \tau) = \tau + A(D_x)$. Note that for the problem (5.4), (5.5) the Green formula (1.10) is of the form

$$\begin{aligned} &\Big(M(D_x,\tau)u, v\Big)_{\mathcal{K}} + \Big(\Gamma_1 u,\ -iT_1 v\Big)_{\partial\mathcal{K}} = \\ &= \Big(u, M(D_x,\overline{\tau})v\Big)_{\mathcal{K}} + \Big(-iT_2 u,\ \Gamma_2 v\Big)_{\partial\mathcal{K}}, \end{aligned} \tag{5.6}$$

so we have the following adjoint problem with respect to this formula:

$$M(D_x, \overline{\tau})u = f, \tag{5.7}$$

$$\Gamma_2 u = g. \tag{5.8}$$

Considering the problem (5.4), (5.5) in the cone $\mathcal{K}$, we can change the variables $(x_1, x_2, x_3) \mapsto \eta = (|\tau|x_1, |\tau|x_2, |\tau|x_3)$. Denote $\tau/|\tau|$ by θ, put $U(\eta,\tau) = u(|\tau|^{-1}\eta, \tau)$, $F(\eta,\tau) = |\tau|^{-1} f(|\tau|^{-1}\eta,\tau)$, and $G(\eta,\tau) = g(|\tau|^{-1}\eta,\tau)$. Thus we arrive at

$$M(D_\eta, \theta)U = F, \tag{5.9}$$

$$\Gamma_1 U = G. \tag{5.10}$$

5.2 Operator pencil

For the problem (5.4), (5.5), we introduce the operator pencil

$$\mathfrak{C}(\lambda)\Phi = r^{1-i\lambda} A(D) r^{i\lambda}\Phi \tag{5.11}$$

with $\Phi = (\vec{\mathcal{U}},\ \vec{\mathcal{V}},\ \mathcal{H},\ \mathcal{Q}) \in H^1(\Xi)$ such that

$$\vec{\nu} \times [\vec{\mathcal{V}} \times \vec{\nu}] + \psi[\vec{\nu} \times \vec{\mathcal{U}}] = 0,\ \ \mathcal{H} = 0,\ \ \mathcal{Q} = 0 \ \text{ on } \partial\Xi,$$

where $\Xi = \mathcal{K} \cap \mathcal{S}^2$ and $\vec{\nu}$ is the unit outward normal to $\partial\mathcal{K}$. Note that these boundary conditions can be rewritten in the form $\Gamma_1\big(r^{i\lambda}\Phi\big) = 0$ on $\partial\mathcal{K}$. Since the problem (5.4), (5.5) is elliptic, the spectrum of the pencil $\mathfrak{C}$ consists of normal eigenvalues. Concerning the notions of eigenvalues, eigenvectors and associated vectors of the pencil, we refer to [20] or [28, Chapter 1, §2].

LEMMA 5.1. *The eigenvalues of the pencil $\mathfrak{C}$ are independent of the impedance ψ.*

Proof. Let $\{\lambda, \Phi_0\}$ with $\Phi_0 = (\vec{\mathcal{U}}_0,\ \vec{\mathcal{V}}_0,\ \mathcal{H}_0,\ \mathcal{Q}_0)$ be an eigenvalue and an eigenvector of the pencil $\mathfrak{C}$ that is $\mathfrak{C}(\lambda)\Phi_0 = 0$. Then $\{\lambda, \Phi_1\}$ with $\Phi_1 = (-\psi\vec{\mathcal{U}}_0,\ \vec{\mathcal{V}}_0,\ \mathcal{H}_0,\ -\psi\mathcal{Q}_0)$ are an eigenvalue and an eigenvector of the pencil $\mathfrak{C}_0$, defined by (5.11) on functions $\Phi = (\vec{\mathcal{U}},\ \vec{\mathcal{V}},\ \mathcal{H},\ \mathcal{Q}) \in H^1(\Xi)$ such that

$$\vec{\nu} \times [\vec{\mathcal{V}} \times \vec{\nu}] - [\vec{\nu} \times \vec{\mathcal{U}}] = 0,\ \ \mathcal{H} = 0,\ \ \mathcal{Q} = 0 \ \text{ on } \partial\mathcal{K}.$$

In the same way one can verify that if λ is an eigenvalue of the pencil $\mathfrak{C}_0$, then λ is an eigenvalue of the pencil $\mathfrak{C}$. Thus the spectrum of $\mathfrak{C}$ coincides with the spectrum of $\mathfrak{C}_0$. It remains to note that the pencil $\mathfrak{C}_0$ is independent of the impedance ψ. $\square$

Now our purpose is to show that the line $\{\lambda \in \mathbb{C} \ : \ \operatorname{Im} \lambda = 1\}$ is free from the spectrum of $\mathfrak{C}$.

LEMMA 5.2. *Let λ be an eigenvalue of the pencil $\mathfrak{C}$, $\operatorname{Im}\lambda \in [0, 1]$, and let $\Phi = (\vec{\mathcal{U}}, \vec{\mathcal{V}}, \mathcal{H}, \mathcal{Q})$ be an eigenvector corresponding to λ. Then $\mathcal{H} \equiv 0$, $\mathcal{Q} \equiv 0$.*

Proof. Since $\{\lambda, \Phi\}$ are an eigenvalue and an eigenvector of the pencil $\mathfrak{C}$, we have $A(D_x)r^{i\lambda}\Phi = 0$. We again apply the operator $A(D_x)$ and obtain $\triangle r^{i\lambda}\Phi = 0$. Thus for the functions $\mathcal{H}$, $\mathcal{Q}$ we have

$$\begin{cases} \triangle r^{i\lambda}\mathcal{H} = 0 \ \text{ in } \ \mathcal{K}, \\ \ \mathcal{H} = 0 \ \text{ on } \ \partial\mathcal{K}, \end{cases} \qquad \begin{cases} \triangle r^{i\lambda}\mathcal{Q} = 0 \ \text{ in } \ \mathcal{K}, \\ \ \mathcal{Q} = 0 \ \text{ on } \ \partial\mathcal{K}. \end{cases}$$

This means that λ is an eigenvalue of the Dirichlet problem pencil for the Laplace operator and $\mathcal{H}$, $\mathcal{Q}$ are the eigenvectors corresponding to λ. Since the strip $\{\lambda \in \mathbb{C} \ : \ \operatorname{Im}\lambda \in [0, 1]\}$ contains no eigenvalues of the pencil of the Dirichlet problem for the Laplace operator (see [33, §3]), we obtain $\mathcal{H} = 0$, $\mathcal{Q} = 0$ $\square$.

PROPOSITION 5.3. *Let $\mathcal{K} \subset \mathbb{R}^3$ be a cone such that $\Xi = \mathcal{K} \cap \mathcal{S}^2$ is a one - connected domain with smooth boundary. Then the line $\{\lambda \in \mathbb{C} \ : \ \operatorname{Im} \ \lambda = 1\}$ contains no points of the spectrum of the pencil $\mathfrak{C}$.*

Proof. According to Lemma 5.1, it suffices to consider the pencil $\mathfrak{C}_0$. Suppose that λ is an eigenvalue of $\mathfrak{C}_0$ such that $\operatorname{Im}\lambda = 1$. Let $\Phi = (\vec{\mathcal{U}}, \vec{\mathcal{V}}, \mathcal{H}, \mathcal{Q})$ be an eigenvector corresponding to λ. From Lemma 5.2 it follows that $\mathcal{H} \equiv 0$, $\mathcal{Q} \equiv 0$. Apply the Green formula

$$\Big(A(\partial_x)u,\, v\Big)_\Omega + \Big(u,\, A(\partial_x)v\Big)_\Omega = \Big(\Gamma_1 u,\, T_1 v\Big)_{\partial\Omega} + \Big(T_2 u,\, \Gamma_2 v\Big)_{\partial\Omega} \tag{5.12}$$

to $u = r^{i\lambda}\Phi$, $v = r^{i\lambda}\Phi$, and $\Omega = \mathcal{K}_{\varepsilon R} = \{x \in \mathcal{K} \ : \ \varepsilon \le |x| \le R\}$ with $0 < \ \varepsilon < \ R$. Since $A(D_x)r^{i\lambda}\Phi = 0$, we see that the right hand side is equal to zero. The integrals over $S_\varepsilon = \{x \in \mathcal{K} \ : \ |x| = \varepsilon\}$ and $S_R = \{x \in \mathcal{K} \ : \ |x| = R\}$ cancel each other. Taking into account the boundary conditions $\vec{\nu} \times [\vec{\mathcal{V}} \times \vec{\nu}] - [\vec{\nu} \times \vec{\mathcal{U}}] = 0$, we arrive at

$$\int\limits_{\partial\mathcal{K}\cap\{\varepsilon\le r\le R\}} \mathrm{d}S \ |\vec{\nu} \times \vec{\mathcal{V}}|^2/r^2 = 0.$$

This formula and the boundary condition imply that $\vec{\nu} \times \vec{\mathcal{V}} = 0$ and $\vec{\nu} \times \vec{\mathcal{U}} = 0$ on $\partial\mathcal{K}$. Note that the boundary condition $\vec{\nu} \times [\vec{\mathcal{V}} \times \vec{\nu}] - [\vec{\nu} \times \vec{\mathcal{U}}] = 0$ splits into the simple independent parts $\vec{\nu} \times \vec{\mathcal{U}} = 0$ and $\vec{\nu} \times \vec{\mathcal{V}} = 0$. Let us show that $\vec{\mathcal{V}} = 0$. We have rot $(r^{i\lambda}\vec{\mathcal{V}}) = 0$, div $(r^{i\lambda}\vec{\mathcal{V}}) = 0$ in $\mathcal{K}$, and $\vec{\nu} \times \vec{\mathcal{V}} = 0$ on $\partial\mathcal{K}$. Recall that Ξ is one-connected. So $r^{i\lambda}\vec{\mathcal{V}} = \nabla\varphi$, where $\triangle\varphi = 0$ in $\mathcal{K}$ and $\varphi = 0$ on $\partial\mathcal{K}$. It is easy to see that $\varphi(x) = r^{i\mu}W(\omega)$ with Im $\mu = 0$. However, the line $\{\lambda \in \mathbb{C} \ : \ \mathrm{Im}\ \lambda = 0\}$ contains no eigenvalues of the pencil of the Dirichlet problem for the Laplace operator. Therefore we get $\varphi = 0$ and $\vec{\mathcal{V}} = 0$. In the same way it can be shown that $\vec{\mathcal{U}} = 0$. $\square$

Let $\mathfrak{C}(\lambda)^*$ be the operator adjoint to $\mathfrak{C}(\lambda)$. Introduce the pencil $\mathbb{C} \ni \lambda \mapsto \mathfrak{C}^*(\lambda) := \mathfrak{C}(\overline{\lambda})^*$. By means of the Green formula (5.12) one can check that $\mathfrak{C}^*(\lambda) = \mathfrak{D}(\lambda{+}2i)$, where $\mathfrak{D}$ is the pencil defined by (5.11) on functions $\Phi = (\vec{\mathcal{U}},\ \vec{\mathcal{V}},\ \mathcal{H},\ \mathcal{Q}) \in H^1(\Xi, \mathbb{C}^8)$ such that

$$\vec{\nu} \times [\vec{\mathcal{V}} \times \vec{\nu}] - \overline{\psi}[\vec{\nu} \times \vec{\mathcal{U}}] = 0, \ \ \mathcal{H} = 0, \ \ \mathcal{Q} = 0 \ \text{ on } \partial\Xi.$$

The pencil $\mathfrak{D}$ corresponds to the adjoint problem (5.7), (5.8). Now we give some facts concerning the pencils $\mathfrak{C}$ and $\mathfrak{C}^*$. Here we restrict ourselves to formulations and refer the reader to [20] or [28, Chapter 1] for the proofs.

If λ_0 is an eigenvalue of $\mathfrak{C}$, then $\overline{\lambda}_0$ is an eigenvalue of $\mathfrak{C}^*$; the geometric and the algebraic multiplicities of λ and $\overline{\lambda}_0$ coincide. The canonical system of Jordan chains $\{\varphi^{(0,j)}, \dots, \varphi^{(\kappa_j-1,j)} \ ; \ j = 1, \dots, J\}$ and $\{\psi^{(0,j)}, \dots, \psi^{(\kappa_j-1,j)} \ ; \ j = 1, \dots, J\}$ corresponding to λ_0 and $\overline{\lambda}_0$ can be chosen to satisfy the orthogonality and normalization conditions

$$\sum_{p=0}^{\nu}\sum_{q=0}^{k} \frac{1}{(\nu+k+1-p-q)!}\Big(\partial_\lambda^{\nu+k+1-p-q}\mathfrak{C}(\lambda_0)\varphi^{(q,\sigma)}, \psi^{(p,\zeta)}\Big)_\Xi = \delta_{\sigma,\zeta} \cdot \delta_{\kappa_\sigma-k-1,\nu},$$

where $\sigma, \zeta = 1, \dots, J$; $\nu = 0, \dots, \kappa_\zeta - 1$, $\ k = 0, \dots, \kappa_\sigma - 1$; $\delta_{p,q}$ is the Kronecker symbol. The functions

$$u^{(k,j)}(x) = r^{i\lambda_0} \sum_{q=0}^{k} \frac{1}{q!}(i \ln r)^q \varphi^{(k-q,j)}(\omega),$$

where $k = 0, \dots, \kappa_j - 1$; $\ j = 1, \dots, J$, form a basis in the space of power solutions corresponding to λ_0 for the problem

$$\begin{cases} A(D_x)u(x) = 0, \ \ x \in \mathcal{K}, \\ \ \ \Gamma_1 u(x) = 0, \ \ x \in \partial\mathcal{K}. \end{cases}$$

Similarly, the functions

$$v^{(k,j)}(x) = r^{i(\overline{\lambda}_0+2i)} \sum_{q=0}^{k} \frac{1}{q!}(i \ln r)^q \psi^{(k-q,j)}(\omega),$$

where $k = 0, \dots, \kappa_j - 1$; $\ j = 1, \dots, J$, form a basis in the space of power solutions corresponding to $\overline{\lambda}_0 + 2i$ for the problem

$$\begin{cases} A(D_x)u(x) = 0, \ \ x \in \mathcal{K}, \\ \ \ \Gamma_2 u(x) = 0, \ \ x \in \partial\mathcal{K}. \end{cases}$$

Substituting $u^{(k,j)}$ in the homogeneous problem (5.4), (5.5) and successively eliminating the discrepancies, we can construct the formal series

$$U^{(k,j)}(x,\tau) = \sum_{q=0}^{+\infty} r^{i\lambda_0+q} P_q^{(k,j)}(\omega, \ln r, \tau), \tag{5.13}$$

satisfying the homogeneous problem (5.4), (5.5); here $P_q^{(k,j)}$ are polynomials in $\ln r$ and τ with coefficients smoothly depending on $\omega \in \Xi$. In the same way, starting from the function $v^{(k,j)}$, we can construct the formal series

$$V^{(k,j)}(x,\tau) = \sum_{q=0}^{+\infty} r^{i(\overline{\lambda}_0+2i)+q} Q_q^{(k,j)}(\omega, \ln r, \tau) \tag{5.14}$$

satisfying the homogeneous problem (5.7), (5.8).

Let u be a function such that $\chi u \in H^1_\beta(\mathcal{K})$, where $\chi \in \mathcal{C}^\infty_c(\overline{\mathcal{K}})$ and $\chi = 1$ near the vertex of the cone $\mathcal{K}$. Assume that

$$\begin{cases} M(D_x,\tau)u(x) = f(x), & x \in \mathcal{K}, \\ \Gamma_1 u(x) = g(x), & x \in \partial\mathcal{K}, \end{cases}$$

where $f \in H^0_\beta(\mathcal{K}) \cap H^0_\gamma(\mathcal{K})$ and $g \in H^{1/2}_\beta(\partial\mathcal{K}) \cap H^{1/2}_\gamma(\partial\mathcal{K})$ with $\gamma < \beta$. Then the asymptotic formula

$$u = \chi \sum c_\mu^{(k,j)} U_{\mu,T}^{(k,j)} + h$$

holds with the remainder h such that $\chi h \in H^1_\gamma(\mathcal{K})$. Here by $U_{\mu,T}^{(k,j)}$ we denote the first T terms of the series (5.13) corresponding to λ_μ. Number T is chosen to be sufficiently large to provide the inclusion $\chi r^{i\lambda_\mu+(T+1)} P_{T+1}^{(k,j)} \in H^1_\gamma(\mathcal{K})$. In the last formula summation is over all $k = 0, \dots, \kappa_{j,\mu}$, all $j = 1, \dots, J_\mu$, and all μ such that $\operatorname{Im}\lambda \in [\gamma + 1/2,\ \beta + 1/2[$.

5.3 Energy estimates

In this section we obtain estimates for solutions to the problem (5.4), (5.5) in domains with smooth boundary and in a class of admissible cones. Our aim is to estimate the "energy" $\|u;\ L_2(\Omega)\|$ by appropriate norms of the right-hand side $\{Mu, \Gamma_1 u\}$.

