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Viktor Kalvine

Scattering and Point Spectra for Elliptic Systems in Domains with Cylindrical Ends

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ABSTRACT

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In the present work we consider a class of dissipative and self-adjoint elliptic problems for systems of differential equations of arbitrary order in domains with finitely many cylindrical ends. One can treat the cylindrical ends as waveguides and introduce families of incoming and outgoing "waves." The amplitudes of these waves may grow with exponential rate at infinity. Taking into account finitely many waves, we introduce (augmented) scattering matrices. The scattering matrices turn out to be of use in many applications. In particular, such effects as surface waves in diffraction gratings, trapped water waves in channels, and bound states in quantum waveguides can be characterized in terms of the matrices. We suggest and justify a numerical method for the computation of the matrices. The results allow to study acoustic and electromagnetic diffraction gratings, quantum waveguides, various problems in elasticity and hydrodynamics.

Keywords: elliptic problems, scattering matrix, augmented scattering matrix, radiation conditions, cylindrical ends, cylindrical outlets, approximation solution, surface waves, trapped modes.

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CONTENTS

1 Introduction and main results

1.1 Introduction

We consider a class of dissipative and self-adjoint elliptic problems for systems of differential equations of arbitrary order in domains with finitely many cylindrical ends. Under certain assumptions on the coefficients of the problem one can treat the cylindrical ends as waveguides and introduce incoming and outgoing "waves". The amplitudes of such waves can grow with power or even exponential rate at infinity. Owing to the classification of waves, the boundary value problems admit well-posed statements with intrinsic radiation conditions; in connection with this the scattering matrices appear. These matrices can take into account not only the oscillating modes but also finitely many modes with growing amplitudes. The norm of such a matrix turns out to be less or equal to one; the scattering matrix of a formally self-adjoint problem is unitary. In terms of scattering matrices one can find the number of linearly independent solutions to the homogeneous problem decreasing at infinity with a given rate.

We suggest and justify a numerical method for computing such scattering matrices. The computation of a row of scattering matrix reduces to the minimization of a quadratic functional. We cut off the cylindrical ends at a finite distance R. Then we use the optimization of the mentioned functional to match a solution in the truncated domain to a predetermined combination of waves at the truncation boundary. At this part of the boundary we set a condition of the impedance type. To obtain the functional, we numerically solve auxiliary boundary value problems in the truncated domain. The minimizer of the quadratic functional tends exponentially to the corresponding row of the scattering matrix as R goes to infinity. We emphasize that the method is developed for augmented scattering matrices which take into account finitely many waves with growing amplitudes.

In applications, the obtained results allow to include into consideration, in particular, acoustic and electromagnetic diffraction gratings, quantum waveguides, various problems in elasticity and hydrodynamics. In the "classical" situation, where the Sommerfeld or Mandelstam radiation principles can be employed, the above conception of the waves coincides with that in the principles, and the scattering matrix coincides with the classical one. The augmented scattering matrices turn out to be of use in different applications. For example, such effects as surface waves in diffraction gratings, trapped water waves in channels, and bound states in quantum waveguides can be characterized in terms of augmented scattering matrices.

Let $G \subset \mathbb{R}^n$ be a domain with cylindrical ends; this means that outside of a large ball the domain G coincides with the union of semi-infinite cylinders $\Pi_+^1, \dots, \Pi_+^N$. The boundary ∂G of G is smooth. In the domain G we consider the general elliptic boundary value problems $\{\mathcal{L}(x, D_x), \mathcal{B}(x, D_x))\}$ with matrix $\mathcal{L}(x, D_x)$ of differential equations and matrix $\mathcal{B}(x, D_x)$ of boundary conditions. The operator $\{\mathcal{L}, \mathcal{B}\}$ is assumed to be dissipative. In particular, operators that are self-adjoint with respect to the Green formula are dissipative. The coefficients of differential operators stabilize at infinity with exponential rate to functions that are constant along the axis of the cylinder Π_+^j, $j = 1, \dots, N$. The limit boundary value problem in the infinite cylinder Π^j is supposed to be self-adjoint with respect to the Green formula. This allows us to extract by a known method (cf. [1]–[5]) two families of asymptotic solutions to the homogeneous equations $\{\mathcal{L}(x, D_x), \mathcal{B}(x, D_x)\}u(x) = 0$ in the semicylinders $\Pi_+^1, \dots, \Pi_+^N$. One of these families plays the role of incoming waves, whereas the other consists of outgoing waves. The families can contain not only bounded asymptotic solutions but also those growing exponentially in amplitude at infinity. Taking into account a finite number of the waves, one can introduce the (augmented) scattering matrices. If the problem is not strictly dissipative then the norm of the scattering matrix is less or equal to one, and the number of linearly independent solutions to the homogeneous problem that decrease at infinity with a given rate can be found in terms of the scattering matrices (existence criterion for the point spectrum). In the case of strictly dissipative problem, the norm of the scattering matrix is strictly less then one, and there are no decreasing solutions to the homogeneous problem. We discuss the problem statement with the so-called radiation conditions. This is a way of choosing solutions (possibly with a certain arbitrariness). The intrinsic radiation conditions (only outgoing "waves" occur in asymptotic formulas for solutions) can be applied anyway. Other admissible radiation conditions for the problem under consideration are connected with the intrinsic ones via scattering matrices.

For formally self-adjoint problems, similar results were obtained in [1]–[5]. The general self-adjoint elliptic systems with "impedance" boundary conditions were considered in [6], where, in particular, properties of scattering matrices were discussed. All these topics are studied in the first part of the thesis; here we follow the paper [7].

The second part of the work is devoted to the method for numerical computation of the augmented scattering matrices. This method was originally suggested for acoustic and electromagnetic non-absorbing diffraction gratings in [8] and there was given an outline of the justification of the method. A numerical implementation was employed in [8, 9] to detect families of surface waves in a number of diffraction gratings. In [6, 10, 11] the method was developed for general elliptic formally self-adjoint boundary value problems in domains with cylindrical ends, some examples of numerical realization can be found in [11, 12]. In the thesis we present and justify the method for computing scattering matrices to the arbitrary dissipative and self-adjoint elliptic problems; this part of the thesis is based on the paper [13]. We give a detailed proof of the convergence of the method. In particular, an essential part of the proof is new for the problems considered in [8, 9]. Let us describe the method.

First we assume that the homogeneous problem $\{\mathcal{L}(x, D_x), \mathcal{B}(x, D_x)\}u(x) = 0$ in the domain G has no nontrivial solutions that exponentially decrease at infinity. To compute a scattering matrix means to find the asymptotics of some solutions to the homogeneous problem that, in general, exponentially grow as $|x| \to \infty$. (Emphasize that the method is developed not only for the "classical" scattering matrices but also for the "augmented" matrices, which take into account finitely many waves with growing amplitudes.) To this end we cut off the cylindrical ends of $\Pi_+^1, \dots, \Pi_+^N$ at a finite distance R. Thus, we obtain the bounded domain G^R. As an approximation of a solution Y to the problem in the domain G we choose a solution Y^R to the problem in G^R such that Y^R satisfies some special boundary conditions with unknown numerical coefficients on the truncation boundary. To compute the coefficients, we construct a quadratic functional J^R. As coefficients we take the components of the minimizer of the functional J^R. We prove that the minimizer tends exponentially to a row of the scattering matrix as $R \to +\infty$.

Now, instead of the above assumption, we suppose that the problem $\{\mathcal{L}(x, D_x), \mathcal{B}(x, D_x)\}u(x) = 0$ in the domain G may have nontrivial solutions that exponentially decrease at infinity. In this case, we can assert that the method converges exponentially provided that these nontrivial solutions decrease "not too rapidly". These additional requirements may be caused not by the method for computing scattering matrices itself but by the way of justification of the convergence of this method. Nevertheless, in the papers [6, 8, 9, 10, 11] it is necessary to assume that there are no nontrivial decreasing solutions to the homogeneous problem. Or, allowing such solutions vanishing exponentially at infinity, we should require, as above, that such solutions decrease "not too rapidly." One of applications of the method is the search of the point spectrum of the operator $\{\mathcal{L}(x, D_x) - \mu, \mathcal{B}(x, D_x)\}$ with spectral parameter μ; see [8, 9, 11]. This is based on the above-mentioned existence criterion for the point spectrum.

In Chapter 2 we study the dissipative problems. In particular, we define the (augmented) scattering matrices. Such matrices were already used to study self-adjoint elliptic problems. The matrices were introduced in [1], [2], and [4] for self-adjoint boundary value problems in a cone. In [1], [2], and [4], an existence criterion for nontrivial solutions to the homogeneous problem was obtained in terms of scattering matrices provided that the behavior of the solutions in a neighborhood of the vertex of the cone is prescribed. These results can be easily adapted for the self-adjoint problems in domains with cylindrical ends [5], [15]. The above-mentioned criterion becomes the existence criterion for the point spectrum of the corresponding problems (trapped modes in waveguides, surface waves in diffraction gratings). In [15] and [16], this criterion was used to prove the existence of surface waves for some diffraction gratings described by the Helmholtz equation (cf. also [5]).

For a dissipative problem, the norm of the matrices is less or equal to one. Although the fact could be expected from an analogy to the classical situation, we have to prove this all the same; recall that we deal with the "nonclassical" scattering matrices and the waves involved may grow in the amplitude. In definition and in the study of the scattering matrices, the relative index formula for the Fredholm operator of the boundary value problem in weighted function spaces plays a crucial role (the operator of the problem acts in weighted spaces; as usual by the relative index is meant the difference between the indices of the operator in the two distinct weight spaces). The dissipation allows us to obtain some additional information about asymptotics of the elements in the kernel.

We study the well-posed statements of boundary value problems with "radiation conditions": in the domain of the operator of the problem only functions with special asymptotics are admissible. For example, this approach is reasonable if the statement of problem in the standard weighted space scales does not lead to the unique solvability (see e.g. [1, 2]). Furthermore, the results are used in Chapter 3, where we justify the method for the computation of scattering matrices.

A substantial step in the justification of the method consists in obtaining an approximate solution to the problem in the truncated domain G^R for large R. We find the approximate solution compounded of solutions to "limit problems". In other words, we employ the compound expansion method (see [17]). The roles of the limit problems are played by the initial boundary value problem in the domain G and some problems in the semi-infinite cylinders $\Pi_+^1, \dots, \Pi_+^N$. If all the limit problems are uniquely solvable in certain classes of functions exponentially decaying at infinity, then the applicability of the compound expansion method is provided by Theorem 5.6.3 in [17]. However, generally the problem in G can have a nontrivial kernel and (or) a nontrivial cokernel, and other limit problems can have nontrivial cokernels. Such a situation occurs, for instance, in the theory of diffraction gratings (the Helmholtz equation with large wave number k). Another example is given by the Stokes system (see [18] where the compound expansions method was used and an estimate for the norm of the inverse operator of the problem in G^R was obtained as $R \to +\infty$). The compound expansions method is applicable provided that the nontrivial solutions to the homogeneous problem in G decrease "not too rapidly", and some system of linear algebraic equations is solvable. (That is why we are forced to require that these solutions decrease "not too rapidly".) The solvability of the system requires a special investigation. The matrix of the system is expressed in terms of scattering matrices of the limit problems. On the truncation boundaries of the domain G^R, we set "impedance"-like boundary conditions. Owing to this fact, the limit problems in the semicylinders $\Pi_+^1, \dots, \Pi_+^N$ turn out to be strictly dissipative and the norms of the corresponding scattering matrices are strictly less than 1. The norm of the scattering matrix of the problem in the domain G does not exceed 1. This implies that the determinant of the above-mentioned algebraic system does not vanish. We construct an approximate solution to the problem in

G^R and prove an estimate for the norm of the inverse operator of the problem in G^R as $R \to +\infty$. Note that this part of the proof is new for the Helmholtz equation as well (the corresponding assertion in [8] was given without proof).

The remaining part of the proof is an adaption of the scheme suggested in [8]. We show that the quadratic functional J^R does not degenerate for any $R < \infty$. (Recall that the computation of a row of scattering matrix reduces to the minimization of J^R.) There exists a unique minimizer $a(R)$. The minimal value $J^R(a(R))$ decreases exponentially as $R \to +\infty$. Having obtained this information and taking account of the asymptotic behavior of the waves, we verify that the difference between $a(R)$ and the corresponding row of the scattering matrix is of order $O(\exp(-\varepsilon R))$ with certain positive ε as $R \to +\infty$.

In Appendix we develop an approach that allows us to extend the theory of elliptic problems in domains with cylindrical ends, well-developed earlier for the case of exponentially stabilizing coefficients (see e.g. [1, 2]), to the case of slowly stabilizing coefficients. These results are obtained in collaboration with Pekka Neittaanmäki and Boris A. Plamenevskii and published for the first time. Some results for a class of formally self-adjoint problems with slowly stabilizing coefficients were presented in [14] without proofs. The approach announced in [14] was developed in [25], where the formally self-adjoint problems were studied.

1.2 Main results

Here we cite some results of the thesis. Consider an elliptic boundary value problem

$$\begin{aligned} \mathcal{L}(x, D_x)u(x) &= f(x), \quad x \in G, \\ \mathcal{B}(x, D_x)u(x) &= g(x), \quad x \in \partial G, \end{aligned} \tag{1.1}$$

where $k \times k$-matrix $\mathcal{L} = \|\mathcal{L}_{ij}\|$ and $m \times k$-matrix $\mathcal{B} = \|\mathcal{B}_{qj}\|$ consist of differential operators; moreover, $\operatorname{ord}\mathcal{L}_{ij} = \tau_i + \tau_j$ and $\operatorname{ord}\mathcal{B}_{qj} = \max \tau_i + \sigma_q + \tau_j$, $\tau_1 + \cdots + \tau_k = m$. As usual, τ_j and σ_q are integers appearing in the definition of an elliptic problem in the sense of Petrovskii, $\tau_j \geqslant 0$, and $\sigma_q < 0$; see e.g. [21]. Let the Green formula

$$(\mathcal{L}u, v)_G + (\mathcal{B}u, \mathcal{Q}v)_{\partial G} - (u, \mathcal{L}_* v)_G - (\mathcal{Q}_* u, \mathcal{B}_* v)_{\partial G} = 0$$

be valid for all $u, v \in C_c^\infty(\overline{G})$. Here, $\mathcal{Q}$, $\mathcal{B}_*$, and $\mathcal{Q}_*$ are some $m \times k$-matrices of differential operators, $\mathcal{L}_*$ is the formally adjoint operator of $\mathcal{L}$. We denote by $(\cdot, \cdot)_G$ and $(\cdot, \cdot)_{\partial G}$ the inner product in the spaces $L_2(G)$ and $L_2(\partial G)$ respectively. Let the problem (1.1) be *dissipative* in the following sense:

$$\operatorname{Im}\{(\mathcal{L}u, u)_G + (\mathcal{B}u, \mathcal{Q}u)_{\partial G}\} \geqslant 0 \quad \forall u \in C_c^\infty(\overline{G}).$$

We assume that in every semicylinder $\Pi_+^r = \{(y^r, t^r) : y^r \in \Omega^r, t^r > 0\}$, the coefficients of the operators $\mathcal{L}(y^r, t^r, D_{y^r}, D_{t^r})$ and $\mathcal{L}_*(y^r, t^r, D_{y^r}, D_{t^r})$ stabilize to the coefficients of a limit operator $L^r(y^r, D_{y^r}, D_{t^r})$ at an exponential rate as $t^r \to +\infty$. The operators $\mathcal{B}$ and $\mathcal{B}_*$ stabilize to B^r ($\mathcal{Q}$ and $\mathcal{Q}_*$ stabilize to Q^r) in the same manner as $\mathcal{L}$ stabilizes to L^r. The limit operator $\{L^r, B^r\}$ in the cylinder

$\Pi^r = \Omega^r \times \mathbb{R}$ is formally self-adjoint, $r = 1, \dots, N$. In the cross-section Ω^r of Π^r we define the operator pencil $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}^r(\lambda) = \{L^r(y, D_y, \lambda), B^r(y, D_y, \lambda)\}$. The spectrum of $\mathfrak{A}^r$ consists of normal eigenvalues and is symmetric about the real axis. The total algebraic multiplicity of the eigenvalues of the pencil in the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < \gamma\}$ is always even and will be denoted by $2M^r$. As is known [1]–[5], in the space of solutions to the homogeneous problem $\{L^r, B^r\}u = 0$ in the cylinder Π^r such that $|u(y^r, t^r)| \leqslant C \exp(\gamma |t^r|)$, there exists a basis $u_j^{\pm}$ satisfying

$$q^r(\chi u_j^{\pm}, \chi u_k^{\pm}) = \mp i \delta_{jk}, \quad q^r(\chi u_j^{\pm}, \chi u_k^{\mp}) = 0, \qquad j, k = 1, 2, \dots, M^r;$$

here $\chi \in C^\infty(\mathbb{R})$, $\chi(t^r) = 1$ for $t^r \geqslant 2$ and $\chi(t^r) = 0$ for $t^r \leqslant 1$, and

$$q^r(u, v) = (L^r u, v)_{\Pi^r} + (B^r u, Q^r v)_{\partial \Pi^r} - (u, L^r v)_{\Pi^r} + (Q^r u, B^r v)_{\partial \Pi^r}$$

(see e.g. [5]). We introduce a through numeration in the union of such bases, first indicating the elements of a basis in Π^1 and then in $\Pi^2, \Pi^3, \dots, \Pi^N$, and obtain the set of functions $\chi u_1^{\pm}, \chi u_2^{\pm}, \dots, \chi u_M^{\pm}$ where $M = M^1 + \dots + M^N$. We extend $\chi u_j^{\pm}$, $j = 1, \dots, M$ by zero to the domain G. The elements χu_j^+ (χu_j^-) are called incoming (outgoing) waves.

Theorem 1.1. *Let $\gamma > 0$ and let $2M$ be the total algebraic multiplicity of the eigenvalues of the pencils $\mathfrak{A}^1, \mathfrak{A}^2, \dots, \mathfrak{A}^N$ in the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < \gamma\}$. Denote by $\mathcal{K}(\gamma)$ the linear space of all solutions to the homogeneous (dissipative) problem (1.1) satisfying $u(x) = O(\exp(\gamma|x|))$ as $|x| \to \infty$. In the space $\mathcal{K}(\gamma)$ there exists a basis $Y_1, Y_2, \dots, Y_M$ modulo $\mathcal{K}(-\gamma)$ such that*

$$Y_j(x) = u_j^+ + \sum_{k=1}^{M} S_{jk} u_k^- + O(\exp(-\gamma|x|)), \qquad j = 1, \dots, M, \tag{1.2}$$

as $|x| \to \infty$. The norm of the matrix $S = \|S_{jk}\|_{j,k=1}^{M}$ is less or equal to one.

The matrix S is called the (augmented) scattering matrix (corresponding to γ). Generally speaking, the $M \times M$ scattering matrix $S(\gamma)$ is not a block of the $M' \times M'$ matrix $S(\gamma')$ for $\gamma' > \gamma$ and $M' > M$. If the operator $\{\mathcal{L}, \mathcal{B}\}$ of the problem (1.1) is formally self-adjoint then the scattering matrix S is unitary (see [1]–[5]).

The following theorem describes the statement of dissipative problem with intrinsic radiation conditions. Here, for simplicity, we assume that the right-hand side $\{f, g\}$ of the problem (1.1) consists of smooth functions with compact supports; Theorem 2.16 contains a general assertion.

Theorem 1.2. *Suppose that*
i. *$\gamma > 0$ and the line $\{\lambda \in \mathbb{C} : \operatorname{Im} \lambda = \gamma\}$ is free from the eigenvalues of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$;*
ii. *the set $\{Z_1, \dots Z_d\}$ consists of all linearly independent solutions to the problem*

$$\begin{aligned} \mathcal{L}_*(x, D_x) z(x) &= 0, \quad x \in G, \\ \mathcal{B}_*(x, D_x) z(x) &= 0, \quad x \in \partial G, \end{aligned}$$

satisfying $Z_j(x) = O(e^{-\gamma|x|})$, $x \in G$*;*
iii. *the right-hand side* $\{f, g\}$, *where* $f \in C_c^\infty(G)$ *and* $g \in C_c^\infty(\partial G)$, *satisfies*

$$(f, X_j)_G + (g, \mathcal{Q}X_j)_{\partial G} = 0, \quad j = 1, \dots, d.$$

Then there exists a solution to the problem (1.1) such that

$$u(x) = a_1 u_1^-(x) + a_2 u_2^-(x) + \dots + a_M u_M^-(x) + O(e^{-\gamma|x|}),\ x \in G,$$

where

$$a_j = -i(f, Z_j)_G - i(g, \mathcal{Q}Z_j)_{\partial G}, \quad j = 1, \dots, M.$$

The solution u, $u(x) = O(e^{\gamma|x|})$, *is determined up to a solution* v *to the homogeneous problem (1.1) such that* $v(x) = O(e^{-\gamma|x|})$.

Now we formulate the method for computing scattering matrices. We introduce the sets

$$\Pi_+^{r,R} = \{(y^r, t^r) \in \Pi^r : t^r > R\}, \quad G^R = G \setminus \cup_{r=1}^N \Pi_+^{r,R}$$

for large R. Then $\partial G^R \setminus \partial G = \Gamma^R = \cup_r \Gamma^{r,R}$, where $\Gamma^{r,R} = \{(y^r, t^r) \in \Pi^r : t^r = R\}$. In addition to the above assumptions we suppose that the operator $\{\mathcal{L}, \mathcal{B}\}$ differs from a formally self-adjoint operator only on a compact set in $\overline{G}$ (in other words $\mathcal{L}_*(x, D_x) = \mathcal{L}(x, D_x)$, $\mathcal{B}_*(x, D_x) = \mathcal{B}(x, D_x)$ and $\mathcal{Q}_*(x, D_x) = \mathcal{Q}(x, D_x)$ for $|x| > R$ with some large R). The scheme can be applied (with obvious modifications) to some other classes of dissipative operators (for example, to operators of the form $\mathcal{L} + q$, where $\mathcal{L} = \mathcal{L}_*$ and q is the operator of multiplication by a complex-valued function q that exponentially decreases at infinity). We have

$$\begin{aligned}(\mathcal{L}u, v)_{G^R} &+ (\mathcal{B}u, \mathcal{Q}v)_{\partial G^R \setminus \Gamma^R} + (\mathcal{N}u, \mathcal{D}v)_{\Gamma^R} \\ &= (u, \mathcal{L}_* v)_{G^R} + (\mathcal{Q}_* u, \mathcal{B}_* v)_{\partial G^R \setminus \Gamma^R} + (\mathcal{D}u, \mathcal{N}v)_{\Gamma^R},\end{aligned} \tag{1.3}$$

where $\mathcal{D}$ and $\mathcal{N}$ stand for $m \times k$-matrices of differential operators, $\mathcal{D}$ being a Dirichlet system (cf. [21]). As an example of the Dirichlet system we can take the matrix consisting of m rows of the form $e^{(j)}\partial_\nu^h$, where $j = 1, \dots, k$, $h = 0, \dots, \tau_j - 1$, $e^{(j)} = (\delta_{1,j}, \dots, \delta_{k,j})$, and ν is the outward normal to Γ^R. If in particular $k = 1$, then $\mathcal{D}$ is the usual Dirichlet system of order τ (see [26]–[29]).

Consider the boundary value problem in the truncated domain G^R:

$$\begin{aligned}\mathcal{L}(x, D_x)\mathcal{X}_l^R &= 0, \quad x \in G^R, \\ \mathcal{B}(x, D_x)\mathcal{X}_l^R &= 0, \quad x \in \partial G^R \setminus \Gamma^R, \\ (\mathcal{N} + i\zeta\mathcal{D})\mathcal{X}_l^R &= (\mathcal{N} + i\zeta\mathcal{D})\Big(u_l^+ + \sum_{j=1}^M a_j u_j^-\Big), \quad x \in \Gamma^R,\end{aligned} \tag{1.4}$$

where $\zeta \in \mathbb{R} \setminus \{0\}$, the complex numbers $a_1, \dots, a_M$ are arbitrary, and the operators $\mathcal{D}$ and $\mathcal{N}$ are the same as in formula (1.3).

As an approximation of the row $(S_{\ell 1}, S_{\ell 2}, \dots, S_{\ell M})$ of the scattering matrix $S = S(\gamma)$, we take the minimizer $a^0(R) = (a_1^0(R), \dots, a_M^0(R))$ of the functional

$$J_l^R(a_1, \dots, a_M) = \Big\| \mathcal{D}(\mathcal{X}_l^R - u_l^+ - \sum_{j=1}^{M} a_j u_j^-)(\cdot, R); L_2(\Gamma^R) \Big\|^2, \tag{1.5}$$

where $X_l^R(\cdot, R)$ is the trace on Γ^R of the solution to the problem (3.6) depending on $(a_1, a_2, \dots, a_M)$.

Let us explain the origin of the problem (1.4). It is obvious that the solution Y_l to the homogeneous problem (1.1) satisfies the first two equations in (1.4). Since the asymptotic equality (1.2) can be differentiated, we have

$$(\mathcal{N} + i\zeta\mathcal{D})Y_\ell = (\mathcal{N} + i\zeta\mathcal{D})\Big(u_\ell^+ + \sum_{j=1}^{M} a_j u_j^-\Big) + O(e^{-\gamma R})$$

for $a_j = S_{\ell j}$, i.e. the function Y_ℓ leaves an exponentially small discrepancy in the last equation of (1.4). Therefore, one can expect that $a_j^0 \to S_{lj}$ with exponential rate as $R \to \infty$ and $j = 1, \dots, M$.

The following theorem justifies the above method:

Theorem 1.3. *Let ζ be a fixed positive number and let the homogeneous problem (1.1) have no nontrivial solutions that exponentially decrease at infinity. Then for any $R > R_0$ there exists a unique vector*

$$a^0(R) = (a_1^0(R), \dots, a_M^0(R))$$

minimizing the functional J_l^R in (1.5). The inequalities

$$|a_j^0(R) - S_{lj}| \leqslant C(\varepsilon) \exp(-\varepsilon R)$$

hold for $j = 1, \dots, M$ with a positive ε and a constant $C(\varepsilon)$ independent of R.

A more general assertion (that includes the case of existence of exponentially decreasing solutions to the homogeneous problem (1.1)) is given in Theorem 3.21.

1.3 Numerical detection of trapped modes

As was mentioned in Introduction, one of the possible applications of (augmented) scattering matrices concerns the use of the existence criterion for exponentially decreasing solutions. Let us discuss this in detail. Let $0 < \gamma < \gamma'$, and let the waves $u_{M+1}^\pm, \dots, u_{M'}^\pm$ be chosen in such a way that $u_j^+ + u_j^- = O(\exp -\gamma|x|)$, $j = M+1, \dots, M'$ (such a choice is possible; cf., for example, [5]). To construct $S(\gamma')$ we complete the set $\{u_1^\pm, \dots, u_M^\pm\}$ with the waves $u_{M+1}^\pm, \dots, u_{M'}^\pm$ and the set $\{Y_1, \dots, Y_M\}$ with the solutions $Y_{M+1}, \dots, Y_{M'}$. In the general case, $S(\gamma)$ does not coincide with the block of the matrix $S(\gamma')$. Denote by $N(\gamma)$ ($N(\gamma')$ respectively) the dimension of the space of solutions u to the homogeneous problem

(1.1) satisfying the estimate $u(x) = O(\exp(-\gamma|x|))$ ($u(x) = O(\exp(-\gamma'|x|))$ respectively) as $|x| \to \infty$. We write the scattering $M' \times M'$-matrix $S(\gamma')$ in the form $S(\gamma') = \|S_{(ij)}\|_{i,j=1,2}$, where the block $S_{(22)}$ is of size $(M'-M) \times (M'-M)$. Then we have

$$N(\gamma) - N(\gamma') = \dim\ker(S_{(22)} - 1), \tag{1.6}$$

where the right-hand side is equal to the dimension of the eigenspace of $S_{(22)}$ corresponding to the eigenvalue 1.

We now explain the equality (1.6). Suppose that there exist M and M' such that $N(\gamma) - N(\gamma') > 0$. Then there exists a solution u to the homogeneous problem (1.1) such that $u(x) = O(\exp(-\gamma|x|))$ as $|x| \to \infty$, and the estimate $u(x) = O(\exp(-\gamma'|x|))$ is not valid. Thus, formula (1.6) yields not only the existence of a nontrivial solution u but also some information about the behavior of this solution at infinity.

Here, as an example, we implement our method to detect a Neumann trapped mode for the Helmholtz equation in a plane domain with two cylindrical ends.

Let G consist of the points $(x, y) \in \mathbb{R}^2$ satisfying the inequalities

$$\begin{aligned} |y| &< d_1/2 && \text{for} \quad x \in (-\infty, -0.5), \\ |y| &< (d_2 - d_1)x/2 + (d_2 + d_1)/4 && \text{for} \quad x \in [-0.5, 0.5], \\ |y| &< d_2/2 && \text{for} \quad x \in (0.5, +\infty), \end{aligned}$$

where d_1 and d_2 are the widths of the left and right outlets, and $d_2 \geqslant d_1$. In the domain G we also put a circle hole B_r of radius $r < d_1/2$, centered at the point $(0, 0)$. Let us consider the problem

$$(\Delta + k^2)u(x, y) = 0, \quad (x, y) \in G \setminus B_r; \quad \frac{\partial u}{\partial \nu}(x, y) = 0, \quad (x, y) \in \partial G \cup \partial B_r, \tag{1.7}$$

where ν is the outward normal to $\partial G \cup \partial B_r$.

We look for a wavenumber k for which there exists an exponentially decaying solution to the problem (1.7), see Fig. 1 and Fig. 2. In particular, in the case $d_1 = d_2$, the results obtained are in good agreement with those in [19]. To solve the auxiliary problems in the truncated domain G^R we employ finite element technique.

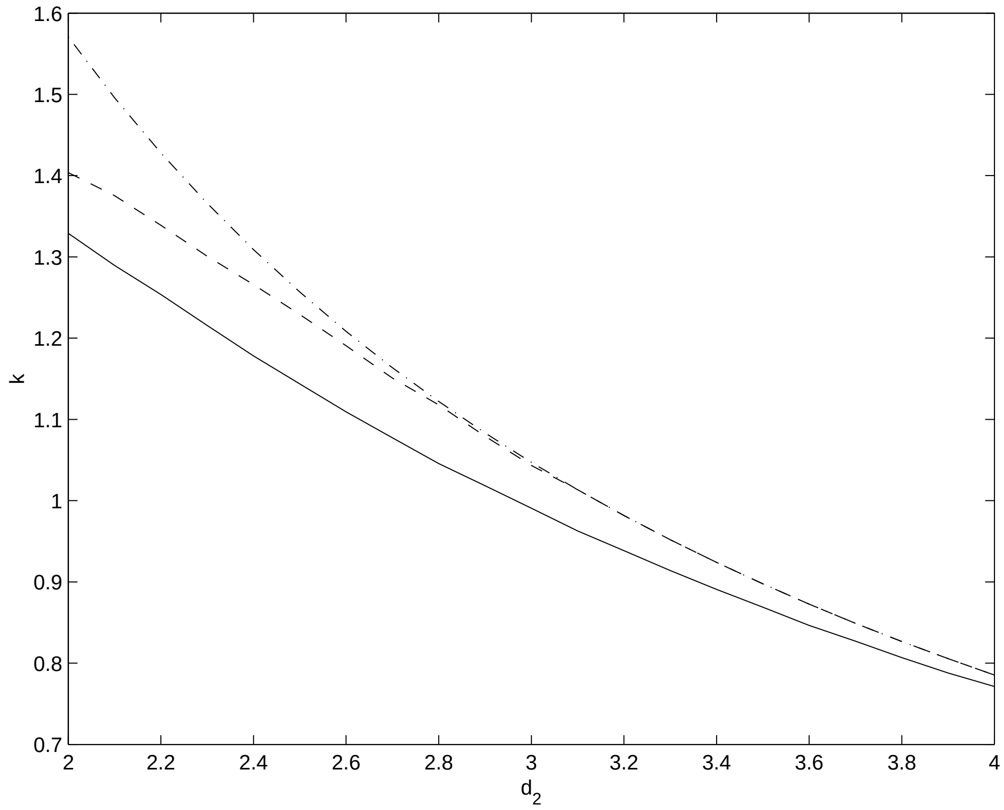

FIGURE 1: Trapped modes for $r = 0.8$ (solid line) and $r = 0.5$ (dashed line) while $d_1 = 2$, wavenumber k and width d_2 are variable. The dash-dot line corresponds to the threshold π/d_2.

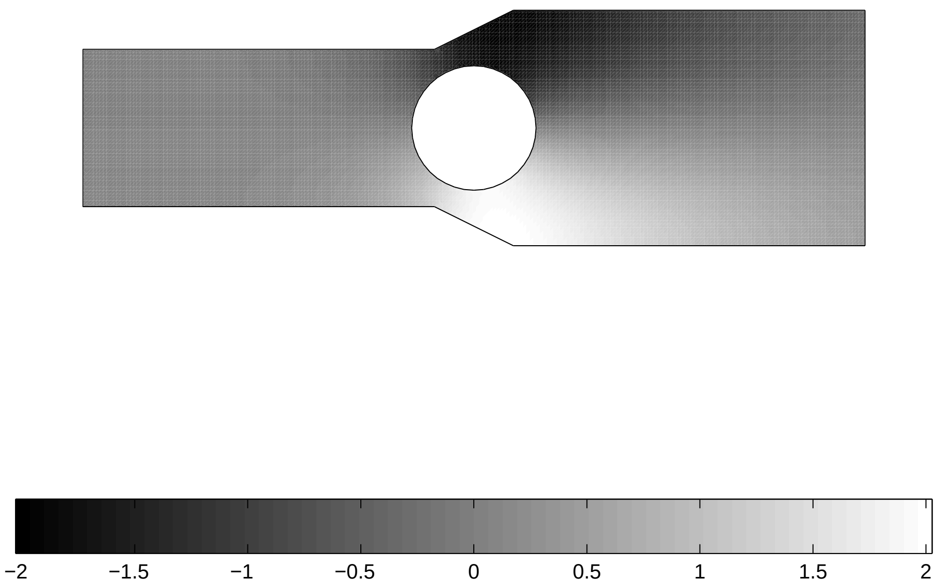

FIGURE 2: Trapped mode for $d_1 = 2$, $d_2 = 3$, $r = 0.8$, and the wavenumber $k = 0.988$.

2 Dissipative and accretive problems, scattering matrices, and radiation conditions

This chapter contains a scattering theory of dissipative elliptic systems. The operator $\{\mathcal{L}, \mathcal{B}\}$ of the problem is supposed to be dissipative (or accretive) in the sense of Definition 2.1 below. The operator acts in weighted spaces of functions exponentially growing at infinity. Thanks to the dissipativity one can obtain some additional information about asymptotics of the elements in the kernel (Lemma 2.10). This allows to prove a relation for the dimensions of the kernel and cokernel of the operator of the problem (Proposition 2.9). Then we prove the existence of a special basis in the kernel of the operator of the problem and thereby we introduce (augmented) scattering matrices (Proposition 2.11 and Definition 2.12). Proposition 2.13 asserts that the matrices are contractions. In terms of matrices we compute the number of linearly independent solutions to the homogeneous problem with a given behavior at infinity (Proposition 2.14).

In Sec. 3, we discuss the statement of a problem with radiation conditions: the domain of the operator contains only functions with special asymptotics (Theorem 2.16). The natural radiation conditions (the solution mainly consists of outgoing waves for dissipative waves and of incoming waves for accretive ones) can be utilized in every case. To verify whether given radiation conditions can be used, it is required to know the scattering matrix (cf. Proposition 2.17). As is shown in Sec. 4, the dissipative problem with intrinsic radiation conditions is associated with the maximal extension of a dissipative operator. The chapter is based on the paper [7].

2.1 Statement of the Problem. Preliminaries

2.1.1 Domain and boundary value problem.

We denote by Π^r, $r = 1, \dots, N$, the cylinder in $\mathbb{R}^{n+1}$ with cross-section Ω^r. In Π^r, we introduce the Cartesian coordinates (y^r, t^r) so that $\Pi^r = \{(y^r, t^r) : y^r \in$

$\Omega^r, t^r \in \mathbb{R}\}$. We suppose that the cross-section Ω^r is a domain in $\mathbb{R}^n$ with compact closure $\overline{\Omega}^r$ and smooth boundary $\partial\Omega^r$. Assume that the domain G in $\mathbb{R}^{n+1}$ has smooth boundary ∂G and, outside a ball of sufficiently large radius, coincides with the union $\Pi^1_+ \cup \ldots \cup \Pi^N_+$ of disjoint semicylinders $\Pi^r_+ = \{(y^r, t^r) \in \Pi^r, t^r > 0\}$.

In the domain G, we consider the elliptic boundary value problem

$$\begin{aligned} \mathcal{L}(x, D_x)u(x) &= f(x), \quad x \in G, \\ \mathcal{B}(x, D_x)u(x) &= g(x), \quad x \in \partial G, \end{aligned} \tag{2.1}$$

where $k \times k$-matrix $\mathcal{L} = \|\mathcal{L}_{ij}\|$ and $m \times k$-matrix $\mathcal{B} = \|\mathcal{B}_{qj}\|$ consist of differential operators; moreover, $\operatorname{ord} \mathcal{L}_{ij} = \tau_i + \tau_j$, $\operatorname{ord} \mathcal{B}_{qj} = \max \tau_i + \sigma_q + \tau_j$, and $\tau_1 + \cdots + \tau_k = m$, where $\tau_j \geqslant 0$ and $\sigma_q < 0$.

We assume that for $u, v \in C^\infty_c(\overline{G})$ the Green formula holds:

$$(\mathcal{L}u, v)_G + (\mathcal{B}u, \mathcal{Q}v)_{\partial G} - (u, \mathcal{L}_* v)_G - (\mathcal{Q}_* u, \mathcal{B}_* v)_{\partial G} = 0. \tag{2.2}$$

Here, $\mathcal{Q}$, $\mathcal{B}_*$, and $\mathcal{Q}_*$ are some $m \times k$-matrices of differential operators, $\mathcal{L}_*$ is the formally adjoint operator of $\mathcal{L}$. We denote by $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_{\partial G}$ the inner product in the spaces $L_2(G)$ and $L_2(\partial G)$ respectively. We assume that the coefficients of all the operators are smooth and the operator $\{\mathcal{L}_*, \mathcal{B}_*\}$ is elliptic.

We equip an operator with superscript r if it is written in the coordinates (y^r, t^r) inside the semicylinder Π^r_+. Assume that there exist matrices L^r, B^r, and Q^r, $r = 1, \ldots, N$, of differential operators whose coefficients are smooth in $\overline{\Omega}^r$ and are constant with respect to t^r; moreover, the coefficients of the differences $\mathcal{L}^r - L^r$, $\mathcal{B}^r - B^r$, $\mathcal{Q}^r - Q^r$, and $\mathcal{L}^r_* - L^r$, $\mathcal{B}^r_* - B^r$, $\mathcal{Q}^r_* - Q^r$, together with all their derivatives, decrease like $O(\exp(-\delta t^r))$, $\delta > 0$, as $t^r \to +\infty$.

Among all operators $\{\mathcal{L}, \mathcal{B}\}$ of the problem (2.1), we extract classes of dissipative and accretive operators in accordance with the following definition.

Definition 2.1. *The operator $\{\mathcal{L}, \mathcal{B}\}$ is said to be dissipative if for all $u \in C^\infty_c(\overline{G})$*

$$\operatorname{Im}\{(\mathcal{L}u, u)_G + (\mathcal{B}u, \mathcal{Q}u)_{\partial G}\} \geqslant 0 \tag{2.3}$$

and accretive if the sign $\leqslant$ is realized in (2.3) instead of $\geqslant$.

Remark 2.2. *The self-adjoint Green formula*

$$(\mathcal{L}v, w)_G + (\mathcal{B}v, \mathcal{Q}w)_{\partial G} - (v, \mathcal{L}w)_G - (\mathcal{Q}v, \mathcal{B}w)_{\partial G} = 0, \quad v, w \in C^\infty_c(\overline{G}), \tag{2.4}$$

holds if and only if

$$\operatorname{Im}\{(\mathcal{L}u, u)_G + (\mathcal{B}u, \mathcal{Q}u)_{\partial G}\} = 0 \tag{2.5}$$

for all $u \in C^\infty_c(\overline{G})$. Indeed, from (2.5) we get

$$(\mathcal{L}u, u)_G + (\mathcal{B}u, \mathcal{Q}u)_{\partial G} - (u, \mathcal{L}u)_G - (\mathcal{Q}u, \mathcal{B}u)_{\partial G} = 0, \quad u \in C^\infty_c(\overline{G}). \tag{2.6}$$

Substituting $v+iw$, $v-iw$, $v+w$, and $v-w$ for u in (2.6) and using the polarization identity

$$\begin{aligned} (\mathcal{L}v, w)_G = 1/4\{&(\mathcal{L}(v+w), (v+w))_G - (\mathcal{L}(v-w), (v-w))_G \\ &+ i(\mathcal{L}(v+iw), (v+iw))_G - i(\mathcal{L}(v-iw), (v-iw))_G\} \end{aligned}$$

(and the similar expressions for $(\mathcal{B}v, \mathcal{Q}w)_{\partial G}$, $(v, \mathcal{L}w)_G$, and $(\mathcal{Q}v, \mathcal{B}w)_{\partial G}$), we arrive at (2.4). The converse assertion follows from (2.4) immediately.

2.1.2 Function spaces

Denote by e_β, $\beta \in \mathbb{R}$, a smooth positive function in $\overline{G}$ that is equal to $\exp \beta t^r$ in the semicylinder Π^r_+. Introduce the space $W^\ell_\beta(G)$ as the completion of the set $C^\infty_c(\overline{G})$ in the norm

$$\|u; W^\ell_\beta(G)\| \equiv \|e_\beta u; H^\ell(G)\|,$$

where $H^\ell(G)$ is a Sobolev space. Let $W^{\ell-1/2}_\beta(\partial G)$ be the trace space on ∂G of functions in $W^\ell_\beta(G)$. We set

$$\begin{aligned} \mathcal{D}^\ell_\beta W(G) &= \prod_{j=1}^{k} W^{\ell+\tau_j}_\beta(G), \\ \mathcal{R}^\ell_\beta W(G) &= \prod_{i=1}^{k} W^{\ell-\tau_i}_\beta(G) \times \prod_{q=1}^{m} W^{\ell-\tau-\sigma_q-1/2}_\beta(\partial G), \end{aligned} \tag{2.7}$$

where $\tau = \max\{\tau_j\}$ and $\ell \geqslant \tau$. By the assumptions in subsection 2.1, we have

$$\operatorname{ord} \mathcal{L}_{ij} = \operatorname{ord} L^r_{ij} = \operatorname{ord} \mathcal{L}_{*ij},$$

$$\operatorname{ord} \mathcal{B}_{ij} = \operatorname{ord} B^r_{ij} = \operatorname{ord} \mathcal{B}_{*ij}.$$

The following mappings are continuous:

$$\mathcal{A}(\beta) = \{\mathcal{L}, \mathcal{B}\} : \mathcal{D}^\ell_\beta W(G) \to \mathcal{R}^\ell_\beta W(G), \tag{2.8}$$

$$\mathcal{A}_*(\beta) = \{\mathcal{L}_*, \mathcal{B}_*\} : \mathcal{D}^\ell_\beta W(G) \to \mathcal{R}^\ell_\beta W(G). \tag{2.9}$$

2.1.3 Operator pencils

From the equality (2.2) we obtain the Green formula in the cylinder Π^r:

$$(L^r u, v)_{\Pi^r} + (B^r u, Q^r v)_{\partial\Pi^r} = (u, L^r v)_{\Pi^r} + (Q^r u, B^r v)_{\partial\Pi^r}, \tag{2.10}$$

where $u, v \in C^\infty_c(\overline{\Pi}^r)$, $r = 1, \dots, N$. We consider the formally self-adjoint operator

$$\mathfrak{A}^r(D_t) = \{L^r(y, D_y, D_t), B^r(y, D_y, D_t)\},$$

where $(y, t) \equiv (y^r, t^r) \in \Pi^r$. Introduce the operator pencil

$$\mathbb{C} \ni \lambda \mapsto \mathfrak{A}^r(\lambda) = \{L^r(y, D_y, \lambda), B^r(y, D_y, \lambda)\} \tag{2.11}$$

in the domain Ω^r. The spectrum of the pencil $\mathfrak{A}^r$ consists of normal eigenvalues and is symmetric with respect to the real axis. Each strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < h < \infty\}$ contains at most finitely many eigenvalues.

Proposition 2.3 (see [1], [2]). *The operator* (2.8) *of the problem* (2.1) *is a Fredholm operator if and only if the line* $\mathbb{R} + i\beta = \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda = \beta\}$ *is free from the spectra of the pencils* $\mathfrak{A}^r$, $r = 1, \dots, N$.

We denote by $\lambda_{-\nu^0}, \dots, \lambda_{\nu_0}$, $\nu_0 \geqslant 0$, all real eigenvalues of the pencil $\mathfrak{A}^r$ (if the number of real eigenvalues is even, then λ_0 is absent). We enumerate nonreal eigenvalues so that

$$0 < \operatorname{Im} \lambda_{\nu^0+1} \leqslant \operatorname{Im} \lambda_{\nu^0+2} \leqslant \dots$$

and $\lambda_\nu = \overline{\lambda}_{-\nu}$, where $\nu = \nu^0 + 1, \nu^0 + 2, \dots$. Suppose that

$$\{\varphi_\nu^{(0,j)}, \dots, \varphi_\nu^{(\varkappa_{j\nu}-1,j)}; j = 1, \dots, J_\nu := \dim \ker \mathfrak{A}^r(\lambda_\nu)\}$$

is a canonical system of Jordan chains of the pencil $\mathfrak{A}^r$ corresponding to the eigenvalue λ_ν. Here, $\varphi_\nu^{(0,j)}$ is an eigenvector and $\varphi_\nu^{(1,j)}, \dots, \varphi_\nu^{(\varkappa_{j\nu}-1,j)}$ are associated vectors [1],[2], [20]. The functions

$$u_\nu^{(\sigma,j)}(y,t) = \exp(i\lambda_\nu t) \sum_{\ell=0}^{\sigma} \frac{1}{\ell!} (it)^\ell \varphi_\nu^{(\sigma-\ell,j)}(y), \tag{2.12}$$

where $\sigma = 0, \dots, \varkappa_{j\nu} - 1$, satisfy the homogeneous problem

$$\begin{aligned} L^r(y, D_y, D_t)v(y,t) &= 0, \quad (y,t) \in \Pi^r, \\ B^r(y, D_y, D_t)v(y,t) &= 0, \quad (y,t) \in \partial\Pi^r. \end{aligned} \tag{2.13}$$

We introduce the form

$$q^r(u,v) := (L^r u, v)_{\Pi^r} + (B^r u, Q^r v)_{\partial\Pi^r} - (u, L^r v)_{\Pi^r} - (Q^r u, B^r v)_{\partial\Pi^r}. \tag{2.14}$$

If $u, v \in C_c^\infty(\overline{\Pi}^r)$, then $q^r(u,v) = 0$ in view of (2.10). Let $\chi \in C^\infty(\mathbb{R})$, $\chi(t) = 1$ for $t \geqslant 2$ and $\chi(t) = 0$ for $t \leqslant 1$. The form q^r is extended to the pairs $\{\chi u_\nu^{(\sigma,j)}, \chi u_\mu^{(\tau,p)}\}$, and the following assertion holds.

Proposition 2.4 (see [1],[2]). *Jordan chains $\{\varphi_\nu^{(\sigma,j)}\}$ can be chosen in such a way that for any cut-off function χ ($\chi \in C^\infty(\mathbb{R})$, $\chi(t) = 1$ for $t \geqslant 2$ and $\chi(t) = 0$ for $t \leqslant 1$) the following conditions are satisfied.*

1. *If $|\nu|, |\mu| > \nu_0$, then $q^r(\chi u_\nu^{(\sigma,j)}, \chi u_\mu^{(\tau,p)}) = i\delta_{-\nu,\mu}\delta_{j,p}\delta_{\varkappa_{j\nu}-1-\sigma,\tau}$.*
2. *If $|\nu|, |\mu| \leqslant \nu_0$, then $q^r(\chi u_\nu^{(\sigma,j)}, \chi u_\mu^{(\tau,p)}) = \pm i\delta_{\nu,\mu}\delta_{j,p}\delta_{\varkappa_{j\nu}-1-\sigma,\tau}$, where the choice of the sign depends on ν and j (but it cannot be chosen arbitrarily).*
3. *If $|\nu| \leqslant \nu_0$ and $|\mu| > \nu_0$, then $q^r(\chi u_\nu^{(\sigma,j)}, \chi u_\mu^{(\tau,p)}) = 0$ for arbitrary choice of Jordan chains and for any superscripts.*

2.1.4 The space of waves

We extend by zero the functions $\chi u_\nu^{(\sigma,j)}$ defined (cf. Sec. 1.3) in the semicylinder Π_+^r to the domain G. Consider the form

$$q(u,v) := (\mathcal{L}u, v)_G + (\mathcal{B}u, \mathcal{Q}v)_{\partial G} - (u, \mathcal{L}_* v)_G - (\mathcal{Q}_* u, \mathcal{B}_* v)_{\partial G}. \tag{2.15}$$

As was shown in [1],[2], the equality $q(\chi u_\nu^{(\sigma,j)}, \chi u_\mu^{(\tau,p)}) = q^r(\chi u_\nu^{(\sigma,j)}, \chi u_\mu^{(\tau,p)})$ is valid if the corresponding eigenvalues λ_ν and λ_μ of the pencil $\mathfrak{A}^r$ belong to the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < \gamma\}$, where $\gamma \leqslant \delta/2$ (the parameter δ is responsible for the stabilization rate of the coefficients of the operator; cf. Sec. 1.1). As is known [1],[2],[4], the total algebraic multiplicity of all the eigenvalues of the pencil that belong to the strip is even. We denote it by $2M^r(\equiv 2M^r_\gamma)$. Let $\mathcal{W}^r_\gamma(G)$ be the linear span of functions representable in the form $\chi u_\nu^{(\sigma,j)} + w$, where $w \in \mathcal{D}^\ell_\gamma W(G)$ and $|\operatorname{Im}\lambda_\nu| < \gamma \leqslant \delta/2$. We define the space of waves $\mathcal{W}_\gamma(G)$ with the indefinite metric $\rho(u,v) := iq(u,v)$ as the lineal spanned by the spaces $\mathcal{W}^1_\gamma(G), \dots, \mathcal{W}^N_\gamma(G)$. It is obvious that $\rho(u,v) = \overline{\rho(v,u)}$ and $\rho(u,u) \in \mathbb{R}$.

The quantity $\rho(u,u)$ plays the role of the total energy flow transferred by the wave u through the infinitely far cross-sections of cylindrical ends. Positive vectors (i.e., such that $\rho(u,u) > 0$) are called *incoming* waves, whereas negative vectors ($\rho(u,v) < 0$) are said to be *outgoing* waves. The space $\mathcal{W}_\gamma(G)$ is degenerate in the sense that $\rho(u,v) = 0$ for any $u \in \mathcal{D}^\ell_\gamma W(G)$ and $v \in \mathcal{W}_\gamma(G)$. (Indeed, the Green formula (2.2) can be extended by continuity to functions $u \in \mathcal{D}^\ell_\gamma W(G)$ and $v \in \mathcal{D}^\ell_{-\gamma} W(G)$; it is clear that $\mathcal{W}_\gamma(G) \subset \mathcal{D}^\ell_{-\gamma} W(G)$.) For the quotient space $\mathcal{W}_\gamma(G)/\mathcal{D}^\ell_\gamma W(G)$ of dimension $2M$, $M = M^1 + \ldots + M^N$, one can choose the basis (cf. [1]–[5])

$$u_1^+, \dots, u_M^+, u_1^-, \dots, u_M^- \tag{2.16}$$

such that

$$\rho(u_h^\pm, u_j^\pm) = \pm\delta_{h,j}, \quad \rho(u_h^\pm, u_j^\mp) = 0, \quad h, j = 1, \dots, M. \tag{2.17}$$

The space $\mathcal{W}_\gamma(G)$ is represented as the ρ-orthogonal direct sum

$$\mathcal{W}_\gamma(G) = S^+[\dot{+}]S^-[\dot{+}]\mathcal{D}^\ell_\gamma W(G), \tag{2.18}$$

where S^+ and S^- are the linear spans of the functions $u_1^+, \dots, u_M^+$ and $u_1^-, \dots, u_M^-$ respectively. The decomposition (2.18) and the choice of the bases (2.16), (2.17) are not unique.

To conclude the section, we formulate the following assertion.

Proposition 2.5 (see [1],[2]). *Let $0 < \gamma < \delta/2$, and let the line $\mathbb{R} + i\gamma = \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda = \gamma\}$ be free from the spectra of the pencils $\mathfrak{A}^r$, $r = 1, \dots, N$. For any solution $u \in \mathcal{D}^\ell_{-\gamma} W(G)$ to the homogeneous problem* (2.1) *the following inclusion holds*:

$$u - \sum_{j=1}^{M} (a_j u_j^+ + b_j u_j^-) \in \mathcal{D}^\ell_\gamma W(G) \tag{2.19}$$

with some complex coefficients a_j and b_j, $j = 1, \dots, M$. (In other words, any element of $\ker \mathcal{A}(-\gamma)$ *of the operator* (2.8) *belongs to the space of waves* $\mathcal{W}_\gamma(G)$.)

2.2 Dissipative and Accretive Problems

2.2.1 Properties of the operator $\{\mathcal{L}, \mathcal{B}\}$

Let us also use another definition of a dissipative operator.

Definition 2.6. *An operator* $\{\mathcal{L},\mathcal{B}\}$ *is said to be dissipative if for all* $u\in C_c^\infty(\overline{G})$

$$\left(\frac{\mathcal{L}-\mathcal{L}_*}{2i}u,u\right)_G+\left(\frac{\mathcal{B}-\mathcal{B}_*}{2i}u,\frac{\mathcal{Q}+\mathcal{Q}_*}{2}u\right)_{\partial G}-\left(\frac{\mathcal{B}+\mathcal{B}_*}{2}u,\frac{\mathcal{Q}-\mathcal{Q}_*}{2i}u\right)_{\partial G}\geqslant 0 \tag{2.20}$$

and accretive if the the left-hand side is nonpositive.

Proposition 2.7. 1. *The definitions* 2.1 *and* 3.1 *are equivalent.*

2. *An operator* $\{\mathcal{L},\mathcal{B}\}$ *is dissipative* (*accretive*) *if and only if the adjoint operator with respect to the Green formula* (2.2) $\{\mathcal{L}_*,\mathcal{B}_*\}$ *is accretive* (*dissipative*).

3. *On elements* u *of the space of waves* $\mathcal{W}_\gamma(G)$, *we have*

$$\begin{aligned}\operatorname{Im}\{(\mathcal{L}u,u)_G+(\mathcal{B}u,\mathcal{Q}u)_{\partial G}\}&=-\frac{1}{2}\rho(u,u)+\left(\frac{\mathcal{L}-\mathcal{L}_*}{2i}u,u\right)_G\\&+\left(\frac{\mathcal{B}-\mathcal{B}_*}{2i}u,\frac{\mathcal{Q}+\mathcal{Q}_*}{2}u\right)_{\partial G}-\left(\frac{\mathcal{B}+\mathcal{B}_*}{2}u,\frac{\mathcal{Q}-\mathcal{Q}_*}{2i}u\right)_{\partial G}.\end{aligned} \tag{2.21}$$

4. *The inequality* (2.20) *is extended by continuity from the set* $C_c^\infty(\overline{G})$ *to the space* $\mathcal{D}^\ell_{-\gamma}W(G)\,(\supset\mathcal{W}_\gamma(G))$.

Proof. Assertions 1–3 immediately follow from definitions and the Green formula (2.2). We prove assertion 4.

The coefficients of the operators $\mathcal{L}-\mathcal{L}_*$, $\mathcal{B}-\mathcal{B}_*$, and $\mathcal{Q}-\mathcal{Q}_*$ decrease, together with all their derivatives, like $O(\exp(-\delta|x|))$ as $|x|\to\infty$. Consequently, the following mappings are continuous:

$$\begin{aligned}&\{\mathcal{L}-\mathcal{L}_*,\mathcal{B}-\mathcal{B}_*\}:\mathcal{D}^\ell_{-\gamma}W(G)\to\mathcal{R}^\ell_{\delta-\gamma}W(G),\\&\{\mathcal{Q}-\mathcal{Q}_*\}:\mathcal{D}^\ell_{-\gamma}W(G)\to\prod_{q=1}^m W^{\ell+\tau+\sigma_q+1/2}_{\delta-\gamma}(\partial G),\end{aligned} \tag{2.22}$$

where

$$\operatorname{ord}\mathcal{Q}_{qj}=\operatorname{ord}Q_{qj}=-\operatorname{ord}\mathcal{B}_{qi}+\tau_i+\tau_j-1=\tau_j-\sigma_q-\tau-1$$

in view of the assumptions in Sec. 1.1. We recall that $0<\gamma<\delta/2$. Let $u\in C_c^\infty(\overline{G})$. Each inner product on the left-hand side of the inequality (2.20) is estimated in terms of $\|u;\mathcal{D}^\ell_{-\gamma}W(G)\|$ (it suffices to use Cauchy–Bunyakowski–Schwarz inequality and take into account the continuity of the mappings (2.22)). To complete the proof, we complete the set $C_c^\infty(\overline{G})$ with respect to the norm of the space $\mathcal{D}^\ell_{-\gamma}W(G)$. □

Corollary 2.8. 1. *Let the operator* $\{\mathcal{L},\mathcal{B}\}$ *of the problem* (2.1) *be dissipative. Then* $\{\mathcal{L},\mathcal{B}\}$ *is dissipative on "nonincoming" waves in the sense that the inequality* (2.3) *holds on the set* $\{u\in\mathcal{W}_\gamma(G):\rho(u,u)\leqslant 0\}$.

2. *Let the operator* $\{\mathcal{L},\mathcal{B}\}$ *be accretive. Then* $\{\mathcal{L},\mathcal{B}\}$ *is accretive on "non-outgoing" waves and the inequality (2.3) holds on* $\{u\in\mathcal{W}_\gamma(G):\rho(u,u)\geqslant 0\}$ *with the sign* $\geqslant$ *replaced by* $\leqslant$.

2.2.2 The kernel of the operator $\mathcal{A}(-\gamma)$. Scattering matrices

We first establish the relation (2.23) for the dimensions of the kernel and cokernel of the operator (2.8). Then we prove the existence of special bases modulo $\mathcal{D}^\ell_\gamma W(G)$ in the space $\ker \mathcal{A}(-\gamma)$ and thereby we introduce the scattering matrices $\mathfrak{S}$ and $\mathfrak{T}$ (cf. Proposition 2.11). In Proposition 2.13, we show that the matrices $\mathfrak{S}$ and $\mathfrak{T}$ are contractions. One can express in terms of the matrices $\mathfrak{S}$ and $\mathfrak{T}$ the dependence of the dimension of $\ker \mathcal{A}(\gamma)$ on the parameter γ (cf. Proposition 2.28). Finally, we clarify how scattering matrices of adjoint operators with respect to the Green formula (2.2) are connected (cf. Proposition 2.29). The arguments of this section are close to those in [6].

Proposition 2.9. *Let $0 < \gamma < \delta/2$, and let the line $\mathbb{R} + i\gamma$ be free from the spectra of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$ in* (2.11). *We assume that the operator $\{\mathcal{L}, \mathcal{B}\}$ of the problem* (2.1) *is dissipative or accretive. Then*

$$\dim \ker \mathcal{A}(-\gamma) - \dim \ker \mathcal{A}(\gamma) = \dim \operatorname{coker} \mathcal{A}(\gamma) - \dim \operatorname{coker} \mathcal{A}(-\gamma) = M, \quad (2.23)$$

where $2M$ is the total algebraic multiplicity of all eigenvalues of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$ in the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < \gamma\}$.

We start by proving the following assertion.

Lemma 2.10. *Let the assumptions of Proposition* 2.9 *be valid, and let, for definiteness, the operator $\{\mathcal{L}, \mathcal{B}\}$ be dissipative. The following assertions hold.*

1. *For any $u \in \ker \mathcal{A}(-\gamma)$ the inequality $\rho(u, u) \geqslant 0$ holds.*
2. *For any $u \in \ker \mathcal{A}_*(-\gamma)$ the inequality $\rho(u, u) \leqslant 0$ holds.*
3. *If*

$$u \in \ker \mathcal{A}(-\gamma), \quad u - \sum_{j=1}^{M} a_j u_j^- \in \mathcal{D}^\ell_\gamma W(G) \quad (2.24)$$

with some complex coefficients a_j, then $a_1 = a_2 = \ldots = a_M = 0$, $u \in \ker \mathcal{A}(\gamma)$.

4. *If*

$$u \in \ker \mathcal{A}_*(-\gamma), \quad u - \sum_{j=1}^{M} b_j u_j^+ \in \mathcal{D}^\ell_\gamma W(G) \quad (2.25)$$

with some $b_j \in \mathbb{C}$, then $b_1 = b_2 = \ldots = b_M = 0$, $u \in \ker \mathcal{A}_*(\gamma)$.

Proof. 1. We note that for $u \in \ker \mathcal{A}(-\gamma)\,(\subset \mathcal{W}_\gamma(G))$ the (indefinite) metric $\rho(u, u)$ is well defined. By Proposition 2.7, the equality (2.21) holds. Assume the contrary: $u \in \ker \mathcal{A}(-\gamma)$, $\rho(u, u) < 0$. Then the left-hand side of the equality (2.21) vanishes, whereas the right-hand side is strictly positive. Indeed, it suffices to note that the estimate (2.20) holds (cf. assertions 1 and 4 of Proposition 2.7). We arrive at a contradiction, which proves assertion 1. The proof of assertion 2 is similar.

3. Let (2.24) hold. We compute $\rho(u,u)$. We use the equality (2.17) and the fact that the lineal $\mathcal{D}^\ell_\gamma W(G)$ is ρ-orthogonal to the space $\mathcal{W}_\gamma(G)$. We have

$$\rho(u,u) = \rho\biggl(\sum_{k=1}^{M} a_k u_k^-, \sum_{j=1}^{M} a_j u_j^-\biggr) = -\sum_{j=1}^{M} |a_j|^2. \tag{2.26}$$

By assertion 1, we have the inequality $\rho(u,u) \geqslant 0$. From (2.26) it follows that $a_1 = a_2 = \ldots = a_M = 0$. Hence

$$u \in \mathcal{D}^\ell_\gamma W(G), \quad u \in \ker \mathcal{A}(\gamma).$$

Assertion 3 is proved. Assertion 4 is proved in a similar way. □

Proof of Proposition 2.9. We consider, for example, the case of a dissipative operator $\{\mathcal{L}, \mathcal{B}\}$. From Proposition 2.5 we have

$$\tau := \dim\ker \mathcal{A}(-\gamma) - \dim\ker \mathcal{A}(\gamma) \leqslant 2M.$$

We show that $\tau \leqslant M$. If $\tau > M$, then there is an element $u \notin \ker \mathcal{A}(\gamma)$ satisfying the inclusions (2.24). We obtain a contradiction with assertion 3 of Lemma 2.10. Similarly, using assertion 4 of Lemma 2.10, one can show that

$$\tau_* := \dim\ker \mathcal{A}_*(-\gamma) - \dim\ker \mathcal{A}_*(\gamma) \leqslant M.$$

As usual, the cokernel of the operator is described in terms of the kernel of the adjoint operator with respect to the Green formula (cf., for example, [1],[2]). We have

$$\begin{aligned} \dim\ker \mathcal{A}_*(-\gamma) &= \dim\operatorname{coker} \mathcal{A}(\gamma), \\ \dim\ker \mathcal{A}_*(\gamma) &= \dim\operatorname{coker} \mathcal{A}(-\gamma). \end{aligned} \tag{2.27}$$

By (2.27), to complete the proof it remains to exclude the possibility $\tau < M$ or $\tau_* < M$. For this purpose, we consider the index

$$\operatorname{Ind} \mathcal{A}(\gamma) = \dim\ker \mathcal{A}(\gamma) - \dim\operatorname{coker} \mathcal{A}(\gamma)$$

of the operator $\mathcal{A}(\gamma)$. As is known [1],[2],

$$\operatorname{Ind} \mathcal{A}(-\gamma) - \operatorname{Ind} \mathcal{A}(\gamma) = 2M.$$

We write the left-hand side of the last equality in the form

$$(\dim\ker \mathcal{A}(-\gamma) - \dim\ker \mathcal{A}(\gamma)) + (\dim\operatorname{coker} \mathcal{A}(\gamma) - \dim\operatorname{coker} \mathcal{A}(-\gamma)) = \tau + \tau_*.$$

Thus, we have $\tau + \tau_* = 2M$, $\tau \leqslant M$, and $\tau_* \leqslant M$. But, in this case, $\tau = \tau_* = M$.

To consider the case of an accretive $\{\mathcal{L}, \mathcal{B}\}$, it suffices to exchange the roles of the operators $\{\mathcal{L}, \mathcal{B}\}$ and $\{\mathcal{L}_*, \mathcal{B}_*\}$ (cf. assertion 2 of Proposition 2.7). □

Proposition 2.11. *Let $0 < \gamma < \delta/2$, and let the line $\mathbb{R} + i\gamma$ be free of the spectra of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$.*

1. *If the operator $\{\mathcal{L}, \mathcal{B}\}$ is dissipative, then the space* $\ker \mathcal{A}(-\gamma)$ *possesses a basis $Z_1, \dots, Z_M$ modulo $\mathcal{D}^\ell_\gamma W(G)$ such that*

$$Z_k - \left(u_k^+ + \sum_{j=1}^{M} \mathfrak{T}_{kj} u_j^-\right) \in \mathcal{D}^\ell_\gamma W(G), \quad k = 1, \dots, M. \tag{2.28}$$

2. *If the operator $\{\mathcal{L}, \mathcal{B}\}$ is accretive, then the space* $\ker \mathcal{A}(-\gamma)$ *possesses a basis $X_1, \dots, X_M$ modulo $\mathcal{D}^\ell_\gamma W(G)$ such that*

$$X_k - \left(u_k^- + \sum_{j=1}^{M} \mathfrak{S}_{kj} u_j^+\right) \in \mathcal{D}^\ell_\gamma W(G), \quad k = 1, \dots, M. \tag{2.29}$$

Proof. Let an operator $\{\mathcal{L}, \mathcal{B}\}$ be dissipative or accretive. By Propositions 2.9 and 2.5, the space $\ker \mathcal{A}(-\gamma)$ possesses a basis $Y_1, \dots, Y_M$ modulo $\mathcal{D}^\ell_\gamma W(G)$ such that

$$Y_k - \sum_{j=1}^{M} (a_{kj} u_j^+ + b_{kj} u_j^-) \in \mathcal{D}^\ell_\gamma W(G), \quad k = 1, \dots, M, \tag{2.30}$$

with some complex coefficients a_{kj} and b_{kj}.

To prove assertion 1, it suffices to show that rank $\|a_{kj}\|_{k,j=1}^M = M$, if $\{\mathcal{L}, \mathcal{B}\}$ is dissipative.

Assume the contrary: rank $\|a_{kj}\|_{k,j=1}^M < M$. Then there exists a linear combination $u = c_1 Y_1 + \ldots + c_M Y_M$ such that $u \notin \ker \mathcal{A}(\gamma)$ and the inclusions (2.24) hold. We arrive at a contradiction with assertion 3 of Lemma 2.10. Assertion 1 is proved.

Assertion 2 is proved in a similar way. Let the operator $\{\mathcal{L}, \mathcal{B}\}$ be accretive, and let rank $\|b_{kj}\|_{k,j=1}^M < M$. There exists a linear combination $u = c_1 Y_1 + \ldots + c_M Y_M$ such that $u \notin \ker \mathcal{A}(\gamma)$ and the inclusions (2.25) hold with $\mathcal{A}_*(-\gamma)$ replaced by $\mathcal{A}(-\gamma)$ and $\mathcal{A}_*(\gamma)$ replaced by $\mathcal{A}(\gamma)$.