5.3.1 The main identity

Now we make some technical preparations for the further work. We prove an integral identity. It will be the starting point for the proof of energy estimates.
In the fixed cartesian coordinates the operator $A(\partial)$ can be written in the form $A(\partial) = \sum_{k=1}^{3} g^k \partial_k$, where $\{g^k\}$ are the constant real symmetric 8×8 matrices. Rewrite (5.3) in the form

$$\sum_{\alpha=0}^{3} \sum_{j=1}^{8} g^\alpha_{ij} \partial_\alpha u_j = f_i,\ i = 1, .., 8;$$

by ∂_0 and g^0 denote ∂_t and I (the identity matrix), respectively. Consider the following identity

$$\partial_\alpha\{r_{ik}\overline{u_k}g^\alpha_{ij}u_j\} = \partial_\alpha(r_{ik}\overline{u_k})g^\alpha_{ij}u_j + r_{ik}\overline{u_k}g^\alpha_{ij}\partial_\alpha u_j,$$

where $R = \{r_{ik}\}$ is a real 8×8 – matrix. Repeated indices imply summation: here $\alpha = 0,..,3$ and $i,j,k = 1,..,8$. Multiplying both sides by $e^{-2\gamma t}$, we get

$$\begin{aligned}\partial_\alpha\{e^{-2\gamma t}r_{ik}\overline{u_k}g^\alpha_{ij}u_j\} + 2\gamma e^{-2\gamma t}r_{ik}\overline{u_k}u_i =\\ = e^{-2\gamma t}\partial_\alpha(r_{ik}\overline{u_k})g^\alpha_{ij}u_j + e^{-2\gamma t}r_{ik}\overline{u_k}g^\alpha_{ij}\partial_\alpha u_j.\end{aligned} \tag{5.15}$$

Let $\Omega\subset\mathbb{R}^3$ be a domain with smooth boundary and let $u\in\mathcal{C}^\infty_c(\overline{\Omega}\times\mathbb{R},\mathbb{C}^8)$. Integrate (5.15) over $\Omega\times\mathbb{R}$, applying the divergence theorem of Gauss to the first term on the left, then we have

$$\begin{aligned}\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t}r_{ik}\overline{u_k}(g^1_{ij}\nu_1 + g^2_{ij}\nu_2 + g^3_{ij}\nu_3)u_j + 2\gamma\int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}r_{ik}\overline{u_k}u_i =\\ = \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\partial_\alpha(r_{ik}\overline{u_k})g^\alpha_{ij}u_j + \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}r_{ik}\overline{u_k}g^\alpha_{ij}\partial_\alpha u_j,\end{aligned}$$

where $\vec{\nu} = (\nu_1,\ \nu_2,\ \nu_3)$ is the unit outward normal to Ω. Introduce the notation

$$A(\vec{\nu}) = \sum_{i=1}^3 g^i\nu_i \ \text{ and } M = \sum_{\alpha=0}^3 g^\alpha\partial_\alpha.$$

Using these notations we rewrite the main identity in the vector form

$$\begin{aligned}\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t}\langle A(\vec{\nu})u, Ru\rangle_8 + 2\gamma\int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle u, Ru\rangle_8 =\\ = \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle u, MRu\rangle_8 + \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle Mu, Ru\rangle_8.\end{aligned} \tag{5.16}$$

5.3.2 Structure of the matrix $A(\vec{\nu})$

From the expression for $A(\partial)$ we get

$$A(\vec{\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 & \nu_3 & -\nu_2 & \nu_1 & 0\\ 0 & 0 & 0 & -\nu_3 & 0 & \nu_1 & \nu_2 & 0\\ 0 & 0 & 0 & \nu_2 & -\nu_1 & 0 & \nu_3 & 0\\ 0 & -\nu_3 & \nu_2 & 0 & 0 & 0 & 0 & \nu_1\\ \nu_3 & 0 & -\nu_1 & 0 & 0 & 0 & 0 & \nu_2\\ -\nu_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 & \nu_3\\ \nu_1 & \nu_2 & \nu_3 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \nu_1 & \nu_2 & \nu_3 & 0 & 0 \end{pmatrix},$$

$A(\vec{\nu})A(\vec{\nu}) = I$, and $A(\vec{\nu})u = (-[\vec{\nu}\times\vec{v}] + \vec{\nu}h,\ [\vec{\nu}\times\vec{u}] + \vec{\nu}q,\ \langle\vec{u},\vec{\nu}\rangle_3,\ \langle\vec{v},\vec{\nu}\rangle_3)^T$ for the vector $u = (\vec{u},\ \vec{v},\ h,\ q)^T$. Consider the 8×8 matrix, defined by the formula

$$P = \begin{pmatrix} 1-\nu_1\nu_1 & -\nu_1\nu_2 & -\nu_1\nu_3 & 0 & 0 & 0 & 0 & 0 \\ -\nu_2\nu_1 & 1-\nu_2\nu_2 & -\nu_2\nu_3 & 0 & 0 & 0 & 0 & 0 \\ -\nu_3\nu_1 & -\nu_3\nu_2 & 1-\nu_3\nu_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\nu_1\nu_1 & -\nu_1\nu_2 & -\nu_1\nu_3 & 0 & 0 \\ 0 & 0 & 0 & -\nu_2\nu_1 & 1-\nu_2\nu_2 & -\nu_2\nu_3 & 0 & 0 \\ 0 & 0 & 0 & -\nu_3\nu_1 & -\nu_3\nu_2 & 1-\nu_3\nu_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that

$$Pu = (\vec{u}_\sigma, \vec{v}_\sigma,\ 0,\ 0)^T,$$

where $\vec{u}_\sigma = \vec{\nu}\times[\vec{u}\times\vec{\nu}]$. Let $Q = I - P$, then

$$Qu = (\vec{\nu}\langle\vec{u},\vec{\nu}\rangle_3,\ \vec{\nu}\langle\vec{v},\vec{\nu}\rangle_3,\ h,\ q)^T.$$

Taking into account the formulas for Pu and Qu, we obtain

$$P + Q = I,\ \langle Pu, Qv\rangle_8 = 0.$$
$$P^2 = P,\ Q^2 = Q.$$

Therefore P, Q are orthogonal projections and the decomposition

$$\mathbb{C}^8 = \mathcal{R}P \oplus \mathcal{R}Q$$

holds, where by $\mathcal{R}L$ the range of L is denoted. From the explicit formulas for $A(\vec{\nu}), P, Q$ we get

$$A(\vec{\nu})Pu = ([\vec{v}\times\vec{\nu}],\ -[\vec{u}\times\vec{\nu}],\ 0, 0)^T,$$
$$A(\vec{\nu})Qu = (h\vec{\nu},\ q\vec{\nu},\ \langle\vec{u},\vec{\nu}\rangle_3,\ \langle\vec{v},\vec{\nu}\rangle_3)^T,$$
$$A(\vec{\nu})Pu \perp \mathcal{R}Q,\ A(\vec{\nu})Qu \perp \mathcal{R}P.$$

These formulas imply the following representation for the matrix $A(\vec{\nu})$ in a basis adapted to the decomposition $\mathbb{C}^8 = \mathcal{R}P \oplus \mathcal{R}Q$:

$$A(\vec{\nu})u = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix}, \quad \text{where}\ \ u = \begin{pmatrix} U \\ V \end{pmatrix},$$
$$Pu = \begin{pmatrix} U \\ 0 \end{pmatrix} \in \mathcal{R}P,\ \ Qu = \begin{pmatrix} 0 \\ V \end{pmatrix} \in \mathcal{R}Q.$$

Since $A(\vec{\nu})$ is a real symmetric matrix and $A(\vec{\nu})A(\vec{\nu}) = I$, it follows that

$$B_1\cdot B_1 = I,\ B_1 = B_1^T,\ B_2\cdot B_2 = I,\ B_2 = B_2^T.$$

Prepare the choice of the matrix R in (5.16). Let $R = R(\vec{\nu})$ such that

$$R(\vec{\nu}) = \begin{pmatrix} -B_1 & 0 \\ 0 & B_2 \end{pmatrix},$$

then we obtain

$$\langle A(\vec{\nu})u, Ru\rangle_8 = \left\langle \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}, \begin{pmatrix} -B_1 & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \right\rangle_8 = \\ = \langle B_2V,\, B_2V\rangle_4 - \langle B_1U,\, B_1U\rangle_4 = |V|^2 - |U|^2. \tag{5.17}$$

Let us find the representation of the matrix $R(\vec{\nu})$ in the initial standard basis. In a basis adapted to the decomposition $\mathbb{C}^8 = \mathcal{R}P \oplus \mathcal{R}Q$ we have

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, A(\vec{\nu}) = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, R(\vec{\nu}) = \begin{pmatrix} -B_1 & 0 \\ 0 & B_2 \end{pmatrix},$$

$$A(\vec{\nu})P = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}, A(\vec{\nu})Q = \begin{pmatrix} 0 & 0 \\ 0 & B_2 \end{pmatrix},$$

therefore $R(\vec{\nu}) = -A(\vec{\nu})P + A(\vec{\nu})Q$. Finally, applying the formulas for $A(\vec{\nu}), P, Q$, we obtain

$$R(\vec{\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 & -\nu_3 & \nu_2 & \nu_1 & 0 \\ 0 & 0 & 0 & \nu_3 & 0 & -\nu_1 & \nu_2 & 0 \\ 0 & 0 & 0 & -\nu_2 & \nu_1 & 0 & \nu_3 & 0 \\ 0 & \nu_3 & -\nu_2 & 0 & 0 & 0 & 0 & \nu_1 \\ -\nu_3 & 0 & \nu_1 & 0 & 0 & 0 & 0 & \nu_2 \\ \nu_2 & -\nu_1 & 0 & 0 & 0 & 0 & 0 & \nu_3 \\ \nu_1 & \nu_2 & \nu_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu_1 & \nu_2 & \nu_3 & 0 & 0 \end{pmatrix}.$$

It can be easily checked by means of the formulas for $A(\vec{\nu})$ and $R(\vec{\nu})$ that

$$A(\vec{a})R(\vec{b}) = R(\vec{b})A(\vec{a}) \tag{5.18}$$

for all $\vec{a}, \vec{b} \in \mathbb{C}^3$. Recall that $A(\vec{\nu}) = g^1\nu_1 + g^2\nu_2 + g^3\nu_3$, so $g^k = A(\vec{e}_k)$, where $\{\vec{e}_i\}_{i=1}^3$ is the standard basis in $\mathbb{R}^3$. From (5.18) it follows that $R(\vec{\nu}) \cdot g^k = g^k \cdot R(\vec{\nu})$ with $k = 1, \ldots, 3$.

5.3.3 Energy estimates in domains with smooth boundary

PROPOSITION 5.4. *Let $\Omega \subset \mathbb{R}^3$ be a domain with smooth boundary. Suppose $\gamma \geq 1$ and $u = (\vec{u},\, \vec{v},\, h,\, q) \in \mathcal{C}_c^\infty(\overline{\Omega} \times \mathbb{R}, \mathbb{C}^8)$; then the inequality*

$$\gamma^2 \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\, e^{-2\gamma t}|u|^2 + \gamma \int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\, e^{-2\gamma t}(|\langle \vec{v}, \vec{\nu}\rangle_3|^2 + |\langle \vec{u}, \vec{\nu}\rangle_3|^2 + |\vec{v}_\sigma|^2) \leq \\ \leq c\Big(\gamma \int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\, e^{-2\gamma t}(|\vec{\Phi}|^2 + |h|^2 + |q|^2) + \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\, e^{-2\gamma t}|Mu|^2\Big) \tag{5.19}$$

holds, where $\vec{\Phi} = \vec{\nu} \times [\vec{v} \times \vec{\nu}] + \psi[\vec{\nu} \times \vec{u}]$. The constant c is independent of the function u.

Proof. Put $R = I$ in (5.16), then we get

$$\begin{gathered}2\gamma \int_{\Omega}\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}|u|^2 = \\ = 2\mathrm{Re} \int_{\Omega}\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle u, Mu\rangle_8 - \int_{\partial\Omega}\int_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t}\langle A(\vec{\nu})u, u\rangle_8\,.\end{gathered} \tag{5.20}$$

Consider the surface integral in (5.20). Taking into account the explicit formula for $A(\vec{\nu})$, we obtain

$$\langle A(\vec{\nu})u, u\rangle_8 = 2\mathrm{Re}\ \langle[\vec{\nu}\times\vec{u}],\ \vec{v}\rangle_3 + 2\mathrm{Re}\ \Big(\overline{h}\cdot\langle\vec{u},\vec{\nu}\rangle_3\Big) + 2\mathrm{Re}\ \Big(\overline{q}\cdot\langle\vec{v},\vec{\nu}\rangle_3\Big).$$

Transform the first term on the right side in order to get the boundary conditions:

$$\begin{gathered}2\ \mathrm{Re}\ \langle[\vec{\nu}\times\vec{u}],\vec{v}\rangle_3 = 2\ \mathrm{Re}\ \langle\psi\cdot[\vec{\nu}\times\vec{u}],\ (1/\overline{\psi})\cdot\vec{v}\rangle_3 = \\ 2\ \mathrm{Re}\ \langle\vec{\nu}\times[\vec{v}\times\vec{\nu}] + \psi\cdot[\vec{\nu}\times\vec{u}] - \vec{\nu}\times[\vec{v}\times\vec{\nu}],\ (1/\overline{\psi})\vec{v}\rangle_3 = \\ 2\ \mathrm{Re}\ \langle(1/\psi)\vec{\Phi},\ \vec{v}_\sigma\rangle_3 + 2\mathrm{Re}(-1/\psi)|\vec{v}_\sigma|^2.\end{gathered}$$

Recall that $\psi = a + ib$, where $a < 0$, then $\alpha = \mathrm{Re}(-1/\psi) = -a/|\psi|^2 > 0$. Finally, we rewrite (5.20) in the form

$$\begin{gathered}\alpha \int_{\partial\Omega}\int_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t}|\vec{v}_\sigma|^2 + \gamma \int_{\Omega}\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}|u|^2 = \\ = \mathrm{Re} \int_{\Omega}\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle u, Mu\rangle_8 - \\ -\mathrm{Re} \int_{\partial\Omega}\int_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t}\Big(\overline{h}\cdot\langle\vec{u},\vec{\nu}\rangle_3 + \overline{q}\cdot\langle\vec{v},\vec{\nu}\rangle_3 + (1/\psi)\langle\vec{\Phi},\vec{v}_\sigma\rangle_3\Big).\end{gathered} \tag{5.21}$$

Note that in the surface integral on the right-hand side the functions $\vec{\Phi}$, h, q correspond to the boundary condition $\Gamma_1 u$. Now it remains to estimate the functions $|\langle\vec{v},\vec{\nu}\rangle_3|$, $|\langle\vec{v},\vec{\nu}\rangle_3|$.

Choose a sufficiently small positive number δ. Let $\zeta \in \mathcal{C}^\infty(\overline{\Omega})$ be a function such that $\zeta(x) = 1$ for all $x \in \{y \in \overline{\Omega}\ :\ \mathrm{dist}(y,\partial\Omega) < \delta\}$ and $\zeta(x) = 0$ for all $x \in \{y \in \Omega\ :\ \mathrm{dist}(y,\partial\Omega) > 2\delta\}$. Let $\vec{n} \in \mathcal{C}^\infty(\overline{\Omega})$ be a smooth vector field such that $\vec{n}|_{\partial\Omega} = \vec{\nu}$. Put $R = \zeta R(\vec{n})$ in (5.16) and recall (5.17). Besides, from (5.18) it follows that

$$\begin{gathered}MRu = R\partial_t u + \sum_{j=1}^{3} g^j(\partial_j R)u + \sum_{j=1}^{3} g^j R\partial_j u = \\ = R\partial_t u + \sum_{j=1}^{3} g^j(\partial_j R)u + \sum_{j=1}^{3} Rg^j\partial_j u = \\ = Su + RMu,\end{gathered}$$

where $S = \sum\limits_{j=1}^{3} g^j(\partial_j R)$. Thus for this choice of the matrix R the identity (5.16) takes the form

$$\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t}(|V|^2 - |U|^2) + 2\gamma \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle u, Ru\rangle_8 =$$
$$= 2\mathrm{Re}\int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle Mu, Ru\rangle_8 + \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}\langle u, Su\rangle_8.$$

Applying the Cauchy inequality to the terms with Su, Mu, Ru, we have

$$\begin{aligned}\gamma\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t}|V|^2 \le c\Big\{\gamma\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\; e^{-2\gamma t}|U|^2 +\\ + \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}|Mu|^2 + (\gamma+\gamma^2)\int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}|u|^2\Big\}.\end{aligned} \tag{5.22}$$

Let us remember that $|U|^2 = |Pu|^2 = |\vec{u}_\sigma|^2 + |\vec{v}_\sigma|^2$ and $|V|^2 = |Qu|^2 = |\langle\vec{u},\vec{\nu}\rangle_3|^2 + |\langle\vec{v},\vec{\nu}\rangle_3|^2 + |h|^2 + |q|^2$. From (5.21) we get the following estimate:

$$\begin{aligned}\gamma\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\;\; e^{-2\gamma t}|\vec{v}_\sigma|^2 + \gamma^2\int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\;\; e^{-2\gamma t}|u|^2 \le\\ \le c\Big\{\int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\;\; e^{-2\gamma t}|Mu|^2 + \gamma\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\;\; e^{-2\gamma t}|\vec{\Phi}|^2 +\\ +\gamma\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\;\; e^{-2\gamma t}(|\langle\vec{u},\vec{\nu}\rangle_3|\cdot|h| + |\langle\vec{v},\vec{\nu}\rangle_3|\cdot|q|)\Big\}.\end{aligned} \tag{5.23}$$

The last term on the right-hand side of (5.23) is majorized by

$$\gamma\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\;\; e^{-2\gamma t}(\varepsilon|\langle\vec{u},\vec{\nu}\rangle_3|^2 + (1/\varepsilon)|h|^2 + \varepsilon|\langle\vec{v},\vec{\nu}\rangle_3|^2 + (1/\varepsilon)|q|^2)$$

The application of (5.22) yields the estimate

$$\begin{aligned}\gamma^2\int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; e^{-2\gamma t}|u|^2 + \gamma\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\;\; e^{-2\gamma t}(\varepsilon|\langle\vec{v},\vec{\nu}\rangle_3|^2 + \varepsilon|\langle\vec{u},\vec{\nu}\rangle_3|^2 + |\vec{v}_\sigma|^2) \le\\ \le c\Big\{\gamma\int\limits_{\partial\Omega}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\;\; e^{-2\gamma t}(\varepsilon|\vec{u}_\sigma|^2 + \varepsilon|\vec{v}_\sigma|^2 + |\vec{\Phi}|^2 + (1/\varepsilon)|h|^2 + (1/\varepsilon)|q|^2) +\\ + \int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\;\; e^{-2\gamma t}|Mu|^2 + \varepsilon(\gamma+\gamma^2)\int\limits_{\Omega}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\;\; e^{-2\gamma t}|u|^2\Big\}.\end{aligned} \tag{5.24}$$

We have

$$|\vec{u}_\sigma|^2 = |\vec{\nu}\times\vec{u}|^2 = (1/|\psi|^2)|\psi\cdot[\vec{\nu}\times\vec{u}]|^2 =$$
$$(1/|\psi|^2)|\vec{\nu}\times\vec{\Phi}\; - \;\vec{\nu}\times[\vec{v}\times\vec{\nu}]|^2 \le (2/|\psi|^2)(|\vec{\Phi}|^2 + |\vec{v}_\sigma|^2).$$

Suppose $\gamma \ge 1$. Choosing the number ε sufficiently small, one can rearrange the right-hand side terms with $|\vec{v}_\sigma|^2$ and $|u|^2$ to the left side. As a result we obtain the required estimate (5.19). $\square$

Proposition 5.5. *Let $\Omega \subset \mathbb{R}^3$ be a domain with smooth boundary. For any function $v \in \mathcal{C}_c^\infty(\overline{\Omega}, \mathbb{C}^8)$ the estimate*

$$\begin{aligned}\gamma^2\|v;\ L_2(\Omega)\|^2 + \gamma\|T_1 v;\ L_2(\partial\Omega)\|^2 \leq \\ \leq c\Big\{\|M(D_x,\tau)v;\ L_2(\Omega)\|^2 + \gamma\|\Gamma_1 v;\ L_2(\partial\Omega)\|^2\Big\}\end{aligned} \tag{5.25}$$

holds, where $\tau = \sigma - i\gamma$ with $\gamma \geq 1$ and $\sigma \in \mathbb{R}$. The constant c is independent of the function v and the parameter τ.

Proof. It can be easily shown that the identity (5.16) and the estimate (5.19) are valid for all functions u such that $u(x,t) = \varkappa(t)v(x)$, where $v \in \mathcal{C}_c^\infty(\overline{\Omega})$ and $\varkappa$, $e^{-\gamma t}\varkappa \in \mathcal{S}(\mathbb{R})$. Applying the Fourier transform $\mathcal{F}_{t\to\tau}$ to the estimate (5.19), we obtain

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{d}\sigma\, |\widehat{\varkappa}(\sigma - i\gamma)|^2\Big(\gamma^2\|v;\ L_2(\Omega)\|^2 + \gamma\|T_1 v;\ L_2(\partial\Omega)\|^2\Big) \leq \\ \leq c\int_{\mathbb{R}} \mathrm{d}\sigma\, |\widehat{\varkappa}(\sigma - i\gamma)|^2\Big(\|M(D_x,\tau)v;\ L_2(\Omega)\|^2 + \gamma\|\Gamma_1 v;\ L_2(\partial\Omega)\|^2\Big).\end{aligned}$$

Since $\varkappa$ is arbitrary, we arrive at the required estimate (5.25). $\square$

5.3.4 Energy estimates in domains with conical points

In this subsection it is shown that the energy estimate (5.19) remains valid for $\mathcal{K}$ instead of G, where $\mathcal{K}$ is an admissible cone in the sense of the following definition.

Definition 5.6. *A cone $\mathcal{K} \subset \mathbb{R}^3$ is called admissible if there exists a constant vector $\vec{f} \in \mathbb{R}^3$ such that $\langle \vec{f}, \vec{\nu}\rangle_3 \geq c_0 > 0$ for all outward normals to $\partial\mathcal{K}$.*

It is not hard to find a nonadmissible cone, considering nonconvex sets $\mathcal{K} \cap \mathcal{S}^2$. In what follows we only deal with admissible cones.

Definition 5.7. *Let $\mathcal{D}(\mathcal{K})$ denote the lineal spanned by the functions $w \in \mathcal{C}_c^\infty(\overline{\mathcal{K}} \setminus \mathcal{O}, \mathbb{C}^8)$ and by the functions of the form $\chi u_\mu^{(k,j)}$ for $\operatorname{Im}\lambda_\mu < 1$. Here $\chi \in \mathcal{C}_c^\infty(\overline{\mathcal{K}})$ is a cut-off function such that $\chi = 1$ near the point $\mathcal{O}$.*

Note that for any function $u \in \mathcal{D}(\mathcal{K})$ the inclusions $u \in L_2(\mathcal{K})$, $u|_{\partial\mathcal{K}} \in L_2(\partial\mathcal{K})$, and $A(D_x)u \in L_2(\mathcal{K})$ hold. Now we prove the first main result of this section.

Proposition 5.8. *Let $\mathcal{K} \subset \mathbb{R}^3$ be an admissible cone with conical point $\mathcal{O}$. For any function $v \in \mathcal{D}(\mathcal{K})$ the estimate*

$$\begin{aligned}\gamma^2\|v; L_2(\mathcal{K})\|^2 + \gamma\|T_1 v; L_2(\partial\mathcal{K})\|^2 \leq \\ \leq c\{\|M(D_x,\tau)v; L_2(\mathcal{K})\|^2 + \gamma\|\Gamma_1 v; L_2(\partial\mathcal{K})\|^2\}\end{aligned} \tag{5.26}$$

holds, where $\tau = \sigma - i\gamma$ with $\sigma \in \mathbb{R}$, $\gamma > 0$. The constant c is independent of the parameter τ and of the function v.

Proof. Let $u(x,t) = \psi(t)v(x)$, where $v \in \mathcal{D}(\mathcal{K})$ and $\psi,\ e^{-\gamma t}\psi \in \mathcal{S}(\mathbb{R})$. The estimate (5.23) in $\mathcal{K}$ is proved in the same way as in Ω. Now we turn to the estimate in $\mathcal{K}$ similar to (5.22). Let $\vec{f} \in \mathbb{R}^3$ be a constant vector from the condition imposed on the cone $\mathcal{K}$ (see Definition 5.6). We put $R = R(\vec{f})$ in (5.16). Note that $R(\vec{f})$ is a constant matrix, hence $\sum\limits_{i=1}^{3} g^i(\partial_i R) = 0$. Therefore $MRu = RMu$ and we get

$$\int\limits_{\partial\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\, e^{-2\gamma t}\langle A(\vec{\nu})u, R(\vec{f})u\rangle_8 + 2\gamma \int\limits_{\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\, e^{-2\gamma t}\langle u, R(\vec{f})u\rangle_8 = \\ = \mathrm{Re} \int\limits_{\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\ \ e^{-2\gamma t}\langle Mu, R(\vec{f})u\rangle_8\,. \tag{5.27}$$

For the vector $\vec{f}$ the decomposition $\vec{f} = s(x)\vec{\nu}(x) + \vec{\sigma}(x)$, holds, where $x \in \partial\mathcal{K} \setminus \mathcal{O}$, $\vec{\nu}$ is the unit outward normal to $\partial\mathcal{K}$, $\vec{\sigma}$ is tangent to $\partial\mathcal{K}$, and the function s satisfies the inequalities $0 < c_0 \le s(x) \le C$ for $x \in \partial\mathcal{K} \setminus \mathcal{O}$. Then we have $R(\vec{f}) = sR(\vec{\nu}) + R(\vec{\sigma})$. Recall the explicit formulas for $A(\vec{\nu})u$ and $R(\vec{\sigma})u$:

$$A(\vec{\nu})u = (-[\vec{\nu}\times\vec{v}] + \vec{\nu}h,\ [\vec{\nu}\times\vec{u}] + \vec{\nu}q,\ \langle\vec{u},\vec{\nu}\rangle_3,\ \langle\vec{v},\vec{\nu}\rangle_3)^T,$$

$$R(\vec{\sigma})u = ([\vec{\sigma}\times\vec{v}] + \vec{\sigma}h,\ -[\vec{\sigma}\times\vec{u}] + \vec{\sigma}q,\ \langle\vec{u},\vec{\sigma}\rangle_3,\ \langle\vec{v},\vec{\sigma}\rangle_3)^T.$$

Taking into account these formulas it is not hard to prove that

$$|\langle A(\vec{\nu})u, R(\vec{\sigma})u\rangle_8| \le c|U|\cdot|V| \le c(\frac{1}{\varepsilon}|U|^2 + \varepsilon|V|^2). \tag{5.28}$$

Combining (5.17), (5.27), and (5.28) we obtain

$$\gamma\int\limits_{\partial\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\, e^{-2\gamma t}|V|^2 \le c\Big(\gamma\int\limits_{\partial\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\ \ e^{-2\gamma t}|U|^2 + \\ + \int\limits_{\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\ \ e^{-2\gamma t}|Mu|^2 + \gamma^2\int\limits_{\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\ \ e^{-2\gamma t}|u|^2\Big). \tag{5.29}$$

This estimate plays the same role as the estimate (5.22) played in the proof of Proposition 5.4. In the same way as we did in Proposition 5.4, from the estimate (5.23) for the cone $\mathcal{K}$ and from the estimate (5.29) we get

$$\gamma^2\int\limits_{\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\ \ e^{-2\gamma t}|u|^2 + \gamma\int\limits_{\partial\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\ \ e^{-2\gamma t}(|\langle\vec{v},\vec{\nu}\rangle_3|^2 + |\langle\vec{u},\vec{\nu}\rangle_3|^2 + |\vec{v}_\sigma|^2) \le \\ c\Big(\gamma\int\limits_{\partial\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}S\,\mathrm{d}t\ \ e^{-2\gamma t}(|\vec{\Phi}|^2 + |h|^2 + |q|^2) + \int\limits_{\mathcal{K}}\int\limits_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\ \ e^{-2\gamma t}|Mu|^2\Big).$$

Let us now apply the Fourier transform $\mathcal{F}_{t\to\tau}$ to the last inequality:

$$\int\limits_{\mathbb{R}} \mathrm{d}\sigma\,|\widehat{\varkappa}(\sigma - i\gamma)|^2\Big(\gamma^2\|v; L_2(\mathcal{K})\|^2 + \gamma\|T_1v; L_2(\partial\mathcal{K})\|^2\Big) \le$$

$$\le c\int\limits_{\mathbb{R}} \mathrm{d}\sigma\,|\widehat{\varkappa}(\sigma - i\gamma)|^2\Big(\|M(D_x,\tau)v; L_2(\mathcal{K})\|^2 + \gamma\|\Gamma_1 v; L_2(\partial\mathcal{K})\|^2\Big).$$

Since $\varkappa$ is arbitrary, we arrive at the required estimate (5.26). $\square$

Finally, we consider the problem (5.4), (5.5) in bounded domains with conical point. Let $G \subset \mathbb{R}^3$ be a domain such that G coincides with an admissible cone $\mathcal{K}$ in a neighborhood of the point $\mathcal{O}$.

Definition 5.9. *Let $\mathcal{D}(G)$ denote the lineal spanned by the functions $w \in \mathcal{C}_c^\infty(\overline{G} \setminus \mathcal{O})$ and by the functions of the form $\chi u_\mu^{(k,j)}$ for $\operatorname{Im} \lambda_\mu < 1$. Here $\chi \in \mathcal{C}^\infty(\overline{G})$ is a cut-off function such that $\chi = 1$ near the point $\mathcal{O}$ and $\chi = 0$ outside a neighborhood where G coincides with $\mathcal{K}$.*

Proposition 5.10. *For any function $v \in \mathcal{D}(G)$ the estimate*

$$\begin{gathered}\gamma^2 \|v; L_2(G)\|^2 + \gamma \|T_1 v; L_2(\partial G)\|^2 \leq \\ \leq c\{\|M(D_x, \tau)v; L_2(G)\|^2 + \gamma \|\Gamma_1 v; L_2(\partial G)\|^2\}\end{gathered} \tag{5.30}$$

holds. Here $\tau = \sigma - i\gamma$, $\sigma \in \mathbb{R}$, $\gamma > \gamma_0$ with sufficiently large γ_0. The constant c is independent of the parameter τ and of the function v.

Proof. Let $\chi + \zeta = 1$ be a partition of unity on G such that $\chi = 1$ near the point $\mathcal{O}$ and $\chi = 0$ outside a neighborhood where G coincides with $\mathcal{K}$. Then we have

$$\begin{gathered}\gamma^2 \|v; L_2(G)\|^2 + \gamma \|T_1 v; L_2(\partial G)\|^2 \leq (\gamma^2 \|\chi v; L_2(\mathcal{K})\|^2 + \gamma \|T_1(\chi v); L_2(\partial \mathcal{K})\|^2) + \\ + (\gamma^2 \|\zeta v; L_2(G)\|^2 + \gamma^2 \|T_1(\zeta v); L_2(\partial G)\|^2).\end{gathered}$$

For the first expression in brackets we use the estimate (5.26). The second expression in brackets is estimated by the inequality (5.25) for domains with smooth boundary. Therefore we obtain

$$\gamma^2 \|v; L_2(G)\|^2 + \gamma \|T_1 v; L_2(\partial G)\|^2 \leq$$

$$\begin{gathered}\leq c\{\|M(D_x, \tau)(\chi v); L_2(\mathcal{K})\|^2 + \gamma \cdot \|\Gamma_1(\chi v); L_2(\partial \mathcal{K})\|^2 + \\ + \|M(D_x, \tau)(\zeta v); L_2(G)\|^2 + \gamma \cdot \|\Gamma_1(\zeta v); L_2(\partial G)\|^2\} \leq\end{gathered}$$

$$\begin{gathered}\leq c\{\|M(D_x, \tau)v; L_2(G)\|^2 + \gamma \cdot \|\Gamma_1 v; L_2(\partial G)\|^2 + \\ + \|[A(D_x), \chi]v; L_2(G)\|^2 + \|[A(D_x), \zeta]v; L_2(G)\|^2\}.\end{gathered}$$

For the commutators we get

$$\|[A(D_x), \chi]v; L_2(G)\|^2 \leq c\|v; L_2(G)\|^2, \quad \|[A(D_x), \zeta]v; L_2(G)\|^2 \leq c\|v; L_2(G)\|^2,$$

then

$$\gamma^2 \|v; L_2(G)\|^2 + \gamma \|T_1 v; L_2(\partial G)\|^2 \leq$$

$$\leq c\{\|M(D_x, \tau)v; L_2(\mathcal{K})\|^2 + \gamma \cdot \|\Gamma_1 v; L_2(\partial \mathcal{K})\|^2 + \|v; L_2(G)\|^2\}.$$

Choosing γ sufficiently large, one can rearrange the term $c\|v; L_2(G)\|^2$ to the left. $\square$

Now we briefly discuss the adjoint problem (5.7), (5.8) in $\mathcal{K}$.

Definition 5.11. *Let $\mathcal{E}(\mathcal{K})$ denote the lineal spanned by the functions $w \in \mathcal{C}_c^\infty(\overline{\mathcal{K}} \setminus \mathcal{O})$ and by the functions of the form $\chi v_\mu^{(k,j)}$ for $\operatorname{Im}\lambda_\mu > 1$. Here $\chi \in \mathcal{C}^\infty(\overline{\mathcal{K}})$ is a cut-off function such that $\chi = 1$ near the vertex $\mathcal{O}$ of the cone.*

Slight modification of the proofs given in this section leads us to the following result for the adjoint problem.

Proposition 5.12. *Let $\mathcal{K} \subset \mathbb{R}^3$ be an admissible cone with conical point $\mathcal{O}$. For any function $v \in \mathcal{E}(\mathcal{K})$ the estimate*

$$\begin{gathered}\gamma^2\|v; L_2(\mathcal{K})\|^2 + \gamma\|T_2 v; L_2(\partial\mathcal{K})\|^2 \le \\ \le c\{\|M(D_x, \overline{\tau})v; L_2(\mathcal{K})\|^2 + \gamma\|\Gamma_2 v; L_2(\partial\mathcal{K})\|^2\}\end{gathered} \tag{5.31}$$

holds, where $\tau = \sigma - i\gamma$ with $\sigma \in \mathbb{R},\ \gamma > 0$. The constant c is independent of the parameter τ and of the function v.

We note that the obtained estimates do not contain the boundary Sobolev spaces of fractional order $H^s(\partial\mathcal{K})$ and $H^s(\partial G)$. One might expect the order $s = 1/2$ considering the elliptic problem or applying some kind of trace theorems. The order $s = -1/2$ could appear if the estimate was proved applying some "duality" technique. But the original problem is not elliptic, it is hyperbolic and we treated it applying completely distinct methods.

5.3.5 Another energy estimate

Here we prove a different energy estimate for the boundary conditions (1.6) with $H = 0$ and $Q = 0$. It will be shown in the last section that such boundary conditions are needed when we "return" to the usual Maxwell system. So if we are mainly interested in the usual Maxwell system, then we can confine ourselves to these boundary conditions. It is important that the new estimate is valid for arbitrary cones. Besides, the proof is simpler than the proof of (5.30).

Proposition 5.13. *Let $v = (\vec{u},\ \vec{v},\ h,\ q) \in \mathcal{D}(G)$ satisfy the augmented impedance boundary conditions (5.2) with $H = 0,\ Q = 0$. Then*

$$\begin{gathered}\gamma^2\|v; L_2(G)\|^2 + (\gamma|\operatorname{Re}\psi|/|\psi|)\cdot\|\vec{v}\times\vec{\nu}; L_2(\partial G)\|^2 \\ \le c\{\|M(D_x,\tau)v; L_2(G)\|^2 + (\gamma|\psi|/|\operatorname{Re}\psi|)\cdot\|\vec{\Phi}\times\vec{\nu}; L_2(\partial G)\|^2\},\end{gathered} \tag{5.32}$$

where the constant c is independent of v and τ.