To arrive at a contradiction, it suffices to formulate assertion 4 of Lemma 2.10 in terms of the operators $\mathcal{A}(\gamma)$ and $\mathcal{A}(-\gamma)$, where $\{\mathcal{L}, \mathcal{B}\}$ is accretive. Thus, rank $\|b_{kj}\|_{k,j=1}^M = M$, and assertion 2 is proved. □

Definition 2.12. *The matrices $\mathfrak{S} = \|\mathfrak{S}_{kj}\|_{k,j=1}^M$ and $\mathfrak{T} = \|\mathfrak{T}_{kj}\|_{k,j=1}^M$ from Proposition 2.11 are called scattering matrices.*

Proposition 2.13. *The scattering matrices $\mathfrak{S}$ and $\mathfrak{T}$ are contractions.*

Proof. Let $\{\mathcal{L}, \mathcal{B}\}$ be a dissipative operator. We denote by $Z_1, \dots, Z_M$ the basis from Proposition 2.11. We set $u = \alpha_1 Z_1 + \ldots + \alpha_M Z_M$, where $\alpha \in \mathbb{C}^M$ and

$\|\alpha;\mathbb{C}^M\| = 1$. We compute $\rho(u,u)$. We have

$$\rho(u,u) = \rho\Big(\sum_k \alpha_k\Big\{u_k^+ + \sum_j \mathfrak{T}_{kj}u_j^-\Big\}, \sum_m \alpha_m\Big\{u_m^+ + \sum_\ell \mathfrak{T}_{m\ell}u_\ell^-\Big\}\Big)$$
$$= 1 + \rho\Big(\sum_j\Big(\sum_k \alpha_k\mathfrak{T}_{kj}\Big)u_j^-, \sum_\ell\Big(\sum_m \alpha_m\mathfrak{T}_{m\ell}\Big)u_\ell^-\Big) = 1 - \|\alpha\mathfrak{T};\mathbb{C}^M\|^2 \quad (2.31)$$

(cf. (2.26) from the proof of Lemma 2.10). It is clear that $u \in \ker\mathcal{A}(-\gamma)$. By assertion 1 of Lemma 2.10, we have $\rho(u,u) \geqslant 0$. Together with (2.31), this leads to the estimate $\|\alpha\mathfrak{T};\mathbb{C}^M\| \leqslant 1$.

Similarly, one can prove that the matrix $\mathfrak{S}$ is a contraction. □

Proposition 2.14. *Let $0 < \gamma < \gamma' < \delta/2$, and let the lines $\mathbb{R} + i\gamma$ and $\mathbb{R} + i\gamma'$ be free from the spectra of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$. We construct the basis* (2.16), (2.17) *by parameter γ and complete it to the basis $\{u_j^\pm\}_j^{M'}$ (corresponding to γ') so that $u_s^+ + u_s^- \in \mathcal{D}_\gamma^\ell W(G)$ for $s = M+1, \dots, M'$ and the relations* (2.17) *hold for $h, j = 1, \dots, M'$* (cf., for example, [5])*. If $\{\mathfrak{L}, \mathfrak{B}\}$ is dissipative, then*

$$\dim\ker\mathcal{A}(\gamma) - \dim\ker\mathcal{A}(\gamma') = \dim\ker(\mathfrak{T}^{2,2} - I),$$

and if $\{\mathfrak{L}, \mathfrak{B}\}$ is accretive, then

$$\dim\ker\mathcal{A}(\gamma) - \dim\ker\mathcal{A}(\gamma') = \dim\ker(\mathfrak{S}^{2,2} - I).$$

Here, $\mathfrak{T}^{2,2}$ and $\mathfrak{S}^{2,2}$ denote $(M' - M) \times (M' - M)$-submatrices of the scattering matrices $\mathfrak{T} = \|\mathfrak{T}^{k,\ell}(\gamma')\|_{k,\ell=1,2}$ and $\mathfrak{S} = \|\mathfrak{S}^{k,\ell}(\gamma')\|_{k,\ell=1,2}$ respectively.

Proof. In fact, the proof is contained in [5], where similar arguments are given for problems that are self-adjoint relative to the Green formulas. In the case of a dissipative operator $\{\mathfrak{L}, \mathfrak{B}\}$, we use the contraction matrix $\mathfrak{T}$ instead of the unitary scattering matrix S. If the operator is accretive, the same arguments are valid for the matrix $\mathfrak{S}$. □

Proposition 2.15. *Let the assumptions of Proposition* 2.11 *hold.*

1. *If the operator $\{\mathfrak{L}, \mathfrak{B}\}$ is dissipative, then in $\ker\mathcal{A}_*(-\gamma)$ of the operator* (2.9) *there exists a basis $X_1, \dots, X_M$ modulo $\mathcal{D}_\gamma^\ell W(G)$ satisfying* (2.29)*; moreover, $\mathfrak{S} = \mathfrak{T}^*$, where $\mathfrak{T}$ is the scattering matrix from Proposition* 2.11*.*

2. *If the operator $\{\mathfrak{L}, \mathfrak{B}\}$ is accretive, then in the space $\ker\mathcal{A}_*(-\gamma)$ there exists a basis $Z_1, \dots, Z_M$ modulo $\mathcal{D}_\gamma^\ell W(G)$ satisfying the relation* (2.28)*; moreover, $\mathfrak{T} = \mathfrak{S}^*$, where $\mathfrak{S}$ is the scattering matrix from Proposition* 2.11*.*

Proof. 1. The existence of a basis of the form (2.29) modulo $\mathcal{D}_\gamma^\ell W(G)$ for the subspace $\ker\mathcal{A}_*(-\gamma)$ follows from the Proposition 2.11 and the fact that the operator $\{\mathfrak{L}_*, \mathfrak{B}_*\}$ is accretive. We prove that $\mathfrak{S} = \mathfrak{T}^*$. Let $\alpha, \beta \in \mathbb{C}^M$. It is clear that

$$\sum \alpha_k Z_k \in \ker\mathcal{A}(-\gamma), \quad \sum \beta_\ell X_\ell \in \ker\mathcal{A}_*(-\gamma).$$

Therefore,

$$\rho\Big(\sum \alpha_k Z_k, \sum \beta_\ell X_\ell\Big) = 0.$$

On the other hand, by (2.28), (2.29), and (2.17), we have

$$\rho\Big(\sum_k \alpha_k Z_k, \sum_\ell \beta_\ell X_\ell\Big) = \rho\Big(\sum_k \alpha_k \Big(u_k^+ + \sum_j \mathfrak{T}_{kj} u_j^-\Big), \sum_\ell \beta_\ell \Big(u_\ell^- + \sum_h \mathfrak{S}_{\ell h} u_h^+\Big)\Big)$$
$$= \sum_{k\ell} \alpha_k \overline{\beta_\ell \mathfrak{S}_{\ell k}} - \sum_{k\ell} \alpha_k \overline{\beta}_\ell \mathfrak{T}_{k\ell} = (\alpha(\mathfrak{S}^* - \mathfrak{T}), \beta)_{\mathbb{C}^M}.$$

Thus, for any $\alpha, \beta \in \mathbb{C}^M$ we have $(\alpha(\mathfrak{S}^* - \mathfrak{T}), \beta)_{\mathbb{C}^M} = 0$. Therefore, $\mathfrak{T} = \mathfrak{S}^*$. The proof of assertion 2 is similar. □

2.3 Problem with Radiation Conditions

As above, we suppose that the line $\mathbb{R}+i\gamma$ is free from the spectra of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$ and $0 < \gamma < \delta/2$. Denote by $\mathcal{W}_\gamma(G)$ the space of waves (2.18). Now, for the quotient space $\mathcal{W}_\gamma(G)/\mathcal{D}^\ell_\gamma W(G)$ we take an arbitrary basis $u_1, u_2, \dots, u_{2M}$. We consider the restriction A of the operator $\mathcal{A}(-\gamma)$ in (2.8) to the subspace $\mathfrak{h}[\dot{+}]\mathcal{D}^\ell_\gamma W(G)$, where $\mathfrak{h}$ is the linear span of the functions $u_1, \dots, u_M$ (cf. (2.18)). The mapping $A : \mathfrak{h}[\dot{+}]\mathcal{D}^\ell_\gamma W(G) \to \mathcal{R}^\ell_\gamma W(G)$ is continuous.

Theorem 2.16. *Suppose that the operator* $\{\mathcal{L}, \mathcal{B}\}$ *of the problem* (2.1) *is dissipative or accretive,* $\eta_1, \dots, \eta_{T_*}$ *is a basis of the kernel* $\ker \mathcal{A}_*(\gamma)$ *of the operator* (2.9), *and the right-hand side* $\{f, g\} \in \mathcal{R}^\ell_\gamma W(G)$ *satisfies the orthogonality conditions* $(f, \eta_j)_G + (g, \mathcal{Q}\eta_j)_{\partial G} = 0$, $j = 1, \dots, T_*$. *We assume that for the space* $\ker \mathcal{A}_*(-\gamma)$ *one can choose a basis* $V_1, \dots, V_M$ *modulo* $\mathcal{D}^\ell_\gamma W(G)$ *that is compatible with the basis* $u_1, \dots, u_{2M}$ *for the quotient space* $\mathcal{W}_\gamma(G)/\mathcal{D}^\ell_\gamma W(G)$ *in the following sense*:

$$\rho(u_j, V_k) = \delta_{kj}, \quad k, j = 1, \dots, M. \tag{2.32}$$

Then the following assertions hold.

1. *There exists a unique up to an arbitrary element of* $\ker \mathcal{A}(\gamma)$ *solution* $u \in \mathfrak{h}[\dot{+}]\mathcal{D}^\ell_\gamma W(G)$ *to the problem* (2.1).

2. *The following inclusion holds*:

$$v \equiv u - a_1 u_1 - a_2 u_2 - \dots - a_M u_M \in \mathcal{D}^\ell_\gamma W(G), \tag{2.33}$$

where $a_j = i(f, V_j)_G + i(g, \mathcal{Q}V_j)_{\partial G}$, $j = 1, \dots, M$.

3. *The solution* u *satisfies the inequality*

$$\|v; \mathcal{D}^\ell_\gamma W(G)\| + |a_1| + |a_2| + \dots + |a_M| \leqslant C(\|\{f, g\}; \mathcal{R}^\ell_\gamma W(G)\| + \|e_\gamma v; L_2(G)\|). \tag{2.34}$$

4. *Let* $\zeta_1, \dots, \zeta_T$ *be a basis for* $\ker \mathcal{A}(\gamma)$. *The solution* u *satisfying the additional conditions* $(u, \zeta_j)_G = 0$, $j = 1, \dots, T$, *is unique and satisfies the estimate* (2.34) *with the right-hand side replaced by the expression* $\|\{f, g\}; \mathcal{R}^\ell_\gamma W(G)\|$.

Proof. The operator $\mathcal{A}(-\gamma)$ is a Fredholm operator (cf. Proposition 2.3). Therefore, the orthogonality conditions $(f, \eta_j)_G + (g, \mathcal{Q}\eta_j)_{\partial G} = 0$, $j = 1, \ldots, T_*$, provide the existence of a solution $u \in \mathcal{D}^\ell_{-\gamma} W(G)$ to the problem (2.1). By Proposition 2.9, the equalities (2.23) hold. Suppose that $\{U_1, \ldots, U_M\}$ is a basis modulo $\mathcal{D}^\ell_\gamma W(G)$ for $\ker \mathcal{A}(-\gamma)$ and $\{\zeta_1, \ldots, \zeta_T\}$ is a basis for $\ker \mathcal{A}(\gamma)$. A general solution to the problem has the form

$$u = u^0 + \sum_{j=1}^{M} c_j U_j + \sum_{k=1}^{T} d_k \zeta_k, \tag{2.35}$$

where u^0 is some special solution in the space of waves $\mathcal{W}_\gamma(G)$, whereas the constants c_j and d_k are arbitrary. The relation (2.33) can be provided by the choice of constants $c_1, \ldots, c_M$. Indeed, in the opposite case, there are constants $A_1, \ldots, A_M$ and $C_1, \ldots, C_M$ such that

$$\sum_{j=1}^{M} A_j U_j - \sum_{j=1}^{M} C_j u_j \in \mathcal{D}^\ell_\gamma W(G); \tag{2.36}$$

moreover, $|A_1| + \ldots + |A_M| > 0$. It is clear that

$$\sum_{j=1}^{M} A_j U_j \in \ker \mathcal{A}(-\gamma)$$

and

$$\rho\Big(\sum_{j=1}^{M} A_j U_j, V_k\Big) = 0, \quad k = 1, \ldots, M. \tag{2.37}$$

On the other hand, the inclusion (2.36) and relation (2.32) imply

$$\rho\Big(\sum_{j=1}^{M} A_j U_j, V_k\Big) = \rho\Big(\sum_{j=1}^{M} C_j u_j, V_k\Big) = C_k, \quad k = 1, \ldots, M, \tag{2.38}$$

where we used the ρ-orthogonality of $\mathcal{W}_\gamma(G)$ and $\mathcal{D}^\ell_\gamma W(G)$. Together with (2.37), this yields the equalities $C_1 = C_2 = \ldots = C_M = 0$, which contradicts the linear independence of $U_1, \ldots, U_M$ modulo $\mathcal{D}^\ell_\gamma W(G)$.

Thus, we have established the inclusion (2.33) with some constants a_j. Assertion 1 is proved. To obtain explicit formulas for the coefficients a_j, we compute $\rho(u, V_j)$. As in the case (2.38), we find

$$\rho(u, V_j) = \rho\Big(\sum_{k=1}^{M} a_k u_k, V_j\Big) = a_j, \quad j = 1, \ldots, M.$$

It remains to note that $\rho(u, V_j) = iq(u, V_j) = i(f, V_j)_G + i(g, \mathcal{Q}V_j)_{\partial G}$. Assertion 2 is proved.

Finally, assertions 3 and 4 are direct consequence of the formulas for a_j (for details we refer to [1, Remark 3.2.2]). $\square$

Inclusions of the form (2.33) are usually called *radiation conditions*. In the classical case, such a definition agrees with the Sommerfeld and Mandelstam principles (energy radiation). By Theorem 2.16, to list all possible radiation conditions is the same as to list all the bases $u_1, \dots, u_{2M}$ for the quotient space $\mathcal{W}_\gamma(G)/\mathcal{D}^\ell_\gamma W(G)$ and bases $V_1, \dots, V_M$ modulo $\mathcal{D}^\ell_\gamma W(G)$ for the subspace $\ker \mathcal{A}_*(-\gamma)$ compatible in the sense of (2.32).

Proposition 2.17. *Suppose that the operator* $\{\mathcal{L}, \mathcal{B}\}$ *of the problem* (2.1) *is dissipative and for the quotient space* $\mathcal{W}_\gamma(G)/\mathcal{D}^\ell_\gamma W(G)$ *the basis* (2.16), (2.17) *is taken. Let* $\{X_1, \dots, X_M\}$ *be a set of solutions to the homogeneous problem* $\{\mathcal{L}_*, \mathcal{B}_*\}v = 0$ *satisfying the inclusions* (2.29). *Then the following assertions hold.*

1. *If* R *is arbitrary and* S *is an invertible operator in* $\mathbb{C}^M$, *then*

$$\begin{aligned} V_k &= \sum_{m=1}^{M} \overline{(S^{-1})_{mk}} X_m, \\ u_j &= \sum_{m=1}^{M} \Big(S_{jm} u_m^- + \sum_{p=1}^{M} R_{jp} \Big\{ u_p^+ + \sum_{i=1}^{M} \mathfrak{S}^*_{pi} u_i^- \Big\} \Big), \end{aligned} \tag{2.39}$$

where $j, k = 1, \dots, M$, *satisfy the condition* (2.32).

2. *If a basis* $V_1, \dots, V_M$ *modulo* $\mathcal{D}^\ell_\gamma W(G)$ *for the space* $\ker \mathcal{A}_*(-\gamma)$ *and a basis* $u_1, \dots, u_{2M}$ *for the quotient space* $\mathcal{W}_\gamma(G)/\mathcal{D}^\ell_\gamma W(G)$ *satisfy* (2.32), *then there exist operators* R *and* S *such that the equalities* (2.39) *hold.*

Proof. Assertion 1 is established by direct computations of the left-hand side of the equalities (2.32) on the elements (2.39). Let us prove assertion 2. Since $V_1, \dots, V_M$ and $X_1, \dots, X_M$ are bases for the same space of solutions to the homogeneous problem $\{\mathcal{L}_*, \mathcal{B}_*\}v = 0$, we obtain the first equality in (2.39) with some invertible matrix S. (The existence of the basis $X_1, \dots, X_M$ is guaranteed by Propositions 2.7 and 2.11.) We set

$$u_j = \sum_{m=1}^{M} S_{jm} u_m^-.$$

It is easy to see that $\rho(u_j, V_k) = \delta_{kj}$, $k, j = 1, \dots, M$. Suppose that $\rho(w_j, V_k) = \delta_{kj}$ for some $w_1, \dots, w_M$. Then $\rho(u_j - w_j, V_k) = 0$ for all $j, k = 1, \dots, M$. In particular,

$$u_j - w_j = \sum_{p=1}^{M} R_{jp} \Big\{ u_p^+ + \sum_{i=1}^{M} \mathfrak{S}^*_{pi} u_i^- \Big\}, \tag{2.40}$$

where R is some matrix. Indeed, the sums $u_p^+ + \sum \mathfrak{S}^*_{pi} u_i^-$ yield asymptotic formulas for the elements Z_p of the basis $Z_1, \dots, Z_M$ modulo $\mathcal{D}^\ell_\gamma W(G)$ for the subspace

$\ker \mathcal{A}(-\gamma)$ (cf. Proposition 2.15) and the following equalities hold:

$$\rho\left(u_p^+ + \sum_{i=1}^{M} \mathfrak{S}_{pi}^* u_i^-, V_k\right) = \rho(Z_p, V_k) = 0, \quad p, k = 1, \ldots, M.$$

The quotient space $\mathcal{W}_\gamma(G)/\mathcal{D}_\gamma^\ell W(G)$ has dimension $2M$. Therefore, the arbitrariness is exhausted by the equality (2.40). □

Remark 2.18. *If the operator $\{\mathcal{L}, \mathcal{B}\}$ of the problem (2.1) is accretive, then all the collections $\{V_1, \ldots, V_M\}$ and $\{u_1, \ldots, u_{2M}\}$ that are compatible in the sense (4.1) are defined by the transformations (2.39) with X_m replaced by Z_m and $\mathfrak{S}^*$ replaced by $\mathfrak{T}^*$. Here, $\{Z_1, \ldots, Z_M\}$ is a family of solutions to the problem $\{\mathcal{L}_*, \mathcal{B}_*\}v = 0$ such that the inclusions (2.28) hold (such a family exists in view of Proposition 2.11). The proof is the same as that of Proposition 2.17.*

Remark 2.19. *If the operator $\{\mathcal{L}, \mathcal{B}\}$ of the problem (2.1) is dissipative and*

$$u_j = \sum_{m=1}^{M} S_{jm} u_m^-, \quad j = 1, \ldots, M,$$

where S is some invertible matrix, then the existence of a compatible in the sense (2.32) basis $V_1, \ldots, V_M$ modulo $\mathcal{D}_\gamma^\ell W(G)$ for the space $\ker \mathcal{A}_*(-\gamma)$ is guaranteed by Propositions 2.17, 2.11 and assertion 2 of Proposition 2.7. Similarly, if $\{\mathcal{L}, \mathcal{B}\}$ is accretive and

$$u_j = \sum_{m=1}^{M} S_{jm} u_m^+, \quad j = 1, \ldots, M,$$

where S is some nonsingular matrix, then there exist $V_1, \ldots, V_M$, compatible in the sense (4.1) (cf. Remark 2.18 and Propositions 2.11, 2.7). Theorem 2.16 justifies the statement of the problem (2.1) with the radiation conditions (2.33). In the general case, in order to verify the existence of suitable $V_1, \ldots, V_M$, it is necessary to know the scattering matrix.

2.4 Dissipative Operators in Spaces with Weight Norm

2.4.1 Relationship with the classical definition of a dissipative operator. The adjoint operator

Here we assume that the elliptic system $\{\mathcal{L}, \mathcal{B}\}$ is homogeneous. In other words, $\tau_1 = \tau_2 = \ldots = \tau_k \equiv \tau$ and $\mathcal{D}_\gamma^\ell W(G) := \prod_{i=1}^k W_\gamma^{2\tau}(G)$. With the problem (2.1) we associate an operator $\mathcal{M}$ with the domain

$$\mathcal{D}(\mathcal{M}) = \{u \in \mathcal{D}_\gamma^\ell W(G) : \mathcal{B}(x, D_x)u(x) = 0, x \in \partial G\}$$

that acts in the Hilbert space

$$L_2(G; e_{-\gamma}) \equiv \prod_{i=1}^{k} W_{-\gamma}^0(G)$$

by the formula

$$(\mathcal{M}u)(x) = e_\gamma(x)^2\mathcal{L}(x, D_x)u(x). \tag{2.41}$$

We denote by $(\cdot,\cdot)_{-\gamma}$ the inner product

$$(u, v)_{-\gamma} = \int\limits_G e_{-\gamma}(x)^2 u(x)\overline{v(x)}\,dx \tag{2.42}$$

in the space $L_2(G; e_{-\gamma})$.

Proposition 2.20. *Suppose that the operator of the problem* (2.1) *is dissipative in accordance with Definition* 2.1 *and the line* $\mathbb{R} + i\gamma$ *does not contain the spectra of the pencils* $\mathfrak{A}^1, \dots, \mathfrak{A}^N$ *and* $0 < \gamma < \delta/2$. *Then the following assertions hold.*

1. *The operator* $\mathcal{M}$ *is closed.*
2. *For elements* $u \in \mathcal{D}(\mathcal{M})$ *the following estimate holds*:

$$\operatorname{Im}(\mathcal{M}u, u)_{-\gamma} \geqslant 0 \tag{2.43}$$

(*in other words, the operator* $\mathcal{M}$ *is dissipative in the classical sense*).

3. *The following equality holds*:

$$\ker \mathcal{M} = \ker \mathcal{A}(\gamma),$$

where $\mathcal{A}(\gamma)$ *is the operator* (2.8).

4. *For the* coker $\mathcal{M}$ *one can choose a basis* $X_1, \dots, X_M$ *modulo* $\mathcal{D}^\ell_\gamma W(G)$ *such that the inclusions* (2.29) *hold.*

Proof. Let us check that the operator $\mathcal{M}$ is closed. Let $\{w^j\}$ and $\{\mathcal{F}^j\}$ be sequences convergent in $L_2(G; e_{-\gamma})$, and let $w^j \in \mathcal{D}(\mathcal{M})$ and $\mathcal{M}w^j = \mathcal{F}^j$. By (2.41), the function w^j satisfies the problem (2.1) with the right-hand side $\{f^j, g^j\} = \{e^2_{-\gamma}\mathcal{F}^j, 0\}$. The estimate (2.34) (where $v = w^j$, $a_1 = a_2 = \ldots = a_M = 0$ and $\{f, g\} = \{f^j, 0\}$) and the convergence of the sequence $\{f^j\}$ in $L_2(G; e_\gamma)$ imply the convergence of $\{w^j\}$ in $\mathcal{D}^\ell_\gamma W(G)$. Therefore, $u = \lim w^j \in \mathcal{D}(\mathcal{M})$ and $\mathcal{M}u = \lim \mathcal{F}^j$. Assertion 1 is proved.

Let us prove assertion 2. We consider (2.41), (2.42) and write the equality

$$\operatorname{Im}(\mathcal{M}u, u)_{-\gamma} = \operatorname{Im}\{(\mathcal{L}u, u)_G + (\mathcal{B}u, \mathcal{Q}u)_{\partial G}\} \tag{2.44}$$

for $u \in \mathcal{D}(\mathcal{M})$. By Corollary 2.8, the inequality (2.3) holds for functions in the set $\{u \in \mathcal{W}_\gamma(G) : \rho(u, u) \geqslant 0\} \supset \mathcal{D}(\mathcal{M})$. Together with (2.44), this leads to the estimate (2.43).

The proof of assertion 3 is obvious, and assertion 4 is a direct consequence of Proposition 2.11. □

Proposition 2.21. *The operator* $\mathcal{M}^*$, *adjoint to* $\mathcal{M}$ *in the space* $L_2(G; e_{-\gamma})$, *is defined on the set*

$$\mathcal{D}(\mathcal{M}^*) = \{v \in \mathcal{W}_\gamma(G) : \mathcal{B}_*(x, D_x)v(x) = 0, x \in \partial G\} \tag{2.45}$$

and acts by the formula

$$(\mathcal{M}^*v)(x) = e_\gamma(x)^2\mathcal{L}_*(x, D_x)u(x). \tag{2.46}$$

Proof. Consider the equality (2.15). Let $u \in \mathcal{D}(\mathcal{M})(\subset \mathcal{D}^\ell_\gamma W(G))$, and let $v \in \mathcal{D}(\mathcal{M}^*)(\subset \mathcal{W}_\gamma(G))$. Then $q(u,v)=0$ and

$$(\mathcal{M}u, v)_{-\gamma} = (u, \mathcal{M}^* v)_{-\gamma}. \tag{2.47}$$

It remains to show that if for $v, h \in L_2(G; e_{-\gamma})$

$$(\mathcal{M}u, v)_{-\gamma} = (u, h)_{-\gamma} \quad \forall\, u \in \mathcal{D}(\mathcal{M}), \tag{2.48}$$

then $v \in \mathcal{D}(\mathcal{M}^*)$ and $\mathcal{M}^* v = h$.

Let $\zeta_1, \dots, \zeta_T$ be a basis for the space $\ker \mathcal{M} = \ker \mathcal{A}(\gamma)$. By (2.48), we have $(\zeta_j, h)_{-\gamma} = 0$, $j = 1, \dots, T$. Therefore, there exists a solution v' to the problem $\{\mathcal{L}_*, \mathcal{B}_*\}v' = \{e^2_{-\gamma} h, 0\}$ in the space of waves $\mathcal{W}_\gamma(G)$. We have

$$(u, h)_{-\gamma} = (u, \mathcal{M}^* v')_{-\gamma} = (\mathcal{M}u, v')_{-\gamma}.$$

Together with (2.48), this leads to the relation

$$(\mathcal{M}u, v - v')_{-\gamma} = 0 \quad \forall\, u \in \mathcal{D}(\mathcal{M}).$$

Consequently, $v - v' \in \operatorname{coker} \mathcal{M} \subset \mathcal{W}_\gamma(G)$ and the function v belongs to the set (2.45). We note that $\mathcal{M}^* v = \mathcal{M}^* v' = h$. □

2.4.2 Maximal extension of a dissipative operator

We extend the dissipative operator $\mathcal{M}$ from Sec. 4.1 by adjoining representatives of waves to the domain. Suppose that the assumptions of Theorem 2.16 hold and $u_1 = u_1^-, u_2 = u_2^-, \dots, u_M = u_M^-$, $\mathfrak{h} = S^-$ (then the relations (2.32) automatically hold; cf. Remark 2.19). With the problem (2.1) with the radiation conditions (2.33) we associate an operator $\mathbb{M}$ with the domain

$$\mathcal{D}(\mathbb{M}) = \{u : u \in S^-[\dot{+}]\mathcal{D}^\ell_\gamma W(G), \mathcal{B}(x, D_x)u(x) = 0, x \in \partial G\} \tag{2.49}$$

that acts by formula (2.41) with $\mathcal{M}$ replaced by $\mathbb{M}$. We recall that the parameter γ is chosen in such a way that the line $\mathbb{R} + i\gamma$ is free from the spectra of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$. We also recall that $\mathcal{A}(\gamma)$ and $\mathcal{A}_*(\gamma)$ denote the operators (2.8) and (2.9) respectively.

Proposition 2.22. *The closed operator* $\mathbb{M}$ *is dissipative* (*i.e.,* $\operatorname{Im}(\mathbb{M}u, u)_{-\gamma} \geqslant 0$ *for all* $u \in \mathcal{D}(\mathbb{M})$); *moreover,* $\ker \mathbb{M} = \ker \mathcal{M} = \ker \mathcal{A}(\gamma)$ *and* $\operatorname{coker} \mathbb{M} = \ker \mathcal{A}_*(\gamma)$.

Proof. The fact that the operator is dissipative is established in the same way as in Proposition 2.20. The remaining assertions follow from Theorem 2.32. □

Proposition 2.23. *The operator* $\mathbb{M}^*$, *adjoint to* $\mathbb{M}$ *in the space* $L_2(G; e_{-\gamma})$, *is defined on the set*

$$\mathcal{D}(\mathbb{M}^*) = \{v : v \in S^+[\dot{+}]\mathcal{D}^\ell_\gamma W(G), \mathcal{B}_*(x, D_x)u(x) = 0, x \in \partial G\}$$

and acts by formula (2.46) *with* $\mathcal{M}^*$ *replaced by* $\mathbb{M}^*$. *The operator* $\mathbb{M}^*$ *is accretive* (*i.e.,* $\operatorname{Im}(\mathbb{M}^* v, v)_{-\gamma} \leqslant 0$ *for all* $v \in \mathcal{D}(\mathbb{M}^*)$).

Proof. The operator $\mathbb{M}^*$ is accretive. By Corollary 2.8, we have

$$\operatorname{Im}(\mathbb{M}^* v, v)_{-\gamma} = \operatorname{Im}\{(\mathcal{L}_* v, v)_G + (\mathcal{B}_* v, \mathcal{Q}_* v)_{\partial G}\} \leqslant 0$$

for $v \in \mathcal{D}(\mathbb{M}^*)(\subset \{v : v \in \mathcal{W}_\gamma(G), \rho(v,v) \geqslant 0\})$. The proof of the fact that $\mathbb{M}^*$ is the adjoint of $\mathbb{M}$ is almost the same as the proof of Proposition 2.21. We only note that

(i) $\rho(u,v) = 0$ for $u \in \mathcal{D}(\mathbb{M})$ and $v \in \mathcal{D}(\mathbb{M}^*)$ (cf. (2.18));

(ii) a solution $v' \in S^+[\dot{+}]\mathcal{D}^\ell_\gamma W(G)$ to the problem

$$\{\mathcal{L}_*, \mathcal{B}_*\}v' = \{e^2_{-\gamma} h, 0\}$$

exists in view of Theorem 2.16. □

Corollary 2.24. *The operator* $\mathbb{M}$ *is maximally dissipative.*

3 On the method for computing scattering matrices

The structure of this chapter is the following. The statement of the problem and preliminaries are given in Section 3.1. In particular, we discuss in detail the method for computing scattering matrices. In Section 3.2, we study auxiliary problems in the truncated domain G^R, the limit problems in the domain G and semi-infinite cylinders Π_+^j, and scattering matrices for these limit problems. In Section 3.3 an approximation solution to the problem in the domain G^R is constructed and an estimate for the norm of the inverse operator of the problem is given. The convergence of the method for computing scattering matrices is established in Section 3.4. In Section 3.5, we briefly discuss applications of the method to problems with spectral parameter.

We rely heavily on the theory of dissipative problems as developed in Chapter 2 and the notions used there in. This chapter is based on the paper [14].

3.1 Statement of the Problem. Preliminaries

3.1.1 Boundary value problem

Let us recall some of the notation. We denote by Π^r, $r = 1, \dots, N$, the cylinder in $\mathbb{R}^{n+1}$ with cross-section Ω^r. In Π^r, we introduce the Cartesian coordinates (y^r, t^r) so that $\Pi^r = \{(y^r, t^r) : y^r \in \Omega^r, t^r \in \mathbb{R}\}$. We assume that Ω^r is a domain in $\mathbb{R}^n$ with compact closure $\overline{\Omega}^r$ and smooth boundary $\partial\Omega^r$. Suppose that the domain G in $\mathbb{R}^{n+1}$ coincides with the union $\Pi_+^1 \cup \dots \cup \Pi_+^N$ of the semicylinders $\Pi_+^r = \{(y^r, t^r) \in \Pi^r : t^r > 0\}$ outside a ball B_R of sufficiently large radius R, the semicylinders Π_+^r do not intersect, and the boundary ∂G of the domain G is smooth. We consider the elliptic boundary value problem

$$\begin{aligned} \mathcal{L}(x, D_x)u(x) &= \mathcal{F}(x), \quad x \in G, \\ \mathcal{B}(x, D_x)u(x) &= \mathcal{G}(x), \quad x \in \partial G, \end{aligned} \tag{3.1}$$

where $\mathcal{L} = \|\mathcal{L}_{ij}\|$ and $\mathcal{B} = \|\mathcal{B}_{qj}\|$ are $k \times k$- and $m \times k$-matrices of differential operators; moreover $\operatorname{ord} \mathcal{L}_{ij} = \tau_i + \tau_j$ and $\operatorname{ord} \mathcal{B}_{qj} = \max \tau_i + \sigma_q + \tau_j$, where τ_j and σ_q are integers, $\tau_j \geqslant 0$, and $\tau_1 + \cdots + \tau_k = m$. It is assumed that the Green formula

$$(\mathcal{L}u, v)_G + (\mathcal{B}u, \mathcal{Q}v)_{\partial G} - (u, \mathcal{L}_* v)_G - (\mathcal{Q}_* u, \mathcal{B}_* v)_{\partial G} = 0 \tag{3.2}$$

holds for $u, v \in C_c^\infty(\overline{G})$ and the operator $\{\mathcal{L}, \mathcal{B}\}$ of the problem (3.1) is dissipative in the sense of Definition 2.1, i.e.

$$\operatorname{Im}\{(\mathcal{L}u, u)_G + (\mathcal{B}u, \mathcal{Q}u)_{\partial G}\} \geqslant 0 \quad \forall u \in C_c^\infty(\overline{G}). \tag{3.3}$$

Here, $\mathcal{Q}$, $\mathcal{B}_*$, and $\mathcal{Q}_*$ are some $m \times k$-matrices of differential operators, $\mathcal{L}_*$ is the formally adjoint to $\mathcal{L}$. The coefficients of all the operators are assumed to be smooth, and the operator $\{\mathcal{L}_*, \mathcal{B}_*\}$ is elliptic.

In addition to the assumptions of the previous chapter we suppose that the operator $\{\mathcal{L}, \mathcal{B}\}$ differs from a formally self-adjoint operator only on a compact set in $\overline{G}$. In other words, $\mathcal{L}_*(x, D_x) = \mathcal{L}(x, D_x)$, $\mathcal{B}_*(x, D_x) = \mathcal{B}(x, D_x)$ and $\mathcal{Q}_*(x, D_x) = \mathcal{Q}(x, D_x)$ outside a fixed compact set. The scheme below can be applied (with obvious modifications) to some other classes of dissipative operators (for example, to operators of the form $\mathcal{L} + q$, where $\mathcal{L} = \mathcal{L}_*$ and q is the operator of multiplication by a complex-valued function q that exponentially decreases at infinity).

We write the superscript r at $\mathcal{L}$, $\mathcal{B}$, and other operators if they are written in the coordinates (y^r, t^r) inside the semicylinder Π_+^r. It is supposed that there exist matrices L^r, B^r,Q^r, $r = 1, \dots, N$, of differential operators with smooth in $\overline{\Omega}^r$ and constant over t^r coefficients such that the coefficients of differences $\mathcal{L}^r - L^r$, $\mathcal{B}^r - B^r$, $\mathcal{Q}^r - Q^r$ together with all the derivatives are of order $O(\exp(-\delta t^r))$ with some $\delta > 0$ as $t^r \to +\infty$.

3.1.2 Method for computing scattering matrices

Let $0 < \gamma < \delta/2$, and let the total algebraic multiplicity of eigenvalues of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$ in the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < \gamma\}$ be equal to $2M > 0$; see section 2.1.3. Suppose that the line $\{\lambda \in \mathbb{C} : \operatorname{Im} \lambda = \gamma\}$ is free from the spectrum of the pencils. As was shown in the previous chapter, in the space of solutions u to the homogeneous problem (3.1) satisfying the estimate $u(x) = O(\exp \gamma |x|)$ as $x \to \infty$, there exist elements $Y_1, \dots, Y_M$ such that

$$Y_j(x) = u_j^+(x) + \sum_{k=1}^{M} S_{jk} u_k^-(x) + O(e^{-\gamma |x|}), \tag{3.4}$$

see Proposition 2.11. By Proposition 2.13 the scattering matrix $S = S(\gamma) = \|S_{jk}\|_{j,k=1}^M$ is a contraction. In the self-adjoint case, the scattering matrix is unitary. The amplitudes of waves $u_j^\pm$ may grow with exponential rate at infinity. However, if the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < \gamma\}$ contains only simple real eigenvalues of the pencils then our definition of s-matrix coincides with the classical one.

We introduce the sets

$$\Pi_+^{r,R} = \{(y^r, t^r) \in \Pi^r : t^r > R\}, \quad G^R = G \setminus \cup_{r=1}^N \Pi_+^{r,R}$$

for large R. Then $\partial G^R \setminus \partial G = \Gamma^R = \cup_r \Gamma^{r,R}$, where $\Gamma^{r,R} = \{(y^r, t^r) \in \Pi^r : t^r = R\}$. Recall that the operator $\{\mathcal{L}, \mathcal{B}\}$ is self-adjoint outside some compact set. Therefore, for large R we have

$$\begin{aligned}(\mathcal{L}u, v)_{G^R} \quad &+ \quad (\mathcal{B}u, \mathcal{Q}v)_{\partial G^R \setminus \Gamma^R} + (\mathcal{N}u, \mathcal{D}v)_{\Gamma^R} \\ &= \quad (u, \mathcal{L}_* v)_{G^R} + (\mathcal{Q}_* u, \mathcal{B}_* v)_{\partial G^R \setminus \Gamma^R} + (\mathcal{D}u, \mathcal{N}v)_{\Gamma^R}, \end{aligned} \tag{3.5}$$

where $\mathcal{D}$ and $\mathcal{N}$ stand for $m \times k$-matrices of differential operators, $\mathcal{D}$ being a Dirichlet system (cf. [21]). As an example of the Dirichlet system we can take the matrix consisting of m rows of the form $e^{(j)}\partial_\nu^h$, where $j = 1, \dots, k$, $h = 0, \dots, \tau_j - 1$, $e^{(j)} = (\delta_{1,j}, \dots, \delta_{k,j})$, and ν is the outward normal to Γ^R.

We look for the l-th row $(S_{l1}, \dots, S_{lM})$ of the scattering matrix $S(= S(\gamma))$. As an approximation of the row we take a vector minimizing some quadratic functional. To construct the functional, we introduce the boundary value problem in the truncated domain G^R:

$$\begin{aligned}\mathcal{L}(x, D_x)\mathcal{X}_l^R &= 0, \quad x \in G^R, \\ \mathcal{B}(x, D_x)\mathcal{X}_l^R &= 0, \quad x \in \partial G^R \setminus \Gamma^R, \\ (\mathcal{N} + i\zeta\mathcal{D})\mathcal{X}_l^R &= (\mathcal{N} + i\zeta\mathcal{D})\Big(u_l^+ + \sum_{j=1}^M a_j u_j^-\Big), \quad x \in \Gamma^R,\end{aligned} \tag{3.6}$$

where $\zeta \in \mathbb{R}\setminus\{0\}$, the complex numbers $a_1, \dots, a_M$ are arbitrary, and the operators $\mathcal{D}$ and $\mathcal{N}$ are the same as in formula (3.5).

As an approximation of the row $(S_{l1}, \dots, S_{lM})$ it is natural to take the minimizer $a^0(R) = (a_1^0(R), \dots, a_M^0(R))$ of the functional

$$J_l^R(a_1, \dots, a_M) = \Big\| \mathcal{D}(\mathcal{X}_l^R - u_l^+ - \sum_{j=1}^M a_j u_j^-); L_2(\Gamma^R) \Big\|^2, \tag{3.7}$$

where X_l^R is a solution to the problem (3.6) (for motivation see Introduction). One can expect that $a_j^0 \to S_{lj}$ with exponential rate as $R \to \infty$ and $j = 1, \dots, M$. In order to explicitly write the dependence of X_l^R on the parameters $a_1, \dots, a_M$, we consider the auxiliary problems

$$\begin{aligned}\mathcal{L}(x, D_x)v_j^\pm &= 0, \quad x \in G^R, \\ \mathcal{B}(x, D_x)v_j^\pm &= 0, \quad x \in \partial G \setminus \Gamma^R, \\ (\mathcal{N} + i\zeta\mathcal{D})v_j^\pm &= (\mathcal{N} + i\zeta\mathcal{D})u_j^\pm, \quad x \in \Gamma^R, \ j = 1, \dots, M.\end{aligned} \tag{3.8}$$

Then X_l^R can be expressed in terms of the solutions $v_j^\pm = v_{j,R}^\pm$ to the problem (3.8). We have $X_l^R = v_{l,R}^+ + \sum_j a_j v_{j,R}^-$. Introduce $M \times M$-matrices $\mathcal{E}$ and $\mathcal{F}$, where

$$\begin{aligned}\mathcal{E}_{ij}^R &= (\mathcal{D}(v_i^- - u_i^-), \mathcal{D}(v_j^- - u_j^-))_{\Gamma^R}, \\ \mathcal{F}_{ij}^R &= (\mathcal{D}(v_i^+ - u_i^+), \mathcal{D}(v_j^- - u_j^-))_{\Gamma^R}.\end{aligned} \tag{3.9}$$

Besides, we put

$$\mathcal{G}_i^R = (\mathcal{D}(v_i^+ - u_i^+), \mathcal{D}(v_i^+ - u_i^+))_{\Gamma^R}.$$

We write the functional (3.7) in the form

$$J_l^R(a) = \langle a\mathcal{E}^R, a\rangle + 2\,\mathrm{Re}\,\langle \mathcal{F}_l^R, a\rangle + \mathcal{G}_l^R,$$

where $\mathcal{F}_l^R$ is the l-th row of the matrix $\mathcal{F}^R$ and $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{C}^M$. Then the minimum is attained at the vector a^0 (row) satisfying the system $a^0(R)\mathcal{E}^R + \mathcal{F}_l^R = 0$. For an approximation S^R of the scattering matrix S we can take the solution to the equation $S^R\mathcal{E}^R + \mathcal{F}^R = 0$.

In the above scheme, we mean that the problem (3.8) is uniquely solvable for $\zeta \in \mathbb{R}\setminus\{0\}$ and large R. Moreover, to justify the algorithm, it is necessary to show that the matrix $\mathcal{E}^R$ is nonsingular and the minimizer $a^0(R)$ of the functional J_l^R tends to the row $(S_{l1},\dots,S_{lM})$ of the scattering matrix as $R\to\infty$.

3.2 Problems in G^R. Limit Problems

We discuss the solvability of the problem

$$\begin{aligned} &\mathcal{L}(x,D_x)u(x) = f(x), \quad x\in G^R,\\ &\mathcal{B}(x,D_x)u(x) = g(x), \quad x\in \partial G^R\setminus\Gamma^R,\\ &(\mathcal{N}(x,D_x) + i\zeta\mathcal{D}(x,D_x))u(x) = h(x), \quad x\in\Gamma^R, \end{aligned} \tag{3.10}$$

where $\zeta\in\mathbb{R}\setminus 0$. With the problem (3.10) we associate "limit" problems that will be studied in this section.

3.2.1 On the unique solvability of the problem in G^R

From the equality (3.5) we obtain the Green formula

$$\begin{aligned}(\mathcal{L}u,v)_{G^R} &+ (\mathcal{B}u,\mathcal{Q}v)_{\partial G^R\setminus\Gamma^R} + ((\mathcal{N}+i\zeta\mathcal{D})u,\mathcal{D}v)_{\Gamma^R}\\ &= (u,\mathcal{L}_*v)_{G^R} + (\mathcal{Q}_*u,\mathcal{B}_*v)_{\partial G^R\setminus\Gamma^R} + (\mathcal{D}u,(\mathcal{N}-i\zeta\mathcal{D})v)_{\Gamma^R}\end{aligned} \tag{3.11}$$

for the problem (3.10). If w is a solution to the homogeneous problem (3.10), then, substituting $u=v=w$ into (3.11), we have

$$-2i\zeta\|(\mathcal{D}w;L_2(\Gamma^R)\| = (w,\mathcal{L}_*w)_{G^R} + (\mathcal{Q}_*w,\mathcal{B}_*w)_{\partial G^R\setminus\Gamma^R}. \tag{3.12}$$

Since the operator $\{\mathcal{L},\mathcal{B}\}$ is dissipative (cf. (3.3)), taking into account the Green formula (3.2), we obtain the inequality

$$\mathrm{Im}\{(u,\mathcal{L}_*u)_G + (\mathcal{Q}_*u,\mathcal{B}_*u)_{\partial G}\} \geqslant 0 \quad \forall u\in C_c^\infty(\overline{G}). \tag{3.13}$$

Since $\mathcal{L}=\mathcal{L}_*$, $\mathcal{B}=\mathcal{B}_*$ and $\mathcal{Q}=\mathcal{Q}_*$ outside some compact set, the integration in the inner products in (3.13) is extended only over this compact set. This yields the inequality

$$\mathrm{Im}\{(w,\mathcal{L}_*w)_{G^R} + (\mathcal{Q}_*w,\mathcal{B}_*w)_{\partial G^R\setminus\Gamma^R}\} \geqslant 0. \tag{3.14}$$

For $\zeta > 0$ from (3.12) and (3.14) it follows that

$$\mathcal{D}(x, D_x)w(x) = 0, \quad x \in \Gamma^R. \tag{3.15}$$

An assertion like the Holmgren theorem, together with the conditions (3.15), provides the uniqueness of a solution to the problem (3.10). It is obvious that the same arguments are valid for the adjoint problem relative to the Green formula (3.11), which allows us to establish the solvability of the problem (3.10).

The boundary of the domain G^R contains edges $\partial\Gamma^R$. Therefore, one has to use either generalized statement of the problem or proper weighted function spaces. The implementation of such a plan is a part of the well-developed theory of elliptic boundary value problems. Further, restricting ourselves to the above remarks, we postulate the unique solvability of the problem (3.10). Let $\mathbb{V}(G^R)$ and $\mathbb{W}(G^R)$ be function spaces where the operator $\mathcal{A}^R$ of the problem (3.10) implements an isomorphism

$$\mathcal{A}^R : \mathbb{V}(G^R) \to \mathbb{W}(G^R). \tag{3.16}$$

The procedure of constructing such spaces is well understood in the theory of elliptic problems in domains with piecewise smooth boundary (cf. [1], [2]). The required spaces (generally speaking, with weight norms) exist for all classical problems in mathematical physics. We give two examples.

Example 3.1. We specify the spaces $\mathbb{V}(G^R)$ and $\mathbb{W}(G^R)$ for the problem

$$\begin{aligned} (\Delta + k^2)u(x) &= \mathcal{F}(x), \quad x \in G, \\ u(x) &= \mathcal{G}(x), \quad x \in \partial G. \end{aligned}$$

Define the space $V^\ell_\beta(G^R)$ as the completion of the set $C_0^\infty(\overline{G^R} \setminus \partial\Gamma^R)$ in the norm

$$\|v; V^\ell_\beta(G^R)\| = \Big(\sum_{|\alpha| \leqslant \ell} \|\rho^{\beta - \ell + |\alpha|} D^\alpha_x v; L_2(G^R)\|^2 \Big)^{1/2},$$

where $\beta \in \mathbb{R}$, $\ell = 0, 1, \dots$, and ρ is a smooth positive function on $\overline{G^R} \setminus \partial\Gamma^R$ coinciding with the distance function from the edge $\partial\Gamma^{r,R}$ in a neighborhood of $\partial\Gamma^{r,R}$. We denote by $V^{\ell+1/2}_\beta(\partial G^R \setminus \Gamma^R)$ and $V^{\ell+1/2}_\beta(\Gamma^R)$ the trace spaces on $\partial G^R \setminus \Gamma^R$ and on Γ^R of functions in $V^{\ell+1}_\beta(G^R)$. Consider the operator

$$\mathcal{A}^R(\beta; \zeta) : V^2_\beta(G^R) \to V^0_\beta(G^R) \times V^{3/2}_\beta(\partial G^R \setminus \Gamma^R) \times V^{1/2}(\Gamma^R) \tag{3.17}$$

of the problem

$$\begin{aligned} &(\Delta + k^2)u(x) = f(x), \quad x \in G^R, \\ &u(x) = g(x), \quad x \in \partial G^R \setminus \Gamma^R, \\ &(\partial_\nu + i\zeta)u(x) = h(x), \quad x \in \Gamma^R, \end{aligned} \tag{3.18}$$

where ν is the outward normal. If u is a solution to the homogeneous problem (3.18), then from the Green formula

$$\begin{aligned} ((\Delta + k^2)v, w)_{G^R} \;&+\; (v, \partial_\nu w)_{\partial G^R \setminus \Gamma^R} + ((\partial_\nu + i\zeta)v, w)_{\Gamma^R} \\ &= \; (v, (\Delta + k^2)w)_{G^R} + (\partial_\nu v, w)_{\partial G^R \setminus \Gamma^R} + (v, (\partial_\nu - i\zeta)w)_{\Gamma^R} \end{aligned}$$

for $v = w = u$ we find

$$u(x) = 0, \quad x \in \Gamma^R. \tag{3.19}$$

By the Holmgren theorem, together with the conditions (3.19), the equality $\ker \mathcal{A}^R(\beta;\zeta) = 0$ holds. The same arguments are valid for the adjoint problem. Therefore, $\ker \mathcal{A}^R(2-\beta;-\zeta) = 0$. If the number $|\beta - 1|$ is sufficiently small, then the operator (3.17) is a Fredholm operator; moreover, $\operatorname{coker} \mathcal{A}^R(\beta;\zeta) = \{(u, \partial_\nu u) : u \in \ker \mathcal{A}^R(2-\beta;-\zeta)\}$ (cf., for example, [1], [2]). Thus, if $|\beta - 1|$ is sufficiently small, then the mapping (3.17) is an isomorphism.

Example 3.2. We give an example of truncated domain with smooth boundary. Consider the Neumann problem

$$\mathcal{L}(x, D_x)u(x) = \mathcal{F}(x), x \in G,$$
$$\mathcal{N}(x, D_x)u(x) = \mathcal{G}(x), x \in \partial G.$$

Denote by $\widetilde{\Pi}^{r,R}$ a domain with smooth boundary such that

$$\{(y^r, t^r) \in \Pi^r : t^r < R\} \subset \widetilde{\Pi}^{r,R} \subset \{(y^r, t^r) \in \Pi^r : t^r < R+1\}.$$

Let G^R be the same domain as in (3.1). We set

$$\widetilde{G}^R = G^R \cup_{r=1}^N \{(y^r, t^R) \in \widetilde{\Pi}^{r,R}, t^r > R\}.$$

We choose a cut-off function χ on $\partial\widetilde{G}^R$ such that $\operatorname{supp} \chi \subset \cup_r \partial\widetilde{\Pi}^{r,R} \setminus \partial G^R$ and $\operatorname{mes} \operatorname{supp} \chi > 0$.

The role of the problem (3.10) is played by the problem

$$\begin{aligned} &\mathcal{L}(x, D_x)u(x) = f(x), \quad x \in \widetilde{G}^R, \\ &(\mathcal{N}(x, D_x) + i\zeta\chi\mathcal{D}(x, D_x))u(x) = g(x), \quad x \in \partial\widetilde{G}^R, \end{aligned} \tag{3.20}$$

in a bounded domain with smooth boundary. The Green formula holds:

$$(\mathcal{L}u, v)_{\widetilde{G}^R} + ((\mathcal{N} + i\zeta\chi\mathcal{D})u, \mathcal{D}v)_{\partial\widetilde{G}^R} = (u, \mathcal{L}v)_{\widetilde{G}^R} + (\mathcal{D}u, (\mathcal{N} - i\zeta\chi\mathcal{D})v)_{\partial\widetilde{G}^R}.$$

Now we can set

$$\begin{aligned} &\mathbb{V}(\widetilde{G}^R) = \prod_{j=1}^k H^{\ell+\tau_j}(\widetilde{G}^R), \\ &\mathbb{W}(\widetilde{G}^R) = \prod_{j=1}^k H^{\ell-\tau_j}(\widetilde{G}^R) \times \prod_{h=1}^m H^{\ell-\tau-\sigma_h-1/2}(\partial\widetilde{G}^R), \end{aligned} \tag{3.21}$$

where $\tau = \max\{\tau_1, \dots, \tau_k\}$, $\operatorname{ord} \mathcal{N}_{hj} = \sigma_h + \tau + \tau_j$ and $\ell > \tau$. From (3.21) it follows that the solution to the homogeneous problem (3.20) satisfies the equation $\chi\mathcal{D}(x, D_x)u(x) = 0$ on $\partial\widetilde{G}^R$. As $\{\mathcal{L}, \mathcal{B}\}$ one can take, among others, the operator $\{E(D_x) + k^2, \mathcal{N}(x, D_x)\}$in elasticity theory. Here,

$$E(D_x) = -\mu\nabla_x \cdot \nabla_x - (\lambda + \mu)\nabla_x\nabla_x \cdot$$

is the Lamé system with parameters λ and μ, $\mathcal{N}(x, D_x)u(x) := \sigma(u; x)\nu(x)$, and $\sigma(u; x)$ is the stress tensor,

$$\sigma_{jk}(u; x) = \mu\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) + \lambda\delta_{jk}\nabla_x \cdot u.$$

The operator

$$\mathcal{A}^R : \mathbb{V}(\widetilde{G}^R) \to \mathbb{W}(\widetilde{G}^R) \tag{3.22}$$

is Fredholm. The Holmgren theorem implies the uniqueness of a solution to the problem (3.20) and to the adjoint problem. The mapping (3.22) is an isomorphism.

3.2.2 Limit problems

To estimate the norm of the inverse operator $(\mathcal{A}^R)^{-1}$, we use the compound expansion method. For this purpose we introduce and study the limit problems in the domain G and semicylinders $\Pi_-^{r,0}$, $r = 1, \dots, N$, where $\Pi_-^{r,R} = \{(y^r, t^r) \in \Pi^r, t^r < R\}$. The role of the limit problem in G is played by the problem (3.1). The limit problem in $\Pi_-^{r,0}$ is as follows:

$$\begin{aligned}
&\mathbf{L}^r(y, D_y, D_t)u(y,t) = F^r(y,t), \quad (y,t) \in \Pi_-^{r,0},\\
&\mathbf{B}^r(y, D_y, D_t)u(y,t) = G^r(y,t), \quad (y,t) \in \partial\Pi_-^{r,0} \setminus \Gamma^{r,0},\\
&(\mathbf{N}^r(y, D_t) + i\zeta\mathbf{D}^r(y, D_t))u(y,t)|_{t=0} = H^r(y), \quad y \in \Omega^r.
\end{aligned} \tag{3.23}$$

We denote by $\mathbf{N}^r$ and $\mathbf{D}^r$ the limit operators for $\mathcal{N}$ and $\mathcal{D}$. The operators $\mathcal{N}$ and $\mathcal{D}$ stabilize to $\mathbf{N}^r$ and $\mathbf{D}^r$ (in the same sense as $\mathcal{L}$ to $\mathbf{L}^r$) as $x \to \infty$ in Π^r.

We note that in Example 3.2, instead of (3.23), we have a problem in the domain $\widetilde{\Pi}_-^{r,0} = \{(y^r, t^r + R) \in \widetilde{\Pi}^{r,R}\}$ with smooth boundary.

3.2.3 Limit problems in semicylinders

In this subsection, we assume that a number $r = 1, \dots, N$ is fixed and omit the corresponding subscripts in the notation $\Pi_-^{r,R}$, $\Gamma^{r,R}$, etc. The spaces $\mathbb{V}(\Pi_-)$ and $\mathbb{W}(\Pi_-)$ satisfy the following conditions. We set

$$\|u; \mathbb{V}_\gamma(\Pi_-)\| = \|e_{-\gamma}u; \mathbb{V}(\Pi_-)\|$$

and

$$\|\{F, G, H\}; \mathbb{W}_\gamma(\Pi_-)\| = \|e_{-\gamma}\{F, G, H\}; \mathbb{W}(\Pi_-)\|,$$

where $e_{-\gamma} : (y,t) \mapsto e_{-\gamma}(y,t) = \exp(-\gamma t)$. The operator

$$A(\gamma, \zeta) : \mathbb{V}_\gamma(\Pi_-) \to \mathbb{W}_\gamma(\Pi_-) \tag{3.24}$$

of the problem (3.23) is a Fredholm operator if and only if the line $\mathbb{R} + i\gamma = \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda = \gamma\}$ is free from the spectrum of the operator pencil $\mathfrak{A}(\lambda) = \{\mathbf{L}(y, D_y, \lambda), \mathbf{B}(y, D_y, \lambda)\}$ (cf. [1],[2]).

Proposition 3.3. *For any real $\zeta \neq 0$ the problem* (3.23) *has at most one solution in the space $\mathbb{V}_\gamma(\Pi_-)$, where $\gamma > 0$.*

Proof. The Green formula

$$(\mathbf{L}u, v)_{\Pi_-} + (\mathbf{B}u, \mathbf{Q}v)_{\partial\Pi_-\setminus\Gamma} + ((\mathbf{N} + i\zeta\mathbf{D})u, \mathbf{D}v)_\Gamma \\ - (u, \mathbf{L}v)_{\Pi_-} - (\mathbf{Q}u, \mathbf{B}v)_{\partial\Pi_-\setminus\Gamma} - (\mathbf{D}u, (\mathbf{N} - i\zeta\mathbf{D})v)_\Gamma = 0 \tag{3.25}$$

holds for all $u, v \in \mathbb{V}_\gamma(\Pi_-)$, $\gamma > 0$. If a function $u \in \mathbb{V}_\gamma(\Pi_-)$ satisfies the homogeneous problem (3.23), then formula (3.25) with $v = u$ implies the equality $\mathbf{D}(y, D_t)u(y, t) = 0$ on Γ. Using the Holmgren theorem, we complete the proof. □

Proposition 3.4. *Let $\gamma < 0$, and let the line $\mathbb{R} + i\gamma$ be free from the spectrum of the pencil $\mathfrak{A}$. Then for any right-hand side $\{F, G, H\} \in \mathbb{W}_\gamma(\Pi_-)$ the problem* (3.23) *has a solution in the space $\mathbb{V}_\gamma(\Pi_-)$.*

Proof. The problem adjoint with respect to the Green formula (3.24) is given by the equation (3.23) with ζ replaced by $-\zeta$. By Proposition 3.3, the kernel of this problem in the space $\mathbb{V}_{-\gamma}(\Pi_-)$, $\gamma < 0$, is trivial. Since the operator (3.24) is Fredholm, it follows that its range coincide with the space $\mathbb{W}_\gamma(\Pi_-)$. □

To the problem (3.23) there corresponds the operator pencil (2.11). Therefore the asymptotics of solutions can be expressed in terms of the functions (2.12). Introduce the form

$$p(u, v) := (\mathbf{L}u, v)_{\Pi_-} + (\mathbf{B}u, \mathbf{Q}v)_{\partial\Pi_-\setminus\Gamma} + (\mathbf{N}u, \mathbf{D}v)_\Gamma \\ - (u, \mathbf{L}v)_{\Pi_-} - (\mathbf{Q}u, \mathbf{B}v)_{\partial\Pi_-\setminus\Gamma} - (\mathbf{D}u, \mathbf{N}v)_\Gamma,$$

which coincides with the left-hand side of (3.25). This enables us to classify the waves as incoming and outgoing in the same way as in the case of the problem (3.1); see section 2.1.4. Denote by W the linear span of the functions $u_\nu^{(\sigma,j)}$ ($\sigma = 0, \dots, \varkappa_{j\nu} - 1; j = 1, \dots, J_\nu$) given in (2.12) and corresponding to the eigenvalues $\lambda_{-\mu}, \dots, \lambda_\mu$ of the pencil $\mathfrak{A}$ in the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \gamma\}$. Let the total algebraic multiplicity of $\lambda_{-\mu}, \dots, \lambda_\mu$ be equal to $2M$. In the space W we choose the basis

$$v_1^+, \dots, v_M^+, v_1^-, \dots, v_M^- \tag{3.26}$$

consisting of incoming and outgoing waves, i.e.

$$p(v_j^\pm, v_h^\pm) = \mp i\delta_{h,j}, \; p(v_j^\pm, v_h^\mp) = 0, \quad j, h = 1, \dots, M. \tag{3.27}$$

Proposition 3.5. *Let $\gamma > 0$ and $\zeta > 0$. Suppose that either*

$$u \in \ker A(-\gamma, -\zeta) \quad \textit{and} \quad u - \sum_{j=1}^M a_j v_j^+ \in \mathbb{V}_\gamma(\Pi_-) \tag{3.28}$$

or

$$u \in \ker A(-\gamma, \zeta) \quad \textit{and} \quad u - \sum_{j=1}^M b_j v_j^- \in \mathbb{V}_\gamma(\Pi_-), \tag{3.29}$$

where a_j and b_j are some complex coefficients. Then $u \equiv 0$.

Proof. Let, for example, the assumption (3.28) be satisfied. Let us compute the value $p(u,u)$. Taking into account (3.27) and the second inclusion in (3.28), we find

$$p(u,u) = p\Big(\sum_{j=1}^{M} a_j v_j^+, \sum_{j=1}^{M} a_j v_j^+\Big) = -i \sum_{j=1}^{M} |a_j|^2. \tag{3.30}$$

The first inclusion in (3.28) implies the equality

$$p(u,u) = -(\mathbf{D}u, 2i\zeta\mathbf{D}u)_\Gamma = 2i\zeta\|\mathbf{D}u; L_2(\Gamma)\|^2. \tag{3.31}$$

Comparing (3.30) and (3.31), we arrive at the relation

$$2\zeta\|\mathbf{D}u; L_2(\Gamma)\|^2 = -\sum_{j=1}^{M} |a_j|^2,$$

where $\zeta > 0$. Consequently, $a_j = 0$ and $u \in \mathbb{V}_\gamma(\Pi_-)$. This means $u \equiv 0$ by Proposition 3.3.

In the case of the assumption (3.29), the assertion is proved in a similar way. □

Proposition 3.6. *Suppose that $\gamma > 0$, the line $\mathbb{R} + i\gamma$ be free from the spectrum of the pencil $\mathfrak{A}$, and the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \gamma\}$ contains the eigenvalues $\lambda_{-\nu}, \dots, \lambda_\nu$ of the pencil $\mathfrak{A}$. The following assertions hold.*

1. $\dim\ker A(-\gamma,\zeta) = M$ *for $\zeta \neq 0$, where $2M$ is the total algebraic multiplicity of $\lambda_{-\nu}, \dots, \lambda_\nu$.*

2. *For an arbitrary basis $Z_1, \dots, Z_M$ for the space* $\ker A(-\gamma,\zeta)$ *we have*

$$Z_k - \sum_{j=1}^{M} a_{kj} v_j^+ - \sum_{j=1}^{M} b_{kj} v_j^- \in \mathbb{V}_\gamma(\Pi_-) \tag{3.32}$$

with some complex coefficients a_{kj} and b_{kj}. Moreover,

$$\begin{aligned} &\operatorname{rank}\ a = M \quad \textit{for}\ \ \zeta > 0,\\ &\operatorname{rank}\ b = M \quad \textit{for}\ \ \zeta < 0, \end{aligned}$$

where $a = \|a_{kj}\|_{k,j=1}^M$ and $b = \|b_{kj}\|_{k,j=1}^M$.

Proof. Any solution $Z \in \mathbb{V}_{-\gamma}(\Pi_-)$ to the homogeneous problem (3.23) satisfies the inclusion

$$Z - \sum_{j=1}^{M} a_j v_j^+ - \sum_{j=1}^{M} b_j v_j^- \in \mathbb{V}_\gamma(\Pi_-).$$

Taking into account Proposition 3.3, we arrive at the inequality $\dim\ker A(-\gamma,\zeta) \leqslant 2M$, the representation (3.32) holds for any basis $Z_1, \dots, Z_M$.

Let us show that $\tau^\pm = \dim\ker A(-\gamma, \mp\zeta) \leqslant M$ (for $\zeta > 0$). In the case $\tau^+ > M$, there is an element $u \neq 0$ satisfying the relations (3.28), which contradicts to Proposition 3.5. The case $\tau^- > M$ can be treated in a similar manner with (3.29) instead of (3.28).

It remains to eliminate the inequalities $\tau^+ < M$ and $\tau^- < M$. For this purpose we consider the index $\operatorname{Ind} A(\gamma,\zeta)$ of the operator $A(\gamma,\zeta)$. We have

$$\begin{aligned}\operatorname{Ind} A(\gamma,\zeta) &= \dim\ker A(\gamma,\zeta) - \dim\operatorname{coker}\, A(\gamma,\zeta)\\ &= \dim\ker A(\gamma,\zeta) - \dim\ker A(-\gamma,-\zeta)\\ &= -\dim\ker A(-\gamma,-\zeta) \end{aligned}\tag{3.33}$$

since $\dim\ker A(\gamma,\zeta)=0$ by Proposition 3.3. Moreover,

$$\begin{aligned}\operatorname{Ind} A(-\gamma,\zeta) &= \dim\ker A(-\gamma,\zeta) - \dim\ker A(\gamma,-\zeta)\\ &= \dim\ker A(-\gamma,\zeta). \end{aligned}\tag{3.34}$$

As is known, $\operatorname{Ind} A(\gamma,\zeta) = \operatorname{Ind} A(-\gamma,\zeta) - 2M$ (cf. [1], [2]). From (3.33) and (3.34) we find

$$\dim\ker A(-\gamma,\zeta) + \dim\ker A(-\gamma,-\zeta) = 2M.$$

As was shown, each term on the left-hand side does not exceed M. Therefore each of them is equal to M. The first assertion of the proposition is proved.

Suppose that $\zeta > 0$ and rank $a < M$. Then there exists a nonzero element u satisfying the inclusions (3.29), which contradicts Proposition 3.5. The equality rank $a = M$ is proved. In a similar way, one can prove that rank $b = M$ for $\zeta < 0$. □

Corollary 3.7. *Let $\zeta > 0$. For the space $\ker A(-\gamma,\zeta)$ there exists a basis $Z_1,\dots,Z_M$ such that*

$$Z_j - \Big(v_j^+ + \sum_{k=1}^{M} \mathfrak{T}_{jk} v_k^-\Big) \in \mathbb{V}_\gamma(\Pi_-).\tag{3.35}$$

In the case $\zeta < 0$, there exists a basis $X_1,\dots,X_M$ such that

$$X_j - \Big(v_j^- + \sum_{k=1}^{M} \mathfrak{S}_{jk} v_k^+\Big) \in \mathbb{V}_\gamma(\Pi_-).\tag{3.36}$$

Definition 3.8. *The matrices $\mathfrak{S} = \|\mathfrak{S}_{kj}\|_{k,j=1}^M$ and $\mathfrak{T} = \|\mathfrak{T}_{kj}\|_{k,j=1}^M$ are called scattering matrices.*

Proposition 3.9. *The matrices*

$$\mathfrak{S} = \|\mathfrak{S}_{kj}\|_{k,j=1}^M, \quad \mathfrak{T} = \|\mathfrak{T}_{kj}\|_{k,j=1}^M$$

are contractions such that

$$\|\alpha\mathfrak{S}\|_{\mathbb{C}^M} < 1, \quad \|\alpha\mathfrak{T}\|_{\mathbb{C}^M} < 1$$

for any row $\alpha \in \mathbb{C}^M$, $\|\alpha\|_{\mathbb{C}^M} = 1$.