Proof. Let $u(x_1, x_2, x_3, t) = \zeta(t)v(x_1, x_2, x_3)$ with $\zeta,\ e^{-\gamma t}\zeta \in \mathcal{S}(\mathbb{R})$. From the Green formula (1.10) it follows that

$$\begin{aligned}\operatorname{Re}\int_G \langle A(\partial)u, u\rangle\, \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 &= \operatorname{Re}\int_{\partial G}\langle \vec{u},\ [\vec{v}\times\vec{\nu}]\rangle_3\,\mathrm{d}S \\ &= \operatorname{Re}\frac{1}{\psi}\int_{\partial G}\langle[\vec{\Phi}\times\vec{\nu}],\ \vec{v}_\sigma\rangle_3\,\mathrm{d}S - \operatorname{Re}\frac{1}{\psi}\int_{\partial G}|\vec{v}_\sigma|^2\,\mathrm{d}S,\end{aligned}$$

where $\vec{v}_\sigma = \vec{\nu} \times [\vec{v} \times \vec{\nu}]$ is the component of $\vec{v}$ tangent to ∂G and $\vec{\nu}$ the unit outward normal. Then

$$\frac{d}{dt}\|u(\cdot,t); L_2(G)\| = 2\mathrm{Re} \int\limits_G \langle u_t + A(\partial)u, u\rangle_8 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 -$$
$$-2\mathrm{Re}\, \frac{1}{\psi} \int\limits_{\partial G} \langle [\vec{\Phi} \times \vec{\nu}], \vec{v}_\sigma\rangle_3 \, \mathrm{d}S + 2\mathrm{Re}\, \frac{1}{\psi} \int\limits_{\partial G} |\vec{v}_\sigma|^2 \, \mathrm{d}S.$$

Taking into account the equality $\psi = a + ib$ and majorizing the right-hand side, we arrive at

$$\frac{d}{dt}\|u(\cdot,t); L_2(G)\| + 2\frac{|a|}{|\psi|} \cdot \|\vec{v}_\sigma; L_2(\partial G)\|^2 \leq$$
$$\leq c\{\|Mu; L_2(G)\| \cdot \|u; L_2(G)\| + \|\vec{\Phi} \times \vec{\nu}; L_2(\partial G)\| \cdot \|\vec{v}_\sigma; L_2(\partial G)\|\}.$$

Integrate over $(-\infty, t)$, multiply the obtained equality by $e^{-2\gamma t}$, and then integrate over $(-\infty, +\infty)$. Changing the order of integration, we have

$$\int\limits_{-\infty}^{+\infty} e^{-2\gamma t}\|u(\cdot,t); L_2(G)\|^2 \, \mathrm{d}t + \frac{|a|}{\gamma|\psi|} \int\limits_{-\infty}^{+\infty} e^{-2\gamma t}\|\vec{v}_\sigma(\cdot,t); L_2(\partial G)\|^2 \, \mathrm{d}t \leq$$
$$\leq \frac{c}{2\gamma} \int\limits_{-\infty}^{+\infty} e^{-2\gamma t}\|Mu(\cdot,t); L_2(G)\| \cdot \|u(\cdot,t); L_2(G)\| \, \mathrm{d}t +$$
$$+\frac{c}{2\gamma} \int\limits_{-\infty}^{+\infty} e^{-2\gamma t}\|\vec{\Phi}(\cdot,t); L_2(\partial G)\| \cdot \|\vec{v}_\sigma(\cdot,t); L_2(\partial G)\| \, \mathrm{d}t.$$

Further,

$$\int\limits_{-\infty}^{+\infty} e^{-2\gamma t}\|u; L_2(G)\|^2 \, \mathrm{d}t + \frac{|a|}{\gamma|\psi|} \int\limits_{-\infty}^{+\infty} e^{-2\gamma t}\|\vec{v}_\sigma; L_2(\partial G)\|^2 \, \mathrm{d}t \leq$$
$$\leq \frac{c}{2\gamma} \int\limits_{-\infty}^{+\infty} e^{-2\gamma t}(\|Mu; L_2(G)\| + \sqrt{\frac{|\psi|\gamma}{|a|}} \cdot \|\vec{\Phi}; L_2(\partial G)\|)$$
$$\times(\|u; L_2(G)\| + \sqrt{\frac{|a|}{\gamma|\psi|}} \cdot \|\vec{v}_\sigma; L_2(\partial G)\|) \, \mathrm{d}t.$$

By the Cauchy inequality,

$$\int\limits_{-\infty}^{+\infty} e^{-2\gamma t}(\|u; L_2(G)\|^2 + \frac{|a|}{\gamma|\psi|} \cdot \|\vec{v}_\sigma; L_2(\partial G)\|^2) \, \mathrm{d}t \leq$$
$$\leq \frac{c}{\gamma^2} \int\limits_{-\infty}^{+\infty} e^{-2\gamma t}(\|Mu; L_2(G)\|^2 + \frac{|\psi|\gamma}{|a|} \cdot \|\vec{\Phi}; L_2(\partial G)\|^2) \, \mathrm{d}t.$$

Since ζ is an arbitrary function, Parseval's equality for the Fourier transform $\mathcal{F}_{t\to\tau}$ leads to (5.32). $\square$

Applying in subsequent considerations the estimate (5.32) instead of (5.30), we can finally obtain the results for the Maxwell system (1.1) with impedance boundary conditions (1.3) in arbitrary cones. However, the energy estimate (5.32) is less informative than (5.30), because in (5.30) the left-hand side contains also the normal components of the electric and magnetic fields (see the formula for $T_1 v$).

5.4 Operator of the problem

In this section we investigate the operator of the problem (5.4), (5.5) in spaces related to (5.26). In what follows we consider only the problem in $\mathcal{K}$. However, all results remain valid for the problem in G as well. At the end of the section we give the necessary remarks. Introduce the function space corresponding to the boundary operator Γ_1 by

$$L_{2,T}(\partial\mathcal{K}) = \\ = \Big\{(\vec{\Phi},\ h,\ q)\ :\ h, q \in L_2(\partial\mathcal{K}); \vec{\Phi} \in L_2(\partial\mathcal{K}, \mathbb{C}^3) \text{ and } \langle\vec{\Phi}\cdot\vec{\nu}\rangle_3 = 0 \text{ a.e. on } \partial\mathcal{K}\Big\}.$$

Further we often use function spaces of vector-valued functions, e.g., $L_2(\mathcal{K}, \mathbb{C}^8)$, $H^1_\beta(\mathcal{K}, |\tau|, \mathbb{C}^8)$. As a rule, we omit the symbol $\mathbb{C}^k$ and keep the simple notations $L_2(\mathcal{K})$, $H^1_\beta(\mathcal{K}, |\tau|)$.

With the problem (5.4), (5.5) in $\mathcal{K}$ we associate the unbounded operator $v \mapsto \mathcal{M}_1(\tau)v := \{M(D_x, \tau)v, \Gamma_1 v\}$ with domain $\mathcal{D}(\mathcal{K})$ acting from $L_2(\mathcal{K})$ to $L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$. We claim that $\mathcal{M}_1(\tau)$ admits closure. Indeed, let $\{v_m\} \subset \mathcal{D}(\mathcal{K})$, $v_m \to 0$ in $L_2(\mathcal{K})$, and $\{M(D_x, \tau)v_m, \Gamma_1 v_m\} \to \{f, g\}$ in $L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$ as $m \to \infty$. Then $(M(D, \tau)v_m, w)_\mathcal{K} = (v_m, M(D_x, \overline{\tau})w)_\mathcal{K}$ for any $w \in \mathcal{C}^\infty_c(\mathcal{K})$. Letting $m \to \infty$, we obtain $(f, w)_\mathcal{K} = 0$, hence $f = 0$. Now let $w \in \mathcal{C}^\infty_c(\overline{\mathcal{K}} \setminus \mathcal{O})$ such that $\Gamma_2 w = 0$. Applying the Green formula (5.6), we get

$$(M(D_x, \tau)v_m, w)_\mathcal{K} + (\Gamma_1 v_m, T_1 w)_{\partial\mathcal{K}} = (v_m, M(D_x, \overline{\tau})w)_\mathcal{K} + (T_2 v_m, \Gamma_2 w)_\mathcal{K}.$$

As $m \to \infty$ we have $(g, T_1 w)_{\partial\mathcal{K}} = 0$, hence $g = 0$. In what follows we deal with the closed operator only, keeping the notations $\mathcal{M}_1(\tau)$ and $\mathcal{D}\mathcal{M}_1(\tau)$ for the closed operator and its domain.

Let $v \in \mathcal{D}\mathcal{M}_1(\tau)$ and $\mathcal{M}_1(\tau)v = \{f, g\}$. There exists a sequence $\{v_k\}_{k=1}^\infty \subset \mathcal{D}(\mathcal{K})$ such that $v_k \to v$ in $L_2(\mathcal{K})$, $M(D_x, \tau)v_k \to f$ in $L_2(\mathcal{K})$, and $\Gamma_1 v_k \to g$ in $L_2(\partial\mathcal{K})$. From the energy estimate (5.26) it follows that there exists $h \in L_2(\partial\mathcal{K})$ such that $T_1 v_k \to h$ in $L_2(\partial\mathcal{K})$. It can be shown that if $v|_\omega \in H^1(\omega)$, where $\omega \subset \overline{\mathcal{K}} \setminus \mathcal{O}$ such that $\partial\omega \cap \partial\mathcal{K} = S$, then $T_1 v = h$ on S. Thus we say that for $v \in \mathcal{D}\mathcal{M}_1(\tau)$ there exists a boundary value $T_1 v = h$. Applying (5.26) we see that

$$\gamma\|T_1 v; L_2(\partial\mathcal{K})\|^2 + \gamma^2\|v; L_2(\mathcal{K})\|^2 \le c\{\|f; L_2(\mathcal{K})\|^2 + \gamma\|g; L_{2,T}(\partial\mathcal{K})\|^2\}.$$

This estimate implies the following result.

Proposition 5.14. *A)* Ker $\mathcal{M}_1(\tau) = \{0\}$.
B) The range $\mathcal{R}\mathcal{M}_1(\tau)$ *is closed in* $L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$.

Let us now prove that the range $\mathcal{R}\mathcal{M}_1(\tau)$ of $\mathcal{M}_1(\tau)$ coincides with $L_2(\mathcal{K})\times L_{2,T}(\partial\mathcal{K})$. To this end we investigate the kernel of the adjoint operator.

Proposition 5.15. $\mathcal{R}\mathcal{M}_1(\tau) = L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$.

Proof. It suffices to verify that Ker $\mathcal{M}_1(\tau)^* = \{0\}$. Suppose that $\{w, z\} \in$ Ker $\mathcal{M}_1(\tau)^*$. Applying the known local properties of solutions to elliptic problems (see [28, Chapter 1, §1]), we get $w \in \mathcal{C}^\infty(\overline{\mathcal{K}} \setminus \mathcal{O})$ and $z = T_1 w$ while w satisfies the homogeneous problem (5.7), (5.8). Moreover the function w admits the asymptotic representation

$$w = \chi \sum_{k,j,\mu} c_\mu^{(k,j)} V_{\mu,N}^{(k,j)} + h, \tag{5.33}$$

where $\chi \in \mathcal{C}_c^\infty(\overline{\mathcal{K}})$ such that $\chi = 1$ near the point $\mathcal{O}$). Since $\{w, T_1 w\} \in L_2(\mathcal{K}) \times L_2(\partial\mathcal{K})$, the asymptotics contains only $V_{\mu,N}^{(k,j)}$ such that $\operatorname{Im}\lambda_\mu < 1$ (this condition implies that $\chi V_{\mu,N}^{(k,j)} \in L_2(\mathcal{K})$ and $\chi V_{\mu,N}^{(k,j)}|_{\partial\mathcal{K}} \in L_2(\partial\mathcal{K})$).
Let $\zeta_n \in \mathcal{C}^\infty(\overline{\mathcal{K}})$ such that $\zeta_n = 1$ for $r < n$ and $\zeta_n = 0$ for $r > n+1$. It is easy to see that we can apply the estimate (5.31) to $\zeta_n w$. So we get

$$\gamma^2 \|\zeta_n w; L_2(\mathcal{K})\|^2 \leq c\|[M(D_x, \overline{\tau}), \zeta_n]w; L_2(\mathcal{K})\|^2.$$

The commutator is estimated as follows

$$\|[M(D_x, \overline{\tau}), \zeta_n]w; L_2(\mathcal{K})\| \leq c\|w; L_2(\mathcal{K} \cap \{n < r < n+1\})\|.$$

Since $w \in L_2(\mathcal{K})$, then $\|w; L_2(\mathcal{K} \cap \{n < r < n+1\})\| \to 0$ as $n \to \infty$. Hence $w = 0$. □

Definition 5.16. *A solution of the equation* $\mathcal{M}_1(\tau)v = \{f, g\}$ *with* $\{f, g\} \in L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$ *is called a strong solution of the problem (5.4), (5.5) in* $\mathcal{K}$.

The next assertion summarizes the results of this section.

Theorem 5.17. *For any* $\{f, g\} \in L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$ *and every* $\tau = \sigma - i\gamma$ *with* $\sigma \in \mathbb{R}$, $\gamma > 0$ *there exists a unique strong solution* v *to the problem (5.4), (5.5) in* $\mathcal{K}$ *with right-hand side* $\{f, g\}$. *Moreover, there exists a boundary value* $T_1 v \in L_2(\partial\mathcal{K})$ *and the estimate*

$$\gamma\|T_1 v;\ L_2(\partial\mathcal{K})\|^2 + \gamma^2\|v;\ L_2(\mathcal{K})\|^2 \leq c\{\|f;\ L_2(\mathcal{K})\|^2 + \gamma\|g;\ L_2(\partial\mathcal{K})\|^2\}$$

holds, where c *is independent of* τ *and of* v.

Remark 5.18. *Let* $\gamma \geq \gamma_0$ *with sufficiently large* γ_0. *Then Theorem 5.17 is true for the problem (5.4), (5.5) in* G *as well.*

Finally we briefly discuss the results concerning the adjoint problem. With the problem (5.7), (5.8) in $\mathcal{K}$ we associate the unbounded operator $v \mapsto \mathcal{M}_2(\overline{\tau})v := \{M(D_x, \overline{\tau})v, \Gamma_2 v\}$ with domain $\mathcal{E}(\mathcal{K})$ acting from $L_2(\mathcal{K})$ to $L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$. This operator admits the closure. The closed operator $\mathcal{M}_2(\overline{\tau})$ has the same properties as the operator $\mathcal{M}_1(\tau)$. Namely, we have $\mathrm{K}er\, \mathcal{M}_2(\overline{\tau}) = \{0\}$ and $\mathcal{R}\mathcal{M}_2(\overline{\tau}) = L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$.

Definition 5.19. *A solution of the equation* $\mathcal{M}_2(\overline{\tau})v = \{f, g\}$ *with* $\{f, g\} \in L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$ *is called a strong solution of the problem (5.7), (5.8) in* $\mathcal{K}$.

Theorem 5.20. *For any* $\{f, g\} \in L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$ *and every* $\tau = \sigma - i\gamma$ *with* $\sigma \in \mathbb{R}$, $\gamma > 0$ *there exists a unique strong solution* v *to the problem (5.7), (5.8) in* $\mathcal{K}$ *with right-hand side* $\{f, g\}$. *Moreover, there exists a boundary value* $T_2 v \in L_2(\partial\mathcal{K})$ *and the estimate*

$$\gamma\|T_2 v;\ L_2(\partial\mathcal{K})\|^2 + \gamma^2\|v;\ L_2(\mathcal{K})\|^2 \leq c\{\|f;\ L_2(\mathcal{K})\|^2 + \gamma\|g;\ L_2(\partial\mathcal{K})\|^2\}$$

holds, where c *is independent of* τ *and of* v.

Remark 5.21. *Let* $\gamma \geq \gamma_0$ *with sufficiently large* γ_0. *Then Theorem 5.20 is true for the problem (5.7), (5.8) in* G *as well.*

5.5 Weighted combined estimate

In this section, we prove a more informative *a priori* estimate for the problem (5.4), (5.5). This estimate will be used in the study of the asymptotics of strong solutions near the conical point $\mathcal{O}$.

Definition 5.22. *Let* $\mathcal{D}_\beta(\mathcal{K})$ *with* $\beta \leq 1$ *stand for the lineal spanned by the functions* $w \in \mathcal{C}_c^\infty(\overline{\mathcal{K}} \setminus \mathcal{O}, \mathbb{C}^8)$ *and by the functions of the form* $\chi u_\mu^{(k,j)}$ *such that* $\mathrm{Im}\,\lambda_\mu < min\{1, \beta + 1/2\}$. *Here* $\chi \in \mathcal{C}_c^\infty(\overline{\mathcal{K}})$ *is a cut-off function such that* $\chi = 1$ *near the point* $\mathcal{O}$.

The lineal $\mathcal{D}_\beta(G)$ is defined in a similar way.

5.5.1 The estimate in the cone $\mathcal{K}$

Proposition 5.23. *Let* $\beta \leq 1$ *and let the line* $\mathrm{Im}\,\lambda = \beta + 1/2$ *be free from the spectrum of* $\mathfrak{C}$. *Then for* $v \in \mathcal{D}_\beta(\mathcal{K})$ *the inequality*

$$\begin{gathered}\gamma\|T_1 v; H_\beta^0(\partial\mathcal{K})\|^2 + \gamma^2\|v; H_\beta^0(\mathcal{K})\|^2 + \|\chi_\tau v; H_\beta^1(\mathcal{K}, |\tau|)\|^2 \leq \\ \leq c\Big(\|M(D_x, \tau)v; H_\beta^0(\mathcal{K})\|^2 + \gamma\|\Gamma_1 v; H_\beta^0(\partial\mathcal{K})\|^2 + \|\chi_\tau \Gamma_1 v; H_\beta^{1/2}(\partial\mathcal{K})\|^2 \\ +(|\tau|^{1-\beta}/\gamma)^2\{\|M(D_x, \tau)v; L_2(\mathcal{K})\|^2 + \gamma\|\Gamma_1 v; L_2(\partial\mathcal{K})\|^2\}\Big)\end{gathered} \tag{5.34}$$

holds, where $\chi_\tau(r) = \chi(|\tau| r)$ *and* $\chi \in \mathcal{C}_c^\infty(\overline{\mathcal{K}})$ *is a fixed cut-off function such that* $\chi = 1$ *near the point* $\mathcal{O}$. *The constant* c *is independent of* v *and* τ.