Proof. Let $\zeta < 0$, and let $X_1, \dots, X_M$ be the basis (3.36). We set $u = \sum \alpha_k X_k \in \ker A(-\gamma, \zeta)$ for $\alpha \in \mathbb{C}^M$, $\|\alpha\|_{\mathbb{C}^M} = 1$. We have

$$p(u,u) = p\Big(\sum_k \alpha_k \{v_k^- + \sum_j \mathfrak{S}_{kj} v_j^+\}, \sum_m \alpha_m \{v_m^- + \sum_\ell \mathfrak{S}_{m\ell} v_\ell^+\}\Big)$$
$$= i + p\Big(\sum_j (\sum_k \alpha_k \mathfrak{S}_{kj}) v_j^+, \sum_\ell (\sum_m \alpha_m \mathfrak{S}_{m\ell}) v_\ell^+\Big) = i - i\|\alpha\mathfrak{S}\|^2_{\mathbb{C}^M}.$$

Moreover, formula (3.31) holds with ζ replaced by $-\zeta$. We obtain the equality

$$1 - \|\alpha\mathfrak{S}\|^2_{\mathbb{C}^M} = -2\zeta \|\mathbf{D}u; L_2(\Gamma)\|^2.$$

If $\mathbf{D}(y, D_t)u(y,t) = 0$ on Γ, then $u \equiv 0$, which contradicts the condition $\|\alpha\|_{\mathbb{C}^M} = 1$. Therefore, $1 - \|\alpha\mathfrak{S}\|^2_{\mathbb{C}^M} > 0$ for all $\alpha \in \mathbb{C}^M$, $\|\alpha\|_{\mathbb{C}^M} = 1$.

Similarly, we can verify that $\mathfrak{T}$ is a contraction for $\zeta > 0$. □

To conclude this subsection, we prove the existence of a unique solution to the problem (3.23) satisfying the radiation conditions. This result will be used in the following subsection for constructing an approximate solution to the problem (3.10) in G^R.

Proposition 3.10. 1. *Suppose that the assumptions of Proposition* 3.6 *hold,* $\{F, G, H\} \in \mathbb{W}_\gamma(\Pi_-)$*, and* $\zeta < 0$*. Then there exists a unique solution* u *to the problem* (3.23) *satisfying the inclusion*

$$u - \sum_{1 \leqslant j \leqslant M} a_j v_j^+ \in \mathbb{V}_\gamma(\Pi_-) \tag{3.37}$$

with some coefficients $a_j \in \mathbb{C}$*. The following inequality holds*:

$$\Big\| u - \sum_{1 \leqslant j \leqslant M} a_j v_j^+; \mathbb{V}_\gamma(\Pi_-)\Big\| + \sum_{1 \leqslant j \leqslant M} |a_j| \leqslant C \|\{F, G, H\}; \mathbb{W}_\gamma(\Pi_-)\|. \tag{3.38}$$

In the case $\zeta > 0$*, formula* (3.37) *is replaced with the inclusion*

$$u - \sum_{1 \leqslant j \leqslant M} b_j v_j^- \in \mathbb{V}_\gamma(\Pi_-), \tag{3.39}$$

the inequality (3.38) *holds for* b_j *and* v_j^- *instead of* a_j *and* v_j^+*; here* a_j *and* b_j *are some complex coefficients,* $j = 1, \dots, M$*.*

2. *If the pencil* $\mathfrak{A}$ *has no real eigenvalues, then for sufficiently small* γ *the operator* (3.24) *implements an isomorphism.*

Proof. Let, for example, $\zeta < 0$. By Proposition 3.4, there exists a solution $u \in \mathbb{V}_{-\gamma}(\Pi_-)$ to the problem (3.23). Since $\{F, G, H\} \in \mathbb{W}_\gamma(\Pi_-)$, we have

$$u - \sum_{1 \leqslant j \leqslant M} (d_j^+ v_j^+ + d_j^- v_j^-) \in \mathbb{V}_\gamma(\Pi_-)$$

with some $d_j^+, d_j^- \in \mathbb{C}$ (cf. [1],[2]). By Corollary 3.7, the function $u + c_1 X_1 + \dots + c_M X_M$ is a solution to the same problem for any $c_j \in \mathbb{C}$. We set $c_j = -d_j^-$ for $j = 1, \dots, M$. Taking into account (3.36), we obtain (3.37) with coefficients $a_j = d_j^+ - \sum d_k^- \mathfrak{S}_{kj}$, where $j = 1, \dots, M$. This proves the existence of a solution. Proposition 3.5 yields the uniqueness. □

3.2.4 On the limit problem (3.1)

Let e_β be a smooth positive function on $\overline{G}$, equal to $\exp(\beta t^r)$ in $\overline{\Pi}^r_+$. We equip the space $W^\ell_\beta(G)$ of functions in G with the norm $\|e_\beta\cdot; H^\ell(G)\|$ and denote by $W^{\ell-1/2}_\beta(\partial G)$ the space of traces on ∂G of functions in $W^\ell_\beta(G)$. Replacing $\widetilde{G}^R$ and H with G and W_β in (3.21) (and using the notation $\sigma_h = \operatorname{ord} \mathcal{B}_{hj} - \tau - \tau_j$), we introduce the spaces $\mathbb{V}_\beta(G)$ and $\mathbb{W}_\beta(G)$ of vector-valued functions. As is known [1], [2], the continuous operator

$$\mathcal{A}(\beta) = \{\mathcal{L}, \mathcal{B}\} : \mathbb{V}_\beta(G) \to \mathbb{W}_\beta(G)$$

of the problem (3.1) is a Fredholm operator if and only if the line $\mathbb{R} + i\beta$ is free from the spectrum of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$; moreover, $\operatorname{coker} \mathcal{A}(\beta) = \{(u, \mathcal{Q}u) : u \in \ker \mathcal{A}_*(-\beta)\}$, where $\mathcal{Q}$ is defined by formula (3.2) and the operator

$$\mathcal{A}_*(-\beta) = \{\mathcal{L}_*, \mathcal{B}_*\} : \mathbb{V}_{-\beta}(G) \to \mathbb{W}_{-\beta}(G)$$

is adjoint to $\mathcal{A}(\beta)$ with respect to the Green formula (3.2).

Recall that the functions $Y_1, \dots, Y_M$ (cf. (3.4)) make up the basis in $\ker \mathcal{A}(-\gamma)$ modulo $\mathbb{V}_\gamma(G)$. By virtue of Proposition 2.15 there exists a basis $Y_{*1}, \dots, Y_{*M}$ in the space $\ker \mathcal{A}_*(-\gamma)$ modulo $\mathbb{V}_\gamma(G)$ such that

$$Y_{*k} - \Big(u_k^- + \sum_{j=1}^{M} S^*_{kj} u_j^+ \Big) \in \mathbb{V}_\gamma(G), \quad k = 1, \dots, M; \tag{3.40}$$

moreover, the matrix S^* is adjoint to the scattering matrix S from (3.4) and is a contraction. As was mentioned in the beginning of this chapter, we will assume that

$$\dim \ker \mathcal{A}(\gamma) = 0, \quad \dim \ker \mathcal{A}_*(\gamma) = 0 \tag{3.41}$$

for some $\gamma < \delta/2$, where the number δ is responsible for the stabilization rate of coefficients. Under this assumption, it is clear that the functions (3.4) and (3.40) form bases of the spaces $\ker \mathcal{A}(-\gamma)$ and $\ker \mathcal{A}_*(-\gamma)$ respectively. We note that the conditions (3.41) are satisfied, for example, if $\ker \mathcal{A}(\gamma) = 0$ and $\ker \mathcal{A}_*(\gamma') = 0$ for some number $\gamma' < \delta/2$. To see this, it suffices to apply Proposition 2.14 to the original problem and to the formally adjoint problem taking into account that their scattering matrices are mutually adjoint.

The problem (3.1) with the right-hand side $\{\mathcal{F}, \mathcal{G}\} \in \mathbb{W}_\gamma(G)$ is (uniquely) solvable in the space $\mathbb{V}_\gamma(G)$ if and only if

$$(\mathcal{F}, Y_{*k})_G + (\mathcal{G}, \mathcal{Q}Y_{*k})_{\partial G} = 0, \quad k = 1, \dots, M. \tag{3.42}$$

3.3 Estimate on the Norm of the Inverse Operator of Problem (3.10)

The estimate is based on the construction of an approximate solution satisfying the problem (3.10) with a discrepancy decaying exponentially as $R \to \infty$.

Such a solution can be obtained by the compound expansion method (cf. [17]): the approximate solution is “composed” from solutions to the limit problems (3.1) and (3.23) in the domain G and in the semicylinders $\Pi_-^{r,0}$, $r = 1, \dots, N$. This scheme is standard, one should only check its availability.

We assume that the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \gamma\}$ contains some eigenvalues of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$. If this strip is free from the spectrum of the pencils, then the arguments below can be modified in an obvious way.

Constructing an approximate solution, we obtain a linear algebraic system. The method can be applied if this system is solvable. The matrix of the system is expressed in terms of the scattering matrices of the limit problems. The norm of the scattering matrix of the problem (3.1) in the domain G does not exceed 1, whereas the norms of the scattering matrices of the problems (3.23) in the semicylinders $\Pi_-^{r,0}$ are strictly less than 1. Owing to this fact, the determinant of the system differs from zero.

3.3.1 Constructing an approximate solution

The procedure comprises of several steps.

1) Calculation of right-hand sides of the limit problems in semicylinders. For a given right-hand side of the problem (3.10) in G^R we find the right-hand sides of the limit problems in the semicylinders.

2) Solving of the problems (3.23) with determined right-hand sides. The problems are considered in the spaces $\mathbb{V}_{-\gamma}(\Pi_-^{r,0})$ with $\gamma > 0$, where the operator $A^r(-\gamma, \zeta)$ (cf. (3.24)) has kernel of dimension M^r (Proposition 3.16).

3) Calculation of right-hand side of the problem (3.1) in the domain G.

4) Verification of the solvability of the problem (3.1) in the class $\mathbb{V}_\gamma(G)$ of functions exponentially decaying at infinity. (Since $\dim\ker A^r(-\gamma,\zeta) = M^r$ and $M = M^1 + \dots + M^N$, it is possible to find the right-hand side at the previous step so that the conditions (3.42) are satisfied and the problem is solvable.)

Let us implement this plan.

1) The right-hand sides of the problems in semicylinders. In the cylinder Π^r, we introduce the shift operator $(U_R u)(y^r, t^r) = u(y^r, t^r - R)$. Let $\rho_R \in C_c^\infty(\overline{G^R})$, $\rho_R(x) = 1$ for $x \in G^{R-2}$ and $\rho_R(x) = 0$ for $x \in G^R \setminus G^{R-1}$. We choose the right-hand side $\{F^r, G^r, H^r\}$ of the problem (3.23) in such a way that $H^r = h|\Gamma^{r,R}$. The function F^r coincides with $U_R^{-1}(1-\rho_R)f$ on the set $\{(y^r, t^r) \in \Pi_-^{r,0}, t^r > -2\}$ and vanishes on the remaining part of the semicylinder $\Pi_-^{r,0}$. The function G^r coincides with $U_R^{-1}(1-\rho_R)g$ on $\{(y^r, t^r) \in \partial\Pi_-^{r,0} \setminus \Gamma^{r,0}, t^r > -2\}$ and vanishes outside this set.

2) Solution of the problems in semicylinders. In Subsection 2.3, we introduced the waves $\{v_j^{r,\pm}\}_{j=1}^{M^r}$ in $\Pi_-^{r,0}$, $r = 1, \dots, N$ (cf. (3.26)), bases $\{X_j^r\}_{j=1}^{M^r}$, and $\{Z_j^r\}_{j=1}^{M^r}$ in $\ker A^r(-\gamma,\zeta)$ (here, $\zeta \lesseqgtr 0$), and scattering matrices $\mathfrak{S}^r$, $\mathfrak{T}^r$ (cf. Corollary 3.7). It is convenient to enumerate the elements of set $\cup_r\{v_j^{r,\pm}\}_{j=1}^{M^r}$ in such a way that the waves $v_1^\pm, \dots, v_{M^1}^\pm$ correspond to the semicylinder $\Pi_-^{1,0}$ and the waves $\{v_j^\pm : j \in \mathfrak{M}^r\}$

with subscripts from the set

$$\mathfrak{M}^r = \Big\{ \sum_{k=1}^{r-1} M^k + 1, \sum_{k=1}^{r-1} M^k + 2, \dots, \sum_{k=1}^{r-1} M^k + M^r \Big\}, \tag{3.43}$$

correspond to the semicylinder $\Pi_-^{r,0}$, $r = 1, \dots, N$. We also enumerate elements of the sets $\cup_r \{X_j^r\}_{j=1}^{M^r}$ and $\cup_r \{Z_j^r\}_{j=1}^{M^r}$ in such a way that

$$\begin{aligned} &X_k \in \ker A^r(-\gamma, -\zeta), \quad X_k - \Big(v_k^+ + \sum_{\ell \in \mathfrak{M}^r} \mathbb{S}^0_{k\ell} v_\ell^- \Big) \in \mathbb{V}_\gamma(\Pi_-^{r,0}), \quad k \in \mathfrak{M}^r, \\ &Z_k \in \ker A^r(-\gamma, \zeta), \quad Z_k - \Big(v_k^- + \sum_{\ell \in \mathfrak{M}^r} \mathbb{T}^0_{k\ell} v_\ell^+ \Big) \in \mathbb{V}_\gamma(\Pi_-^{r,0}), \quad k \in \mathfrak{M}^r, \end{aligned} \tag{3.44}$$

where $\zeta > 0$, $r = 1, \dots, N$, and

$$\begin{aligned} \mathbb{S}^0 &= \operatorname{diag}\{\mathfrak{S}^1, \mathfrak{S}^2, \dots, \mathfrak{S}^N\}, \\ \mathbb{T}^0 &= \operatorname{diag}\{\mathfrak{T}^1, \mathfrak{T}^2, \dots, \mathfrak{T}^N\} \end{aligned} \tag{3.45}$$

(cf. (3.44), (3.35), and (3.36)). It is clear that the set $\{U_R Z_j : j \in \mathfrak{M}^r\}$ for $\zeta > 0$ ($\{U_R X_j : j \in \mathfrak{M}^r\}$ for $\zeta < 0$) forms a basis in the kernel of the shifted limit problem

$$\begin{aligned} &\mathbf{L}^r(y, D_y, D_t) u(y,t) = (U_R F^r)(y,t), \quad (y,t) \in \Pi_-^{r,R}, \\ &\mathbf{B}^r(y, D_y, D_t) u(y,t) = (U_R G^r)(y,t), \quad (y,t) \in \partial\Pi_-^{r,R} \setminus \Gamma^{r,R}, \\ &(\mathbf{N}^r(y, D_t) + i\zeta \mathbf{D}^r(y, D_t)) u(y,t)|_{t=R} = H^r(t), \quad y \in \Omega^r, \end{aligned} \tag{3.46}$$

in the space $\mathbb{V}_{-\gamma}(\Pi_-^{r,R})$ equipped with the norm $\|\cdot; \mathbb{V}_{-\gamma}(\Pi_-^{r,R})\| = \|U_R^{-1}\cdot; \mathbb{V}_{-\gamma}(\Pi_-^{r,0})\|$. For the same kernel we choose the basis $\{Z_j^R : j \in \mathfrak{M}^r\}$ if $\zeta > 0$ (the basis $\{X_j^R : j \in \mathfrak{M}^r\}$ if $\zeta < 0$); moreover,

$$\begin{aligned} Z_j^R - \Big(v_j^+ + \sum_{k \in \mathfrak{M}^r} \mathbb{T}^R_{jk} v_k^- \Big) \in \mathbb{V}_\gamma(\Pi_-^{r,R}), \quad j \in \mathfrak{M}^r, \\ X_j^R - \Big(v_j^- + \sum_{k \in \mathfrak{M}^r} \mathbb{S}^R_{jk} v_k^+ \Big) \in \mathbb{V}_\gamma(\Pi_-^{r,R}), \quad j \in \mathfrak{M}^r, \end{aligned} \tag{3.47}$$

where the block-diagonal $M \times M$-matrices $\mathbb{T}^R$ and $\mathbb{S}^R$ consist of the scattering matrices $\mathfrak{T}^{r,R}$ and $\mathfrak{S}^{r,R}$ of the shifted limit problems (3.46) (as in the case of (3.45)). To prove the existence of such bases, it suffices to repeat the arguments from Subsection 2.3 replacing (3.23) with (3.46). The bases $\{Z_j^R : j \in \mathfrak{M}^r\}$, $\{X_j^R : j \in \mathfrak{M}^r\}$ and the matrices $\mathfrak{T}^{r,R}$, $\mathfrak{S}^{r,R}$ depend on R. The norms $\|\mathfrak{T}^{r,R}\|$ and $\|\mathfrak{S}^{r,R}\|$ are less than 1 for all R.

Let, for example, $\zeta > 0$. The general solution $w^r \in \mathbb{V}_{-\gamma}(\Pi_-^{r,0})$ to the limit problem (3.23) with the right-hand side $\{F^r, G^r, H^r\}$ admits the representation

$$w^r = u^r + \sum_{j \in \mathfrak{M}^r} c_j U_R^{-1} Z_j^R, \tag{3.48}$$

where the coefficients c_j are arbitrary and the partial solution $u^r \in \mathbb{V}_{-\gamma}(\Pi_-^{r,0})$ satisfies the condition (3.39), which takes the form

$$u^r - \sum_{k \in \mathfrak{M}^r} b_k v_k^- \in \mathbb{V}_\gamma(\Pi_-^{r,0}) \tag{3.49}$$

due to the new enumeration of the waves.

3) A right-hand side for the limit problem (3.1) in the domain G. Introduce a cut-off function $\eta_R \in C^\infty(\mathbb{R})$ such that $\eta_R(t) = 0$ for $t < R/2$ and $\eta_R(t) = 1$ for $t > R/2+1$. We clarify the contribution of the function w^r from (3.48) to the right-hand side of the problem (3.10). We extend the restriction $(\eta_R U_R w^r)|(G^R \cap \Pi_-^{r,R})$ by zero to G^R and preserve the same notation for the extension. We have

$$\{\mathcal{L}, \mathcal{B}\} \sum_r (\eta_R U_R w^r) = \sum_r \{\mathcal{L}^r, \mathcal{B}^r\} \eta_R \Big(U_R u^r + \sum_{j \in \mathfrak{M}^r} c_j Z_j^R \Big) \tag{3.50}$$

(the right-hand side is assumed to be given on $G^R \times \{\partial G^R \backslash \Gamma^R\}$ since $\operatorname{supp} \eta_R U_R u^r \subset G^R \cap \Pi_-^{r,R}$).

Denote by $\{\mathcal{F}, \mathcal{G}\}$ the vector $\{f, g\}$ minus the right-hand side of the equality (3.50), where f and g are taken from (3.10). Suppose that the function $\{\mathcal{F}, \mathcal{G}\}$ is extended to the domain G; moreover, $\{\mathcal{F}, \mathcal{G}\} \in \mathbb{W}_\gamma(G)$. If the problem (3.1) with the right-hand side $\{\mathcal{F}, \mathcal{G}\}$ has a solution $\mathcal{Y} \in \mathbb{V}_\gamma(G)$, then the sum $\mathcal{Y} + \sum \eta_R U_R w^r$ satisfies the problem (3.10), perhaps, except for the equation on Γ^R. The discrepancy on Γ^R exponentially decreases as $R \to \infty$. The above solution $\mathcal{Y}$ exists only if the orthogonality conditions (3.42) are satisfied. We intend to provide the validity of these conditions by an appropriate choice of the coefficients $c_1, \ldots, c_M$. To simplify the linear algebraic system for $\{c_j\}$, we eliminate from $\{\mathcal{F}, \mathcal{G}\}$ some "nonessential" terms such that the new approximate solution

$$\mathcal{Y}^R = \mathcal{Y} + \sum_r \eta_R U_R w^r \tag{3.51}$$

leaves an exponentially decaying (as $R \to \infty$) discrepancy not only in the third equation of (3.10) but also in the first and second equations. In Proposition 3.11, we estimate the eliminated terms. Recall that e_β denotes a smooth positive function on $\overline{G}$ that coincides with $x = (y^r, t^r) \mapsto \exp(\beta t^r)$ in Π_+^r for $r = 1, \ldots, N$.

Proposition 3.11. *Suppose that the assumptions of Proposition* 3.6 *are satisfied and*

$$\operatorname{Im} \lambda_{-\nu} \leqslant \operatorname{Im} \lambda_{-\nu+1} \leqslant \cdots \leqslant \operatorname{Im} \lambda_\nu.$$

Let $u^r \in \mathbb{V}_{-\gamma}(\Pi_-^{r,0})$ *be the solution to the problem* (3.23) (*with* $\zeta > 0$) *subjected to* *the condition* (3.49) *and let the set* $\{Z_j^R, j \in \mathfrak{M}^r\}$ *be the bases* (3.47). *Then*

$$\Big\|e_\beta\{\mathcal{L}^r, \mathcal{B}^r, 0\}\eta_R U_R\Big(u^r - \sum_{\ell\in\mathfrak{M}^r} b_\ell v_\ell^-\Big) - e_\beta U_R\{F^r, G^r, 0\}; \mathbb{W}(G^R)\Big\|$$
$$\leqslant C\exp(-\mu R)\|\{F^r, G^r, H^r\}; \mathbb{W}_\gamma(\Pi_-^{r,0})\|, \tag{3.52}$$
$$\Big\|e_\beta\{\mathcal{L}^r, \mathcal{B}^r, 0\}\eta_R\Big(Z_j^R - v_j^+ - \sum_{k\in\mathfrak{M}^r} \mathbb{T}_{jk}^R v_k^-\Big); \mathbb{W}(G^R)\Big\|$$
$$\leqslant C\exp(-\mu R + \operatorname{Im}\lambda_\nu R)R^{\varkappa-1}, \tag{3.53}$$

where $\mu = (\gamma - \beta)/2 > 0$ *and* $\varkappa$ *denotes the length of the longest Jordan chain among those corresponding to the eigenvalues* $\{\lambda : \operatorname{Im}\lambda = \operatorname{Im}\lambda_\nu\}$ *of the pencil* $\mathfrak{A}^r$.

In the case $\zeta < 0$, *we have* (3.52) *and* (3.53) *with* b_j, v_j^+, v_j^-, Z_j^R, $\mathbb{T}^R$ *replaced with* a_j, v_j^-, v_j^+, X_j^R, $\mathbb{S}^R$ *respectively.*

Proof. We have

$$\begin{aligned}\{\mathcal{L}^r, \mathcal{B}^r\}\eta_R U_R u^r &= \eta_R\{\mathcal{L}^r, \mathcal{B}^r\}U_R u^r + [\{\mathcal{L}^r, \mathcal{B}^r\}, \eta_R]U_R u^r\\ &= \eta_R\{\mathcal{L}^r - \mathbf{L}^r, \mathcal{B}^r - \mathbf{B}^r\}U_R u^r + U_R\{F^r, G^r\} + [\{\mathcal{L}^r, \mathcal{B}^r\}, \eta_R]U_R u^r.\end{aligned} \tag{3.54}$$

Recall that $\{\mathbf{L}^r, \mathbf{B}^r\}v_j^- = 0$ for $j \in \mathfrak{M}^r$. Hence

$$\begin{aligned}[\{\mathcal{L}^r, \mathcal{B}^r\}, \eta_R]U_R u^r &= [\{\mathcal{L}^r, \mathcal{B}^r\}, \eta_R]U_R\Big(u^r - \sum_{\ell\in\mathfrak{M}^r} b_\ell v_\ell^-\Big)\\ &+ \{\mathcal{L}^r, \mathcal{B}^r\}\eta_R U_R \sum_{\ell\in\mathfrak{M}^r} b_\ell v_\ell^- - \eta_R\{\mathcal{L}^r - \mathbf{L}^r, \mathcal{B}^r - \mathbf{B}^r\}U_R \sum_{\ell\in\mathfrak{M}^r} b_\ell v_\ell^-.\end{aligned} \tag{3.55}$$

The coefficients of the operator $\eta_R\{\mathcal{L}^r - \mathbf{L}^r, \mathcal{B}^r - \mathbf{B}^r\}$ are decreasing of order $O(e^{-\delta R/2})$ as $R \to +\infty$. Further,

$$\begin{aligned}\Big\|e_\beta\eta_R\{\mathcal{L}^r - \mathbf{L}^r, \mathcal{B}^r - \mathbf{B}^r, 0\}U_R\Big(u^r - \sum_{\ell\in\mathfrak{M}^r} b_\ell v_\ell^-\Big); \mathbb{W}(G^R)\Big\|&\\ \leqslant Ce^{(\beta-\delta)R/2}\Big\|u^r - \sum_{\ell\in\mathfrak{M}^r} b_\ell v_\ell^-; \mathbb{V}_\gamma(\Pi_-^{r,0})\Big\|&\\ \leqslant Ce^{(\beta-\delta)R/2}\|\{F^r, G^r, H^r\}; \mathbb{W}_\gamma(\Pi_-^{r,0})\|&\end{aligned} \tag{3.56}$$

(at the last step, we used the inequality from Proposition 3.10).

We estimate the first term on the right-hand side of (3.55). The supports of the coefficients of the operator $[\{\mathcal{L}^r, \mathcal{B}^r\}, \eta_R]$ are located inside the set $\{(y^r, t^r) \in \Pi^r : R/2 \leqslant t^r \leqslant R/2 + 1\}$. Using the inequality from Proposition 3.10, we find

$$\begin{aligned}\Big\|e_\beta[\{\mathcal{L}^r, \mathcal{B}^r, 0\}, \eta_R]U_R\Big(u_r - \sum_{\ell\in\mathfrak{M}^r} b_\ell v_\ell^-\Big); \mathbb{W}(G^R)\Big\|&\\ \leqslant Ce^{(\beta-\gamma)R/2}\Big\|u^r - \sum_{\ell\in\mathfrak{M}^r} b_\ell v_\ell^-; \mathbb{V}_\gamma(\Pi_-^{r,0})\Big\|&\\ \leqslant Ce^{(\beta-\gamma)R/2}\|\{F^r, G^r, H^r\}; \mathbb{W}_\gamma(\Pi_-^{r,0})\|.&\end{aligned} \tag{3.57}$$

By (3.54)–(3.57), we obtain (3.52), where $\mu = 1/2 \min\{\gamma - \beta, \delta - \beta\} = (\gamma - \beta)/2$.

We proceed by proving the estimate (3.53). We set

$$\mathcal{Z}_j^R = Z_j^R - v_j^+ - \sum_{k \in \mathfrak{M}^r} \mathbb{T}_{jk}^R v_k^-, \quad j \in \mathfrak{M}^r. \tag{3.58}$$

We have $\mathcal{Z}_j^R \in \mathbb{V}_\gamma(\Pi_-^{r,R})$ (cf. (3.47)). Note that

$$\{\mathbf{L}^r, \mathbf{B}^r\}\Big(v_j^+ + \sum_{k \in \mathfrak{M}^r} \mathbb{T}_{jk}^R v_k^-\Big) = 0.$$

We have

$$\begin{aligned}\{\mathcal{L}^r, \mathcal{B}^r\}\eta_R Z_j^R &= \{\mathcal{L}^r, \mathcal{B}^r\}\eta_R\Big(v_j^+ + \sum_{k \in \mathfrak{M}^r} \mathbb{T}_{jk}^R v_k^-\Big) \\ &\quad + \{\mathcal{L}^r - \mathbf{L}^r, \mathcal{B}^r - \mathbf{B}^r\}\eta_R \mathcal{Z}_j^R + [\{\mathbf{L}^r, \mathbf{B}^r\}, \eta_R]\mathcal{Z}_j^R.\end{aligned} \tag{3.59}$$

To repeat the arguments used in (3.56) and (3.57), it is necessary to estimate the norm $\|\mathcal{Z}_j^R; \mathbb{V}_\gamma(\Pi_-^{r,R})\|$ as $R \to +\infty$. For this purpose we note that the function $\mathcal{Z}_j^R$ satisfies the problem (3.46), where $F^r = 0$, $G^r = 0$, and $H^r = (\mathbf{N}^r + i\zeta\mathbf{D}^r)(v_j^+ + \sum_{k \in \mathfrak{M}^r} \mathbb{T}_{jk}^R v_k^-)$. The operator of the problem (3.46) is a Fredholm operator, its kernel in the space $\mathbb{V}_\gamma(\Pi_-^{r,R})$ is trivial (Proposition 3.3). Consequently,

$$\|\mathcal{Z}_j^R; \mathbb{V}_\gamma(\Pi_-^{r,R})\| \leqslant c\Big\|\{0, 0, (\mathbf{N}^r + i\zeta\mathbf{D}^r)\Big(v_j^- + \sum_{k \in \mathfrak{M}^r} \mathbb{T}_{jk}^R v_k^+\Big)\}; \mathbb{W}_\gamma(\Pi_-^{r,R})\Big\|, \tag{3.60}$$

where the constant c is independent of R. We emphasize that the waves $v_j^\pm$ are linear combinations (with coefficients independent of R) of the functions (2.12) and the norm of the matrix $\mathbb{T}^R$ is less than 1 for all $R > 0$. Therefore, the right-hand side of (3.60) does not exceed $C \exp(\operatorname{Im} \lambda_\nu R) R^{\varkappa - 1}$ (here, λ_ν and $\varkappa$ are the same as in the formulation of the proposition). Repeating the arguments used in the proof of the estimate (3.52), we find

$$\|e_\beta\{\mathcal{L}^r - \mathbf{L}^r, \mathcal{B}^r - \mathbf{B}^r, 0\}\eta_R \mathcal{Z}_j^R; \mathbb{W}(G^R)\| \leqslant c \exp((\beta - \delta)R/2 + \operatorname{Im} \lambda_\nu R) R^{\varkappa - 1}. \tag{3.61}$$

Moreover,

$$\begin{aligned}\|e_\beta[\{\mathbf{L}^r, \mathbf{B}^r, 0\}, \eta_R]\mathcal{Z}_j^R; \mathbb{W}(G^R)\| &\leqslant c e^{(\beta - \gamma)R/2} \|\mathcal{Z}_j^R; \mathbb{V}_\gamma(\Pi_-^{r,R})\| \\ &\leqslant c \exp((\beta - \delta)R/2 + \operatorname{Im} \lambda_\nu R) R^{\varkappa - 1}.\end{aligned} \tag{3.62}$$

The inequalities (3.59), (3.61), and (3.62) lead to the estimate (3.53). □

In the case $\zeta > 0$, for the right-hand side of the limit problem (3.1) we take

$$\{\mathcal{F}, \mathcal{G}\} = \rho_R\{f, g\} - \{\mathcal{L}, \mathcal{B}\} \sum_{r=1}^{N} \eta_R\Big(\sum_{\ell \in \mathfrak{M}^r} b_\ell U_R v_\ell^- + \sum_{j \in \mathfrak{M}^r} c_j\Big(v_j^+ + \sum_{k \in \mathfrak{M}^r} \mathbb{T}_{jk}^R v_k^-\Big)\Big), \tag{3.63}$$

where the function $\rho_R\{f,g\}$ is extended by zero to the set $G \times \partial G$. Necessary modifications in the case $\zeta < 0$ are listed in Proposition 3.11.

4) Solvability of the problem (3.1) in the class $\mathbb{V}_\gamma(G)$. (Verification of the orthogonality conditions (3.42) for the right-hand side (3.63).) Substituting (3.63) into (3.42), we obtain the following system for the constants $c_1, \dots, c_M$:

$$q\Big(\sum_{r=1}^{N} \eta_R \sum_{j\in\mathfrak{M}^r} c_j\Big(v_j^+ + \sum_{k\in\mathfrak{M}^r} \mathbb{T}_{jk}^R v_k^-\Big), Y_{*\ell}\Big) = (\rho_R f, Y_{*\ell})_G + (\rho_R g, QY_{*\ell})_{\partial G}$$
$$-q\Big(\sum_{r=1}^{N} \eta_R \sum_{k\in\mathfrak{M}^r} b_k U_R v_k^-, Y_{*\ell}\Big), \quad \ell = 1, \dots, M; \tag{3.64}$$

the definition and properties of the form q are given in Subsection 2.1.4. By (3.40), the system (3.64) can be written in the form

$$q\Big(\sum_{r=1}^{N} \eta_R \sum_{j\in\mathfrak{M}^r} c_j\Big(v_j^+ + \sum_{k\in\mathfrak{M}^r} \mathbb{T}_{jk}^R v_k^-\Big), u_\ell^- + \sum_{m=1}^{M} S_{\ell m}^* u_m^+\Big) = (\rho_R f, Y_{*\ell})_G$$
$$+(\rho_R g, QY_{*\ell})_{\partial G} - q\Big(\sum_{r=1}^{N} \eta_R \sum_{k\in\mathfrak{M}^r} b_k U_R v_k^-, u_\ell^- + \sum_{m=1}^{M} S_{\ell m}^* u_m^+\Big). \tag{3.65}$$

In the case $\zeta < 0$, instead of (3.65) we obtain the system

$$q\Big(\sum_{r=1}^{N} \eta_R \sum_{j\in\mathfrak{M}^r} c_j\Big(v_j^- + \sum_{k\in\mathfrak{M}^r} \mathbb{S}_{jk}^R v_k^+\Big), u_\ell^- + \sum_{m=1}^{M} S_{\ell m}^* u_m^+\Big) = (\rho_R f, Y_{*\ell})_G$$
$$+(\rho_R g, QY_{*\ell})_{\partial G} - q\Big(\sum_{r=1}^{N} \eta_R \sum_{k\in\mathfrak{M}^r} a_k U_R v_k^+, u_\ell^- + \sum_{m=1}^{M} S_{\ell m}^* u_m^+\Big), \tag{3.66}$$

where $\ell = 1, \dots, M$. Thus, to verify if the compound expansion method can be used, it suffices to check the solvability of the systems (3.65) and (3.66) for the coefficients $c_1, \dots, c_M$.