Proof. Step 1. An estimate near the vertex of $\mathcal{K}$. According to Proposition 1.2, the problem $\{A(D_\eta),\ \Gamma_1\}$ is elliptic. Therefore, if the line $\operatorname{Im}\lambda = \beta + 1/2$ contains no eigenvalues of the pencil $\mathfrak{C}$, then a function $U \in H^1_\beta(\mathcal{K}, 1)$ satisfies the inequality

$$\|\chi U; H^1_\beta(\mathcal{K})\|^2 \le c\{\|A(D_\eta)\chi U; H^0_\beta(\mathcal{K})\|^2 + \|\Gamma_1\chi U; H^{1/2}_\beta(\partial\mathcal{K})\|^2\}.$$

Since $A(D_\eta)\chi U = \chi A(D_\eta)U + [A(D_\eta), \chi]U$ and $M(D_\eta, \theta) = \theta + A(D_\eta)$, the last inequality takes the form

$$\begin{gathered}\|\chi U; H^1_\beta(\mathcal{K}, 1)\|^2 \le c\{\|\chi M(D_\eta, \theta)U; H^0_\beta(\mathcal{K})\|^2 \\ +\|\Gamma_1\chi U; H^{1/2}_\beta(\partial\mathcal{K})\|^2 + \|\zeta U; H^0_\beta(\mathcal{K})\|^2\},\end{gathered} \tag{5.35}$$

where $\zeta \in \mathcal{C}^\infty_c(\overline{\mathcal{K}}),\ \chi\zeta = \chi$.

Step 2. An estimate far from the vertex. On this step we prove the inequality

$$\begin{gathered}\frac{\gamma}{|\tau|}\|\kappa_\infty T_1 U; H^0_\beta(\partial\mathcal{K})\|^2 + \frac{\gamma^2}{|\tau|^2}\|\kappa_\infty U; H^0_\beta(\mathcal{K})\|^2 \le \\ \le c\{\|\kappa_\infty M(D_\eta, \theta)U; H^0_\beta(\mathcal{K})\|^2 + \|\zeta_\infty U; H^0_{\beta-1}(\mathcal{K})\|^2 + \\ + \frac{\gamma}{|\tau|}\|\kappa_\infty \Gamma_1 U; H^0_\beta(\partial\mathcal{K})\|^2\}\end{gathered} \tag{5.36}$$

for any $\beta \in \mathbb{R}$ and every $U \in H^1_\beta(\mathcal{K}, 1)$, where the constant c is independent of U and τ. The functions κ_∞ and ζ_∞ are smooth in $\mathcal{K}$, equal to 0 near the vertex and 1 in a neighborhood of infinity, while $\kappa_\infty\zeta_\infty = \kappa_\infty$.

Let $\kappa, \zeta \in \mathcal{C}^\infty(\overline{\mathcal{K}})$ such that $\operatorname{supp}\kappa \subset \{x \in \mathcal{K} : 1/2 < |x| < 2\}$, $\operatorname{supp}\zeta \subset \{x \in \mathcal{K} : 1/4 < |x| < 4\}$, and $\kappa\zeta = \kappa$. The application of (5.26) yields

$$\begin{gathered}\gamma\|\kappa T_1 v; L_2(\partial\mathcal{K})\|^2 + \gamma^2\|\kappa v; L_2(\mathcal{K})\|^2 \le \\ \le \{\|\kappa M(D_x, \tau)v; L_2(\mathcal{K})\|^2 + \|\zeta v; L_2(\mathcal{K})\|^2 + \gamma\|\kappa\Gamma_1 v; L_2(\partial\mathcal{K})\|^2\}.\end{gathered}$$

If we replace v by the function $(x_1, x_2, x_3) \mapsto U^\varepsilon(x_1, x_2, x_3) = U(x_1/\varepsilon, x_2/\varepsilon, x_3/\varepsilon)$, and change τ for $\tau/(|\tau|\varepsilon)$ with $\varepsilon > 0$, we obtain

$$\begin{gathered}(\gamma/|\tau|\varepsilon)\|\kappa T_1 U^\varepsilon; L_2(\partial\mathcal{K})\|^2 + (\gamma/|\tau|\varepsilon)^2\|\kappa U^\varepsilon; L_2(\mathcal{K})\|^2 \le \\ \le c\{\|\kappa M(D_x, \tau/|\tau|\varepsilon)U^\varepsilon; L_2(\mathcal{K})\|^2 + \|\zeta U^\varepsilon; L_2(\mathcal{K})\|^2 + \\ +(\gamma/|\tau|\varepsilon)\|\kappa\Gamma_1 U^\varepsilon; L_2(\partial\mathcal{K})\|^2\}.\end{gathered}$$

Change the variables $(x_1, x_2, x_3) \mapsto (\eta_1, \eta_2, \eta_3) = (x_1/\varepsilon, x_2/\varepsilon, x_3/\varepsilon)$. Then we arrive at the estimate

$$\begin{gathered}\frac{\gamma}{|\tau|}\|\kappa_\varepsilon T_1 U; L_2(\partial\mathcal{K})\|^2 + \frac{\gamma^2}{|\tau|^2}\|\kappa_\varepsilon U; L_2(\mathcal{K})\|^2 \le c\{\|\kappa_\varepsilon M(D_\eta, \theta)U; L_2(\mathcal{K})\|^2 \\ +\varepsilon^2\|\zeta_\varepsilon U; L_2(\mathcal{K})\|^2 + \frac{\gamma}{|\tau|}\|\kappa_\varepsilon\Gamma_1 U; L_2(\partial\mathcal{K})\|^2\},\end{gathered}$$

where $\kappa_\varepsilon(\eta) = \kappa(\varepsilon\eta),\ \zeta_\varepsilon(\eta) = \zeta(\varepsilon\eta)$. Multiplying the inequality by $\varepsilon^{-2\beta}$, putting $\varepsilon = 2^{-j}, j = 1, 2, 3, ...$, and adding all these inequalities, we obtain (5.36).

Step 3. An estimate in intermediate zone. Let $\kappa_\infty = 1$ outside the support of χ. Since

$$(\gamma/|\tau|)\|\chi U; H^0_\beta(\mathcal{K})\| \le \|\chi U; H^0_\beta(\mathcal{K})\| \le \|\chi U; H^1_\beta(\mathcal{K}, 1)\|$$

and

$$(\gamma/|\tau|)\|\chi T_1 U; H^0_\beta(\partial\mathcal{K})\| \le \|\chi U; H^{1/2}_\beta(\partial\mathcal{K}, 1)\| \le \|\chi U; H^1_\beta(\mathcal{K}, 1)\|,$$

then, summing (5.35) and (5.36), we obtain the inequality

$$\begin{gathered}\frac{\gamma}{|\tau|}\|\kappa_\infty T_1 U; H^0_\beta(\partial\mathcal{K})\|^2 + \frac{\gamma^2}{|\tau|^2}\|U; H^0_\beta(\mathcal{K})\|^2 + \|\chi U; H^1_\beta(\mathcal{K}, 1)\|^2 \le \\ \le c\{\|M(D_\eta, \theta)U; H^0_\beta(\mathcal{K})\|^2 + \|\zeta_\infty U; H^0_{\beta-1}(\mathcal{K})\|^2 + \|\zeta U; H^0_\beta(\mathcal{K})\|^2 + \\ + \frac{\gamma}{|\tau|}\|\kappa_\infty \Gamma_1 U; H^0_\beta(\partial\mathcal{K})\|^2 + \|\chi\Gamma_1 U; H^{1/2}_\beta(\partial\mathcal{K})\|^2\}.\end{gathered}$$

Now we estimate the term on the right

$$\begin{aligned}\|\zeta U; H^0_\beta(\mathcal{K})\|^2 &\le \int\limits_{|\eta|<a} |\eta|^{2\beta}|U|^2\,\mathrm{d}\eta = \\ &= (\int\limits_{0\le|\eta|\le\varepsilon} + \int\limits_{\varepsilon\le|\eta|\le a})|\eta|^{2\beta}|U|^2\,\mathrm{d}\eta.\end{aligned}$$

The first integral is majorized by $c\varepsilon^2\|\chi U; H^1_\beta(\mathcal{K})\|^2$. We can rearrange it to the left side of the inequality, choosing ε sufficiently small. The second integral does not exceed $c\|\zeta_\infty U; H^0_{\beta-1}(\mathcal{K})\|^2$. Now the estimate can be rewritten in the form

$$\begin{gathered}\frac{\gamma}{|\tau|}\|T_1 U; H^0_\beta(\partial\mathcal{K})\|^2 + \frac{\gamma^2}{|\tau|^2}\|U; H^0_\beta(\mathcal{K})\|^2 + \|\chi U; H^1_\beta(\mathcal{K}, 1)\|^2 \le \\ \le c\{\|M(D_\eta, \theta)U; H^0_\beta(\mathcal{K})\|^2 + \|\zeta_\infty U; H^0_{\beta-1}(\mathcal{K})\|^2 + \\ + \frac{\gamma}{|\tau|}\|\kappa_\infty \Gamma_1 U; H^0_\beta(\partial\mathcal{K})\|^2 + \|\chi\Gamma_1 U; H^{1/2}_\beta(\partial\mathcal{K})\|^2\}.\end{gathered}$$

After the change of variables $(\eta_1, \eta_2, \eta_3) \mapsto (x_1, x_2, x_3) = (|\tau|^{-1}\eta_1, |\tau|^{-1}\eta_2, |\tau|^{-1}\eta_3)$ we obtain

$$\begin{gathered}\gamma\|T_1 v; H^0_\beta(\partial\mathcal{K})\|^2 + \gamma^2\|v; H^0_\beta(\mathcal{K})\|^2 + \|\chi_\tau v; H^1_\beta(\mathcal{K}, |\tau|)\|^2 \le \\ \le c\{\|M(D_x, \tau)v; H^0_\beta(\mathcal{K})\|^2 + \|\zeta_{\infty,\tau} v; H^0_{\beta-1}(\mathcal{K})\|^2 + \\ + \gamma\|\kappa_{\infty,\tau}\Gamma_1 v; H^0_\beta(\partial\mathcal{K})\|^2 + \|\chi_\tau \Gamma_1 v; H^{1/2}_\beta(\partial\mathcal{K})\|^2\},\end{gathered}$$

where $\zeta_{\infty,\tau}(x) = \zeta_\infty(|\tau|x)$, $\chi_\tau(x) = \chi(|\tau|x), v(x_1, x_2, x_3) = U(|\tau|x_1, |\tau|x_2, |\tau|x_3)$ Note that

$$\|\kappa_{\infty,\tau}\Gamma_1 v; H^0_\beta(\partial\mathcal{K})\| \le \|\Gamma_1 v; H^0_\beta(\partial\mathcal{K})\|.$$

Taking the inequalities (5.26) and $\beta \le 1$ into account, we get

$$\|\zeta_{\infty,\tau} v; H^0_{\beta-1}(\mathcal{K})\|^2 \le \int\limits_{b/|\tau|<r} r^{2(\beta-1)}|v|^2\,\mathrm{d}x \le$$

$$\le c\,|\tau|^{2(1-\beta)}\int\limits_{\mathcal{K}} |v|^2\,\mathrm{d}x \le c\frac{|\tau|^{2(1-\beta)}}{\gamma^2}\Big(\|M(D_x, \tau)v; L_2(\mathcal{K})\|^2 + \gamma\|\Gamma_1 v; L_2(\partial\mathcal{K})\|^2\Big),$$

which leads to (5.34).□

By $\mathcal{D}H_\beta(\mathcal{K},\tau)$ we denote the completion of the set $\mathcal{C}_c^\infty(\overline{\mathcal{K}}\setminus\mathcal{O},\mathbb{C}^8)$ with respect to the norm

$$\|v;\mathcal{D}H_\beta(\mathcal{K},\tau)\| = \left(\gamma^2\|v;H_\beta^0(\mathcal{K})\|^2 + \|\chi_\tau v;H_\beta^1(\mathcal{K},|\tau|)\|^2\right)^{1/2}.$$

By $\mathcal{R}H_\beta(\mathcal{K},\tau)$ we denote the completion of the set $\mathcal{C}_c^\infty(\overline{\mathcal{K}}\setminus\mathcal{O},\mathbb{C}^8)\times\mathcal{C}_c^\infty(\overline{\mathcal{K}}\setminus\mathcal{O},\mathbb{C}^5)$ with respect to the norm

$$\begin{gathered}\|\{f,g\};\mathcal{R}H_\beta(\mathcal{K},\tau)\| = \Big(\|f;H_\beta^0(\mathcal{K})\|^2+\\ +\gamma\|g;H_\beta^0(\partial\mathcal{K})\|^2 + \|\chi_\tau g;H_\beta^{1/2}(\partial\mathcal{K})\|^2+\\ +(|\tau|^{1-\beta}/\gamma)^2(\|f;L_2(\mathcal{K})\|^2+\gamma\|g;L_2(\partial\mathcal{K})\|^2)\Big)^{1/2}.\end{gathered}$$

Here $\chi_\tau(x_1,x_2,x_3)=\chi(|\tau|x_1,|\tau|x_2,|\tau|x_3)$ and $\chi\in\mathcal{C}_c^\infty(\overline{\mathcal{K}})$ is a cut-off function that is equal to 1 near the conical point $\mathcal{O}$. The spaces $\mathcal{D}H_\beta(G,\tau)$ and $\mathcal{R}H_\beta(G,\tau)$ are defined in a similar way. Taking into account these new notations, one can rewrite the estimate (5.34) in the form

$$\begin{gathered}\gamma\|T_1v;H_\beta^0(\partial\mathcal{K})\|^2+\|v;\mathcal{D}H_\beta(\mathcal{K},\tau)\|^2\leq\\ \leq c\,\|\{M(D_x,\tau)v,\Gamma_1 v\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|^2.\end{gathered} \tag{5.37}$$

5.5.2 The estimate in G

PROPOSITION 5.24. *Let $\beta\leq 1$ and let the line* $\operatorname{Im}\lambda=\beta+1/2$ *contain no eigenvalues of the pencil $\mathfrak{C}$. Assume that $\gamma>\gamma_0$ with sufficiently large γ_0. Then the inequality*

$$\begin{gathered}\gamma\|T_1v;H_\beta^0(\partial G)\|^2+\|v;\mathcal{D}H_\beta(G,\tau)\|^2\leq\\ \leq c\|\{M(D_x,\tau)v,\Gamma_1 v\};\mathcal{R}H_\beta(G,\tau)\|^2\end{gathered} \tag{5.38}$$

holds for any $v\in\mathcal{D}_\beta(G)$ with a constant c independent of v and τ.

Proof. Let $\zeta\in\mathcal{C}^\infty(\overline{G})$ be a cut-off function that is equal to 1 near $\mathcal{O}$ and vanishes outside the neighborhood where G coincides with $\mathcal{K}$. Since $v=\zeta v+(1-\zeta)v$, we have

$$\|v;\mathcal{D}H_\beta(G,\tau)\|\leq\|\zeta v;\mathcal{D}H_\beta(\mathcal{K},\tau)\|+\|(1-\zeta)v;\mathcal{D}H_\beta(G,\tau)\|. \tag{5.39}$$

Estimate the first term on the right. From (5.37) it follows that

$$\begin{gathered}\gamma^{1/2}\|\zeta T_1v;H_\beta^0(\partial\mathcal{K})\|+\|\zeta v;\mathcal{D}H_\beta(\mathcal{K},\tau)\|\leq\|\{M\zeta v,\Gamma_1\zeta v\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|\leq\\ \leq c\{\|\{\zeta Mv,\zeta\Gamma_1 v\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|+\|\{[M,\zeta]v,0\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|\},\end{gathered}$$

where M stands for $M(D_x,\tau)$. Clearly

$$\|\{\zeta Mv,\zeta\Gamma_1 v\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|\leq\|\{Mv,\Gamma_1 v\};\mathcal{R}H_\beta(G,\tau)\|.$$

The commutator can be estimated as follows

$$\|\{[M,\zeta]v,0\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|^2=\|[M,\zeta]v;H^0_\beta(\mathcal{K})\|^2+$$
$$+\left(|\tau|^{1-\beta}/\gamma\right)^2\|[M,\zeta]v;L_2(\mathcal{K})\|^2\le c\{\|v;H^0_\beta(G)\|^2+\left(|\tau|^{1-\beta}/\gamma\right)^2\|v;L_2(G)\|^2\}.$$

Taking the estimate (5.30) into account we arrive at

$$\|\{[M,\zeta]v,0\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|^2\le$$
$$\le c\Big(\|v;H^0_\beta(G)\|^2+\left(|\tau|^{1-\beta}/\gamma\right)^2\cdot(1/\gamma^2)\{\|Mv;L_2(G)\|^2+\gamma\|\Gamma_1 v;L_2(\partial G)\|^2\}\Big)$$
$$\le c\{\|v;H^0_\beta(G)\|^2+(1/\gamma^2)\|\{Mv,\Gamma_1 v\};\mathcal{R}H_\beta(G,\tau)\|^2\}.$$

Finally, for the first term in the right-hand side of (5.39) we get

$$\gamma\|\zeta T_1 v;H^0_\beta(\partial\mathcal{K})\|^2+\|\zeta v;\mathcal{D}H_\beta(\mathcal{K},\tau)\|^2\le$$
$$\le c\Big((1+1/\gamma^2)\|\{Mv,\Gamma_1 v\};\mathcal{R}H_\beta(G,\tau)\|^2+\|v;H^0_\beta(G)\|^2\Big).$$

Consider the second term in the right-hand side of (5.39). From the definition of the norm in $\mathcal{D}H_\beta(G,\tau)$ it follows that

$$\|(1-\zeta)v;\mathcal{D}H_\beta(G,\tau)\|^2=\gamma^2\|(1-\zeta)v;H^0_\beta(G)\|^2+\|\chi_\tau(1-\zeta)v;H^1_\beta(G,|\tau|)\|^2.$$

For sufficiently large γ we have $\chi_\tau(1-\zeta)\equiv 0$ because the supports of the factors do not overlap. Then

$$\gamma\|(1-\zeta)T_1 v;H^0_\beta(\partial G)\|^2+\gamma^2\|(1-\zeta)v;H^0_\beta(G)\|^2\le$$
$$\le c\Big(\gamma\|(1-\zeta)T_1 v;L_2(\partial G)\|^2+\gamma^2\|(1-\zeta)v;L_2(G)\|^2\Big)\le$$
$$\le c\Big(\|M(1-\zeta)v;L_2(G)\|^2+\gamma\|(1-\zeta)\Gamma_1 v;L_2(\partial G)\|^2\Big)$$
$$\le c\Big(\|(1-\zeta)Mv;H^0_\beta(G)\|^2+\gamma\|(1-\zeta)\Gamma_1 v;H^0_\beta(\partial G)\|^2+\|[M,(1-\zeta)]v;L_2(G)\|^2\Big)$$
$$\le c\Big(\|\{Mv,\Gamma_1 v\};\mathcal{R}H_\beta(G,\tau)\|^2+\|v;H^0_\beta(G)\|^2\Big).$$

Combining the obtained estimates, we rewrite (5.39) in the form

$$\gamma\|T_1 v;H^0_\beta(\partial G)\|^2+\|v;\mathcal{D}H_\beta(G,\tau)\|^2\le$$
$$\le c\Big((1+1/\gamma^2)\|\{Mv,\Gamma_1 v\};\mathcal{R}H_\beta(G,\tau)\|^2+\|v;H^0_\beta(G)\|^2\Big).$$

Since $\|v;H^0_\beta(G)\|\le(1/\gamma)\|v;\mathcal{D}H_\beta(G,\tau)\|$, then

$$\gamma\|T_1 v;H^0_\beta(\partial G)\|^2+\|v;\mathcal{D}H_\beta(G,\tau)\|^2\le$$
$$\le\ c(1+1/\gamma^2)\|\{Mv,\Gamma_1 v\};\mathcal{R}H_\beta(G,\tau)\|^2+(c/\gamma^2)\|v;\mathcal{D}H_\beta(G,\tau)\|^2.$$

One can rearrange the term $(1/\gamma)\|v;\mathcal{D}H_\beta(G,\tau)\|$ to the left, choosing γ sufficiently large. As a result, we obtain (5.38). $\square$

Introduce a subspace of $\mathcal{R}H_\beta(\mathcal{K},\tau)$ by

$$\mathcal{R}H_{\beta,T}(\mathcal{K},\tau)=\Big\{\{f,g\}\in\mathcal{R}H_\beta(\mathcal{K},\tau)\text{ with }g=(\vec{\Phi},h,q)\\ \text{such that }\langle\vec{\Phi},\vec{\nu}\rangle_3=0\text{ a.e. on }\partial\mathcal{K}\Big\}.$$

It is easy to see that for any $v\in\mathcal{D}_\beta(\mathcal{K})$ we have the inclusion $\{M(D_x,\tau)v,\Gamma v\}\in\mathcal{R}H_{\beta,T}(\mathcal{K},\tau)$.