Recall that the wave $u_j^\pm$ in Π_+ is defined by the equality $u_j^\pm = \chi w_j^\pm$, where $w_j^\pm$ is a linear combination of the functions (2.12) and χ is a cut-off function. The bases $\{w_j^\pm\}$ and $\{v_j^\pm\}$ were introduced independently. Now, to compute the left-hand sides of the equalities (3.65) and (3.66), we choose consistently $\{w_j^\pm\}$ and $\{v_j^\pm\}$. Namely, let $w_j^\pm$ be chosen in such a way that $w_j^\pm = v_j^\mp$. Then the left-hand side of (3.65) takes the form

$$q\Big(\sum_{r=1}^{N} \eta_R \sum_{j\in\mathfrak{M}^r} c_j\Big(u_j^- + \sum_{k\in\mathfrak{M}^r} \mathbb{T}_{jk}^R u_k^+\Big), u_\ell^- + \sum_{m=1}^{M} S_{\ell m}^* u_m^+\Big). \tag{3.67}$$

From the Green formula (3.2) it follows that the value of the expression (3.67) remains unchanged if we omit the cut-off function η_R. By (2.17) (where $\rho = iq$) we have

$$q\Big(\sum_{r=1}^{N}\sum_{j\in\mathfrak{M}^r} c_j\Big(u_j^- + \sum_{k\in\mathfrak{M}^r}\mathbb{T}^R_{jk}u_k^+\Big), u_\ell^- + \sum_{m=1}^{M} S^*_{\ell m}u_m^+\Big) = ic_\ell - i\sum_{k=1}^{M} S_{k\ell}\sum_{j=1}^{M} c_j\mathbb{T}^R_{jk}. \tag{3.68}$$

Comparing (3.65) and (3.68), we arrive at a linear system relative to the constants $c_j = c_j(R)$ in the case $\zeta > 0$:

$$i(c_1, \dots, c_M)(I - \mathbb{T}^R S) = \mathbb{H}, \tag{3.69}$$

where the row $\mathbb{H} = (\mathbb{H}_1(R), \dots, \mathbb{H}_M(R))$ is composed of the values of the right-hand side of (3.65) for $\ell = 1, \dots, M$. For $\zeta < 0$ we obtain, instead of (3.69), the system

$$i(c_1, \dots, c_M)(\mathbb{S}^R - S) = \mathbb{H}. \tag{3.70}$$

The norms of the matrices $\mathbb{S}^R$ and $\mathbb{T}^R$ are less than 1 (it suffices to note that the norms of the blocks $\mathfrak{S}^{r,R}$ and $\mathfrak{T}^{r,R}$ of the block-diagonal matrices $\mathbb{S}^R$ and $\mathbb{T}^R$ are less than 1 for $R < \infty$, cf. Proposition 3.9). The norm of the scattering matrix S does not exceed 1. Therefore, the matrix $I - \mathbb{T}^R S$ is nonsingular for all $R < \infty$. In the general case, we can say nothing about the matrix $\mathbb{S}^R - S$. For example, if the matrix S turns out to be unitary, then $\det(\mathbb{S}^R - S) \neq 0$ (in particular, the scattering matrix of the self-adjoint problem is unitary, cf. [1]–[5]). If the limit problem (3.1) coincides with the problem (3.23) after the replacement of ζ with $-\zeta$ ($\zeta < 0$), then $\mathbb{S}^R - S = 0$. Thus, we have proved the following assertion.

Proposition 3.12. *The determinant of the system* (3.69) *does not vanish, and there exists a solution* $\mathcal{Y} \in \mathbb{V}_\gamma(G)$ *to the problem* (3.1) *with the right-hand side* (3.63)*, where the coefficients* $c_1, \dots, c_M$ *satisfy Eq.* (3.69)*. Therefore, the compound expansions method can be used in the case* $\zeta > 0$*.*

Thus, we have obtained an approximate solution (3.51) to the problem (3.10) with the right-hand side $\{f, g, h\}$ and positive ζ. In the case $\zeta < 0$, the coefficients $c_1, \dots, c_M$ in formula (3.63), where the replacements have been made in accordance with Proposition 3.11, must satisfy the system (3.70) whose determinant may vanish. We do not know if the compound expansions method can be used in this case. In order to estimate the discrepancies left in Eqs. (3.10), we study the behavior of solution $(c_1(R), \dots, c_M(R))$ to the system (3.69) as $R \to \infty$.

3.3.2 The behavior of the solution $(c_1(R), \dots, c_M(R))$ to the system (3.69) as $R \to +\infty$

We estimate the norm of the inverse operator $(I - \mathbb{T}^R S)^{-1}$ and the norm of the right-hand side $\|\mathbb{H}(R)\|$ of the system (3.69) as $R \to +\infty$. Thereby we estimate the vector $(c_1(R), \dots, c_M(R))$.

We first assume that the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \gamma\}$ contains only simple real eigenvalues of the pencils.

Proposition 3.13. *Let the strip* $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \gamma\}$ *contain only simple real eigenvalues of the pencils* $\mathfrak{A}^1, \dots, \mathfrak{A}^N$*. Then the solution*

$$(c_1(R), \dots, c_M(R))$$

to the system (3.69) *for* $\zeta > 0$ *admits the estimate*

$$|c_1(R)| + \dots + |c_M(R)| \leqslant C\|\{e_\beta f, e_\beta g, h\}; \mathbb{W}(G^R)\|$$

with any $\beta \in (0, \gamma)$.

Proof. Denote by $\lambda_1^+, \dots, \lambda_M^+$ the eigenvalues of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$ corresponding to the incoming waves $v_j^+(y,t) = \exp(i\lambda_j^+ t)\varphi_j(y)$, $(y,t) \in \Pi^r$, $j \in \mathfrak{M}^r$. Let $\lambda_1^-, \dots, \lambda_M^-$ correspond to outgoing waves $v_j^-(y,t) = \exp(i\lambda_j^- t)\varphi_j(y)$ (the definition of the set $\mathfrak{M}^r$ see in (3.43)). It is clear that $U_R v_j^\pm = \exp(-i\lambda_j^\pm R)v_j^\pm$. Introduce the matrices

$$V_R^+ = \operatorname{diag}\{e^{-i\lambda_1^+ R}, \dots, e^{-i\lambda_M^+ R}\}, \quad V_R^- = \operatorname{diag}\{e^{-i\lambda_1^- R}, \dots, e^{-i\lambda_M^- R}\}.$$

By (3.44) and (3.47), we find

$$\mathbb{T}^R = (V_R^-)^* \mathbb{T}^0 V_R^+.$$

Recall that $\|\mathbb{T}^0\| < 1$ (cf. (3.45) and Proposition 3.9) and $\|S\| \leqslant 1$. The following inequality holds:

$$\|(I - \mathbb{T}^R S)^{-1}\| < C < \infty.$$

We estimate the right-hand side of the system (3.69). We have

$$|\mathbb{H}_\ell| \leqslant \Big| q\Big(\sum_{r=1}^N \eta_R \sum_{k \in \mathfrak{M}^r} b_k e^{-i\lambda_k^- R} u_k^+, u_\ell^- + \sum_{m=1}^M S_{\ell m}^* u_m^+ \Big) \Big|$$
$$+ |(\rho_R f, Y_{*\ell})| + |(\rho_R g, Q Y_{*\ell})| \leqslant C\Big(\|\{e_\beta f, e_\beta g, 0\}; \mathbb{W}(G^R)\| + \sum_{k=1}^M |b_k| \Big) \quad (3.71)$$

(the form q is immediately computed as in (3.68)). Taking into account Proposition 3.10 and the fact that $\operatorname{supp}\{F^r, G^r, H^r\}$ is bounded and is independent of R, we find

$$\sum |b_k| \leqslant C\|\{F^r, G^r, H^r\}; \mathbb{W}_\gamma(\Pi_-^{r,0})\| \leqslant C\|\{F^r, G^r, H^r\}; \mathbb{W}_0(\Pi_-^{r,0})\|$$
$$\leqslant C\|\{f, g, h\}; \mathbb{W}(G^R)\| \leqslant C\|\{e_\beta f, e_\beta g, h\}; \mathbb{W}(G^R)\|. \quad (3.72)$$

Now, formulas (3.71) and (3.72) lead to the estimate

$$\|\mathbb{H}\| \leqslant C\|\{e_\beta f, e_\beta g, h\}; \mathbb{W}(G^R)\|, \quad (3.73)$$

where the constant C is independent of R. □

In the general case, the following assertion holds.

Proposition 3.14. *The solution* $(c_1(R), \dots, c_M(R))$ *to the system* (3.69) *satisfies the estimate*

$$|c_1(R)| + \cdots + |c_M(R)| \leqslant c \exp\{3\aleph R\} R^{2\varkappa-2} \|\{e_\beta f, e_\beta g, h\}; \mathbb{W}(G^R)\|. \qquad (3.74)$$

Here, $\aleph = \operatorname{Im} \lambda_{\max}(\gamma)$ *and* $\lambda_{\max}(\gamma)$ *is the eigenvalue with the largest imaginary part among all the eigenvalues of the pencils* $\mathfrak{A}^1, \dots, \mathfrak{A}^N$ *in the strip* $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < \gamma\}$ *and* $\varkappa$ *denotes the number of elements of the longest Jordan chain corresponding to the eigenvalues located on the line* $\{\lambda : \operatorname{Im} \lambda = \aleph\}$*. Finally,* $\beta \in (\aleph, \gamma)$*.*

Proof. We first assume that the strip $\{\lambda \in \mathbb{C} \,|\operatorname{Im} \lambda| < \gamma\}$ contains a single eigenvalue of the pencil; moreover, it is an eigenvalue of the pencil $\mathfrak{A}^1$. Since the spectrum is symmetric, this eigenvalue is real. We assume that there is a unique Jordan chain $\{\varphi^{(0)}, \dots, \varphi^{(2m-1)}\}$ corresponding to this eigenvalue. Under these assumptions, the matrix $\mathbb{T}^R$ coincides with the block $\mathfrak{T}^{1,R}$.

We estimate the norm of the inverse matrix of the system (3.69). We outline a further consideration. Let the functions $Z_1^1, \dots, Z_M^1$ satisfy the inclusions (3.35) and form a basis for the space of solutions to the homogeneous problem (3.23) in $\Pi_-^{1,0}$. As above, $(U_R\varphi)(t) = \varphi(t - R)$. It is clear that

$$U_R Z_n^1 \sim U_R v_n^+ + \sum_j \mathfrak{T}_{nj}^1 U_R v_j^- \qquad (3.75)$$

(i.e. the left-hand side coincides with the right-hand side up to a summand in $\mathbb{V}_{-\gamma}(\Pi_-^{1,R})$; in the sequel, the notation $\sim$ is understood in the same sense). The collection $\{U_R Z_n^1\}_{n=1}^M$ forms a basis for the space of solutions to the homogeneous problem (3.46); moreover,

$$U_R Z_n^1 = \sum_{j=1}^M \alpha_{nj}(R) Z_j^{1,R}, \quad n = 1, \dots, M, \qquad (3.76)$$

where $Z_j^{1,R}$ are solutions to the same problem with asymptotics

$$Z_j^{1,R} \sim v_j^+ + \sum \mathfrak{T}_{jk}^{1,R} v_k^-. \qquad (3.77)$$

By (3.77), we have

$$\sum_{j=1}^M \alpha_{nj}(R) Z_j^{1,R} \sim \sum_{j=1}^M \alpha_{nj}(R)(v_j^+ + \sum_{k=1}^M \mathfrak{T}_{jk}^{1,R} v_k^-). \qquad (3.78)$$

The functions $U_R v_n^+$ and $U_R v_n^-$ are linearly expressed in terms of $v_n^\pm$; moreover, the role of coefficients are played by polynomials in R. We write formula (3.75) in the form

$$U_R Z_n^1 \sim \sum_k P_{nk}(R) v_k^+ + \sum_k Q_{nk}(R) v_k^-, \qquad (3.79)$$

where $P_{nk}(R)$ and $Q_{nk}(R)$ are some polynomials. From formulas (3.76)–(3.79) it follows that

$$\sum_{k=1}^{M} P_{nk}(R)v_k^+ + \sum_{k=1}^{M} Q_{nk}(R)v_k^- \sim \sum_{j=1}^{M} \alpha_{nj}(R)(v_j^+ + \sum_{\ell=1}^{M} \mathfrak{T}_{j\ell}^{1,R} v_\ell^-).$$

The vectors $\{\alpha_n(R) = (\alpha_{n1}(R), \dots, \alpha_{n\,M}(R)) : n = 1, \dots, M\}$ in (3.76) are linearly independent for all finite R. Therefore, to estimate the norm

$$\|\mathfrak{T}^{1,R}\| = \sup_{\alpha} \frac{\|\alpha \mathfrak{T}^{1,R}\|}{\|\alpha\|} \tag{3.80}$$

as $R \to +\infty$, we can represent the vector $\alpha \in \mathbb{C}^m$ in the form $\alpha = \sum a^n \alpha_n$. Since $\alpha_{nk} = P_{nk}$ and $(\alpha_n \mathfrak{T}^{1,R})_k = Q_{nk}$, the right-hand side of the equality (3.80) as $R \to +\infty$ behaves itself as the ratio of two polynomials in R. To prove the relation $(1 - \|\mathfrak{T}^{1,R}\|)^{-1} = O(R^\nu)$ as $R \to +\infty$ with some $\nu > 0$, it suffices to find a vector $\alpha = \alpha(R)$ such that $\|\alpha \mathfrak{T}^{1,R}\| / \|\alpha\| \to 1$ (recall that $\|\mathfrak{T}^{1,R}\| < 1$ for all $R < \infty$). This leads to the estimate

$$\|(I - \mathbb{T}^R S)^{-1}\| \leqslant (1 - \|\mathbb{T}^R\|)^{-1} \leqslant cR^\nu$$

for the inverse matrix of the system (3.69). Now, we follow the above scheme.

To express $U_R v_n^+$ and $U_R v_n^-$ in terms of $\{v_k^\pm\}$, we represent the waves as linear combinations of the functions (2.12). By the above assumptions, the canonical system of Jordan chains contains only one chain $\{\varphi^0, \dots, \varphi^{2m-1}\}$. The functions (2.12) have the form

$$u^{(\sigma)}(y,t) = \exp(i\lambda t) \sum_{\ell=0}^{\sigma} \frac{1}{\ell!}(it)^\ell \varphi^{(\sigma-\ell)}(y), \quad \sigma = 0, \dots, 2m-1. \tag{3.81}$$

As is known [1], [2], a Jordan chain can be chosen in such a way that

$$p(u^{(\sigma)}, u^{(\tau)}) = \pm i\delta_{2m-1-\tau,\sigma}. \tag{3.82}$$

We set

$$V_n^\pm = 2^{-1/2}(u^{(n-1)} \pm u^{(2M-n)}), \tag{3.83}$$

where $n = 1, \dots, M$ and $M = m$. By (3.82), we have $p(V_k^+, V_n^+) = \pm i\delta_{k,n}$ (the sign is the same as in (3.82)) and $p(V_k^-, V_n^-) = \mp i\delta_{k,n}$ (the sign is opposite to the sign in (3.82)). We set $v_n^+ = V_n^+$ if we have the sign "$-$" in (3.82) (for $k = n$) and $v_n^+ = V_n^-$ if we have the sign "$+$" in (3.82). Finally, $v_n^- = V_n^-$ ($v_n^- = V_n^+$ respectively) if the sign "$+$" ("$-$" respectively) takes place in (3.82) Thus, $\{v_n^+\}_{n=1}^M$ consists of incoming waves and $\{v_n^-\}_{n=1}^M$ consists of outgoing waves. Recalling (3.81), we find

$$\begin{aligned}
(U_R u^{(\sigma)})(y,t) &= \exp(i\lambda(t-R)) \sum_{\ell=0}^{\sigma} \frac{1}{\ell!} i^\ell (t-R)^\ell \varphi^{(\sigma-\ell)}(y) \\
&= \exp(-i\lambda R)\exp(i\lambda t) \sum_{k=0}^{\sigma} \frac{(-iR)^k}{k!} \sum_{n=0}^{\sigma-k} \frac{(it)^n}{n!} \varphi^{(\sigma-n-k)}(y) \\
&= \exp(-i\lambda R) \sum_{k=0}^{\sigma} \frac{(-iR)^k}{k!} u^{(\sigma-k)}(y,t).
\end{aligned}$$

Let, for example, $v_n^+ = V_n^+$ and $v_n^- = V_n^-$ for $n = 1, \dots, M$ (otherwise, obvious modifications of the formulas are required). By (3.83),

$$U_R v_n^\pm = 2^{-1/2} \exp(-i\lambda R)\Big(\sum_{k=0}^{n-1} \frac{(-iR)^k}{k!} u^{(n-k-1)}(y,t) \pm \sum_{k=0}^{2M-n} \frac{(-iR)^k}{k!} u^{(2M-n-k)}(y,t)\Big). \tag{3.84}$$

We note that

$$u^{(n-1)} = 2^{-1/2}(v_n^+ + v_n^-), \quad n = 1, \dots, M,$$
$$u^{(n-1)} = 2^{-1/2}(v_{2M-n+1}^+ - v_{2M-n+1}^-), \quad n = M+1, \dots, 2M.$$

Now, we rewrite (3.84) in the form

$$\begin{aligned} U_R v_n^\pm &= \frac{1}{2}\exp(-i\lambda R)\Big(\sum_{k=0}^{n-1} \frac{(-iR)^k}{k!}(v_{n-k}^+ + v_{n-k}^-) \pm \sum_{k=0}^{M-n} \frac{(-iR)^k}{k!}(v_{n+k}^+ - v_{n+k}^-) \\ &\quad \pm \sum_{k=M-n+1}^{2M-n} \frac{(-iR)^k}{k!}(v_{2M-n-k+1}^+ + v_{2M-n-k+1}^-)\Big) \\ &= \frac{1}{2}\exp(-i\lambda R)\Big(\sum_{k=1}^{n} \frac{(-iR)^{n-k}}{(n-k)!}(v_k^+ + v_k^-) \pm \sum_{k=n}^{M} \frac{(-iR)^{k-n}}{(k-n)!}(v_k^+ - v_k^-) \\ &\quad \pm \sum_{k=1}^{M} \frac{(-iR)^{2m-n-k+1}}{(2M-n-k+1)!}(v_k^+ + v_k^-)\Big). \end{aligned} \tag{3.85}$$

Substituting (3.85) into (3.75), we arrive at the relation

$$\begin{aligned} U_R Z_n^1 &= \frac{1}{2}\exp(-i\lambda R)\Big\{\sum_{k=1}^{n} \frac{(-iR)^{n-k}}{(n-k)!} v_k^+ + \sum_{k=n}^{M} \frac{(-iR)^{k-n}}{(k-n)!} v_k^+ \\ &+ \sum_{k=1}^{M} \frac{(-iR)^{2M-n-k+1}}{(2M-n-k+1)!} v_k^+ + \sum_{j=1}^{M} \mathfrak{T}_{nj}^1 \Big(\sum_{k=1}^{j} \frac{(-iR)^{j-k}}{(j-k)!} v_k^+ - \sum_{k=j}^{M} \frac{(-iR)^{k-j}}{(k-j)!} v_k^+ \\ &- \sum_{k=1}^{M} \frac{(-iR)^{2M-j-k+1}}{(2M-j-k+1)!} v_k^+\Big) + \sum_{k=1}^{n} \frac{(-iR)^{n-k}}{(n-k)!} v_k^- - \sum_{k=n}^{M} \frac{(-iR)^{k-n}}{(k-n)!} v_k^- \\ &+ \sum_{k=1}^{M} \frac{(-iR)^{2M-n-k+1}}{(2M-n-k+1)!} v_k^- + \sum_{j=1}^{M} \mathfrak{T}_{nj}^1 \Big(\sum_{k=1}^{j} \frac{(-iR)^{j-k}}{(j-k)!} v_k^- + \sum_{k=j}^{M} \frac{(-iR)^{k-j}}{(k-j)!} v_k^- \\ &- \sum_{k=1}^{M} \frac{(-iR)^{2M-j-k+1}}{(2M-j-k+1)!} v_k^-\Big)\Big\}. \end{aligned} \tag{3.86}$$

From (3.76), (3.78), and (3.86) it follows that the right-hand side of (3.78) differs from the right-hand side of (3.86) by a summand in $\mathbb{V}_{-\gamma}(\Pi_-^{1,R})$.

It remains to find a vector α such that $\|\alpha\mathfrak{T}^{1,R}\|/\|\alpha\| \to 1$ as $R \to \infty$. Let $\alpha = \alpha_1 = (\alpha_{11}, \dots, \alpha_{1M})$. By (3.86), the coefficient at R^{2M-1} in α_{11} is equal to $\exp(-i\lambda R)(1-\mathfrak{T}^1_{11})/2(2M-1)!$. Since $\|\mathfrak{T}^1\| < 1$, this coefficient does not vanish. Using (3.86) again, we see that the coefficient at the same power R^{2M-1} in the component $(\alpha_1\mathfrak{T}^{1,R})_1$ of the vector $\alpha_1\mathfrak{T}^{1,R} = ((\alpha_1\mathfrak{T}^{1,R})_1, \dots, (\alpha_1\mathfrak{T}^{1,R})_M)$ coincides with the coefficient obtained for α_{11}. We have $\|\alpha_1\mathfrak{T}^{1,R}\|/\|\alpha_1\| \to 1$ as $R \to \infty$ and

$$(1 - \|\mathfrak{T}^{1,R}\|)^{-1} = O(R^{2m-1}).$$

For simple eigenvalues an estimate for the right-hand side $\mathbb{H} = \mathbb{H}(R)$ of the system (3.69) is obtained via the inequality (3.73). In the case of multiple eigenvalues, one can use formula (3.85) in order to obtain the following inequalities instead of (3.71) – (3.73):

$$\begin{aligned} |\mathbb{H}_\ell| &\leqslant |(\rho_R f, Y_{*\ell})_G| + |(\rho_R g, Y_{*\ell})_{\partial G}| + CR^{\varkappa-1} \sum_{1\leqslant k\leqslant M} |b_k| \\ &\leqslant CR^{\varkappa-1} \|\{e_\beta f, e_\beta g, h\}; \mathbb{W}(G^R)\|. \end{aligned} \tag{3.87}$$

Thus, the estimate (3.74) is proved in the case where the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \gamma\}$ contains a single eigenvalue of the pencils and this eigenvalue is real. In the case of several real eigenvalues, the necessary modifications are obvious.

Now, we assume that the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \gamma\}$ can contain nonreal eigenvalues of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$. For the sake of simplicity, we consider the case where only the eigenvalues λ_1 and $\lambda_{-1}(= \overline{\lambda}_1 \notin \mathbb{R})$ of the pencil $\mathfrak{A}^1$ lie in the strip. Assume that with λ_1 (and, consequently, with λ_{-1}) only one Jordan chain is associated. We argue as above (cf. formulas (3.75)–(3.80) and comments), but the coefficients in the expressions of $U_R v_n^\pm$ in terms of $v_n^\pm$, as well as $P_{nk}(R)$ and $Q_{nk}(R)$, are not polynomials. We describe the necessary modifications. Let a Jordan chain be chosen in such a way that

$$p(u_\nu^{(\sigma)}, u_\mu^{(\tau)}) = i\delta_{-\nu,\mu}\delta_{\varkappa-\sigma-1,\tau},$$

where $u_\nu^{(\sigma)}$ are the functions in (2.12) (the subscript j is omitted); cf. [1]–[5]. Introduce $2M(=2\varkappa)$ waves by the equalities

$$v_n^+ = 2^{-1/2}(u_1^{(n-1)} - u_{-1}^{(M-n)}), \quad v_n^- = 2^{-1/2}(u_1^{(n-1)} + u_{-1}^{(M-n)}), \quad n = 1, \dots, M.$$

Instead of (3.85), we have

$$U_R v_n^\pm = \frac{1}{2}\Big(e^{-i\lambda_1 R} \sum_{\ell=1}^{n} \frac{(-iR)^{n-\ell}}{(n-\ell)!}(v_\ell^+ + v_\ell^-) \pm e^{-i\overline{\lambda}_1 R} \sum_{\ell=n}^{M} \frac{(-iR)^{\ell-n}}{(\ell-n)!}(v_\ell^+ - v_\ell^-)\Big). \tag{3.88}$$

Substituting (3.88) into (3.75), we obtain the relation

$$\begin{aligned} U_R Z_n^1 \sim \frac{1}{2}\Big\{ & e^{-i\lambda_1 R}\sum_{\ell=1}^{n}\frac{(-iR)^{n-\ell}}{(n-\ell)!}(v_\ell^+ + v_\ell^-) \\ & + e^{-i\bar\lambda_1 R}\sum_{\ell=n}^{M}\frac{(-iR)^{\ell-n}}{(\ell-n)!}(v_\ell^+ - v_\ell^-) \\ + \sum_{j=1}^{M}\mathfrak{T}_{nj}^1\Big(& e^{-i\lambda_1 R}\sum_{\ell=1}^{j}\frac{(-iR)^{j-\ell}}{(j-\ell)!}(v_\ell^+ + v_\ell^-) \\ & - e^{-i\bar\lambda_1 R}\sum_{\ell=j}^{M}\frac{(-iR)^{\ell-j}}{(\ell-j)!}(v_\ell^+ - v_\ell^-)\Big)\Big\}. \end{aligned} \tag{3.89}$$

Let, for example, $\lambda_1 = \lambda_{\max}$. By (3.89), the coefficient at

$$\exp(-i\lambda_1 R)(-iR)^{M-1}/(M-1)!$$

in α_{M1} and $(\alpha_M \mathfrak{T}^{1,R})_1$ is equal to $1 + \mathfrak{T}_{MM}^1$ and does not vanish. Therefore, $\|\alpha_1 \mathfrak{T}^{1,R}\|/\|\alpha_1\| \to 1$ as $R \to \infty$ and

$$(1 - \|\mathfrak{T}^{1,R}\|)^{-1} = O(\exp\{2\aleph R\}R^{\varkappa-1}).$$

Instead of (3.87), we have

$$\|\mathbb{H}_\ell\| \leqslant C\exp\{\aleph R\}R^{\varkappa-1}\|\{e_\beta f, e_\beta g, h\}; \mathbb{W}(G^R)\|.$$

The proposition is proved. □

3.3.3 Example

Let $\mathcal{P}$ be a "half-plane" in $\mathbb{R}^2$ with smooth periodic boundary $\partial\mathcal{P}$, i.e., $x = (x_1, x_2) \in \partial\mathcal{P}$ if and only if $(x_1 + 2\pi, x_2) \in \partial\mathcal{P}$. We assume that $\partial\mathcal{P} \subset \{x \in \mathbb{R}^2 : |x_2| < a\}$ for some positive a. Let $G = \{x \in \mathcal{P} : (x_1, x_2), |x_1| < \pi\}$ be the "periodicity cell." We set $\Gamma^\pm = \{x \in \mathcal{P} : x_1 = \pm\pi\}$ and $\Gamma^0 = \partial G \setminus \{\Gamma^+ \cup \Gamma^-\}$. Instead of (x_1, x_2) in G, we write (y, t).

Consider the Helmholtz equation

$$(\Delta + k^2)u(y,t) = 0, \quad (y,t) \in G, \tag{3.90}$$

with the quasiperiodicity condition

$$\partial_y^j u(\pi, t) = e^{2\pi i\alpha}\partial_y^j u(-\pi, t), \quad (\pm\pi, t) \in \Gamma^\pm, \quad j = 0, 1, \tag{3.91}$$

and the boundary condition

$$\mathcal{B}u(y,t) = 0, \quad (y,t) \in \Gamma_0; \tag{3.92}$$

here, k and α are real parameters, $\partial_y^j = \partial^j/\partial y^j$, and $\mathcal{B}$ is an operator with smooth coefficients such that $\operatorname{ord}\mathcal{B} \leqslant 1$ and the boundary value problem (3.90)–(3.92) is elliptic and formally self-adjoint. As is known, such a problem appears in the theory of diffraction gratings. We derive an explicit formulas for waves and scattering matrices $\mathfrak{S}^R$ and $\mathfrak{T}^R$ which were discussed in Subsection 3.1.

In the interval $-\pi < y < \pi$, we introduce the operator pencil $\lambda \mapsto \mathfrak{A}(\lambda) = \{d^2/dy^2 + (k^2 - \lambda^2)\}$ for functions v satisfying the quasiperiodicity condition $\partial_y^j v(\pi) = e^{2\pi i\alpha}\partial_y^j v(-\pi)$, $j = 0, 1$. The spectrum of the pencil $\mathfrak{A}$ consists of eigenvalues $\pm(k^2 - (n+\alpha)^2)^{1/2}$ for $n = 0, \pm 1, \dots$. As above, we denote by $\lambda_{-\nu^0}, \dots, \lambda_{\nu^0}$, where $\nu^0 \geqslant 0$, all the real eigenvalues of the pencil $\mathfrak{A}$; moreover, $\lambda_{-\nu^0} < \lambda_{-\nu^0+1} < \dots < \lambda_{\nu^0}$, $\lambda_0 = 0$ and λ_0 are absent if zero is an eigenvalue. We enumerate nonreal eigenvalues in such a way that $0 < \operatorname{Im}\lambda_{\nu^0+1} < \operatorname{Im}\lambda_{\nu^0+2} < \dots$ and $\lambda_\nu = \overline{\lambda}_{-\nu}$, where $\nu = \nu^0+1, \nu^0+2, \dots$. The eigenvalue $\lambda_{\pm\nu} = \pm(k^2 - (n+\alpha)^2)^{1/2}$, $\nu \neq 0$, is simple and the eigenvector $\varphi_{\pm\nu}$, $\varphi_{\pm\nu}(y) = \exp(i(n+\alpha)y)$ corresponds to this eigenvalue.

The eigenvalue $\lambda_0 = 0$ appears in the case $k^2 = (n+\alpha)^2$ and has algebraic multiplicity 2. With this eigenvalue we associate the eigenvector φ_0^0 and the adjoined vector φ_0^1, $\varphi_0^0(y) = \exp(i(n+\alpha)y)$, $\varphi_0^1(y) \equiv 0$. We set

$$
\begin{aligned}
&u_\nu(y,t) = (4\pi|\lambda_\nu|)^{-1/2}\exp(i\lambda_\nu t)\varphi_\nu(y), \quad \nu \geqslant -\nu_0, \nu \neq 0,\\
&u_\nu(y,t) = -i(4\pi|\lambda_\nu|)^{-1/2}\exp(i\lambda_\nu t)\varphi_\nu(y), \quad \nu < -\nu_0,\\
&u_0^0(y,t) = (2\pi)^{-1/2}\varphi_0^0(y), \quad u_0^1(y,t) = -i(2\pi)^{-1/2} t\varphi_0^0(y)
\end{aligned}
\tag{3.93}
$$

(cf. (2.12)). The functions (3.93) satisfy the homogeneous Helmholtz equation in the strip $\{(y,t) \in \mathbb{R}^2 : -\pi < y < \pi\}$ and the quasiperiodicity condition. Moreover, these functions satisfy the relations in Proposition 2.4, where for q^r we take the form

$$
q(u,v) = ((\Delta + k^2)u, v)_G + (\mathcal{B}u, \mathcal{Q}v)_{\Gamma_0} - (u, (\Delta + k^2)v)_G - (\mathcal{Q}u, \mathcal{B}v)_{\Gamma_0}. \tag{3.94}
$$

We introduce incoming and outgoing waves $\{u_\nu^\pm\}$. We first introduce the functions

$$
\begin{aligned}
&w_\nu^\pm = u_{\pm\nu}, \quad 0 < \nu \leqslant \nu_0; \quad w_\nu^\pm = 2^{-1/2}(u_\nu \mp u_{-\nu}), \quad \nu > \nu_0,\\
&w_0^\pm = 2^{-1/2}(u_0^0 \mp u_0^1).
\end{aligned}
$$

Take $\chi \in C^\infty(\mathbb{R})$ such that $\chi(t) = 1$ for $t \geqslant C + 1$ and $\chi(t) = 0$ for $t \leqslant C$, where the constant C is chosen in such a way that $\chi(t) = 0$ for $(y,t) \in \Gamma_0$. The waves $u_\nu^\pm = \chi w_\nu^\pm$ satisfy the conditions (2.17) with $\rho = iq$ (q is given by (3.94)).

The role of the problem (3.10) in the bounded domain G^R is played by the problem

$$
\begin{aligned}
&(\Delta + k^2)u(x) = f(x), \quad x \in G^R,\\
&\mathcal{B}u(x) = g(x), \quad x \in \Gamma_0,\\
&\partial_y^j u(\pi,t) = e^{2\pi i\alpha}\partial_y^j u(-\pi,t), \quad x = (\pm\pi,t) \in \Gamma^\pm \cap G^R,\\
&(\partial_t + i\zeta)u(y,t) = h(y,t), \quad x = (y,t) \in \Gamma^R.
\end{aligned}
\tag{3.95}
$$

For $f \in L_2(G^R)$, $g \in H^\mu(\Gamma_0)$, where $\mu = 3/2 - \operatorname{ord}\mathcal{B}$, and $h \in H^{1/2}(\Gamma^R)$ there exists a unique solution $u \in H^2(\Gamma^R)$ to the problem (3.95) (cf., for example, [22]).