With the problem (5.4), (5.5) in the cone $\mathcal{K}$ we associate the operator $v\mapsto\mathcal{M}_1(\tau,\beta)v:=\{M(D_x,\tau)v,\Gamma_1 v\}$ with domain $\mathcal{D}_\beta(\mathcal{K})$ such that

$$\mathcal{M}_1(\tau,\beta)\ :\ \mathcal{D}H_\beta(\mathcal{K},\tau)\mapsto\mathcal{R}H_{\beta,T}(\mathcal{K},\tau).$$

It can be proved that the operator $\mathcal{M}_1(\tau,\beta)$ admits closure. We keep the notations $\mathcal{M}_1(\tau,\beta)$ and $\mathcal{D}\mathcal{M}_1(\tau,\beta)$ for the closed operator and its domain.

Let the line $\operatorname{Im}\lambda=\beta+1/2$ contain no eigenvalues of the pencil $\mathfrak{C}$ and $\beta\le 1$. Applying the same arguments as in Section 5.4, we say that for $v\in\mathcal{D}\mathcal{M}_1(\tau,\beta)$ there exists a boundary value $T_1v\in H^0_\beta(\partial\mathcal{K})$ and the estimate

$$\gamma\|T_1v;H^0_\beta(\partial\mathcal{K})\|^2+\|v,\mathcal{D}H_\beta(\mathcal{K},\tau)\|^2\le c\,\|\mathcal{M}_1(\tau,\beta)v,\mathcal{R}H_\beta(\mathcal{K},\tau)\|^2$$

holds. The next proposition immediately follows from this inequality.

Proposition 5.26. *Let $\beta\le 1$ and let the line $\operatorname{Im}\lambda=\beta+1/2$ be free from the eigenvalues of the pencil $\mathfrak{C}$. Then the kernel $\operatorname{Ker}\mathcal{M}_1(\tau,\beta)$ is trivial and the range $\mathcal{R}\mathcal{M}_1(\tau,\beta)$ is closed in $\mathcal{R}H_{\beta,T}(\mathcal{K},\tau)$.*

Let $1/2>\beta_1>\beta_2>\dots$ be all numbers from $]-\infty,1/2[$ such that the line $\operatorname{Im}\lambda=\beta_k+1/2$ contains an eigenvalue of $\mathfrak{C}$. Denote by S_m the number of the eigenvalues of $\mathfrak{C}$ (counted with multiplicity) in the strip $\{\lambda\in\mathbb{C}\ :\ \operatorname{Im}\lambda\in[\beta_m+1/2,\ \beta_1+1/2]\}$.

Definition 5.27. *A solution to the equation $\mathcal{M}_1(\tau,\beta)v=\{f,g\}$, where $\{f,g\}\in\mathcal{R}H_{\beta,T}(\mathcal{K},\tau)$, is called a strong β-solution to the problem (5.4), (5.5) in $\mathcal{K}$ with right-hand side $\{f,g\}$.*

Theorem 5.28. A) *Let $\beta\in]\beta_1,\ 1]$ and let the line $\operatorname{Im}\lambda=\beta+1/2$ contain no eigenvalues of the pencil $\mathfrak{C}$. Then for any $\{f,g\}\in\mathcal{R}H_{\beta,T}(\mathcal{K},\tau)$ there exists a unique strong β-solution v of the problem (5.4), (5.5) with right-hand side $\{f,g\}$. Moreover, there exists a boundary value $T_1v\in H^0_\beta(\partial\mathcal{K})$ and the estimate*

$$\gamma^{1/2}\|T_1v;\ H^0_\beta(\partial\mathcal{K})\|+\|v;\mathcal{D}H_\beta(\mathcal{K},\tau)\|\le c\|\{f,g\};\mathcal{R}H_\beta(\mathcal{K},\tau)\|$$

holds. The constant c is independent of v and of τ.
B) *Let $\beta\in]\beta_{m+1},\beta_m[$. A strong β-solution of the problem (5.4), (5.5) exists only for the right-hand sides $\{f,g\}\in\mathcal{R}H_{\beta,T}(\mathcal{K},\tau)$ satisfying the conditions*

$$(f,w^{(k,j)}_\mu)_{\mathcal{K}}+(g,-iT_1w^{(k,j)}_\mu)_{\partial\mathcal{K}}=0,$$

for all functions $w_\mu^{(k,j)}$ *corresponding to the eigenvalues of the pencil* $\mathfrak{C}$ *in the strip* $\{\lambda \in \mathbb{C} \;:\; \mathrm{Im}\,\lambda \in [\beta_m + 1/2,\;\beta_1 + 1/2]\}$. *A strong* β*–solution* v *is unique. Moreover, there exists a boundary value* $T_1 v \in H^0_\beta(\partial\mathcal{K})$ *and the estimate*

$$\gamma^{1/2}\|T_1 v;\; H^0_\beta(\partial\mathcal{K})\| + \|v; \mathcal{D}H_\beta(\mathcal{K},\tau)\| \le c\|\{f,g\}; \mathcal{R}H_\beta(\mathcal{K},\tau)\|$$

holds. The constant c *is independent of* v *and* τ.

Proof. A) Suppose that $\{w, z\} \in \mathrm{Ker}\,\mathcal{M}_1(\tau,\beta)^*$, where $\mathcal{M}_1(\tau,\beta)^*$ is the adjoint operator to $\mathcal{M}_1(\tau,\beta)$ with respect to the extension of the inner product in $L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$. Applying the known local properties of solutions to elliptic problems (see [28, Chapter 1, §1]), we get $w \in \mathcal{C}^\infty(\overline{\mathcal{K}} \setminus \mathcal{O})$ and $z = -iT_1 w$, while w satisfies (5.7), (5.8). Moreover, the asymptotic representation

$$w = \chi \sum_{k,j,\mu} c_\mu^{(k,j)} V_{\mu,T}^{(k,j)} + \rho \tag{5.41}$$

holds. Since $\{w, -iT_1 w\} \in \mathcal{R}H_\beta(\mathcal{K},\tau)^*$, then for a fixed parameter τ we see that

$$\int\limits_{\mathcal{K}} dx\, |w|^2 (1 + r^{2\beta})^{-1} \le c\|\{w, -iT_1 w\}; \mathcal{R}H_\beta(\mathcal{K},\tau)^*\|^2. \tag{5.42}$$

The estimate (5.36) with $\overline{\tau}$ instead of τ and Γ_2 instead of Γ_1 implies that w decays more rapidly than any power of r as $r \to \infty$.

Assume that $\beta > 0$, hence $w \in L_2(\mathcal{K})$. Let the summation in the asymptotic representation (5.41) be over all μ such that $\mathrm{Im}\,\lambda_\mu \in]1/2,\;1[$, then $\rho(x,\tau) = O(|x|^h)$ with $h = \min\{-\mathrm{Im}\,\lambda_\mu \;:\; \mathrm{Im}\,\lambda_\mu < 1\}$ as $|x| \to 0$. The coefficients in (5.41) are defined by the formula (see [25] or [28, Chapter 3 §5 and Chapter 4 §3]):

$$c_\mu^{(k,j)} = i\Big(w,\; M(D_x,\tau)\chi u_\mu^{(k,j)}\Big)_{\mathcal{K}} + i\Big(-iT_1 w, \Gamma_1 \chi u_\mu^{(k,j)}, w\Big)_{\partial\mathcal{K}}.$$

Taking into account the inclusions $\chi u_\mu^{(k,j)} \in \mathcal{D}\mathcal{M}_1(\tau,\beta)$ and $\{w, -iT_1 w|_{\partial\mathcal{K}}\} \in \mathrm{Ker}\,\mathcal{M}_1(\tau,\beta)^*$, we get

$$c_\mu^{(k,j)} = \Big(\{w, -iT_1 w|_{\partial\mathcal{K}}\},\; \mathcal{M}_1(\tau,\beta)\chi u_\mu^{(k,j)}\Big)_{L_2(\mathcal{K})\times L_2(\partial\mathcal{K})} =$$

$$= \Big(\mathcal{M}_1(\tau,\beta)^*\{w, -iT_1 w|_{\partial\mathcal{K}}\}, \chi u_\mu^{(k,j)}\Big)_{L_2(\mathcal{K})} = 0.$$

So we see that $w \in \mathcal{C}^\infty(\mathcal{K} \setminus \mathcal{O})$, $(1-\chi)w \in H^0_\beta(\mathcal{K})$ for all $\beta \in \mathbb{R}$, and $w = O(|x|^h)$ with $h = \min\{-\mathrm{Im}\,\lambda_\mu \;:\; \mathrm{Im}\,\lambda_\mu < 1\}$ as $|x| \to 0$. Therefore the estimate (5.31) is valid for w. Applying this estimate, we obtain $w \equiv 0$.

Assume that $\beta < 0$. In order to show that $w \equiv 0$ we follow the same scheme as for $\beta > 0$. In this case the proof is even simpler : since $\beta_1 < 0$, then the strip $\{\lambda \in \mathbb{C} \;:\; \mathrm{Im}\,\lambda \in [1/2,\,1]\}$ contains no eigenvalues of the pencil $\mathfrak{C}$. Thus the sum is absent in (5.41) and we immediately obtain the estimate $w = O(|x|^h)$ with $h = \min\{-\mathrm{Im}\,\lambda_\mu \;:\; \mathrm{Im}\,\lambda_\mu < 1\}$ as $|x| \to 0$.

5.6 Operator in the scale of weighted spaces

In this section we study the problem (5.4), (5.5) in the function spaces related to the estimate (5.34). In the first proposition we construct solutions to the homogeneous problem (5.7), (5.8) growing near the vertex of the cone.

Proposition 5.25. *A) Let λ_μ be an eigenvalue of the pencil $\mathfrak{C}$ such that* $\operatorname{Im}\lambda_\mu < 1$. *Then there exist solutions $x \mapsto w_\mu^{(k,j)}(x,\overline{\tau})$ to the homogeneous problem (5.7), (5.8) such that $w_\mu^{(k,j)} \in \mathcal{C}^\infty(\overline{\mathcal{K}} \setminus \mathcal{O})$ and*

$$w_\mu^{(k,j)}(x,\overline{\tau}) = \chi V_{\mu,T}^{(k,j)}(x,\overline{\tau}) + \rho(x,\overline{\tau}). \tag{5.40}$$

By $\chi \in \mathcal{C}_c^\infty(\overline{\mathcal{K}})$ we denote a cut-off function that is equal to 1 near the vertex $\mathcal{O}$ and by $V_{\mu,T}^{(k,j)}$ the first T terms of the series (5.14). The number T is sufficiently large to provide the estimate $\operatorname{Im}(\overline{\lambda}_\mu + 2i) - (T+1) < 1$. *The remainder ρ satisfies the conditions $\rho \in \mathcal{C}^\infty(\overline{\mathcal{K}} \setminus \mathcal{O})$ and $\rho(x,\tau) = O(|x|^h)$ with $h = \min\{-\operatorname{Im}\lambda_\mu \,:\, \operatorname{Im}\lambda_\mu < 1\}$ as $|x| \to 0$ and the parameter τ is fixed. Besides, the inclusion $(1-\chi)\rho \in H_\beta^0(\mathcal{K})$ holds for all $\beta \in \mathbb{R}$. The functions $w_\mu^{(k,j)}$ depend neither on T nor on χ.*

Proof. 1) Let us show that there exists a unique function $w_\mu^{(k,j)}$ with stated properties. Assume that there exists a function $\widetilde{w}_\mu^{(k,j)}$ such that

$$\widetilde{w}_\mu^{(k,j)} = \widetilde{\chi} V_{\mu,\widetilde{T}}^{(k,j)} + \widetilde{\rho}$$

with the same properties as $w_\mu^{(k,j)}$. Then it is easily seen that the estimate (5.31) is valid for the function $v = \widetilde{w}_\mu^{(k,j)} - w_\mu^{(k,j)}$. From this estimate we obtain $\widetilde{w}_\mu^{(k,j)} = w_\mu^{(k,j)}$
2). Let us prove the existence of the functions $w_\mu^{(k,j)}$. We put

$$f_T := M(D_x,\overline{\tau})\chi V_{\mu,T}^{(k,j)} = \chi M(D_x,\overline{\tau}) V_{\mu,T}^{(k,j)} + [M,\chi] V_{\mu,T}^{(k,j)}.$$

The support $\operatorname{supp} f_T \subset \overline{\mathcal{K}}$ is compact and $f_T = O(r^{-\operatorname{Im}(\overline{\lambda}_\mu + 2i) + (T+1)})$ as $|y| \to 0$. Therefore for sufficiently large T the inclusion $f_T \in L_2(\mathcal{K})$ holds. According to the Theorem 5.20, there exists a unique solution ρ to the problem $\mathcal{M}_2(\overline{\tau})\rho = \{-f_T, 0\}$ such that $\rho \in \mathcal{DM}_2(\overline{\tau})$. Put $w_\mu^{(k,j)} = \chi V_{\mu,T}^{(k,j)} + \rho$.
3). Let us show now that the functions $w_\mu^{(k,j)}$ satisfy all the stated properties. Since the problem (5.7), (5.8) with a fixed parameter $\overline{\tau}$ is elliptic and $f_T \in \mathcal{C}^\infty(\overline{\mathcal{K}} \setminus \mathcal{O})$, we obtain $\rho \in \mathcal{C}^\infty(\overline{\mathcal{K}} \setminus \mathcal{O})$. Starting at $\beta = 1$, we iterate the estimate (5.36) with $\overline{\tau}$ instead of τ and Γ_2 instead of Γ_1 for the function $w_\mu^{(k,j)}$. Then we get $(1-\chi)\rho \in H_\beta^0(\mathcal{K})$ for all $\beta \in \mathbb{R}$.

Since ρ is a solution to the elliptic problem with the right-hand side decaying rapidly near the vertex of the cone, we obtain the asymptotics of the form $\rho \sim \sum c_\mu^{(k,j)} V_\mu^{(k,j)}$. This asymptotics and the inclusion $\rho \in \mathcal{DM}_2(\overline{\tau})$ imply the required estimate $\rho = O(|y|^h)$ with $h = \min\{-\operatorname{Im}\lambda_\mu \,:\, \operatorname{Im}\lambda_\mu < 1\}$ as $|x| \to 0$ and the parameter τ is fixed. $\square$

B). Assume that $\beta \in]\beta_{m+1},\ \beta_m[$. Let $\{f, g\} \in \mathcal{R}H_{\beta,T}(\mathcal{K}, \tau)$. From the inclusion $\mathcal{R}H_{\beta,T}(\mathcal{K}, \tau) \subset L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$ and Theorem 5.17 it follows that there exists $v \in L_2(\mathcal{K})$ satisfying $\mathcal{M}_1(\tau)v = \{f, g\}$. The known results concerning the elliptic problems in domains with conical points (see [28, Chapters 3, 4]) and the inequality (5.36) imply that the function v admits the asymptotic representation

$$v = \chi \sum_{\mu,k,j} d_\mu^{(k,j)} U_{\mu,T}^{(k,j)} + h \tag{5.43}$$

with $h \in \mathcal{D}H_\beta(\mathcal{K}, \tau)$. Here the sum is over all μ such that $\operatorname{Im}\lambda_\mu \in [\beta_m + 1/2,\ \beta_1 + 1/2]$. The coefficients are defined by the formula (see [25] or [28, Chapter 3 §5 and Chapter 4 §3])

$$d_\mu^{(k,j)} = i(f, w_\mu^{(k,j)})_{\mathcal{K}} + i(g, -iT_1 w_\mu^{(k,j)})_{\partial\mathcal{K}},$$

where $w_\mu^{(k,j)}$ were constructed in Proposition 5.25. If $\{f, g\} \subset \mathcal{R}\mathcal{M}_1(\tau, \beta)$ then the conditions $d_\mu^{(k,j)} = 0$ in (5.43) are necessary for the inclusion $v \in \mathcal{D}H_\beta(\mathcal{K}, \tau)$. Therefore $\{w_\mu^{(k,j)}, -iT_1 w_\mu^{(k,j)}\} \in \operatorname{K}er\,\mathcal{M}_1(\tau, \beta)^*$ for all μ such that $\operatorname{Im}\lambda_\mu \in [\beta_m + 1/2,\ \beta_1 + 1/2]$.