The functional (3.7) takes the form

$$J_\ell^R(a_1,\dots,a_M) = \Big\|\mathfrak{X}_\ell^R - u_\ell^+ - \sum_{j=1}^M a_j u_j^-; L_2(\Gamma^R)\Big\|^2,$$

where $\mathfrak{X}_\ell^R$ is a solution to the problem (3.95) with the right-hand side

$$\Big\{0,0,(\partial_t + i\zeta)\Big(u_\ell^+ + \sum_{j=1}^M a_j u_j^-\Big)\Big\}.$$

Instead of (3.46), we have

$$\begin{aligned}
&(\Delta + k^2)u(x) = f(x), \quad x \in \Pi_-^R,\\
&\partial_y^j u(\pi,t) = e^{2\pi i\alpha}\partial_y^j u(-\pi,t), \quad x = (\pm\pi,t) \in \partial\Pi_-^R \setminus \Gamma^R,\\
&(\partial_t + i\zeta)u(y,t) = h(y), \quad x = (y,t) \in \Gamma^R.
\end{aligned} \tag{3.96}$$

Suppose that the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \gamma\}$ contains only real eigenvalues of the pencil $\mathfrak{A}$. The total algebraic multiplicity of all such numbers is denoted by $2T$. For the problem (3.96) it is easy to compute explicitly the $T\times T$-matrices $\mathfrak{S}^R$ and $\mathfrak{T}^R$. Indeed, assume that λ_0 does not belong to the spectrum of the pencil $\mathfrak{A}$ and $T = \nu_0$. Note that

$$(\partial_t + i\zeta)w_\nu^+|_{\Gamma^R} = \frac{\zeta+\lambda_\nu}{\zeta-\lambda_\nu}e^{2i\lambda_\nu R}(\partial_t + i\zeta)w_\nu^-|_{\Gamma^R}, \quad \nu = 1,\dots,\nu_0. \tag{3.97}$$

Therefore, the functions

$$\begin{aligned}
X_\nu^R &= w_\nu^+ - \frac{\zeta+\lambda_\nu}{\zeta-\lambda_\nu}e^{2i\lambda_\nu R}w_\nu^-, \quad \nu = 1,\dots,T,\\
Z_\nu^R &= w_\nu^- - \frac{\zeta-\lambda_\nu}{\zeta+\lambda_\nu}e^{-2i\lambda_\nu R}w_\nu^+, \quad \nu = 1,\dots,T,
\end{aligned} \tag{3.98}$$

are solutions to the homogeneous problem (3.96). We choose the basis (3.26) in such a way that $v_\nu^\pm = w_\nu^\mp$. Comparing (3.47) with the first row in (3.98), we find

$$\mathfrak{S}^R = \operatorname{diag}\left(\frac{\lambda_1+\zeta}{\lambda_1-\zeta}e^{2i\lambda_1 R}, \frac{\lambda_2+\zeta}{\lambda_2-\zeta}e^{2i\lambda_2 R},\dots,\frac{\lambda_T+\zeta}{\lambda_T-\zeta}e^{2i\lambda_T R}\right). \tag{3.99}$$

Similarly, we introduce the equality

$$\mathfrak{T}^R = \operatorname{diag}\left(\frac{\lambda_1-\zeta}{\lambda_1+\zeta}e^{-2i\lambda_1 R}, \frac{\lambda_2-\zeta}{\lambda_2+\zeta}e^{-2i\lambda_2 R},\dots,\frac{\lambda_T-\zeta}{\lambda_T+\zeta}e^{-2i\lambda_T R}\right). \tag{3.100}$$

If there is the zero eigenvalue of the pencil $\mathfrak{A}$ then, in addition to the equalities (3.97), we have

$$(\partial_t + i\zeta)w_0^+|_{\Gamma^R} = \frac{i + i\zeta - \zeta R}{-i - i\zeta + \zeta R}(\partial_t + i\zeta)w_0^-|_{\Gamma^R}.$$

Then $T = \nu_0 + 1$ and

$$\mathfrak{S}^R = \operatorname{diag}\left(\frac{i+i\zeta-\zeta R}{i-i\zeta-\zeta R}, \frac{\lambda_1+\zeta}{\lambda_1-\zeta}e^{2i\lambda_1 R}, \dots, \frac{\lambda_T+\zeta}{\lambda_T-\zeta}e^{2i\lambda_T R}\right),$$
$$\mathfrak{T}^R = \operatorname{diag}\left(\frac{i-i\zeta-\zeta R}{i+i\zeta-\zeta R}, \frac{\lambda_1-\zeta}{\lambda_1+\zeta}e^{-2i\lambda_1 R}, \dots, \frac{\lambda_T-\zeta}{\lambda_T+\zeta}e^{-2i\lambda_T R}\right). \tag{3.101}$$

Let the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \gamma\}$ contain some imaginary eigenvalues of the pencil $\mathfrak{A}$. The total algebraic multiplicity of these eigenvalues in the strip is denoted by $2M$. For $\nu > \nu_0$ we have

$$(\partial_t + i\zeta)w_\nu^+|_{\Gamma^R} = \frac{i(\lambda_\nu+\zeta)e^{2i\lambda_\nu R} + (\lambda_\nu-\zeta)}{i(\lambda_\nu+\zeta)e^{2i\lambda_\nu R} - (\lambda_\nu-\zeta)}(\partial_t + i\zeta)w_\nu^-|_{\Gamma^R}. \tag{3.102}$$

Therefore, the scattering $M \times M$-matrices $\mathfrak{S}^R$ and $\mathfrak{T}^R$ are diagonal. If zero is not an eigenvalue of the pencil $\mathfrak{A}$, then the first ν_0 diagonal entries of the matrix $\mathfrak{S}^R$ are the same as in formula (3.99), where $T = \nu_0$ (or the same as on the right-hand side of the first equality in (3.101), where $T = \nu_0$, if zero is an eigenvalue of the pencil). From (3.102) it follows that the remaining diagonal entries of $\mathfrak{S}^R$ are as follows:

$$\frac{\lambda_{\nu_0+1}-\zeta+i(\lambda_{\nu_0+1}+\zeta)e^{2i\lambda_{\nu_0+1}R}}{\lambda_{\nu_0+1}-\zeta-i(\lambda_{\nu_0+1}+\zeta)e^{2i\lambda_{\nu_0+1}R}}, \dots, \frac{\lambda_M-\zeta+i(\lambda_M+\zeta)e^{2i\lambda_M R}}{\lambda_M-\zeta-i(\lambda_M+\zeta)e^{2i\lambda_M R}}. \tag{3.103}$$

Similarly, using (3.100) and the second row in (3.101), we can construct the matrix $\mathfrak{T}^R$; its diagonal entries corresponding to the waves $v_{\nu_0+1}^-, \dots, v_M^-$ are obtained from (3.103) by replacing the fractions with the inverse ones.

By the inequalities $\lambda_\nu > 0$, $\nu = 1, \dots, \nu_0$, the norm of the matrix (3.99) does not exceed $q < 1$ for all R and $\zeta < 0$, the number q is independent of R. The same assertion is true for the matrix (3.100) for $\zeta > 0$ (instead of $\zeta < 0$).

From the formulas obtained we see that if the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \gamma\}$ contains multiple and (or) imaginary eigenvalues of the pencil $\mathfrak{A}$, then the norm of the matrix $\mathfrak{S}^R$ is less than 1 only for finite values of R and tends to 1 as $R \to \infty$ and $\zeta < 0$; the same is true for the matrix $\mathfrak{T}^R$ in the case $\zeta > 0$.

3.3.4 Estimating the discrepancy of the approximate solution to the problem (3.10).

We prove the following assertion.

Proposition 3.15. *Let γ be such that the following conditions hold:*

1) $\dim\ker\mathcal{A}(\gamma) = \dim\ker\mathcal{A}_*(\gamma) = 0$,

2) $\delta/2 > \gamma > 11\aleph$ *(here, $\aleph = \aleph(\gamma)$, cf. Proposition* 3.14*),*

3) *the line $\mathbb{R} + i\gamma$ is free from the spectrum of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$.*

Then for $\zeta > 0$ the function (3.51) *is an approximate solution to the problem* (3.10) *in the sense that for $\beta \in (3\aleph, \gamma - 8\aleph)$ the following estimate holds:*

$$\|\{e_\beta\mathcal{L}, e_\beta\mathcal{B}, \mathcal{N} + i\zeta\mathcal{D}\}\mathcal{Y}^R - \{e_\beta f, e_\beta g, h\}; \mathbb{W}(G^R)\| \leqslant ce^{-\mu R}\|\{e_\beta f, e_\beta g, h\}; \mathbb{W}(G^R)\|, \tag{3.104}$$

where $\mu > 0$.

Remark 3.16. *Let us explain the assumptions of Proposition 3.15, for example, in the case of self-adjoint problems under the assumption that the domain G has a single end at infinity, Π_+.*

The number γ satisfying the assumptions of Proposition 3.15 can be chosen in the case where all the nontrivial solutions to the homogenous problem $\{\mathcal{L}, \mathcal{B}\}u = 0$ in the domain G decrease at infinity not too rapidly. In the best situation, where there are no nontrivial decreasing solutions, we set $\aleph = \operatorname{Im}\lambda_{\max} = 0$. The assumptions of Proposition 3.15 are satisfied for sufficiently small positive γ.

Let $u_1, \dots, u_T$ be all linearly independent decreasing solutions to the homogeneous problem

$$u_j(y,t) = e^{i\lambda_j t} P_j(y,t) + o(\exp - \operatorname{Im}\lambda_j t),$$

where $(y,t) \in \Pi_+$, $t \to +\infty$, and $P_j(y,t)$ is a nonzero polynomial in t whose coefficients depend on y. Let $\operatorname{Im}\lambda_T \geqslant \operatorname{Im}\lambda_j$, $j = 1, \dots, T-1$. The parameter γ satisfying the conditions of Proposition 3.15 can be found provided that there exists an eigenvalue $\lambda_{\max}$ of the pencil $\mathfrak{A}$ such that $\operatorname{Im}\lambda_{\max} \geqslant \operatorname{Im}\lambda_T$ and the inequality $\operatorname{Im}\lambda > 11 \operatorname{Im}\lambda_{\max}$ holds for all those eigenvalues λ of $\mathfrak{A}$ that are located above the line $\{\nu \in \mathbb{C} : \operatorname{Im}\nu = \aleph\}$, $\aleph = \operatorname{Im}\lambda_{\max}$.

Proof of Proposition 3.15. Recall that (cf. Subsection 3.1) the discrepancy $\{\mathfrak{F}, \mathfrak{G}, \mathfrak{H}\}$ of the function (3.51) in the problem (3.10) satisfies

$$\begin{aligned} \{\mathfrak{F}, \mathfrak{G}\} &= \{\mathcal{L}, \mathcal{B}\} \sum_r \eta_R \Big(U_R \Big(u^r - \sum_{\ell \in \mathfrak{M}^r} b_\ell v_\ell^- \Big) \\ &\qquad + \sum_{j \in \mathfrak{M}^r} c_j \Big(Z_j^R - v_j^- - \sum_{k \in \mathfrak{M}^r} \mathbb{T}_{jk}^R v_k^+ \Big) \Big) - \sum_r U_R \{F^r, G^r\}, \\ \mathfrak{H} &= \sum_r (\mathcal{N}^r - \mathbf{N}^r + i\zeta(\mathcal{D}^r - \mathbf{D}^r)) \Big(U_R u^r + \sum c_j Z_j^R \Big) \\ &\qquad\qquad + (\mathcal{N} + i\zeta\mathcal{D})\mathcal{Y} \quad \text{on} \quad \Gamma^R. \end{aligned} \tag{3.105}$$

By Propositions 3.11 and 3.14, we have

$$\begin{aligned} &\|e_\beta \{\mathfrak{F}, \mathfrak{G}, 0\}; \mathbb{W}(G^R)\| \\ &\qquad \leqslant C e^{-(\gamma-\beta)R/2} \exp\{4\aleph R\} (R^{\varkappa - 1})^3 \|\{e_\beta f, e_\beta g, h\}; \mathbb{W}(G^R)\|. \end{aligned}$$

We now turn to (3.105). The strip $\{\lambda \in \mathbb{C} : \beta < \operatorname{Im}\lambda < \gamma\}$ is free from the spectrum of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$. Hence the operator $\mathcal{A}(\beta)$ has the bounded left inverse operator ($\ker \mathcal{A}(\gamma) = \ker \mathcal{A}(\beta)$ and $\operatorname{coker} \mathcal{A}(\gamma) = \operatorname{coker} \mathcal{A}(\beta)$; cf. [1],[2]) and $\mathcal{Y} \in \mathbb{V}_\beta(G)$. The following estimates hold:

$$\|\{0, 0, \mathcal{N} + i\zeta\mathcal{D}\}\mathcal{Y}; \mathbb{W}(G^R)\| \leqslant c e^{-\beta R} \|\mathcal{Y}; \mathbb{V}_\beta(G)\| \leqslant c e^{-\beta R} \|\{\mathcal{F}, \mathcal{G}\}; \mathbb{W}_\beta(G)\|. \tag{3.106}$$

Now, (3.63), (3.85), (3.88) and Proposition 3.14 lead to the estimate

$$\|\{\mathcal{F}, \mathcal{G}\}; \mathbb{W}_\beta(G)\| \leqslant C \exp\{3\aleph R\} (R^{\varkappa - 1})^2 \|\{e_\beta f, e_\beta g, h\}; \mathbb{W}(G^R)\|. \tag{3.107}$$

Repeating the arguments of the proof of Proposition 3.11, we obtain the inequality

$$\Big\|\Big\{0,0,\sum_r(\mathcal{N}^r-\mathbf{N}^r+i\zeta(\mathcal{D}^r-\mathbf{D}^r))\Big(U_Ru^r+\sum_{j\in\mathfrak{M}^r}c_j(R)Z_j^R\Big)\Big\}\Big\|$$
$$\leqslant ce^{-\delta R}\exp\{4\aleph R\}(R^{\varkappa-1})^3\|\{e_\beta f,e_\beta g,h\};\mathbb{W}(G^R)\|.$$

We can assume that the number μ in the estimate (3.104) is positive provided that $(\gamma-\beta)/2-4\aleph>0$ and $\beta-3\aleph>0$. Since $\gamma>11\aleph$, both these conditions can be satisfied in view of the choice of β. □

3.3.5 Estimating the norm of the inverse operator $(\mathcal{A}^R)^{-1}$ of problem (3.10) as $R\to\infty$

Let $\zeta>0$, and let the operator $\Upsilon:\mathbb{W}(G^R)\to\mathbb{V}(G^R)$ associate with the triple $\{f,g,h\}$ an approximate solution $\mathcal{Y}^R$ to the problem (3.10), cf. (3.51). We consider the composition $\mathcal{A}^R\Upsilon:\mathbb{W}(G^R)\to\mathbb{W}(G^R)$. We set

$$\|\{f,g,h\};\mathbb{W}(G^R;\beta)\|:=\|\{e_\beta f,e_\beta g,h\};\mathbb{W}(G^R)\|$$

for elements in $\mathbb{W}(G^R)$. It is clear that for a fixed R the norms $\|\cdot;\mathbb{W}(G^R)\|$ and $\|\cdot;\mathbb{W}(G^R;\beta)\|$ are equivalent. Let the assumptions of Proposition 3.15 be satisfied. By this proposition, for $\zeta>0$ and $R>R_0$ we have

$$\|I-\mathcal{A}^R\Upsilon;\mathbb{W}(G^R;\beta)\to\mathbb{W}(G^R;\beta)\|\leqslant Ce^{-\mu R},$$

where the constant C is independent of R and $\mu>0$. Therefore, for sufficiently large R the norm of the continuous mapping

$$(\mathcal{A}^R\Upsilon)^{-1}:\mathbb{W}(G^R;\beta)\to\mathbb{W}(G^R;\beta)$$

is uniformly bounded with respect to R. The operator (3.16) is invertible for $\zeta>0$. Hence there exists Υ^{-1}. It is clear that $(\mathcal{A}^R)^{-1}=\Upsilon(\mathcal{A}^R\Upsilon)^{-1}$. Denote by $\mathbb{W}(G^R)_0$ the subspace of $\mathbb{W}(G^R)$ formed by triples $\{0,0,h\}$. Let $(\mathcal{A}^R)_0^{-1}$ be the restriction of $(\mathcal{A}^R)^{-1}$ to $\mathbb{W}(G^R)_0$.

Theorem 3.17. *Let the assumptions of Proposition* 3.15 *be satisfied. Then for* $\zeta>0$ *and sufficiently large* R *the following estimate holds*:

$$\|(\mathcal{A}^R)_0^{-1};\mathbb{W}(G^R)_0\to\mathbb{V}(G^R)\|\leqslant C\exp\{4\aleph R\}R^{3\varkappa-2},\tag{3.108}$$

where the constant C *is independent of* R *and the numbers* $\aleph$ *and* $\varkappa$ *are the same as in Proposition* 3.14.

If for some number $\gamma\in(0,\delta/2)$ *conditions* 1 *and* 3 *of Proposition* 3.15 *are satisfied and the strip* $\{\lambda\in\mathbb{C}:|\operatorname{Im}\lambda|<\gamma\}$ *does not contain the spectrum of the pencils* $\mathfrak{A}^1,\dots,\mathfrak{A}^N$*, then for* $\zeta>0$ *and sufficiently large* R *the following estimate holds:*

$$\|(\mathcal{A}^R)_0^{-1};\mathbb{W}(G^R)_0\to\mathbb{V}(G^R)\|\leqslant C,$$

where C *is independent of* R (*cf.* [17, Theorem 5.6.3]).

Proof. Let us prove the inequality (3.108). We have

$$\begin{aligned}
&\|(\mathcal{A}^R)_0^{-1}; \mathbb{W}(G^R)_0 \to \mathbb{V}(G^R)\| \\
&\qquad \leqslant \|\Upsilon; \mathbb{W}(G^R;\beta) \to \mathbb{V}(G^R)\| \|(\mathcal{A}^R\Upsilon)^{-1}; \mathbb{W}(G^R)_0 \to \mathbb{W}(G^R;\beta)\| \\
&\qquad\qquad \leqslant C\|\Upsilon; \mathbb{W}(G^R;\beta) \to \mathbb{V}(G^R)\|.
\end{aligned} \tag{3.109}$$

It remains to estimate the norm of the operator Υ. Using (3.51), (3.106), (3.107) and the inequality

$$\|\mathcal{Y}; \mathbb{V}(G^R)\| \leqslant \|\mathcal{Y}; \mathbb{V}_\beta(G^R)\|,$$

we obtain the estimate

$$\|\mathcal{Y}; \mathbb{V}(G^R)\| \leqslant C \exp\{3\aleph R\}(R^{\varkappa-1})^2 \|\{f,g,h\}; \mathbb{W}(G^R;\beta)\|.$$

Moreover, using (3.48) and (3.49), we find

$$\begin{aligned}
\Big\|\sum_r \eta_R U_R w^r; \mathbb{V}(G^R)\Big\| &\leqslant \Big\|\sum_{j\in\mathfrak{M}^r} \eta_R c_j(R) Z_j^R; \mathbb{V}(G^R)\Big\| \\
&+ \sum_r \Big(\Big\|u^r - \sum_{j\in\mathfrak{M}^r} b_j v_j^-; \mathbb{V}_\gamma(\Pi_-^{r,0})\Big\| + \Big\|\sum_{j\in\mathfrak{M}^r} \eta_R U_R b_j v_j^-; \mathbb{V}(G^R)\Big\| \Big).
\end{aligned} \tag{3.110}$$

Consider the first term on the right-hand side. The coefficients c_j were estimated in Proposition 3.14. To estimate Z_j^R, we use (3.58), (3.60) and the arguments after formula (3.60). For $j \in \mathfrak{M}^r$ we find

$$\begin{aligned}
\|\eta_R Z_j^R; \mathbb{V}(G^R)\| &\leqslant \Big\|\eta_R\Big(v_j + \sum_{k\in\mathfrak{M}^r} \mathbb{T}_{jk}^R v_k^+\Big); \mathbb{V}(\Pi_-^{r,R})\Big\| + \|\mathcal{Z}_j^R; \mathbb{V}(\Pi_-^{r,R})\| \\
&\leqslant C \exp\{\aleph R\} R^{\varkappa}.
\end{aligned} \tag{3.111}$$

Thus,

$$\Big\|\sum_{j\in\mathfrak{M}^r} \eta_R c_j(R) Z_j^R; \mathbb{V}(G^R)\Big\| \leqslant C \exp\{4\aleph R\} R^{3\varkappa-2} \|\{f,g,h\}; \mathbb{W}(G^R;\beta)\|. \tag{3.112}$$

The estimate

$$\Big\|u^r - \sum_{j\in\mathfrak{M}^r} b_j v_j^-; \mathbb{V}_\gamma(\Pi_-^{r,0})\Big\| \leqslant C\|\{f,g,h\}; \mathbb{W}(G^R;\beta)\| \tag{3.113}$$

for the second term in (3.110) is verified with the help of Proposition 3.10 and the same arguments as in (3.72). Finally, we estimate the last term

$$\begin{aligned}
\Big\|\sum_{j\in\mathfrak{M}^r} \eta_R U_R b_j v_j^-; \mathbb{V}(G^R)\Big\| &\leqslant C \exp\{\aleph R\} R^{\varkappa} \sum_{j\in\mathfrak{M}^r} |b_j| \\
&\leqslant C \exp\{\aleph R\} R^{\varkappa} \|\{f,g,h\}; \mathbb{W}(G^R;\beta)\|
\end{aligned} \tag{3.114}$$

(we again used (3.72)). From (3.51), (3.109)-(3.114) we obtain the inequality

$$\|\Upsilon; \mathbb{W}(G^R;\beta) \to \mathbb{V}(G^R)\| \leqslant C \exp\{4\aleph R\} R^{3\varkappa-2}$$

which, together with (3.109), yields (3.108). □

3.4 Justification of the Algorithm for Computing Scattering Matrices

To justify the method, it remains to verify the nonsingularity of the matrix $\mathcal{E}^R$ (cf. (3.9)) and stabilization of the vector $a(R)$ minimizing the functional (3.7) to the row of the scattering matrix S as $R \to \infty$.

Proposition 3.18. *The matrix $\mathcal{E}^R$ in formula* (3.9) *is nonsingular for all $R > R^0$.*

Proof. Suppose that the assertion fails. Then for any R^0 there is a number $R > R^0$ such that the matrix $\mathcal{E}^R$ is singular and the functions $\mathcal{U} := \sum c_j u_j^-$ and $\mathcal{V} := \sum c_j v_j^-$ satisfy the equality

$$\mathcal{D}\mathcal{U} = \mathcal{D}\mathcal{V} \quad \text{on} \quad \Gamma^R, \tag{3.115}$$

where v_j^- are solutions to the problems (3.8), $c = (c_1, \dots, c_M)$, and $|c| = 1$. We use the equation on Γ^R from (3.8). We have

$$\mathcal{N}\mathcal{U} = \mathcal{N}\mathcal{V} \quad \text{on} \quad \Gamma^R. \tag{3.116}$$

In the Green formula (3.5), we set $u = v = \mathcal{V}$. Taking into account the first two equations in (3.8), the inequality (3.14) for $w = v_j^-$, and the equalities (3.115), (3.116), we find

$$\operatorname{Im}\{(\mathcal{N}\mathcal{U}, \mathcal{D}\mathcal{U})_{\Gamma^R} - (\mathcal{D}\mathcal{U}, \mathcal{N}\mathcal{U})_{\Gamma^R}\} \geqslant 0. \tag{3.117}$$

On the other hand, since the coefficients of the operators $\mathcal{N} - N^r$ and $\mathcal{D} - D^r$ decrease as $O(\exp(-\delta t^r))$ in Π_+^r and $\mathcal{U} = O(\exp(\gamma t^r))$ as $t^r \to \infty$ and $\gamma < \delta/2$, we have

$$(\mathcal{N}\mathcal{U}, \mathcal{D}\mathcal{U})_{\Gamma^R} - (\mathcal{D}\mathcal{U}, \mathcal{N}\mathcal{U})_{\Gamma^R} = \sum_{r=1}^{N}\{(\mathbf{N}^r\mathcal{U}, \mathbf{D}^r\mathcal{U})_{\Gamma^{r,R}} - (\mathbf{D}^r\mathcal{U}, \mathbf{N}^r\mathcal{U})_{\Gamma^{r,R}}\} + o(1) \tag{3.118}$$

as $R \to \infty$. A direct computation shows that

$$\sum_{r=1}^{N}\{(\mathbf{N}^r\mathcal{U}, \mathbf{D}^r\mathcal{U})_{\Gamma^{r,R}} - (\mathbf{D}^r\mathcal{U}, \mathbf{N}^r\mathcal{U})_{\Gamma^{r,R}}\} = -i\sum_{j=1}^{M}|c_j|^2 = -i. \tag{3.119}$$

From (3.117), (3.118), and (3.119) it follows that $-1 \geqslant o(1)$. A contradiction obtained completes the proof. □

Recall that $\aleph(\gamma) = \operatorname{Im}\lambda_{\max}(\gamma)$, where $\lambda_{\max}(\gamma)$ is the eigenvalue of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$ in the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \gamma\}$ with the maximal imaginary part.

Proposition 3.19. *Suppose that for some number γ' the assumptions of Proposition* 3.15 *are satisfied and the parameter ζ in the problem* (3.10) *is positive. Assume that the vector $a(R) = (a_1(R), \dots, a_M(R))$ is a minimizer of the functional J_m^R in* (3.7)*. The following assertions hold.*

1. *The following estimate holds:*

$$J_m^R(a(R)) = O(\exp\{-2\Lambda R\}), \quad R \to \infty,$$

with any $\Lambda \in (0, \gamma - 4\aleph(\gamma'))$, *where* γ *is the number taken from formula* (3.4).

2. *If* $\gamma > \aleph(\gamma) + 4\aleph(\gamma')$, *then for all* $R > R_0$ *the components of the vector* $a(R)$ *are uniformly bounded,*

$$|a_j(R)| \leqslant \text{const} < \infty, \quad j = 1, \dots, M.$$

Proof. 1. Denote by Y_m^R the solution to the problem (3.6), where for a_j, $j = 1, \dots, M$, we take the entries S_{mj} of the scattering matrix $S = S(\gamma)$ of the problem (3.1). Since we can differentiate (3.4), we have

$$(\mathcal{N} + i\zeta\mathcal{D})(Y_m^R - Y_m)|_{\Gamma^R} = O(e^{-\gamma R}).$$

By Theorem 3.17, we find

$$\begin{aligned}\|\mathcal{D}(Y_m^R - Y_m); L_2(\Gamma^R)\| &\leqslant c\|Y_m^R - Y_m; \mathbb{V}(G^R)\| \\ &\leqslant \|(\mathcal{A}^R)_0^{-1}\| \|\{0, 0, \mathcal{N} + i\zeta\mathcal{D}\}(Y_m^R - Y_m); \mathbb{W}(G^R)\| \leqslant ce^{-\Lambda R}.\end{aligned}$$

Together with (3.4), this leads to the estimate

$$J_m^R(S_m) = \Big\|\mathcal{D}\Big(Y_m^R - \Big(u_m^+ + \sum_{j=1}^M S_{mj}u_j^-\Big)\Big); L_2(\Gamma^R)\Big\|^2 \leqslant ce^{-2\Lambda R},$$

where the constant c is independent of R. It remains to note that $J_m^R(a(R)) \leqslant J_m^R(S_m)$. Assertion 1 is proved.

2. Let Z_m^R be the solution to the problem (3.6) corresponding to the vector $a(R) = (a_1(R), \dots, a_M(R))$. In the Green formula (3.5), we set $u = v = Z_m^R$. We have

$$\operatorname{Im}\{(\mathcal{N}Z_m^R, \mathcal{D}Z_m^R)_{\Gamma^R} - (\mathcal{D}Z_m^R, \mathcal{N}Z_m^R)_{\Gamma^R}\} \geqslant 0. \tag{3.120}$$

By Assertion 1, we have

$$\Big\|\mathcal{D}\Big(Z_m^R - \Big(u_m^+ + \sum a_j u_j^-\Big)\Big); L_2(\Gamma^R)\Big\| = O(e^{-\Lambda R}), \quad R \to \infty. \tag{3.121}$$

Since

$$(\mathcal{N} + i\zeta\mathcal{D})Z_m^R|_{\Gamma^R} = (\mathcal{N} + i\zeta\mathcal{D})\Big(u_m^+ + \sum_{j=1}^M a_j u_j^-\Big)|_{\Gamma^R},$$

from (3.121) we find

$$\Big\|\mathcal{N}\Big(Z_m^R - \Big(u_m^+ + \sum_{j=1}^M a_j u_j^-\Big)\Big); L_2(\Gamma^R)\Big\| = O(e^{-\Lambda R}) \tag{3.122}$$

as $R \to \infty$. Using (3.121) and (3.122), we reduce (3.120) to the form

$$\operatorname{Im}\{(\mathcal{N}\varphi_m, \mathcal{D}\varphi_m)_{\Gamma^R} - (\mathcal{D}\varphi_m, \mathcal{N}\varphi_m)_{\Gamma^R}\} + O(e^{-\varepsilon R})\Big(\sum |a_j(R)|^2 + 1\Big) \geqslant 0,$$

where $\varepsilon = \gamma - \aleph(\gamma) - 4\aleph(\gamma')$ and $\varphi_m = u_m^+ + \sum_{n=1}^{M} a_n u_n^-$. Taking into account the stabilization of the coefficients of the operators $\mathcal{N}$ and $\mathcal{D}$, we obtain the inequality

$$\operatorname{Im}\Big\{ \sum_{r=1}^{N} \{(\mathbf{N}^r\varphi_m, \mathbf{D}^r\varphi_m)_{\Gamma^{r,R}} - (\mathbf{D}^r\varphi_m, \mathbf{N}^r\varphi_m)_{\Gamma^{r,R}}\}\Big\} + O(e^{-\varepsilon R})\Big(\sum_j |a_j(R)|^2 + 1\Big) \geqslant 0.$$

Here, the sum over r is immediately calculated and is equal to $i - i\sum_j |a_j|^2$. Thus,

$$\sum_{j=1}^{M} |a_j(R)|^2 = 1 + O(e^{-\varepsilon R}), \quad R \to \infty, \tag{3.123}$$

where $\varepsilon > 0$. □

Remark 3.20. *If the domain G has more than one end at infinity, we can weaken the restriction $\gamma > \aleph(\gamma) + 4\aleph(\gamma')$ by defining the parameter ε in (3.123) for each of the semicylinders $\Pi_+^1, \dots, \Pi_+^N$. Namely, let $\lambda_{\max}^r$ be an eigenvalue of the pencil $\mathfrak{A}^r$ such that $\operatorname{Im}\lambda \leqslant \operatorname{Im}\lambda_{\max}^r \leqslant \gamma$ for all eigenvalues λ of the pencil $\mathfrak{A}^r$ in the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leqslant \gamma\}$. We set $\varepsilon^r = \min\{\operatorname{Im}(\lambda - \lambda_{\max}^r) : \operatorname{Im}\lambda > \gamma\}$. For ε in (3.123) we can take the number*

$$\varepsilon < \varepsilon^0 = \min_r \min\{\varepsilon^r, \delta/2 - \lambda_{\max}^r\} - 4\aleph(\gamma'). \tag{3.124}$$

As above, the number δ characterizes the stabilization rate of the coefficients of the problem (3.1) (cf. Subsection 1.1).

We pass to the proof of the main result.

Theorem 3.21. *Let $\zeta > 0$, and let the number γ in* (3.4) *satisfy the following conditions*:

1) *there exists a number $\gamma' \leqslant \gamma$ such that* $\dim\ker\mathcal{A}(\gamma') = \dim\ker\mathcal{A}_*(\gamma') = 0$, *the line $\mathbb{R} + i\gamma'$ is free from the spectrum of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$ and* $\gamma' > 11\aleph(\gamma')$,

2) $\gamma - \aleph(\gamma) - 4\aleph(\gamma') > 0$.

Then for all $R > R_0$ there exists a unique vector

$$a(R) = (a_1(R), \dots, a_M(R))$$

that minimizes the functional J_l^R in (3.7). *The following estimates hold*:

$$|a_j(R) - S_{lj}| \leqslant C(\varepsilon)\exp(-\varepsilon R), \quad j = 1, \dots, M,$$

where $\varepsilon > 0$ *satisfies* (3.124) *and the constant* $C(\varepsilon)$ *is independent of* R.

If the homogeneous problem has no nontrivial decreasing solutions, then the conditions 1 *and* 2 *are automatically satisfied for sufficiently small* $\gamma' > 0$, $\aleph(\gamma') = 0$.

Condition 2 *follows from* 1 *if* $\gamma' = \gamma$.

Proof. As above, we denote by Z_m^R a solution to the problem (3.6) corresponding to $(a_1(R), \dots, a_M(R))$. Recall that Y_m has the asymptotic behavior (3.4) and satisfies the homogeneous problem (3.1). We substitute $u = v = U_m$, where $U_m = Y_m - Z_m^R$, into the Green formula (3.5). Since U_m satisfies the first two equations in (3.6), we have

$$\operatorname{Im}\{(\mathcal{N}U_m, \mathcal{D}U_m)_{\Gamma^R} - (\mathcal{D}U_m, \mathcal{N}U_m)_{\Gamma^R}\} \geqslant 0. \tag{3.125}$$

We set

$$\varphi_m = u_m^+ + \sum_{n=1}^{M} a_n(R) u_n^-, \quad \psi_m = u_m^+ + \sum_{n=1}^{M} S_{mn} u_n^- \tag{3.126}$$

and write U_m in the form

$$U_m = Y_m - Z_m^R = (Y_m - \psi_m) + (\psi_m - \varphi_m) + (\varphi_m - Z_m^R).$$

We note that $(Y_m - \psi_m)|_{\Gamma^R} = O(\exp(-\gamma R))$ in view of (3.4). Taking into account the estimates (3.121), (3.122) and Proposition 3.19, we can pass from (3.125) to the inequality

$$\begin{aligned} \operatorname{Im}\{(\mathcal{N}(\psi_m - \varphi_m), \mathcal{D}(\psi_m - \varphi_m))_{\Gamma^R} \\ - (\mathcal{D}(\psi_m - \varphi_m), \mathcal{N}(\psi_m - \varphi_m))_{\Gamma^R}\} \geqslant O(\exp(-\varepsilon R)) \end{aligned} \tag{3.127}$$

with the same ε as in Remark 3.20. The coefficients of the operators $\mathcal{N} - N^r$ and $\mathcal{D} - D^r$ decrease in the semicylinder Π_r^+ as $O(\exp(-\delta t^r))$, where $\delta > 2\gamma$. Consequently, (3.127) implies the estimate

$$\begin{aligned} \operatorname{Im}\Big\{ \sum_{r=1}^{N} \{(\mathbf{N}^r(\psi_m - \varphi_m), \mathbf{D}^r(\psi_m - \varphi_m))_{\Gamma^{r,R}} \\ - (\mathbf{D}^r(\psi_m - \varphi_m), \mathbf{N}^r(\psi_m - \varphi_m))_{\Gamma^{r,R}}\Big\} \geqslant O(\exp(-\varepsilon R)). \end{aligned} \tag{3.128}$$

Here, the left-hand side is immediately calculated and is equal to $(-\sum_{n=1}^{M} |a_n(R) - S_{mn}|^2)$ (it suffices to use the representations (3.126) and the definition of the waves $u_j^\pm$). Finally, we have

$$\sum_{n=1}^{M} |a_n(R) - S_{mn}|^2 = O(\exp(-\varepsilon R)).$$

The theorem is proved. □

3.5 Operators Depending on Parameter

We now turn to formally self-adjoint problems $\{\mathcal{L}(x, D_x) - \mu, \mathcal{B}(x, D_x)\}$ with parameter μ. A number μ is an *eigenvalue* of the operator $\{\mathcal{L}, \mathcal{B}\}$ if there exists a function u in $L_2(G)$ that is smooth in $\overline{G}$ and satisfies the homogeneous problem $\{\mathcal{L}(x, D_x) - \mu, \mathcal{B}(x, D_x)\}u(x) = 0$. The authors do not know if eigenvalues can accumulate at a finite distance under our assumptions on the operator $\{\mathcal{L}, \mathcal{B}\}$ (the coefficients stabilize at infinity at exponential rate). However, from [14] (where the operators $\{\mathcal{L}, \mathcal{B}\}$ are considered under less rigid assumptions on the stabilization of the coefficients) it follows that accumulation points can be only "threshold" values of the parameter μ. We give some consequences of the results of [14]. For the sake of simplicity, we assume that the domain G has a single end at infinity, Π_+. The number μ_0 is said to be *threshold* for the operator $\{\mathcal{L}, \mathcal{B}\}$ if there exists a sequence $\{\mu_k\} \subset \mathbb{R}$, $\mu_k \to \mu_0$, and a sequence $\{\lambda(\mu_k)\} \subset \mathbb{C}$ such that

(a) $\lambda(\mu_k)$ is an eigenvalue of the pencil $\lambda \mapsto \mathfrak{A}(\lambda; \mu) := \{\mathbf{L}(\lambda) - \mu, \mathbf{B}(\lambda)\}$,

(b) $\operatorname{Im} \lambda(\mu_k) > 0$,

(c) $\operatorname{Im} \lambda(\mu_k) \to 0$ as $k \to \infty$.