Show that these functions form a basis in $\operatorname{K}er\,\mathcal{M}_1(\tau, \beta)^*$. Let $\{w, -iT_1 w\} \in \operatorname{K}er\,\mathcal{M}_1(\tau, \beta)^*$, then the representation

$$w = \chi \sum c_\mu^{(k,j)} V_{\mu,T}^{(k,j)} + \rho$$

holds, where the sum is over all μ such that $\operatorname{Im}\lambda_\mu \in [\beta_m + 1/2,\ \beta_1 + 1/2]$ and for the remainder we have $\rho = O(|x|^h)$ with $h = \min\{-\operatorname{Im}\lambda_\mu\ :\ \operatorname{Im}\lambda_\mu < 1\}$ as $|x| \to 0$. Put

$$z = w - \sum c_\mu^{(k,j)} w_\mu^{(k,j)}.$$

We see that $\{z, -iT_1 z\} \in L_2(\mathcal{K}) \times L_2(\partial\mathcal{K})$, $M(D_x, \overline{\tau})z = 0$, and $\Gamma_2 z = 0$. It is not hard to prove that estimate (5.31) is valid for z. Applying this estimate, we get $z = 0$ and $w = \sum c_\mu^{(k,j)} w_\mu^{(k,j)}$. $\square$

Remark 5.29. *Assume that $\gamma > \gamma_0$ with sufficiently large γ_0. Then Theorem 5.28 is valid for the problem (5.4), (5.5) in G.*

5.7 Asymptotics of solutions

Let $\{f, g\} \in \mathcal{R}H_{\beta,T}(\mathcal{K}, \tau)$ and let $\beta \in]\beta_{m+1}, \beta_m[$. Since $\mathcal{R}H_{\beta,T}(\mathcal{K}, \tau) \subset L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$, then there exists a unique strong solution u to the problem (5.4), (5.5) in $\mathcal{K}$ with the right-hand side $\{f, g\}$. According to Theorem 5.28, this solution is in $\mathcal{D}H_\beta(\mathcal{K}, \tau)$ provided

$$(f, w_\mu^{(k,j)})_{\mathcal{K}} + (g, -iT_1 w_\mu^{(k,j)})_{\partial\mathcal{K}} = 0 \tag{5.44}$$

for all functions $w_\mu^{(k,j)}$ corresponding to the eigenvalues of the pencil $\mathfrak{C}$ in the strip $\{\lambda \in \mathbb{C} \,:\, \operatorname{Im}\lambda \in [\beta_m + 1/2,\ \beta_1 + 1/2]\}$. Now we show that if the conditions (5.44) fail, then the strong solution admits the asymptotics modulo $\mathcal{D}H_\beta(\mathcal{K},\tau)$. For the sake of simplicity we make several additional assumptions. These assumptions assure that the asymptotics of solutions contains only the leading term corresponding to λ_m.

PROPOSITION 5.30. *Let $\gamma > 0$, $\beta \in]\beta_{m+1},\ \beta_m[$ such that $\beta < \beta_m < 0$ and $\beta_m - \beta < 1$. Suppose $\{f,g\} \in \mathcal{R}H_{\beta,T}(\mathcal{K},\tau)$ and the conditions*

$$\Big(f, w_\mu^{(k,j)}(\cdot,\xi,\overline{\tau})\Big)_{\mathcal{K}} + \Big(g, -iT_1 w_\mu^{(k,j)}(\cdot,\xi,\overline{\tau})\Big)_{\partial\mathcal{K}} = 0 \tag{5.45}$$

hold for all functions $w_\mu^{(k,j)}(\cdot,\xi,\overline{\tau})$ corresponding to the eigenvalues λ_μ of the pencil $\mathfrak{C}$ in the strip $\{\lambda \in \mathbb{C} \,:\, \operatorname{Im}\lambda \in [\beta_{m-1} + 1/2,\ \beta_1 + 1/2]\}$.

Then the strong solution u to the problem (5.4), (5.5) in $\mathcal{K}$ with the right-hand side $\{f,g\}$ admits the representation

$$u(x,\tau) = \chi(|\tau|x) \sum_{j=1}^{J_m} \sum_{k=0}^{\kappa_{j,m}-1} d_m^{(k,j)}(\tau) u_m^{(k,j)}(x) + w(x,\tau), \tag{5.46}$$

where $w \in \mathcal{D}H_\beta(\mathcal{K},\tau)$ and $\chi \in \mathcal{C}_c^\infty(\mathcal{K})$ such that $\chi = 1$ near the vertex $\mathcal{O}$ of the cone $\mathcal{K}$. The coefficients are defined by the formula

$$d_m^{(k,j)} = i\Big(f, w_m^{(k,j)}\Big)_{\mathcal{K}} + i\Big(g, -iT_1 w_m^{(k,j)}\Big)_{\partial\mathcal{K}}. \tag{5.47}$$

Besides, for the remainder and the coefficients the estimates

$$\begin{gathered} \|w;\ \mathcal{D}H_\beta(\mathcal{K},\tau)\| \le c\|\{f,g\};\ \mathcal{R}H_\beta(\mathcal{K},\tau)\|, \\ |d_m^{(k,j)}| \le c \cdot |\tau|^{\beta+1/2-\operatorname{Im}\lambda_m} \Big(\sum_{r=0}^{\kappa_{j,m}-1-k} (\ln|\tau|)^r \frac{1}{r!} \Big) \cdot \|\{f,g\};\ \mathcal{R}H_\beta(\mathcal{K},\tau)\| \end{gathered} \tag{5.48}$$

hold. The constant c is independent of τ, f, and g.

Proof. The inclusion $\mathcal{R}H_{\beta,T}(\mathcal{K},\tau) \subset L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$ and Theorem 5.17 imply that there exists a unique strong solution u to the problem (5.4), (5.5) in $\mathcal{K}$ with the right-hand side $\{f,g\}$. Since the problem (5.4), (5.5) with a fixed parameter τ is elliptic, then we can deduce the asymptotic representation (5.46) and the formulas (5.47) from the results concerning the elliptic problems in domains with singularities. It remains to prove the estimates (5.48). From (5.47) we easily obtain

$$|d_m^{(k,j)}| \le \|\{f,g\};\ \mathcal{R}H_\beta(\mathcal{K},\tau)\| \cdot \|\{w_m^{(k,j)}, -iT_1 w_m^{(k,j)}\};\ \mathcal{R}H_\beta(\mathcal{K},\tau)^*\|.$$

However, it is not an easy task to obtain the dependance of the norm $\|\{w_m^{(k,j)}, -iT_1 w_m^{(k,j)}\};\ \mathcal{R}H_\beta(\mathcal{K},\tau)^*\|$ on the parameter τ. That is why we prove the estimates (5.48) in a different way. Change the variables $x \mapsto \eta = |\tau| x$ and consider the problem (5.9), (5.10) depending on the parameter $\theta = \tau/|\tau|$. Since the problem (5.9), (5.10) with a fixed parameter θ is elliptic, we have the following asymptotic representation for the strong solution U:

$$U(\eta,\theta) = \chi(\eta) \sum_{\mu=1}^{m} \sum_{j=1}^{J_\mu} \sum_{k=0}^{\kappa_{j,\mu}-1} c_\mu^{(k,j)}(\theta) u_\mu^{(k,j)}(\eta) + W(\eta,\theta), \tag{5.49}$$

where $\chi W \in H^1_\beta(\mathcal{K})$ and

$$c_\mu^{(k,j)} = i\Big(F, \mathcal{W}_\mu^{(k,j)}(\cdot,\overline{\theta})\Big)_{\mathcal{K}} + i\Big(G, -iT_1 \mathcal{W}_\mu^{(k,j)}(\cdot,\overline{\theta})\Big)_{\partial\mathcal{K}}. \tag{5.50}$$

By $\mathcal{W}_\mu^{(k,j)}(\cdot,\overline{\theta})$ we denote the solutions to the problem (5.9), (5.10) with $\overline{\theta}$ instead of θ similar to the functions $w_\mu^{(k,j)}$, constructed in Proposition 5.25. Applying the conditions (5.45) we rewrite (5.49) as follows

$$U(\eta,\theta) = \chi(\eta) \sum_{j=1}^{J_m} \sum_{k=0}^{\kappa_{j,m}-1} c_m^{(k,j)}(\theta) u_m^{(k,j)}(\eta) + W(\eta,\theta). \tag{5.51}$$

The formula (5.50) yields the inequality

$$|c_m^{(k,j)}| \le \|\{F,G\};\ \mathcal{R}H_\beta(\mathcal{K},1)\| \cdot \|\{\mathcal{W}_m^{(k,j)}, -iT_1\mathcal{W}_m^{(k,j)}\};\ \mathcal{R}H_\beta(\mathcal{K},1)^*\|.$$

Since the functions $\mathcal{W}_m^{(k,j)}$ depend only on θ then we can estimate the norm

$$\|\{\mathcal{W}_m^{(k,j)}, -iT_1\mathcal{W}_m^{(k,j)}\};\ \mathcal{R}H_\beta(\mathcal{K},\tau)^*\|$$

by the constant c independent of $|\tau|$. Thus we get

$$|c_m^{(k,j)}| \le c\|\{F,G\};\ \mathcal{R}H_{\beta,T}(\mathcal{K},\tau)\|. \tag{5.52}$$

Let us now estimate the remainder in the formula (5.51). Consider the problem (5.9), (5.10) with the right-hand side

$$\{F', G'\} = \{F - M(D_\eta,\theta)\big(\chi \sum c_m^{(k,j)} u_m^{(k,j)}\big), G - \Gamma_1\big(\chi \sum c_m^{(k,j)} u_m^{(k,j)}\big)\}.$$

It is easily shown that

$$\Big(F', \mathcal{W}_\mu^{(k,j)}(\cdot,\overline{\theta})\Big)_{\mathcal{K}} + \Big(G', -iT_1\mathcal{W}_\mu^{(k,j)}(\cdot,\overline{\theta})\Big)_{\partial\mathcal{K}} = 0$$

for $\mathcal{W}_\mu^{(k,j)}$ such that $\operatorname{Im}\lambda_\mu \in [\beta_m + 1/2,\ \beta_1 + 1/2]$. Then Theorem 5.28 leads to the inclusion $W \in \mathcal{D}H_\beta(\mathcal{K},1)$ and the estimate

$$\begin{aligned} \|W;\ \mathcal{D}H_\beta(\mathcal{K},1)\| &\le c\|\{F',G'\};\ \mathcal{R}H_\beta(\mathcal{K},1)\| \le \\ &\le c\|\{F,G\};\ \mathcal{R}H_\beta(\mathcal{K},1)\|. \end{aligned} \tag{5.53}$$

Change the variables $\eta \mapsto x = \eta/|\tau|$ and return to the problem (5.4), (5.5). Then we have

$$\sum_{k=0}^{\kappa_{j,m}-1} c_m^{(k,j)} u_m^{(k,j)}(\eta) = \sum_{k=0}^{\kappa_{j,m}-1} c_m^{(k,j)} \sum_{q=0}^{k} \frac{1}{q!} |\eta|^{i\lambda_m} (i \ln|\eta|)^q \varphi_m^{(k-q,j)}(\omega) =$$
$$\sum_{k=0}^{\kappa_{j,m}-1} c_m^{(k,j)} |x|^{i\lambda_m} |\tau|^{i\lambda_m} \sum_{q=0}^{k} \frac{1}{q!} (i \ln|x| + i \ln|\tau|)^q \varphi_m^{(k-q,j)}(\omega) =$$
$$\sum_{k=0}^{\kappa_{j,m}-1} c_m^{(k,j)} |x|^{i\lambda_m} |\tau|^{i\lambda_m} \sum_{q=0}^{k} \sum_{l=0}^{q} \frac{1}{q!} \cdot \frac{q!}{l!(q-l)!} (i \ln|x|)^{q-l} (i \ln|\tau|)^l \varphi_m^{(k-q,j)}(\omega) =$$
$$\sum_{k=0}^{\kappa_{j,m}-1} c_m^{(k,j)} |x|^{i\lambda_m} |\tau|^{i\lambda_m} \sum_{l=0}^{k} (i \ln|\tau|)^l \frac{1}{l!} \sum_{q=l}^{k} \frac{1}{(q-l)!} (i \ln|x|)^{q-l} \varphi_m^{(k-q,j)}(\omega) =$$
$$\sum_{k=0}^{\kappa_{j,m}-1} c_m^{(k,j)} |x|^{i\lambda_m} |\tau|^{i\lambda_m} \sum_{l=0}^{k} (i \ln|\tau|)^l \frac{1}{l!} \sum_{r=0}^{k-l} \frac{1}{r!} (i \ln|x|)^r \varphi_m^{((k-l)-r,j)}(\omega) =$$
$$\sum_{k=0}^{\kappa_{j,m}-1} c_m^{(k,j)} |\tau|^{i\lambda_m} \sum_{l=0}^{k} (i \ln|\tau|)^l \frac{1}{l!} u_m^{(k-l,j)}(x) =$$
$$\sum_{k=0}^{\kappa_{j,m}-1} c_m^{(k,j)} |\tau|^{i\lambda_m} \sum_{s=0}^{k} (i \ln|\tau|)^{k-s} \frac{1}{(k-s)!} u_m^{(s,j)}(x) =$$
$$\sum_{s=0}^{\kappa_{j,m}-1} u_m^{(s,j)}(x) \Big\{ |\tau|^{i\lambda_m} \sum_{k=0}^{\kappa_{j,m}-1-s} (i \ln|\tau|)^k \frac{1}{k!} c_m^{(k+s,j)} \Big\}.$$

Finally, the asymptotics (5.51) takes the form

$$u(x,\tau) = \chi(|\tau|x) \sum_{j=1}^{J_m} \sum_{k=0}^{\kappa_{j,m}-1} d_m^{(k,j)}(\tau) \cdot u_m^{(k,j)}(x) + w(x,\tau)$$

with

$$d_m^{(k,j)}(\tau) = |\tau|^{i\lambda_m} \sum_{r=0}^{\kappa_{j,m}-1-k} (i \ln|\tau|)^r \frac{1}{r!} \cdot c_m^{(k+r,j)}.$$

Changing the variables in (5.52), (5.53) we arrive at the required estimates (5.48). $\square$

REMARK 5.31. *Let* $\{\varphi_m^{(0,j)}\}_{j=1}^{J_m}$ *be the eigenvectors corresponding to* λ_m. *Suppose that there are no associated vectors. Then the asymptotics (5.46) takes the form*

$$u(x,\tau) = \chi(|\tau|x) \sum_{j=1}^{J_m} d_m^{(0,j)}(\tau) r^{i\lambda_m} \varphi_m^{(0,j)}(\omega) + w(x,\tau).$$

REMARK 5.32. *Let* $\gamma > \gamma_0$ *with sufficiently large* $\gamma_0 > 0$. *Then Proposition 5.30 is valid for the problem (5.4), (5.5) in* G.

5.8 Nonstationary problem in cylinders

Applying the inverse Fourier transform $\mathcal{F}^{-1}_{\tau\to t}$, we pass from the problem (5.4), (5.5) to problem the (5.1), (5.2). Fix a cut-off function $\chi \in \mathcal{C}^\infty(\overline{\mathcal{K}})$ such that $\chi = 1$ near the vertex $\mathcal{O}$ of the cone $\mathcal{K}$. Put

$$Xu(x,t) = \mathcal{F}^{-1}_{\tau\to t}\chi(|\tau|x)\mathcal{F}_{t'\to\tau}u(x,t'),$$

$$\Lambda^\mu u(x,t) = \mathcal{F}^{-1}_{\tau\to t}|\tau|^\mu \mathcal{F}_{t'\to\tau}u(x,t'),$$

$$(\ln\Lambda)^\mu u(x,t) = \mathcal{F}^{-1}_{\tau\to t}(\ln|\tau|)^\mu \mathcal{F}_{t'\to\tau}u(x,t').$$

Introduce the spaces $\mathcal{D}V_\beta(\mathcal{Q},\gamma)$, $\mathcal{R}V(\mathcal{Q},\gamma)$, and $\mathcal{R}V_\beta(\mathcal{Q},\gamma)$ equipped with the norms

$$\|u;\mathcal{D}V_\beta(\mathcal{Q},\gamma)\| = \Big(\gamma^2\|u;V^0_\beta(\mathcal{Q},\gamma)\|^2 + \|Xu;V^1_\beta(\mathcal{Q},\gamma)\|^2\Big)^{1/2},$$

$$\|\{f,g\};\mathcal{R}V(\mathcal{Q},\gamma)\| = \Big(\|f;V^0_0(\mathcal{Q},\gamma)\|^2 + \gamma\|g;V^0_0(\partial\mathcal{Q},\gamma)\|^2\Big)^{1/2},$$

$$\|\{f,g\};\mathcal{R}V_\beta(\mathcal{Q},\gamma)\| = \Big(\|f;V^0_\beta(\mathcal{Q},\gamma)\|^2 + (1/\gamma^2)\|\Lambda^{1-\beta}f;V^0_0(\mathcal{Q},\gamma)\|^2$$
$$+\gamma\|g;V^0_\beta(\partial\mathcal{Q},\gamma)\|^2 + (1/\gamma)\|\Lambda^{1-\beta}g;V^0_0(\partial\mathcal{Q},\gamma)\|^2 + \|Xg;V^{1/2}_\beta(\partial\mathcal{Q},\gamma)\|^2\Big)^{1/2}.$$

Denote by $\mathcal{R}V_T(\mathcal{Q},\gamma)$ and $\mathcal{R}V_{\beta,T}(\mathcal{Q},\gamma)$ the subspaces of $\mathcal{R}V(\mathcal{Q},\gamma)$ and $\mathcal{R}V_\beta(\mathcal{Q},\gamma)$ respectively such that for any $\{f,g\} \in \mathcal{R}V_T(\mathcal{Q},\gamma)$ ($\mathcal{R}V_{\beta,T}(\mathcal{Q},\gamma)$) with $g = (\vec{\Phi},h,q)$ we have $\langle\vec{\Phi}\cdot\vec{\nu}\rangle_3 = 0$ a.e. on $\partial\mathcal{Q}$. The spaces in the cylinder $\mathcal{T}$ are defined in a similar way.

Definition 5.33. *Let $\{f,g\} \in \mathcal{R}V_T(\mathcal{Q},\gamma)$ and let $\widehat{u}(x,\tau)$ be the strong solution to the problem (5.4), (5.5) in $\mathcal{K}$ with right-hand side $\{-i\widehat{f},\widehat{g}\}$, where $\widehat{f}(x,\tau) = \mathcal{F}_{t\to\tau}f(x,t)$, $\widehat{g}(x,\tau) = \mathcal{F}_{t\to\tau}g(x,t)$. The function u, defined by $u(x,t) = \mathcal{F}^{-1}_{\tau\to t}\widehat{u}(x,\tau)$, is called a strong solution to the problem (5.1), (5.2) in the cylinder $\mathcal{Q}$ with right-hand side $\{f,g\}$.*

The next result follows from Theorem 5.17.

Theorem 5.34. *For every $\{f,g\} \in \mathcal{R}V_T(\mathcal{Q},\gamma)$ and for any $\gamma > 0$ there exists a strong solution v to the problem (5.1), (5.2) in $\mathcal{Q}$ with right-hand side $\{f,g\}$. Moreover, there exists a boundary value $T_1v \in V^0_0(\partial\mathcal{Q},\gamma)$ and the estimate*

$$\gamma\|T_1v;V^0_0(\partial\mathcal{Q},\gamma)\|^2 + \gamma^2\|v;V^0_0(\mathcal{Q},\gamma)\|^2 \le \|\{f,g\};\mathcal{R}V(\mathcal{Q},\gamma)\|^2$$

holds.

Definition 5.35. *Let $\{f,g\} \in \mathcal{R}V_{\beta,T}(\mathcal{Q},\gamma)$ and let $\widehat{u}(x,\tau)$ be the strong β-solution to the problem (5.4), (5.5) in $\mathcal{K}$ with right-hand side $\{-i\widehat{f},\widehat{g}\}$, where $\widehat{f}(x,\tau) = \mathcal{F}_{t\to\tau}f(x,t)$, $\widehat{g}(x,\tau) = \mathcal{F}_{t\to\tau}g(x,t)$. The function u, defined by $u(x,t) = \mathcal{F}^{-1}_{\tau\to t}\widehat{u}(x,\tau)$, is called a strong β-solution to the problem (5.1), (5.2) in the cylinder $\mathcal{Q}$ with right-hand side $\{f,g\}$.*

The next result follows from Theorem 5.28.

THEOREM 5.36. *1) Let $\beta \in]\beta_1,\ 1]$ and let the line* $\operatorname{Im}\lambda = \beta + 1/2$ *be free from the spectrum of* $\mathfrak{C}$. *Then there exists a unique strong β-solution v to the problem (5.1), (5.2) in $\mathcal{Q}$ with any right-hand side $\{f, g\} \in \mathcal{R}V_{\beta,T}(\mathcal{Q}, \gamma)$. Moreover, there exists a boundary value $T_1 v \in V^0_\beta(\partial\mathcal{Q}, \gamma)$ and the estimate*

$$\gamma\|T_1 v; V^0_\beta(\partial\mathcal{Q}, \gamma)\|^2 + \|v; \mathcal{D}V_\beta(\mathcal{Q}, \gamma)\|^2 \le c\|\{f, g\}; \mathcal{R}V_\beta(\mathcal{Q}, \gamma)\|^2$$

holds.
2) Let $\beta \in]\beta_{m+1},\ \beta_m[$. A strong β-solution to the problem (5.1), (5.2) in $\mathcal{Q}$ with right-hand side $\{f, g\} \in \mathcal{R}V_{\beta,T}(\mathcal{Q}, \gamma)$ exists (and is unique) if the conditions

$$\Big(-i\widehat{f}(\cdot, \tau), w^{(k,j)}_\mu(\cdot, \overline{\tau})\Big)_{\mathcal{K}} + \Big(\widehat{g}(\cdot, \tau), -iT_1 w^{(k,j)}_\mu(\cdot, \overline{\tau})\Big)_{\partial\mathcal{K}} = 0$$

hold for all $\tau = \sigma - i\gamma$ ($\sigma \in \mathbb{R},\ \gamma > 0$) and for all $w^{(k,j)}_\mu$ corresponding to the eigenvalues of the pencil $\mathfrak{C}$ in the strip $\{\lambda \in \mathbb{C}\ :\ \operatorname{Im}\lambda \in [\beta_m + 1/2,\ \beta_1 + 1/2]\}$. If such a solution exists, then it has a boundary value $T_1 v \in V^0_\beta(\partial\mathcal{Q}, \gamma)$ and the estimate from the first part of the theorem holds.

The next theorem follows from Proposition 5.30.

THEOREM 5.37. *Assume that $\{f, g\} \in \mathcal{R}V_{\beta,T}(\mathcal{Q}, \gamma)$ and $\beta \in]\beta_{m+1}, \beta_m[$ such that $\beta < \beta_m < 0$, $\beta_m - \beta < 1$. Then the strong solution to the problem (5.1), (5.2) admits the representation*

$$u(x, t) = \sum_{j=1}^{J_m} \sum_{k=0}^{\kappa_{j,m}-1} (X\widetilde{d}^{(k,j)}_m)(x, t) u^{(k,j)}_m(x) + w(x, t),$$

where $w \in \mathcal{D}V_\beta(\mathcal{Q}, \gamma)$. The coefficients are defined by

$$\widetilde{d}^{(k,j)}_m(t) = \mathcal{F}^{-1}_{\tau\to t} d^{(k,j)}_m(\tau)$$

with

$$d^{(k,j)}_m(\tau) = i\big(-i\widehat{f}(\cdot, \tau), w^{(k,j)}_m(\cdot, \overline{\tau})\big)_{\mathcal{K}} + i\big(\widehat{g}(\cdot, \tau), -iT_1 w^{(k,j)}_m(\cdot, \overline{\tau})\big)_{\partial\mathcal{K}}.$$

Moreover, the following estimates hold

$$\begin{gathered}\|e^{-\gamma t}\widetilde{d}^{(k,j)}_m(\cdot); H^{\operatorname{Im}\lambda_k - \beta - 1/2}(\mathbb{R})\| \le c\|(\ln\Lambda)^{\kappa_{j,m}-1-k}\{f, g\}; \mathcal{R}V_\beta(\mathcal{Q}, \gamma)\|,\\ \|w; \mathcal{D}V_\beta(\mathcal{Q}, \gamma)\| \le c\|\{f, g\}; \mathcal{R}V_\beta(\mathcal{Q}, \gamma)\|.\end{gathered}$$

Strong solutions and strong β-solutions to the problem (5.1), (5.2) in $\mathcal{T}$ can be defined in the same way as those to the problem in the cylinder $\mathcal{Q}$ in Definition 5.33 and Definition 5.35.

REMARK 5.38. *All the theorems in this section are valid for the problem (5.1), (5.2) in $\mathcal{T}$ if $\gamma > \gamma_0$ with sufficiently large $\gamma_0 > 0$.*

5.9 Connection between the augmented and non-augmented Maxwell systems

Up to this point the discussion was related to the augmented Maxwell system. In this section we prove that under some conditions on the right-hand side of such a system, its solutions have the form $u = (\vec{u}, \vec{v}, 0, 0)$, therefore $(\vec{u}, \vec{v})$ satisfy the usual (non-augmented) Maxwell system. The mentioned conditions on the right-hand side are derived from the compatibility of the usual Maxwell system

$$\begin{cases} \partial\vec{E}/\partial t - \mathrm{rot}\ \vec{B} = -\vec{J}, \\ \partial\vec{B}/\partial t + \mathrm{rot}\ \vec{E} = -\vec{F}, \\ \mathrm{div}\ \vec{E} = \rho, \ \ \mathrm{div}\ \vec{B} = \mu \end{cases}$$

with boundary conditions

$$\vec{\nu} \times [\vec{v} \times \vec{\nu}] + \psi[\vec{\nu} \times \vec{u}] = \vec{\Phi}.$$

Namely, for sufficiently smooth functions from the first and the third equations of this system we obtain

$$\partial\rho/\partial t + \mathrm{div}\ \vec{J} = 0. \tag{5.54}$$

Similarly, from the second and the forth equations we get

$$\partial\mu/\partial t + \mathrm{div}\ \vec{F} = 0. \tag{5.55}$$

Consider the problem (5.4), (5.5) in the cone $\mathcal{K}$ with the right-hand side $\{f, g\} \in L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$, where $f = (\vec{A},\ \vec{B},\ \alpha,\ \beta)$ and $g = (\vec{\Phi},\ 0,\ 0)$. Since the right-hand side is not smooth, the conditions (5.54), (5.55) should be understood in a proper way. For this purpose we recall the following definitions.

Let $\vec{u}$ and v be functions such that for any $\phi \in \mathcal{C}_c^\infty(\mathcal{K})$ we have

$$\int\limits_{\mathcal{K}} \mathrm{d}x\ \langle \vec{u}, \nabla\phi \rangle_3 + \int\limits_{\mathcal{K}} \mathrm{d}x\ v \cdot \overline{\phi} = 0.$$

Then we say that $v = \mathrm{div}\ \vec{u}$ in the distributional sense. By $H(\mathrm{div}, \mathcal{K})$ we denote the space of functions $\vec{u} \in L_2(\mathcal{K})$ such that $\mathrm{div}\ \vec{u} \in L_2(\mathcal{K})$, where the divergency is understood in the distributional sense. We endow this space with the norm

$$\|\vec{u}; H(\mathrm{div}, \mathcal{K})\| = \Big(\|\vec{u}; L_2(\mathcal{K})\|^2 + \|\mathrm{div}\ \vec{u}; L_2(\mathcal{K})\|^2 \Big)^{1/2}.$$

With the obvious inner product, $H(\mathrm{div}, \mathcal{K})$ is a Hilbert space.

Applying the Fourier transform $\mathcal{F}_{t\to\tau}$, we rewrite (5.54), (5.55) in the form

$$\begin{aligned} &\vec{A} \in H(\mathrm{div}, \mathcal{K}),\ \alpha \in L_2(\mathcal{K}),\ \mathrm{div}\ \vec{A} = i\tau\alpha, \\ &\vec{B} \in H(\mathrm{div}, \mathcal{K}),\ \beta \in L_2(\mathcal{K}),\ \mathrm{div}\ \vec{B} = i\tau\beta. \end{aligned} \tag{5.56}$$

For the proof of theorem on connection between solutions to the augmented and usual Maxwell systems, we need a lineal. Let $\{\mu_k,\ w_k\}$ be the set of eigenvalues and eigenfunctions of the Dirichlet problem pencils for the Laplace equation in $\mathcal{K}$. Denote by L_D the lineal spanned by the functions in $\mathcal{C}_c^\infty(\mathcal{K})$ and by functions of the form $\chi r^{i\mu_k} w_k$ with $\operatorname{Im}\mu_k < 0$, where $\chi \in \mathcal{C}_c^\infty(\mathcal{K})$ is a cut-off function equal to 1 near the vertex $\mathcal{O}$ of the cone $\mathcal{K}$. According to [33, §3], the range of the Helmholtz operator $\tau^2 + \triangle$ with $\tau = \sigma - i\gamma$ ($\gamma \neq 0$) given on L_D is dense in $L_2(\mathcal{K})$.

Theorem 5.39. *Consider the problem (5.4), (5.5) in the cone $\mathcal{K}$ with the right-hand side $\{f, g\}$, where $g = (\vec{\Phi}, 0, 0)^T$ and the function $f = (\vec{A}, \vec{B}, \alpha, \beta)^T$ is subject to the conditions (5.56). Then the strong solution u is of the form $u = (\vec{u}, \vec{v}, 0, 0)$.*

Proof. Since $\mathcal{M}_1(\tau)$ is the closure of the operator $\{M(D_x, \tau), \Gamma_1\}$ given on $\mathcal{D}(\mathcal{K})$, there exists a sequence $\Big\{u_k = (\vec{u}_k, \vec{v}_k, h_k, q_k)^T\Big\} \subset \mathcal{D}(\mathcal{K})$ such that $u_k \to u$ (the convergence in $L_2(\mathcal{K})$) and $\Big\{M(D_x, \tau)u_k, \Gamma_1 u_k\Big\} \to \{f, g\}$ (the convergence in $L_2(\mathcal{K}) \times L_{2,T}(\partial\mathcal{K})$). We have $u_k \in \mathcal{C}^\infty(\overline{\mathcal{K}} \setminus \mathcal{O}))$ so the system (5.4), (5.5) can be understood as usual. In particular,

$$\begin{gathered} i\tau\vec{u}_k - \operatorname{rot}\vec{v}_k + \nabla h_k = \vec{A}_k, \\ i\tau h_k + \operatorname{div}\vec{u}_k = \alpha_k. \end{gathered}$$

We shall prove that $h = 0$. Assume that $\phi \in L_D$. Multiply the first equality by $\nabla\overline{\phi}$, the second one by $i\tau\overline{\phi}$, and integrate over $\mathcal{K}$. Then

$$i\tau\int_{\mathcal{K}} \langle\vec{u}_k, \nabla\phi\rangle_3\,\mathrm{d}x - \int_{\mathcal{K}} \langle\operatorname{rot}\vec{v}_k, \nabla\phi\rangle_3\,\mathrm{d}x + \int_{\mathcal{K}} \langle\nabla h_k, \nabla\phi\rangle_3\,\mathrm{d}x = \int_{\mathcal{K}} \langle\vec{A}_k, \nabla\phi\rangle_3\,\mathrm{d}x\ ,$$
$$-\tau^2\int_{\mathcal{K}} h_k\cdot\overline{\phi}\,\mathrm{d}x + i\tau\int_{\mathcal{K}} \operatorname{div}\vec{u}_k\cdot\overline{\phi}\,\mathrm{d}x = i\tau\int_{\mathcal{K}} \alpha_k\cdot\overline{\phi}\,\mathrm{d}x.$$

We integrate by parts in the two first terms of the first equality, add the first and the second equalities, and obtain

$$-\tau^2\int_{\mathcal{K}} h_k\cdot\overline{\phi}\,\mathrm{d}x + \int_{\mathcal{K}} \langle\nabla h_k, \nabla\phi\rangle_3\,\mathrm{d}x = i\tau\int_{\mathcal{K}} \alpha_k\cdot\overline{\phi}\,\mathrm{d}x + \int_{\mathcal{K}} \langle\vec{A}_k, \nabla\phi\rangle_3\,\mathrm{d}x.$$

Integrate by parts the second term on the left. Then we get

$$-\tau^2\int_{\mathcal{K}} h_k\cdot\overline{\phi}\,\mathrm{d}x - \int_{\mathcal{K}} h_k\cdot\overline{\triangle\phi}\,\mathrm{d}x + \int_{\partial\mathcal{K}} h_k\cdot\overline{\partial\phi/\partial\nu}\,\mathrm{d}S = i\tau\int_{\mathcal{K}} \alpha_k\cdot\overline{\phi}\,\mathrm{d}x + \int_{\mathcal{K}} \langle\vec{A}_k, \nabla\phi\rangle_3\,\mathrm{d}x.$$

As $k \to \infty$, taking into account the boundary condition for h and the property (5.56), we arrive at the formula

$$\int_{\mathcal{K}} h\cdot\overline{(\overline{\tau}^2 + \triangle)\phi}\,\mathrm{d}x = 0.$$

Therefore $h = 0$ because the range of the operator $\overline{\tau}^2 + \triangle$ given on L_D is dense in $L_2(\mathcal{K})$. By the same argument one can prove that $q = 0$ $\square$.

6 CONCLUSIONS

In this thesis we investigated the Maxwell system in domains with conical points and edges on the boundary. We considered conductive and impedance boundary conditions. The main purpose was to study the behavior of solutions near the singularities on the boundary. We derived and justified the asymptotic expansions of solutions near the singularities and obtained the explicit formulas for the coefficients in the asymptotics. We obtained the explicit formulas for the coefficients in the asymptotics in terms of singular solutions to the adjoint problem and studied the properties of the coefficients.

The suggested methods and obtained results can be applied in numerical treatment of the nonstationary Maxwell system in nonsmooth domains and on different problems of electrodynamics and mathematical physics.

YHTEENVETO (FINNISH SUMMARY)

Väitöskirjassa tutkitaan ei-stationaarista Maxwell-yhtälöä alueissa, joissa on kartiomaisia kärkiä ja teräviä särmiä. Yhtälöihin liittyy johtavuus- ja impedanssityyppisiä, homogeenisia ja epähomogeenisia reunaehtoja. Työn keskeisenä tavoitteena on tutkia yhtälön ratkaisujen käyttäytymistä lähellä ns. singulaaripisteitä, joita epäsileät reunat aiheuttavat. Työssä johdetaan ratkaisuille eksplisiittiset, asymptoottiset laajennukset alueen kartiomaisten kärkien ja terävien särmien ympäristössä. Näitä asymptoottisia esityksiä voidaan hyödyntää useissa sovelluksissa. Tuloksia voidaan erityisesti soveltaa useiden ei-stationaaristen sähkömagnetiikan yhtälöiden matemaattisessa ja numeerisessa analysoinnissa epäsileissä alueissa.

Avainsanat: ei-stationaarinen Maxwell-yhtälö, epäsileä reuna, asymptoottinen laajennus, energiaestimaatti

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