1) Eigenvalues of the operator $\{\mathcal{L}, \mathcal{B}\}$ can accumulate only in a neighborhood of the thresholds and at infinity.

2) Let μ_0 be an accumulation point of the sequence $\{\mu_k\}$ of eigenvalues of the operator $\{\mathcal{L}, \mathcal{B}\}$. Then for μ_k, where k is sufficiently large, there exist numbers $\lambda_k = \lambda(\mu_k)$ satisfying conditions (a)–(c) of the definition of a threshold.

3) For the sake of simplicity, we additionally assume that the sequence λ_k possessing the above properties is unique for a given sequence of eigenvalues μ_k. Then the eigenfunction u_k corresponding to the eigenvalues μ_k admits the following representation in Π_+:

$$u_k(y, t) = e^{i\lambda_k t} P_k(y, t) + o(\exp\{-\operatorname{Im} \lambda_k t\})$$

as $t \to +\infty$. Here, $P_k(y, t)$ denotes a nonzero polynomial in t whose coefficients depend on y.

We say that $\mu \in \mathbb{R}$ is a *number of the first kind* if the operator $\{\mathcal{L} - \mu, \mathcal{B}\}$ satisfies the assumptions of Theorem 3.21 (cf. also Remark 3.16); otherwise, μ is called a *number of the second kind*. In particular, μ is a number of the first kind if there are no nontrivial decreasing solutions to the homogeneous problem $\{\mathcal{L} - \mu, \mathcal{B}\}u = 0$. By Theorem 3.21, the method converges at a point of the first kind.

Let μ_0 be a threshold value of the operator $\{\mathcal{L}, \mathcal{B}\}$. If μ_0 is a number of the first (second) kind, there exists a neighborhood of the point μ_0 (a punctured neighborhood of μ_0) containing only numbers of the first kind. Therefore, numbers of the second kind can accumulate only at infinity.

4 Appendix: Elliptic problems with slowly stabilizing characteristics

In domain $G \subset \mathbb{R}^{n+1}$ with finitely many cylindrical ends we consider general elliptic boundary value problems. As $x \to \infty$, $x \in G$, the coefficients of the problem tend to limits too slow to allow obtaining an asymptotic of solution at infinity. Using the results of the paper [23] (see also [24, Section 8.5]), one can get some "structure" of solutions to the problem: far from the origin a solution is represented as a linear combination of functional series plus a remainder. The coefficients in the linear combination remain unknown. Here we suggest an approach that allows to derive the expressions for the coefficients in the structure of solutions. This enables one to extend the theory of elliptic problems in domains with cylindrical ends, well-developed earlier for the case of exponentially stabilizing coefficients (see e.g. [1, 2, 24]), to the case of slowly stabilizing coefficients. These results are obtained in collaboration with Pekka Neittaanmäki and Boris A. Plamenevskii and published for the first time. Some results for a class of formally self-adjoint problems with slowly stabilizing coefficients were announced in [14]. The formally self-adjoint problems were studied in [25].

4.1 Statement of the problem and preliminaries

Let us recall some of the notation. We denote by $G \subset \mathbb{R}^{n+1}$ a domain with smooth boundary ∂G coinciding, outside a large ball, with the union $\Pi_+^1 \cup \dots \cup \Pi_+^N$ of non-overlapping semicylinders; here $\Pi_+^r = \{(y^r, t^r) : y^r \in \Omega^r, t^r > 0\}$, (y^r, t^r) are local coordinates, and the section Ω^r is a bounded domain in $\mathbb{R}^n$.

In domain G we consider the elliptic boundary value problem

$$\begin{aligned} \mathcal{L}(x, D_x)u(x) &= f(x), \quad x \in G, \\ \mathcal{B}(x, D_x)u(x) &= g(x), \quad x \in \partial G, \end{aligned} \tag{4.1}$$

where $\mathcal{L} = \|\mathcal{L}_{ij}\|$ and $\mathcal{B} = \|\mathcal{B}_{qj}\|$ are $k \times k$ and $m \times k$ matrices of differential operators respectively, and $\operatorname{ord} \mathcal{L}_{ij} = s_i + t_j$, $\operatorname{ord} \mathcal{B}_{qj} = \sigma_q + t_j$, $2m = s_1 + t_1 + \dots +$

$s_k + t_k$. (As usual, s_i, t_j, and σ_q are the integers appearing in the definition of an elliptic problem in the sense of Douglis-Nirenberg.)

Let $\{L^r, B^r\}$ be the operator of a problem in the cylinder $\Pi^r = \Omega^r \times \mathbb{R}$ with coefficients independent of t^r, $\operatorname{ord} L^r_{ij} = \operatorname{ord} \mathcal{L}_{ij}$, and $\operatorname{ord} B^r_{qj} = \operatorname{ord} \mathcal{B}_{qj}$. We denote by $W^\ell_\beta(\Pi^r)$ the space with norm $\|e_\beta\cdot; H^\ell(\Pi^r)\|$, where $H^\ell(\Pi^r)$ is the Sobolev space, $e_\beta : (y^r, t^r) \mapsto \exp \beta t^r$, and $\beta \in \mathbb{R}$. Set

$$\begin{aligned} \mathcal{D}^\ell_\beta W(G) &= \prod_{j=1}^{k} W^{\ell+t_j}_\beta(G), \\ \mathcal{R}^\ell_\beta W(G) &= \prod_{i=1}^{k} W^{\ell-s_i}_\beta(G) \times \prod_{q=1}^{m} W^{\ell-\sigma_q-1/2}_\beta(\partial G), \end{aligned} \tag{4.2}$$

where $\ell \geqslant \max\{-t_j, s_i, \sigma_q + 1\}$. The map $\{L^r, B^r\} : \mathcal{D}^\ell_\beta W(G) \to \mathcal{R}^\ell_\beta W(G)$ is continuous. The coefficients of $\{\mathcal{L}, \mathcal{B}\}$ are supposed to be smooth in $\overline{G}$ and stabilizing at infinity in the sense that

$$\lim_{T\to\infty} \|\eta_T(\{\mathcal{L}, \mathcal{B}\} - \{L^r, B^r\}); \mathcal{D}^\ell_\beta(\Pi^r) \to \mathcal{R}^\ell_\beta(\Pi^r)\| = 0, \tag{4.3}$$

where $\eta_T(t^r) = \eta(t^r - T)$, $\eta \in C^\infty(\mathbb{R})$, $\eta(t^r) = 0$ for $t^r < 1$, and $\eta(t^r) = 1$ for $t^r > 2$. In particular it follows that the operator

$$\mathcal{A}(\beta) = \{\mathcal{L}, \mathcal{B}\} : \mathcal{D}^\ell_\beta(G) \to \mathcal{R}^\ell_\beta(G)$$

is continuous, the spaces $\mathcal{D}^\ell_\beta(G)$ and $\mathcal{R}^\ell_\beta(G)$ being defined by (4.2) with Π^r replaced by G; the space $W^\ell_\beta(G)$ is endowed with the norm $\|e_\beta\cdot; H^\ell(G)\|$, where e_β is a smooth positive function in $\overline{G}$ equal to $\exp \beta t^r$ in Π^r, $r = 1, \dots, N$. We suppose that the operators $\{L^r, B^r\}$ are elliptic. Let the Green formula

$$(\mathcal{L}u, v)_G + (\mathcal{B}u, \mathcal{Q}v)_{\partial G} - (u, \mathcal{L}_* v)_G - (\mathcal{Q}_* u, \mathcal{B}_* v)_{\partial G} = 0 \tag{4.4}$$

be valid for all $u, v \in C^\infty_c(\overline{G})$; here $\mathcal{L}_*$ is the operator formally adjoint to $\mathcal{L}$, while $\mathcal{Q}$, $\mathcal{Q}_*$, and $\mathcal{B}_*$ are certain $m \times k$ matrices of differential operators, $\operatorname{ord} \mathcal{B}_{*qj} = s_i + \rho_q$, $\operatorname{ord} \mathcal{Q}_{*qj} = t_j - \rho_q - 1$, and $\operatorname{ord} \mathcal{Q}_{qi} = s_i - \sigma_q - 1$. We assume that the problem $\{\mathcal{L}_*, \mathcal{B}_*\}$ adjoint to $\{\mathcal{L}, \mathcal{B}\}$ with respect to the Green formula (4.4) has all the properties of the problem (4.1). The correspondent limit operators will be denoted by $\{L^r_*, B^r_*\}$, $r = 1, \dots, N$. In particular the coefficients of $\{\mathcal{L}_*, \mathcal{B}_*\}$ are stabilizing at infinity in the sense that

$$\lim_{T\to\infty} \|\eta_T(\{\mathcal{L}_*, \mathcal{B}_*\} - \{L^r_*, B^r_*\}); \mathcal{D}^\ell_\beta(\Pi^r)_* \to \mathcal{R}^\ell_\beta(\Pi^r)_*\| = 0, \tag{4.5}$$

where

$$\begin{aligned} \mathcal{D}^\ell_\beta W(\Pi)_* &= \prod_{i=1}^{k} W^{\ell+s_i}_{-\beta}(\Pi), \\ \mathcal{R}^\ell_\beta W(\Pi)_* &= \prod_{j=1}^{k} W^{\ell-t_j}_{-\beta}(\Pi) \times \prod_{q=1}^{m} W^{\ell-\rho_q-1/2}_{-\beta}(\partial\Pi) \end{aligned}$$

with $\ell \geqslant \max\{t_j, -s_i, \rho_q + 1\}$.

Introduce the operator pencil

$$\mathbb{C} \ni \lambda \mapsto \mathfrak{A}^r(\lambda) = \{L^r(y, D_y, \lambda), B^r(y, D_y, \lambda)\} \tag{4.6}$$

in the domain Ω^r. It is known [30] that the spectrum of the pencil $\mathfrak{A}^r$ consists of normal eigenvalues, which are located on the set $\{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| < c|\operatorname{Im}\lambda|\}$ with finitely many exceptions. From now on we choose a certain r and do not indicate it in notation. Let

$$\{\varphi_\nu^{(0,j)}, \dots, \varphi_\nu^{(\varkappa_{j\nu}-1,j)}; j = 1, \dots, J_\nu := \dim\ker \mathfrak{A}^r(\lambda_\nu)\}$$

be a canonical system of Jordan chains of the pencil $\mathfrak{A}$ corresponding to the eigenvalue λ_ν. Here, $\varphi_\nu^{(0,j)}$ is an eigenvector and $\varphi_\nu^{(1,j)}, \dots, \varphi_\nu^{(\varkappa_{j\nu}-1,j)}$ are associated vectors [1],[2], [20]. The functions

$$u_\nu^{(\sigma,j)}(y,t) = \exp(i\lambda_\nu t)\sum_{\ell=0}^{\sigma} \frac{1}{\ell!}(it)^\ell \varphi_\nu^{(\sigma-\ell,j)}(y), \tag{4.7}$$

where $\sigma = 0, \dots, \varkappa_{j\nu} - 1$, satisfy the homogeneous problem

$$\{L(y, D_y, D_t), B(y, D_y, D_t)\}u(y,t) = 0, \quad (y,t) \in \Pi.$$

Proposition 4.1 (see [1, 2]). *The map*

$$A(\beta) = \{L, B\} : \mathfrak{D}^\ell_\beta(\Pi) \to \mathfrak{R}^\ell_\beta(\Pi) \tag{4.8}$$

is an isomorphism if and only if the line $\mathbb{R} + i\beta = \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda = \beta\}$ is free from the spectrum of the pencil $\mathfrak{A}$.

In the cylinder Π the Green formula

$$(Lu, v)_\Pi + (Bu, Qv)_{\partial\Pi} - (u, L_* v)_\Pi - (Q_* u, B_* v)_{\partial\Pi} = 0 \tag{4.9}$$

holds for $u, v \in C_c^\infty(\overline{\Pi})$. For the limit operator $\{L_*, B_*\}$ we define the pencil $\lambda \mapsto \mathfrak{A}_*(\lambda)$, $\mathfrak{A}_*(\lambda) = \{L_*(y, D_y, \lambda), B_*(y, D_y, \lambda)\}$. If λ_ν is an eigenvalue of $\mathfrak{A}$ then $\overline{\lambda}_\nu$ is an eigenvalue of the pencil $\mathfrak{A}_*$. Moreover, $\dim\ker\mathfrak{A}(\lambda_\nu) = \dim\ker\mathfrak{A}_*(\overline{\lambda}_\nu)$, and the partial zero multiplicities of the eigenvalues are coincident; see [1, 2]. Let $\{\psi_\nu^{(0,j)}, \dots, \psi_\nu^{(\varkappa_{j\nu}-1,j)}; j = 1, \dots, J_\nu\}$ be the canonical system of Jordan chains of the pencil $\mathfrak{A}_*$. We introduce the solutions $v_\nu^{\sigma,j}$ to the homogeneous problem $\{L_*, B_*\}v = 0$ in Π by the formula (4.7), where u, φ, and λ_ν are replaced by v, ψ, and $\overline{\lambda}_\nu$ respectively.

Denote by $q(u,v)$ the left-hand side of (4.9). As is known [1, 2] the chains $\{\psi_\nu^{(\sigma,j)}\}$ can be chosen such that the relations

$$q(\chi u_\nu^{(\sigma,j)}, \chi v_\mu^{(\tau,p)}) = i\delta_{\mu,\nu}\delta_{j,p}\delta_{\varkappa_{\mu p}-\tau-1,\sigma} \tag{4.10}$$

are fulfilled, where δ is the Kronecker symbol, and $\chi \in C^\infty(\mathbb{R})$, $\chi(t) = 0$ for $t < 0$ and $\chi(t) = 1$ for $t > 1$. Emphasize that the left-hand side of (4.10) does not depend on the choice of χ in view of (4.9).

4.2 Structure of solutions to the model problem in a cylinder, formulas for the coefficients

Let $\{\mathfrak{L}_T, \mathfrak{B}_T\}$ and $\{\mathfrak{L}_{T*}, \mathfrak{B}_{T*}\}$ be differential operators depending on the parameter T such that the following conditions are satisfied: (i) operator $\{\mathfrak{L}_T, \mathfrak{B}_T\}$ (operator $\{\mathfrak{L}_{T*}, \mathfrak{B}_{T*}\}$) coincides with $\{\mathcal{L}, \mathcal{B}\}$ (with $\{\mathcal{L}_*, \mathcal{B}_*\}$) on the set $\{(y,t) \in \overline{\Pi} : t > T+3\}$ and with $\{L, B\}$ (with $\{L_*, B_*\}$) on the set $\{(y,t) \in \overline{\Pi} : t < T\}$; (ii) the norms $\|\Delta_T; \mathcal{D}^l_\beta(\Pi) \to \mathcal{R}^l_\beta(\Pi)\|$ and $\|\Delta_{T*}; \mathcal{D}^l_\beta(\Pi)_* \to \mathcal{R}^l_\beta(\Pi)_*\|$ of the operators

$$\Delta_T = \{L, B\} - \{\mathfrak{L}_T, \mathfrak{B}_T\}, \quad \Delta_{T*} = \{L_*, B_*\} - \{\mathfrak{L}_{T*}, \mathfrak{B}_{T*}\} \tag{4.11}$$

tend to zero as $T \to +\infty$; (iii) for $u, v \in C^\infty_c(\overline{\Pi})$ and sufficiently large T the Green Formula

$$(\mathfrak{L}_T u, v)_\Pi + (\mathfrak{B}_T u, \mathfrak{Q}_T v)_{\partial\Pi} - (u, \mathfrak{L}_{T*} v)_\Pi - (\mathfrak{Q}_{T*} u, \mathfrak{B}_{T*} v)_{\partial\Pi} = 0 \tag{4.12}$$

holds, where $\mathfrak{Q}_T$ and $\mathfrak{Q}_{T*}$ are some $m \times k$-matrices of differential operators such that $\mathfrak{Q}_T$ ($\mathfrak{Q}_{T*}$) coincides with $\mathcal{Q}$ (with $\mathcal{Q}_*$) on the set $\{(y,t) \in \overline{\Pi} : t > T+3\}$ and with Q (with Q_*) on the set $\{(y,t) \in \overline{\Pi} : t < T\}$.

In particular, if $s_1 = s_2 = \cdots = s_k = 0$ and the matrix $\mathcal{B}$ of boundary conditions is t-normal (see e.g. [21]) then the Green formula (4.4) is fulfilled, the existence of the operators $\{\mathfrak{L}_T, \mathfrak{B}_T\}$ and $\{\mathfrak{L}_{T*}, \mathfrak{B}_{T*}\}$ is guarantied by the following (rather strong) stabilization conditions:

$$\begin{gathered}\lim_{T\to+\infty} \|\eta_T(\ell^{\nu\mu}_{ij} - \mathbf{l}^{\nu\mu}_{ij}); C^\infty(\overline{\Pi}_+)\| = 0, \quad |\nu| + \mu \leqslant t_j,\\ \lim_{T\to+\infty} \|\eta_T(b^{\nu\mu}_{qj} - \mathbf{b}^{\nu\mu}_{qj}); C^\infty(\partial\Omega \times \mathbb{R}_+)\| = 0, \quad |\nu| + \mu \leqslant \sigma_q + \tau_j,\end{gathered} \tag{4.13}$$

where $\ell^{\nu\mu}_{ij}$ and $b^{\nu\mu}_{qj}$ are the coefficients of the operators

$$\begin{aligned}\mathcal{L}_{ij}(y,t,D_y,D_t) &= \sum_{|\eta|+\mu\leqslant\tau_j} \ell^{\eta\mu}_{ij}(y,t) D^\eta_y D^\mu_t,\\ B_{qj}(y,t,D_y,D_t) &= \sum_{|\eta|+\mu\leqslant\sigma_q+\tau_j} b^{\eta\mu}_{qj}(y,t) D^\eta_y D^\mu_t,\end{aligned}$$

and $\mathbf{l}^{\nu\mu}_{ij}$, $\mathbf{b}^{\nu\mu}_{qj}$ are the correspondent limit coefficients. The formally self-adjoint problems under the stabilization conditions (4.13) are studied in [25]; the standard method for the Green formula construction permits one to get needed $\{\mathfrak{L}_T, \mathfrak{B}_T\}$ and $\{\mathfrak{L}_{T*}, \mathfrak{B}_{T*}\}$ in the nonself-adjoint case (see e.g. [26, 27, 28, 29]). We do not describe the conditions ensuring the existence of $\{\mathfrak{L}_T, \mathfrak{B}_T\}$ and $\{\mathfrak{L}_{T*}, \mathfrak{B}_{T*}\}$ in a more general situation. In what follows we suppose that such operators exist (and the above conditions (i)–(iii) are fulfilled). If the initial problem (4.1) is formally self-adjoint (i.e. $\mathcal{L} = \mathcal{L}_*$, $\mathcal{B} = \mathcal{B}_*$, and $\mathcal{Q} = \mathcal{Q}_*$) then we assume in addition that the operator $\{\mathfrak{L}_T, \mathfrak{B}_T\}$ is formally self-adjoint.

In the cylinder Π we consider the model problem

$$\begin{aligned}\mathfrak{L}_T(y,t,D_y,D_t)u(y,t) &= \mathfrak{F}(y,t), \quad (y,t) \in \Pi,\\ \mathfrak{B}_T(y,t,D_y,D_t)u(y,t) &= \mathfrak{G}(y,t), \quad (y,t) \in \partial\Pi.\end{aligned} \tag{4.14}$$

Proposition 4.2. *Let the line $\mathbb{R} + i\beta$ contains no eigenvalues of the pencil $\mathfrak{A}$. Then for sufficiently large T the operator $\{\mathfrak{L}_T, \mathfrak{B}_T\} : \mathcal{D}^\ell_\beta(\Pi) \to \mathcal{R}^\ell_\beta(\Pi)$ of the problem (4.14) implements an isomorphism.*

Proof. Since the norm of the operator Δ_T tends to zero as $T \to +\infty$, the assertion follows from Proposition 4.1. □

For every eigenvalue λ_ν of $\mathfrak{A}$ we choose a real number α_ν such that the strip $\alpha_\nu \leqslant \operatorname{Im}\lambda < \operatorname{Im}\lambda_\nu$ is free from the spectrum of $\mathfrak{A}$. For sufficiently large T we set

$$z_\nu^{(\sigma,j)} = u_\nu^{(\sigma,j)} + \sum_{q=1}^{\infty} (A(\alpha_\nu)^{-1}\Delta_T)^q u_\nu^{(\sigma,j)}, \tag{4.15}$$

where the functions $u_\nu^{(\sigma,j)}$ are given in (4.7). Let us discuss the equality (4.15). Note that $\eta_{T-2}u_j^\pm \in \mathcal{D}^l_{\alpha_\nu}(\Pi)$ and $\Delta_T u_j^\pm = \Delta_T \eta_{T-2} u_j^\pm$. The norm of operator $A(\alpha_\nu)^{-1}\Delta_T : \mathcal{D}^l_{\alpha_\nu}(\Pi) \to \mathcal{D}^l_{\alpha_\nu}(\Pi)$ is small. The series $\sum_{q=1}^{\infty}(A(\alpha_\nu)^{-1}\Delta_T)^q u_j^\pm$ converges in the norm of $\mathcal{D}^l_{\alpha_\nu}(\Pi)$. Consequently,

$$z_\nu^{(\sigma,j)} = u_\nu^{(\sigma,j)} \mod \mathcal{D}^l_{\alpha_\nu}(\Pi). \tag{4.16}$$

The functions $z_\nu^{(\sigma,j)}$ do not depend on the choice of α_ν; indeed, $A(\alpha)^{-1}\{F, G\} = A(\beta)^{-1}\{F, G\}$ provided that the strip $\alpha \leqslant \operatorname{Im}\lambda \leqslant \beta$ is free from the spectrum of the pencil $\mathfrak{A}$ and $\{F, G\} \in \mathcal{R}^l_\alpha(\Pi) \cap \mathcal{R}^l_\beta(\Pi)$ (see e.g. [2, Proposition 3.1.4]).

The next assertion is a version of Theorem 6.2 from [23]; see also [24, Theorem 8.5.7].

Theorem 4.3. *(i) The functions $z_\nu^{(\sigma,j)}$ defined by (4.15) are linearly independent solutions to the homogeneous problem (4.14).*
(ii) Assume that $\beta > \gamma$ and the lines $\mathbb{R} + i\beta$ and $\mathbb{R} + i\gamma$ contain no eigenvalues of the pencil $\mathfrak{A}$. Let $\lambda_1, \dots, \lambda_M$ be all the eigenvalues of $\mathfrak{A}$ in the strip $\{\lambda \in \mathbb{C} : \gamma < \operatorname{Im}\lambda < \beta\}$. Then a solution $u \in \mathcal{D}^\ell_\gamma(\Pi)$ to the problem (4.18) with right-hand side $\{\mathfrak{F}, \mathfrak{G}\} \in \mathcal{R}^\ell_\beta(\Pi) \cap \mathcal{R}^\ell_\gamma(\Pi)$ admits the representation

$$u = \sum_{\nu=1}^{M} \sum_{j=1}^{J_\nu} \sum_{\sigma=0}^{\varkappa_{j\nu}-1} d_\nu^{(\sigma,j)}\, z_\nu^{(\sigma,j)} + v, \tag{4.17}$$

where v is a solution to the same problem in $\mathcal{D}^l_\beta(\Pi)$, and $d_\nu^{(\sigma,j)}$ are some complex coefficients.

Define the solution $w_\nu^{(\sigma,j)}$ to the homogeneous problem $\{\mathfrak{L}_{T*}, \mathfrak{B}_{T*}\}v = 0$ in Π by the formula

$$w_\nu^{(\sigma,j)} = v_\nu^{(\sigma,j)} + \sum_{q=1}^{\infty} (A_*(-\tau_\nu)^{-1}\Delta_{T*})^q v_\nu^{(\sigma,j)}, \tag{4.18}$$

where $A_*(\beta) = \{L_*, B_*\} : \mathcal{D}^\ell_\beta(\Pi)_* \to \mathcal{R}^\ell_\beta(\Pi)_*$, and τ_ν is such a number that the strip $\tau_\nu \leqslant \operatorname{Im}\lambda < \operatorname{Im}\overline{\lambda}_\nu$ is free from the spectrum of $\mathfrak{A}_*$ ($\overline{\lambda}_\nu$ is an eigenvalue of $\mathfrak{A}_*$). Denote by $p_T(u, v)$ the left-hand side of the Green formula (4.12).

Lemma 4.4. *Let the functions* $u_\nu^{(\sigma,j)}$ *and* $v_\nu^{(\sigma,j)}$ *in (4.15) and (4.18) be subjected to the conditions (4.10). Then*

$$p_T(\chi z_\nu^{(\sigma,j)}, \chi w_\mu^{(\tau,p)}) = i\delta_{\mu,\nu}\delta_{j,p}\delta_{\varkappa_{p\mu}-\tau-1,\sigma}, \tag{4.19}$$

where $\chi \in C^\infty(\mathbb{R})$, $\chi(t) = 0$ *for* $t < 0$ *and* $\chi(t) = 1$ *for* $t > 1$. *The equalities (4.19) do not depend on the choice of* χ.

Proof. First we prove that $p_T(\chi z_\nu^{(\sigma,j)}, \chi w_\mu^{(\tau,p)}) = 0$ if $\operatorname{Im}(\lambda_\nu + \overline{\lambda}_\mu) \neq 0$.

Let $\operatorname{Im}(\lambda_\nu + \overline{\lambda}_\mu) > 0$. In this case one can choose the numbers α_ν and τ_μ in (4.15) and (4.18) such that $\alpha_\nu + \tau_\mu > 0$. It is easy to see that $\chi z_\nu^{(\sigma,j)} \in \mathcal{D}^l_{-\tau_\mu}(\Pi)$ and $\chi w_\mu^{(\tau,p)} \in \mathcal{D}^l_{-\tau_\mu}(\Pi)_*$. Since the Green formula (4.12) can be extended by continuity to the functions $u \in \mathcal{D}^l_{-\tau_\mu}(\Pi)$ and $v \in \mathcal{D}^l_{-\tau_\mu}(\Pi)_*$, we have $p_T(\chi z_\nu^{(\sigma,j)}, \chi w_\mu^{(\tau,p)}) = 0$.

Let us consider the case $\operatorname{Im}(\lambda_\nu + \overline{\lambda}_\mu) < 0$. We choose β_ν and β_μ such that $\beta_\nu > \operatorname{Im}\lambda_\nu$, $\beta_\mu > \operatorname{Im}\overline{\lambda}_\mu$, and $\beta_\nu + \beta_\mu < 0$. Then the inclusions $(1-\chi)z_\nu^{(\sigma,j)} \in \mathcal{D}^l_{-\beta_\mu}(\Pi)$ and $(1-\chi)w_\mu^{(\tau,p)} \in \mathcal{D}^l_{-\beta_\mu}(\Pi)_*$ are fulfilled; see (4.16). For $u := (1-\chi)z_\nu^{(\sigma,j)}$ and $v := (1-\chi)w_\mu^{(\tau,p)}$ the Green formula (4.12) holds. This implies

$$p_T((1-\chi)z_\nu^{(\sigma,j)}, (1-\chi)w_\mu^{(\tau,p)}) = 0. \tag{4.20}$$

Using the equality $p_T(z_\nu^{(\sigma,j)}, w_\mu^{(\tau,p)}) = 0$ and the Green formula (4.12), we transform the left-hand side of (4.20). We have

$$p_T((1-\chi)z_\nu^{(\sigma,j)}, (1-\chi)w_\mu^{(\tau,p)}) = p_T((1-\chi)z_\nu^{(\sigma,j)}, w_\mu^{(\tau,p)}) = -p_T(\chi z_\nu^{(\sigma,j)}, \chi w_\mu^{(\tau,p)}). \tag{4.21}$$

Thus $p_T(\chi z_\nu^{(\sigma,j)}, \chi w_\mu^{(\tau,p)}) = 0$ if $\operatorname{Im}(\lambda_\nu + \overline{\lambda}_\mu) \neq 0$.

Now we suppose that $\operatorname{Im}(\lambda_\nu + \overline{\lambda}_\mu) = 0$. It is easily seen that

$$\begin{aligned} &\mathcal{D}^\ell_{-\tau_\mu}(\Pi) \ni (1-\chi)z_\nu^{(\sigma,j)} = (1-\chi)u_\nu^{(\sigma,j)} \quad \operatorname{mod}\ \mathcal{D}^\ell_{\alpha_\nu}(\Pi), \\ &\mathcal{D}^\ell_{\alpha_\nu}(\Pi)_* \ni (1-\chi)w_\mu^{(\tau,p)} = (1-\chi)v_\mu^{(\tau,p)} \quad \operatorname{mod}\ \mathcal{D}^\ell_{-\tau_\mu}(\Pi)_*. \end{aligned}$$

From these relations it follows that

$$q((1-\chi)z_\nu^{(\sigma,j)}, (1-\chi)w_\mu^{(\tau,p)}) = q((1-\chi)u_\nu^{(\sigma,j)}, (1-\chi)v_\mu^{(\tau,p)}). \tag{4.22}$$

Owing to the equality $q(u_\nu^{(\sigma,j)}, v_\mu^{(\tau,p)}) = 0$, the right-hand side of (4.22) is equal to $-q(\chi u_\nu^{(\sigma,j)}, \chi v_\mu^{(\tau,p)})$. Taking into account (4.10), we have

$$q((1-\chi)z_\nu^{(\sigma,j)}, (1-\chi)w_\mu^{(\tau,p)}) = -i\delta_{\mu,\nu}\delta_{j,p}\delta_{\varkappa_{p\mu}-\tau-1,\sigma}. \tag{4.23}$$

Note that the left-hand side of (4.23) coincides with the left-hand side of (4.21) (because every differential operator in (4.12) coincides with correspondent operator from (4.9) on the support of $(1-\chi)z_\nu^{(\sigma,j)}$). The equalities (4.19) are readily apparent from (4.23) and (4.21). □

Corollary 4.5. *The coefficients* $d_\nu^{(\sigma,j)}$ *in (4.17) can be found by the formula*

$$d_\nu^{(\sigma,j)} = -i\{(\mathfrak{F}, w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)})_\Pi + (\mathfrak{G}, \mathfrak{Q}_T w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)})_{\partial\Pi}\}, \quad \nu = 1, \dots, M, \quad (4.24)$$

where $w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)}$ *is defined by (4.18), and* $\mathfrak{Q}_T$ *is the same as in the Green formula (4.12).*

Proof. Due to the equality $\{\mathfrak{L}_{T*}, \mathfrak{B}_{T*}\}w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)} = 0$ we get

$$(\mathfrak{F}, w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)})_\Pi + (\mathfrak{G}, \mathfrak{Q}_T w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)})_{\partial\Pi} = p_T(u, w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)}), \quad (4.25)$$

where u is from (4.17). The inclusions $(1-\chi)w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)} \in \mathfrak{D}_\gamma^\ell(\Pi)_*$ and $u \in \mathfrak{D}_\gamma^\ell(\Pi)$ imply $p_T(u, (1-\chi)w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)}) = 0$; here χ is the same as in Lemma 4.4. Together with (4.17) this allows us to write the right-hand side of (4.25) in the form

$$p_T(\sum_{\nu=1}^{M}\sum_{j=1}^{J_\nu}\sum_{\sigma=0}^{\varkappa_{j\nu}-1} d_\nu^{(\sigma,j)}\, z_\nu^{(\sigma,j)} + v, \chi w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)}).$$

Note that $p_T(v, \chi w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)}) = 0$ as far as $v \in \mathfrak{D}_\beta^\ell(\Pi)$ and $\chi w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)} \in \mathfrak{D}_\beta^\ell(\Pi)_*$. Finally we have

$$\begin{aligned}(\mathfrak{F}, w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)})_\Pi &+ (\mathfrak{G}, \mathfrak{Q}_T w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)})_{\partial\Pi} \\ &= p_T(\sum_{\nu=1}^{M}\sum_{j=1}^{J_\nu}\sum_{\sigma=0}^{\varkappa_{j\nu}-1} d_\nu^{(\sigma,j)}\, z_\nu^{(\sigma,j)}, \chi w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)}) \\ &= p_T(\sum_{\nu=1}^{M}\sum_{j=1}^{J_\nu}\sum_{\sigma=0}^{\varkappa_{j\nu}-1} d_\nu^{(\sigma,j)}\, \chi z_\nu^{(\sigma,j)}, \chi w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)}).\end{aligned}$$

By applying Lemma 4.4, we complete the proof. □

As was mentioned in the beginning of Appendix, the results of Theorem 4.3, Lemma 4.4, and Corollary 4.5 allow us to extend the theory of elliptic problems with exponentially stabilizing coefficients (see e.g. [1, 2]) to the case of stabilization in the sense of (4.3), (4.5). The idea is to use the structure of solutions (4.17) and the formulas for the coefficients (4.24) instead of the asymptotic representations and the corresponding formulas for the coefficients in the asymptotics. Thus all the assertions and their proofs from [1, 2] can be easily adapted for the problems with slowly stabilizing coefficients.

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